Money and the Markets

Quasar Chunawala

Abstract

A short note on money and financial markets

---

1 Introduction.

We first lay the mathematical foundation necessary to model certain transactions in the world of finance. Our goal is to provide a complete self-contained mathematical background to the Black-Scholes formula for pricing a call option. That involves two cultures, mathematics and finance, each having its own internal intuitions, concepts, rules and language. In finance, we confine ourselves to the minimal background necessary to achieve our purpose. This involves concepts such as interest rates, present worth or value, discounted value, hedging, risk, bonds, stocks, shares, options, expected return and arbitrage. In the first two chapters we explore these concepts and begin the process of interpreting them mathematically. To illustrate certain points we use examples, artificial from a finance perspective, but as we progress we make them more realistic.

We suppose the reader has some acquaintance with techniques of one variable differential and integral calculus. All other mathematics required, for example, set theory, integration theory and probability theory, are developed ab initio as we proceed. History shows that intuition generally precedes rigour in mathematics, and guided by this principle, we adopt an intuitive approach at first. Afterwards, we introduce necessary rigorous mathematical definitions and provide proofs. The mathematical examples given are often elementary and are provided to improve our basic understanding of complicated concepts. Complicated mathematical formulae and equations often turn out to be nothing more than clever combinations of simple well-known mathematical facts.

2 Money.

In ancient times, trade was conducted exchanging goods, a system known as bartering. To simplify this process, a fixed amount of a single commodity, often silver or gold, was chosen as a unit of value and goods were valued in units of this standard. We call this *standard money*. Silver and gold are maintenance free and easily divided and thus suitable choices. Life would have been more complicated if the unit chosen would have been a live chicken. Money's original role as a *medium of exchange* led to the separation of the acts of buying and selling, and it assumed a further role as a *store of value* as people realized its potential to be used *when* it suited them. Thus, began the relationship between money and time.

When prices are stable, those with money feel financially secure. However, prices do change depending on the *supply* and *demand*. The rate of change over time in the price of a commodity or a number of commodities is called *inflation*. If product A costs \$10 this time last year while today it costs \$12, then the percentage increase in price over the year is $\frac{12-10}{10} \times 100 = 20\%$ and the product A has a 20% annual rate of inflation. The inflation rate for a country is obtained by taking the weighted average of a basket of goods in the overall economy. If we call the real,in contrast to the nominal, value of money what it is capable of buying, then the presence of inflation means that the real value of money is a function of time.

Inflation is a problem for those with money. In its absence, they can estimate their financial obligations and requirements. The presence of inflation reduces their financial security and forces them to confront an intrinsic problem: *how to maintain the future real value of money*? Money securedly locked away is safe, but may be losing value. On the other hand there are others who need money to buy houses, to set up businesses etc. To cater to these, renting money became a business and successful moneylenders prospered and became respectable bankers. Those with money and no immediate need of it rented it to the bank, and those who needed money rented it from the bank. The price of renting money is called interest.

Money deposited in a savings account grows at the prevailing rate of interest, and as most deposits are insured and often guaranteed by governments, they are for all practical purposes, a risk free way of maintaining some growth. Any other way such as investing in a business venture involves risk. Interest rates and inflation rates are distinct processes, one increasing the nominal value of money, the other reducing its real value. However, its often observed in economies that interest rates tend to be slightly greater than inflation rates. It seems savers generally demand a positive real interest rate and borrowers are generally willing to pay for it. We can also identify two groups with different approaches to the management of money. Hedgers are those who wish to eliminate risk as much as possible, while speculators are willing to take risks in the expectation of higher profits.

3 Interest Rates.

We now discuss interest rates and at the same time review some important results from one variable calculus. Interest rates are presented in several forms: simple interest, compound interest, continuously compounded interest, effective rate of interest etc. with charges usually given as annual percentage rates, say 5%, 10%. Since all involve the same basic concept they are comparable. We show how to compare them and having done so, settle on one and use it more or less exclusively afterwards. We let t denote the time variable, $t = 0$ will denote the present, while $t = 10$ will be 10 units of time, usually measured in years into the future. Interest rates vary with time, but initially we assume they are constant.

We begin with the simplest case, simple interest. Ten percent *simple interest* on a loan of \$1,000 for five years means that 10% of the amount borrowed, the principal, is charged for one year of the loan. Thus, the interest charged is $10/100 \times 1,000 \times 5 = 500$. The general formula for calculating simple interest is straightforward: if an amount A is borrowed or saved for T years at a rate r of simple interest, then the repayment due at time T is

$$A + ArT = A(1 + rT)$$

Simple interest is rarely used by banks and it is easy to see why. If \$1,000 is deposited for 2 years at the rate of 10% simple interest, then the amount accumulated at the end of two years, the maturity date, would be \$1,200. If, however, at the end of year one, the amount accumulated at that time, \$1,100 is withdrawn and immediately deposited for a further year at the same rate of simple interest, then the amount accumulated would be \$1,210, a gain of \$10 on the previous amount. If simple interest was the norm, people would be in and out of banks regularly withdrawing and immediately re-depositing their savings. For this reason a different method of calculating interest is normally used. This is called compound interest and is based on applying simple interest over regular pre-assigned periods of the savings or loan to the amount accumulated at the beginning of each period. If a savings account offers 5% interest per annum compounded every 6 months, then the amount accumulated by \$2,000 deposited for two years is calculated as follows. The simple interest rule applied to the first six months period shows that the amount will earn \$50 interest, and the amount deposited will have increased to \$2,050 at the end of 6 months. During the second six months, the \$2,050 will grow to $2,050(1 + \frac{5}{100} \times \frac{1}{2}) = 2,101.25$, during the next period the amount will reach 2,153.8, and in the final six months' period the amount will reach 2,207.63.

Interest can of course be compounded at various other intervals of time, and the more frequent the compounding, the greater the interest earned. Suppose an amount A is borrowed for T years at a rate r per annum compounded at n equally spaced intervals of time per year. Each interval of time $1/n$ has a simple interest of r/n. Thus, after the first time interval, the amount due has grown $A(1+\frac{r}{n})$, after two intervals it becomes $A(1+r/n)(1+r/n) = A(1+r/n)^2$ and so on. Since there are a total of nT intervals of time, the total repayment at the end of T years will be $A(1 + r/n)^{nT}$.

We compare different interest rates by finding their effective rate of interest. This is the rate of simple interest which would give the same return over one year. One thousand dollars borrowed for one year at a rate of 10% per annum compounded every six months would result in a repayment of \$1,102.50 at the end of the year. If the same amount is borrowed for one year at 10.25% simple interest, then the amount due would also be \$1,102.50. Thus, we say that the rate of 10% per annum compounded every six months has a 10.25% effective rate of interest. It is clear that the more frequent the compounding, the higher the effective rate of interest.

Example 3.1. By comparing the