A Short Introduction to Monte Carlo Methods in Financial Mathematics

Workshop Part 1

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[1]This talk represents the views of the author alone, and not the views of BofA Securities, Inc., Citigroup, or any of her previous employers.

Contents

Objectives

For the short course:

- Get you interested on financial mathematics and maybe on pursuing a career as a Quant after graduating or after postgraduate education
- Show you the type of problems we encounter so you can get a flavour of the job that quants do everyday. In particular, we aim to price an European option under the Black-Scholes model by two methods: obtaining the analytical formula, and using Monte Carlo
- Show you what kind of mathematical tools are required and illustrate how these concepts are translated into code in Python

For today:

- Recall concepts on Probaility and Stochastic Processes Theory
  - Random Variables, Stochastic Processes
  - Brownian Motion
  - Ito's formula
  - Geometric Brownian Motion
- See how these concepts are translated into code in Python
- Take a first look at financial time series in Python

Basic Concepts

Random Variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (S, Σ) a measurable space. Then, an (S, Σ) random variable is a measurable function

$$X : \Omega \to S,$$

which means that, for every subset $B \in \Sigma$, its pre-image is $\mathcal{F}-$measurable, i.e.;

$$X^{-1}(B) \in \mathcal{F},$$

where

$$X^{-1}(B) = \{\omega : X(\omega) \in B\}.$$

- Tipically, $S = \mathbb{R}^d$ for some $d \geq 1$, and $\Sigma = \mathcal{B}(\mathbb{R}^d)$ is the corresponding Borel sigma-algebra.
- Examples
  - Discrete random variables: Bernoulli, Binomial, Poisson
  - Continuous random variables: Uniform, Gaussian, Log-normal, t-Student

Python Time

Stochastic Processes

Stochastic Process

For given probability space $(\Omega, \mathcal{F}, P)$ and measurable space (S, Σ), a stochastic process is a collection of S-valued random variables

$$\{X_t : t \in I\},$$

where the set I is called index set.

- Tipically, $S = \mathbb{R}^d$ for some $d \geq 1$ and $\Sigma = \mathcal{B}(\mathbb{R}^d)$ is the corresponding Borel sigma-algebra
- The index set can be discrete, e.g. $I = \mathbb{N}$, or continuous e.g. $I = [0, T]$ for some $T \geq 0$.

Stochastic Processes

Brownian Motion or Wiener process

A standard Brownian motion, or Wiener process, is a stochastic process $\{W_t : t \geq 0\}$ charaterised by the following four properties:

1. $W_0 = 0$
2. W_t has independent increments
3. $W_t - W_s \sim \mathcal{N}(0, t-s)$ for any $0 \leq s \leq t$
4. W_t is almost surely continuous

Here $\mathcal{N}(\mu, \sigma^2)$ denotes the normal or Gaussian distribution with given mean μ and variance σ.

Python Time

Stochastic Processes

Ito's Lemma

Suppose that $X = \{X_t : t \geq 0\}$ is a stochastic process which satisfies the following stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad t \geq 0,$$

i.e.

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \qquad t \geq 0,$$

where W denotes a standard Brownian motion. Let $f(t, X)$ be a twice differentiable function. Then the process $Y = \{Y_t = f(t, X_t), t \geq 0\}$ satisfies the following SDE

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t,$$

i.e.

$$f(t, X_t) = f(0, X_0) + \int_0^t \left(\frac{\partial f}{\partial s} + \mu_s \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_s^2 \frac{\partial^2 f}{\partial x^2}\right) ds + \int_0^t \sigma_s \frac{\partial f}{\partial x} dW_s, \qquad t \geq 0.$$

Stochastic Processes

Geometric Brownian Motion

A geometric Brownian motion is a stochastic process defined by the following SDE

$$dS_t = \mu S_t dt + \sigma S_t W_t, \quad t > 0, \tag{1}$$

where $S_0 = s_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, are constants; and W is a standard Brownian motion.

Solution: Let us set a new process as $X_t = log(S_t)$. Using Ito's formula, we obtain

$$X_t = X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t,$$

or equivalently

$$log(S_t) = log(s_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t.$$

Note that the last expression implies that $\log(S_t)$ follows a normal distribution $\mathcal{N}\left(\log(s_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right)$. This, in turn implies that

$$S_t = s_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\}, \quad \forall t > 0,$$

follows a log-normal distribution.

Python Time

To finish this session let's take a look at financial time series in Python

Many thanks for your attention
See you tomorrow!