

# A Short Introduction to Monte Carlo Methods in Financial Mathematics

## Workshop Part 1

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<sup>1</sup>This talk represents the views of the author alone, and not the views of BofA Securities, Inc., Citigroup, or any of her previous employers.

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2. Basic Concepts

3. Stochastic Processes

# Objectives

For the short course:

- Get you interested on financial mathematics and maybe on pursuing a career as a Quant after graduating or after postgraduate education
- Show you the type of problems we encounter so you can get a flavour of the job that quants do everyday. In particular, we aim to price an European option under the Black-Scholes model by two methods: obtaining the analytical formula, and using Monte Carlo
- Show you what kind of mathematical tools are required and illustrate how these concepts are translated into code in Python

For today:

- Recall concepts on Probability and Stochastic Processes Theory
  - ▶ Random Variables, Stochastic Processes
  - ▶ Brownian Motion
  - ▶ Ito's formula
  - ▶ Geometric Brownian Motion
- See how these concepts are translated into code in Python
- Take a first look at financial time series in Python

## Random Variables

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a **probability space** and  $(S, \Sigma)$  a **measurable space**. Then, an  $(S, \Sigma)$  **random variable** is a **measurable function**

$$X : \Omega \rightarrow S,$$

which means that, for every subset  $B \in \Sigma$ , its pre-image is  $\mathcal{F}$ -measurable, i.e.;

$$X^{-1}(B) \in \mathcal{F},$$

where

$$X^{-1}(B) = \{\omega : X(\omega) \in B\}.$$

- Typically,  $S = \mathbb{R}^d$  for some  $d \geq 1$ , and  $\Sigma = \mathcal{B}(\mathbb{R}^d)$  is the corresponding Borel sigma-algebra.
- Examples
  - ▶ Discrete random variables: **Bernoulli, Binomial, Poisson**
  - ▶ Continuous random variables: **Uniform, Gaussian, Log-normal, t-Student**

# Python Time

## Stochastic Process

For given **probability space**  $(\Omega, \mathcal{F}, P)$  and **measurable space**  $(S, \Sigma)$ , a **stochastic process** is a collection of  $S$ -valued random variables

$$\{X_t : t \in I\},$$

where the set  $I$  is called index set.

- Typically,  $S = \mathbb{R}^d$  for some  $d \geq 1$  and  $\Sigma = \mathcal{B}(\mathbb{R}^d)$  is the corresponding Borel sigma-algebra
- The index set can be discrete, e.g.  $I = \mathbb{N}$ , or continuous e.g.  $I = [0, T]$  for some  $T \geq 0$ .

## Brownian Motion or Wiener process

A **standard Brownian motion**, or **Wiener process**, is a stochastic process  $\{W_t : t \geq 0\}$  characterised by the following four properties:

- 1  $W_0 = 0$
- 2  $W_t$  has independent increments
- 3  $W_t - W_s \sim \mathcal{N}(0, t - s)$  for any  $0 \leq s \leq t$
- 4  $W_t$  is almost surely continuous

Here  $\mathcal{N}(\mu, \sigma^2)$  denotes the normal or Gaussian distribution with given mean  $\mu$  and variance  $\sigma$ .

# Python Time



## Ito's Lemma

Suppose that  $X = \{X_t : t \geq 0\}$  is a stochastic **process** which satisfies the following **stochastic differential equation (SDE)**

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad t \geq 0,$$

i.e.

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \geq 0,$$

where  $W$  denotes a standard Brownian motion. Let  **$f(t, X)$  be a twice differentiable function**. Then the process  $Y = \{Y_t = f(t, X_t), t \geq 0\}$  satisfies the following SDE

$$df(t, X_t) = \left( \frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t,$$

i.e.

$$f(t, X_t) = f(0, X_0) + \int_0^t \left( \frac{\partial f}{\partial s} + \mu_s \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2} \right) ds + \int_0^t \sigma_s \frac{\partial f}{\partial x} dW_s, \quad t \geq 0.$$

## Geometric Brownian Motion

A **geometric Brownian motion** is a stochastic process defined by the following SDE

$$dS_t = \mu S_t dt + \sigma S_t W_t, \quad t > 0, \quad (1)$$

where  $S_0 = s_0 > 0$ ,  $\mu \in \mathbb{R}$ ,  $\sigma > 0$ , are constants; and  $W$  is a standard Brownian motion.

**Solution:** Let us set a new process as  $X_t = \log(S_t)$ . Using Ito's formula, we obtain

$$X_t = X_0 + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t,$$

or equivalently

$$\log(S_t) = \log(s_0) + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t.$$

Note that the last expression implies that  $\log(S_t)$  follows a **normal distribution**  $\mathcal{N} \left( \log(s_0) + \left( \mu - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right)$ . This, in turn implies that

$$S_t = s_0 \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right\}, \quad \forall t > 0,$$

follows a **log-normal distribution**.

# Python Time

To finish this session let's take a look at financial time series in Python

Many thanks for your attention  
See you tomorrow!