

A Short Introduction to Monte Carlo Methods in Financial Mathematics

Workshop

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¹This talk represents the views of the author alone, and not the views of BofA Securities, Inc., Citigroup, or any of her previous employers.

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Objectives

For the short course:

- Get you interested on financial mathematics and maybe on pursuing a career as a Quant after graduating or after postgraduate education
- Show you the type of problems we encounter so you can get a flavour of the job that quants do everyday
- Show you what kind of mathematical tools are required

For today:

- Review the Black-Scholes Model for option pricing
- Review the definition of Monte Carlo
- See how these concepts are translated into code in Python
- Calculate the price of an Asian option using Monte Carlo

European Options

- A **European Option** is a contract which conveys to its owner the right, but not the obligation, to buy (**Call**) or sell (**Put**) an **underlying asset or instrument** at a specified **strike** price K on a **expiry** date T .
- The **payoff** is given by

$$\text{Call Payoff} = (S_T - K)^+$$

$$\text{Put Payoff} = (K - S_T)^+,$$

where K is the strike price, and S_T is the price of the underlying asset at expiry.

Note: The key difference between American and European options relates to when the options can be exercised: An American option may be exercised either at expiry or at any time before it.

Black-Scholes Model

Assumptions

- 1 The risk-free interest rate r is known and constant through time
- 2 The stock price S follows a geometric Brownian motion with constant parameters μ and σ
- 3 Stock pays no dividends
- 4 The option can only be exercised at expiration i.e. it is of European type
- 5 There are no transaction costs
- 6 Fractional trading is possible

Black-Scholes Equation

The idea is to construct a **self-financing portfolio** with an option and Δ units of the underlying **stock**, i.e.

$$\Pi_t = V_t + \Delta S_t, \quad t \geq 0,$$

which satisfies the following two conditions:

- The portfolio is riskless
- The portfolio earns the risk free rate

Under these assumptions, we obtain a second order parabolic partial differential equation (PDE) with boundary condition

$$\begin{aligned} \frac{\partial V}{\partial t} + r^d S_t \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r V_t &= 0, & t \in [0, T] \\ V_T &= (S_T - K)^+. \end{aligned}$$

We will see the full derivation in the Jupyter Notebook.

Solution or Black-Scholes Formula

The solution, which is known as **Black and Scholes formula**, is given by

$$V(S_t, t) = \Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)},$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right]$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$

and Φ denotes the cumulative distribution function (cdf) of a standard normal distribution.

Python Time

Monte Carlo Integration

Let X be either a discrete random variable taking values in a countable or finite set Ω , with probability mass function (p.m.f.) f_X , or a continuous random variable taking values in $\Omega = \mathbb{R}^d$, with probability density function (p.d.f) f_X . Consider

$$\theta = \mathbb{E}[\phi(X)] = \begin{cases} \sum_{x \in \Omega} \phi(x) f_X(x) & \text{if } X \text{ is discrete} \\ \int_{\mathbb{R}^d} \phi(x) f_X(x) dx & \text{if } X \text{ is continuous,} \end{cases} \quad (1)$$

where $\phi : \Omega \rightarrow \mathbb{R}$. Let X_1, \dots, X_n be i.i.d. random variables with p.m.f. (or p.d.f. in the continuous case) f_X . Then

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n \phi(X_i), \quad (2)$$

is called the **Monte Carlo estimator** of the expectation θ .

Monte Carlo methods can be thought of as a stochastic way to approximate integrals.

Algorithm 1 Monte Carlo

- 1: Simulate independent X_1, \dots, X_n from a random variable with distribution f_X
 - 2: Return $\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n \phi(X_i)$.
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Monte Carlo Estimate Properties

If $\theta = \mathbb{E}[\phi(X)]$ exists, then the Monte Carlo estimator $\hat{\theta}_n$ has the following properties

- Unbiasedness

$$\mathbb{E}[\hat{\theta}_n] = \theta,$$

- Strong consistency

$$\hat{\theta}_n \rightarrow \theta \text{ almost surely as } n \rightarrow \infty.$$

Proof: Note that the linearity of expectation implies that

$$\mathbb{E}[\hat{\theta}_n] = \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \phi(X_i) \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\phi(X_i)] = \theta.$$

Strong consistency is a consequence of the strong law of large numbers applied to $Y_i = \phi(X_i)$.

How Good is the Estimate?

- Chebyshev's inequality implies

$$\mathbb{P}\left(|\hat{\theta}_n - \theta| > c \frac{\sigma}{\sqrt{n}}\right) \leq \frac{n\mathbb{V}[\hat{\theta}_n]}{c^2\sigma^2} = \frac{1}{c^2} \quad (3)$$

- The Central Limit Theorem implies that for large n

$$\frac{\sqrt{n}}{\sigma}(\hat{\theta}_n - \theta) \sim \mathcal{N}(0, 1),$$

and then

$$\mathbb{P}\left(|\hat{\theta}_n - \theta| > c \frac{\sigma}{\sqrt{n}}\right) \approx 2(1 - \Phi(c)).$$

Hence, by choosing c_α such that $2(1 - \Phi(c)) = \alpha$, we obtain an approximate $(1 - \alpha)100\%$ confidence interval for θ as follows

$$\left(\hat{\theta}_n \pm c_\alpha \frac{\sigma}{\sqrt{n}}\right) \approx \left(\hat{\theta}_n \pm c_\alpha \frac{S_{\phi(X)}}{\sqrt{n}}\right)$$

Python time

Monte Carlo in Action

Why do we need Monte Carlo if we have a closed formula in the B-S model?

- Multi-asset instruments
- More complex products such as **Asian options**
- Going beyond the Black-Scholes simple model. See for example the **various models** for interest rates.

Many thanks for your attention!