

A Short Introduction to Monte Carlo Methods in Financial Mathematics

Workshop

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¹This talk represents the views of the author alone, and not the views of BofA Securities, Inc., Citigroup, or any of her previous employers.

Contents

1. Welcome

2. Basic Concepts

3. Stochastic Processes

Objectives

For the short course:

- Get you interested on financial mathematics and maybe on pursuing a career as a Quant after graduating or after postgraduate education
- Show you the type of problems we encounter so you can get a flavour of the job that quants do everyday
- Show you what kind of mathematical tools are required

For today:

- Recall concepts on Probability and Stochastic Processes Theory
 - ▶ Random Variables, Stochastic Processes
 - ▶ Brownian Motion
 - ▶ Ito's formula
 - ▶ Geometric Brownian Motion
- See how these concepts are translated into code in Python
- Take a first look at financial time series in Python

Random Variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a **probability space** and (S, Σ) a **measurable space**. Then, an (S, Σ) **random variable** is a **measurable function**

$$X : \Omega \rightarrow S,$$

which means that, for every subset $B \in \Sigma$, its pre-image is \mathcal{F} -measurable, i.e.;

$$X^{-1}(B) \in \mathcal{F},$$

where

$$X^{-1}(B) = \{\omega : X(\omega) \in B\}.$$

- Typically, $S = \mathbb{R}^d$ for some $d \geq 1$, and $\Sigma = \mathcal{B}(\mathbb{R}^d)$ is the corresponding Borel sigma-algebra.
- Examples
 - ▶ Discrete random variables: **Bernoulli, Binomial, Poisson**
 - ▶ Continuous random variables: **Uniform, Gaussian, Log-normal, t-Student**

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Stochastic Process

For given **probability space** (Ω, \mathcal{F}, P) and **measurable space** (S, Σ) , a **stochastic process** is a collection of S -valued random variables

$$\{X_t : t \in I\},$$

where the set I is called index set.

- Typically, $S = \mathbb{R}^d$ for some $d \geq 1$ and $\Sigma = \mathcal{B}(\mathbb{R}^d)$ is the corresponding Borel sigma-algebra
- The index set can be discrete, e.g. $I = \mathbb{N}$, or continuous e.g. $I = [0, T]$ for some $T \geq 0$.

Brownian Motion or Wiener process

A **standard Brownian motion**, or **Wiener process**, is a stochastic process $\{W_t : t \geq 0\}$ characterised by the following four properties:

- 1 $W_0 = 0$
- 2 W_t has independent increments
- 3 $W_t - W_s \sim \mathcal{N}(0, t - s)$ for any $0 \leq s \leq t$
- 4 W_t is almost surely continuous

Here $\mathcal{N}(\mu, \sigma^2)$ denotes the normal or Gaussian distribution with given mean μ and variance σ .

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Ito's Lemma

Suppose that $X = \{X_t : t \geq 0\}$ is a stochastic **process** which satisfies the following **stochastic differential equation (SDE)**

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \geq 0,$$

i.e.

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \geq 0,$$

where W denotes a standard Brownian motion. Let **$f(t, X)$ be a twice differentiable function**. Then the process $Y = \{Y_t = f(t, X_t), t \geq 0\}$ satisfies the following SDE

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t,$$

i.e.

$$f(t, X_t) = f(0, X_0) + \int_0^t \left(\frac{\partial f}{\partial s} + \mu_s \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2} \right) ds + \int_0^t \sigma_s \frac{\partial f}{\partial x} dW_s, \quad t \geq 0.$$

Geometric Brownian Motion

A **geometric Brownian motion** is a stochastic process defined by the following SDE

$$dS_t = \mu S_t dt + \sigma S_t W_t, \quad t > 0, \quad (1)$$

where $S_0 = s_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, are constants; and W is a standard Brownian motion.

Solution: Let us set a new process as $X_t = \log(S_t)$. Using Ito's formula, we obtain

$$X_t = X_0 + \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t,$$

or equivalently

$$\log(S_t) = \log(s_0) + \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t.$$

Note that the last expression implies that $\log(S_t)$ follows a **normal distribution** $\mathcal{N} \left(\log(s_0) + \left(\mu - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right)$. This, in turn implies that

$$S_t = s_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}, \quad \forall t > 0,$$

follows a **log-normal distribution**.

Python Time

To finish this session let's take a look at financial time series in **Python**

Many thanks for your attention
See you tomorrow!