Introduction to Probability Theory and its applications, Volume I

Solution of exercise problems.

Quasar Chunawala

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Abstract
This is a solution manual for Introduction to Probability Theory and its applications, Volume I, by William Feller.

Chapter 2. Combinatorial Analysis.

1. [Feller, 2.1] How many different sets of initials can be formed if every person has one surname and (a) exactly two given names (b) at most two given names? (c) at most three given names?

Solution.

(a) We have 26 ways to choose an alphabet. So, 26^3 strings of length $= 3$ are possible.
(b) Initials of length 2 or more can be formed in $26^2 + 26^3$ distinguishable ways.
(c) Initials of length 2 or more can be formed in $26^2 + 26^3 + 26^4$

2. [Feller. 2.2] In how many ways, can two rooks of different colors be put on a chessboard so they can take each other out?

Solution.

The rook is a piece in the game of chess resembling a castle. The white rooks start on squares a1 and h1, while the black rooks start at a8 and h8.

Two rooks can take each other if and only if they are in a straight line. The number of squares available for the first rook is 64. For each such choice, the second rook can be placed in 49 squares. So, the total number of ways, that the rooks can take each other equals $64 \cdot 49$.

3. [Feller. 2.3] Letters in a morse code are formed by a succession of dashes and dots with repetitions permitted. How many letters is it possible to form with ten symbols or less?

Solution.
We want to form an ordered sample of size $r \leq 10$ from a population of size $n = 2$. The number of letters that can be formed with ten symbols or less are,

$$2^1 + 2^2 + \ldots 2^{10} = 2(2^{10} - 1)$$

4. [Feller. 2.4] Each domino piece is marked by two numbers. The pieces are symmetrical so that the number-pair is not ordered. How many different pieces can be made using the numbers $1, 2, 3, \ldots, n$?

Solution.

There are $\binom{n}{2}$ number-pairs. An additional n pieces can be made that have identical numbers. Hence, number of pieces equal $\binom{n}{2}+n$.

5. [Feller. 2.5] The numbers $1,2,\ldots,n$ are arranged in random order. Find the probability that the digits (a) 1 and 2 (b) $1,2,$ and 3 appear as neighbours in the order named.

Solution.

(a) Gluing 1 and 2 together as a single super-digit '12', we now have $(n-1)$ numbers. These can be arranged in $(n-1)!$ ways. So, the required probability is

$$p=\frac{(n-1)!}{n!}=\frac{1}{n}$$

(b) $p=\dfrac{(n-2)!}{n!}=\dfrac{1}{n(n-1)}$.

6. [Feller. 2.6] (a) Find the probability that among three random digits there occur $2,1,$ or 0 repetitions. (b) Do the same for four random digits.

Solution.

(a) The total number of ordered-strings of length 3, made from digits $0,1,\ldots,9$ with replacement equal 10^3.

If there are two repetitions, number of strings $=10$.

If there is one repetition, number of strings are $\binom{3}{2}\cdot 10\cdot 9$.

If there are zero repetitions, number of strings are $10\cdot 9\cdot 8$.

So, the required probabilities are $0.1, 0.27, 0.72$.

(b)

If there are three repetitions, number of strings $=10$.

If there are two repetitions, either three digits are identical or two pairs of digits are identical. The number of 4-strings with 3 identical digits $=\binom{4}{3}\cdot 10\cdot 9$. The number of 4-strings with two pairs of identical digits are $3\cdot 10\cdot 9$.

If there is one repetition, the number of 4-strings are,

$$\binom{4}{2}\cdot 10\cdot 9\cdot 8$$

If there are no repetitions, the number of 4-strings are

$$10\cdot 9\cdot 8\cdot 7$$

So, the corresponding probabilities for strings of length 4 are,

$0.001, 0.063, 0.432, 0.504$.

7. [Feller, 2.7] Find the probabilities p_r that in a sample r random digits no two are equal. Estimate the numerical value of p_{10} using the Stirling's formula.

Solution.

$$p_r = \frac{10 \cdot 9 \cdots (10-r+1)}{10^r} = \frac{(10)_r}{10^r}$$

8. [Feller, 2.8] What is the probability that among k random digits

(a) 0 does not appear;

(b) 1 does not appear

(c) neither 0 nor 1 appears

(d) Let A and B represent the events in (a) and (b). Express the other events in terms of A and B.

Solution.