A Short Introduction to Monte Carlo Methods in Financial Mathematics

Workshop Part 2

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[1] This talk represents the views of the author alone, and not the views of BofA Securities, Inc., Citigroup, or any of her previous employers.

Contents

Objectives

For the short course:

- Get you interested on financial mathematics and maybe on pursuing a career as a Quant after graduating or after postgraduate education
- Show you the type of problems we encounter so you can get a flavour of the job that quants do everyday. In particular, we aim to price an European option under the Black-Scholes model by two methods: obtaining the analytical formula, and using Monte Carlo
- Show you what kind of mathematical tools are required and illustrate how these concepts are translated into code in Python

For today:

- Review the Black-Scholes Model for option pricing
- Review the definition of Monte Carlo
- See how these concepts are translated into code in Python
- Calculate the price of an Asian option using Monte Carlo

Options

European Options

- A European Option is a contract which conveys to its owner the right, but not the obligation, to buy (Call) or sell (Put) an underlying asset or instrument at a specified strike price K on a expiry date T.
- The payoff is given by

$$Call\ \ Payoff = (S_T - K)^+$$

$$Put\ \ Payoff = (K - S_T)^+,$$

where K is the strike price, and S_T is the price of the underlying asset at expiry.

Note: The key difference between American and European options relates to when the options can be exercised: An American option may be exercised either at epiry or at any time before it.

Black-Scholes Model

Assumptions

1. The risk-free interest rate r is known and constant through time
2. The stock price S follows a geometric Brownian motion with constant parameters μ and σ
3. Stock pays no dividends
4. The option can only be exercised at expiration i.e. it is of European type
5. There are no transaction costs
6. Fractional trading is possible

Black-Scholes Equation

The idea is to construct a self-financing portfolio with an option and Δ units of the underlying stock, i.e.

$$\Pi_t = V_t + \Delta S_t, \qquad t \geq 0,$$

which satisfies the following two conditions:

- The portfolio is riskless
- The portfolio earns the risk free rate

Under these assumptions, we obtain a second order parabollic partial differential equation (PDE) with boundary condition

$$\frac{\partial V}{\partial t} + r^d S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV_t = 0, \qquad t \in [0, T]$$
$$V_T = (S_T - K)^+.$$

We will see the full derivation inthe Jupyter Notebook.

Solution or Black-Scholes Formula

The solution, which is known as Black and Scholes formula, is given by

$$V(S_t, t) = \Phi(d_1)S_t - \Phi(d_2)Ke^{-r(T-t)},$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$

and Φ denotes the cumulative distribution function (cdf) of a standard normal distribution.

Python Time

Monte Carlo 101

Monte Carlo Integration

Let X be either a discrete random variable taking values in a countable or finite set Ω, with probability mass function (p.m.f.) f_X, or a continuous random variable taking values in $\Omega = \mathbb{R}^d$, with probability density function (p.d.f) f_X. Consider

$$\theta = \mathbb{E}[\phi(X)] = \begin{cases} \sum_{x\in\Omega} \phi(x) f_X(x) & \text{if X is discrete} \\ \int_{\mathbb{R}^d} \phi(x) f_x(x) dx & \text{if X is continuous,} \end{cases} \tag{1}$$

where $\phi : \Omega \to \mathbb{R}$. Let $X_1, \cdots, X_n$ be i.i.d. random variables with p.m.f. (or p.d.f. in the continuous case) f_X. Then

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} \phi(X_i), \tag{2}$$

is called the Monte Carlo estimator of the expectation θ.

Monte Carlo 101

Monte Carlo methods can be thought of as a stochastic way to approximate integrals.

Algorithm 1 Monte Carlo

1: Simulate independent $X_1, \cdots, X_n$ from a random variable with distribution f_X
2: Return $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)$.

Monte Carlo Estimate Properties

If $\theta = \mathbb{E}[\phi(X)]$ exists, then the Monte Carlo estimator $\hat{\theta}_n$ has the following properties

- Unbiasedness
$$\mathbb{E}[\hat{\theta}_n] = \theta,$$
- Strong consistency
$$\hat{\theta}_n \to \theta \text{ almost surely as } n \to \infty.$$

Proof: Note that the linerity of ht expectation implies that

$$\mathbb{E}[\hat{\theta}_n] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(X_i)\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\phi(X_i)\right] = \theta.$$

Strong consistency is a consequence of the strong law of large numbers applied to $Y_i = \phi(X_i)$.

Monte Carlo 101

How Good is the Estimate?

- Chebyshev's inequality implies

$$\mathbb{P}\left(|\hat{\theta}_n - \theta| > c\frac{\sigma}{\sqrt{n}}\right) \leq \frac{n\mathbb{V}[\hat{\theta}_n]}{c^2\sigma^2} = \frac{1}{c^2} \tag{3}$$

- The Central Limit Theorem implies that for large n

$$\frac{\sqrt{n}}{\sigma}(\hat{\theta}_n - \theta) \sim \mathcal{N}(0,1),$$

and then

$$\mathbb{P}\left(|\hat{\theta}_n - \theta| > c\frac{\sigma}{\sqrt{n}}\right) \approx 2(1-\Phi(c)).$$

Hence, by choosing c_α such that $2(1-\Phi(c)) = \alpha$, we obtain an approximate $(1-\alpha)100\%$ confidence interval for θ as follows

$$\left(\hat{\theta}_n \pm c_\alpha\frac{\sigma}{\sqrt{n}}\right) \approx \left(\hat{\theta}_n \pm c_\alpha\frac{S_{\phi(X)}}{\sqrt{n}}\right)$$

Python time

Monte Carlo in Action

Why do we need Monte Carlo if we have a closed formula in the B-S model?

- Multi-asset instruments
- More complex products such as Asian options
- Going beyond the Black-Scholes simple model. See for example the various models for interest rates.

Many thanks for your attention!