Exercise set – The sample space 02-October-2021

1. Among the digits 1,2,3,4,5 first one is chosen, and then a second selection is made among the remaining 4 digits. Assume that all possible twenty results have the same probability. Find the probability that an odd digit will be selected (a) the first time (b) the second time (c) both times.

Solution

$$P(\text{Odd digit is selected the first time}) = \frac{12}{20}$$

$$P(\text{Odd digit is selected the second time})$$
$$= P(\text{Odd digit is selected the first time}) = \frac{3\cdot 4}{20}.$$

$$P(\text{Odd digits is selected both times}) = \frac{3\cdot 2}{20} = \frac{3}{10}.$$

2. In the sample space of example (2·a) attach equal probabilities to all 27 points. Using the notation of the example ~~(2·a)~~, attach ~~equal probabilitities~~ (4·d), verify formula (7.4) for the two events $A_1 = S_1$ and $A_2 = S_2$. How many points does S_1S_2 contain?

Solution.

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1A_2).$$

$$P(A_1 \cup A_2) = \frac{18}{27} \quad P(A_1) = \frac{12}{27} \quad P(A_2) = \frac{12}{27} \quad P(A_1A_2) = \frac{6}{27}$$

$$P(A_1 \cup A_2) = \frac{12}{27} + \frac{12}{27} - \frac{6}{27}$$
$$= \frac{18}{27}$$

This completes the proof. S_1S_2 contains 6 points.

3. Consider the 24 possible (permutations) of the symbols 1234 and attach to each the probability 1/24. Let A_i be the event that the digit i appears at its natural place (where $i = 1,2,3,4$). Verify formula (7.4).

Solution.

$$P(A_1) = \frac{1\cdot 3\cdot 2\cdot 1}{24} = \frac{6}{24} = \frac{1}{4}$$

$$P(A_2) = \frac{3\cdot 1\cdot 2\cdot 1}{24} = \frac{6}{24} = \frac{1}{4}$$

$$P(A_1A_2) = \frac{2}{24}$$

$$P(A_1 \cup A_2) = 1 - \frac{14}{24} = \frac{10}{24}$$

$$P(A_1) + P(A_2) - P(A_1A_2)$$
$$= \frac{6}{24} + \frac{6}{24} - \frac{2}{24} = \frac{10}{24}.$$

$$\left.\begin{array}{l} 2\,1\,\_\,\_ \\ 2\,\_\,1\,\_ \\ 2\,\_\,\_\,1 \\ \_\,1\,2\,\_ \\ \_\,1\,\_\,2 \\ \_\,1\,1\,2 \\ \_\,2\,2\,1 \end{array}\right\} 7\times 2 = 14$$

4. A coin is tossed until for the first time the same result appears twice in succession. To every possible outcome requiring n tosses attribute the probability $1/2^n$. Describe the sample space. Find the probability of the following events

(a) the experiment ends before the sixth toss.

(b) an even number of tosses is required.

Solution.

The countably infinite sample space Ω consists of the following sample points:

$$\Omega = \{HH, TT, THH, HTT, HTHH, THTT, THTHH, HTHTT, \ldots\}$$

(a) Since all outcomes are mutually exclusive $P\{\omega_i \cap \omega_j\} = 0$. So, the chance that experiment ends before the sixth toss is –

$$P\{HH\} + P\{TT\} + P\{THH\} + P\{HTT\} + P\{HTHH\} + P\{THTT\}$$
$$+ P\{THTHH\} + P\{HTHTT\}$$
$$= \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^5}$$
$$= 2\left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5}\right) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{15}{16}.$$

(b) P(an even number of tosses is required)

$$= P\{HH\} + P\{TT\} + P\{HTHH\} + P\{THTT\} + \cdots$$
$$= 2\left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \cdots\right)$$
$$= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots \quad \{\text{The infinite geometric series is convergent for } |r| < 1\}$$
$$= \frac{1/2}{1 - (1/2)^2} = \frac{1/2}{3/4} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3}.$$

5. In the sample space of examples (5·b) let us attribute to each point of (*) containing exactly k letters the probability $(1/2^k)$. In other words, aa and bb carry a probability of 1/4, acb has probability 1/8 etc.

(b) Show that the probability that (a) wins is $\frac{5}{14}$. The probability of b winning the same and c has probability $\frac{2}{7}$ of winning.

(a) Show that the probabilities of the points of (*) add upto unity, whence the two points (**) receive probability zero.

(c) the probability that no decision is reached at or before the kth game is $1/2^{k-1}$.

Solution

(a) If we add up the probabilities of points in (*) we get,

$$2\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots\right) = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = \frac{1/2}{1-1/2} = 1.$$

Whence (**) receives the probability zero.

(b) $P(\text{'a' wins}) = P\{aa\} + P\{acbaa\} + P\{acbacbaa\} + \cdots$

$$= \cancel{\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^8}}+\cdots \quad \{\text{Infinite geometric series}\}.$$

$$= \cancel{\frac{1/2^2}{1-1/2^3}} = \cancel{\frac{1/4}{7/8}} =$$

$+ P\{bcaa\} + P\{bcabcaa\} + \cdots$

$$= \left(\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^8}+\cdots\right) + \left(\frac{1}{2^4}+\frac{1}{2^7}+\frac{1}{2^{10}}+\cdots\right)$$

$$= \frac{1/2^2}{1-1/2^3} + \frac{1/2^4}{1-1/2^3}$$

$$= \frac{1/4}{7/8} + \frac{1/16}{7/8} = \frac{5/16}{7/8} = \frac{5}{16_2}\cdot\frac{8}{7} = \frac{5}{14}.$$

$P(\text{'b' wins}) = P\{bb\} + P\{bcabb\} + P\{bcabcabb\} + \cdots$
$+ P\{acbb\} + P\{acbacbb\} + \cdots$

$$= \left(\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^8}+\cdots\right) + \left(\frac{1}{2^4}+\frac{1}{2^7}+\frac{1}{2^{10}}\right) = \frac{5}{14}$$

$$P(\text{'c' wins}) = 1 - P(\text{'a' wins}) - P(\text{'b' wins})$$
$$= 1 - \frac{10}{14} = \frac{4}{14} = \frac{2}{7}.$$

(c) The probability that no decision is reached at or before the kth turn is $= P(\underbrace{acbacb\ldots}_{k\text{ turns}}) + P(\underbrace{bcabca\ldots}_{k\text{ turns}})$

$$= \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}}.$$

6. TODO.

7. In problem 3, show that $A_1 A_2 A_3 \subset A_4$ and $A_1 A_2 A_3^c \subset A_4^c$.

Solution:

$$A_1 A_2 A_3 = \{1234\}$$
$$A_4 = \{1234, 2134, 1324, 3214, 2314, 3124\}$$

$\therefore A_1 A_2 A_3 \subset A_4$

Also,

$$A_1 A_2 A_3^c = \{1243\}$$
$$A_4^c = \{4123, 4132, 4213, 4231, 4312, 4321, 1423, 1432, 2413, 2431, 3412, 3421, 1243, 1342, 2143, 2341, 3142, 3241\}.$$

$\therefore A_1 A_2 A_3^c \subset A_4^c$.

8. Using the notations of example (4·d) show that (a) $S_1 S_2 D_3 = 0$ (b) $S_1 D_2 \subset E_3$ (c) $E_3 - D_2 S_1 \supset S_2 D_1$.

Solution.

(a) $S_1 S_2 D_3 = 0$, because there are only three balls.

~~$S_1 S_2 = \{a + b c$~~

$$S_1 S_2 = \{(a, bc,), (b, ac,), (c, ab,)\}$$
$$D_3 = \{(a, , bc), (, a, bc), (, b, ac), (b, , ac), (c, , ab), (, c, ab)\}.$$

Hence, $S_1 S_2 D_3 = \phi$.

(b) $S_1 D_2 = \{(a, bc,), (b, ac,), (c, ab,)\}$

$$E_3 = \{(abc, ,), (, abc,), (a, bc,), (b, ac,), (c, ab,), (ab, c,), (bc, a,), (ca, b,)\}.$$

$\therefore S_1 D_2 \subset E_3$.

(c) $E_3 - D_2 S_1 = \{(abc, ,), (, abc,), (ab, c,), (bc, a,), (ca, b,)\}$

$$S_2 D_1 = \{(ab, c,), (bc, a,), (ca, b,)\}.$$

$\therefore E_3 - D_2 S_1 \supset S_2 D_1$.

9. Two dice are thrown. Let A be the event that the sum of the faces is odd, B the event ~~that~~ of atleast one six. Describe the events $AB, A\cup B, AB^c$. Find their probabilities assuming that all 36 sample points have equal probabilities.

Solution.

$AB :=$ The event that the sum of the faces is odd and atleast one six is rolled simultaneously.

$A\cup B :=$ Atleast one of the below occurs
(i) The sum of the faces is odd eg: 3, 5, 7, 9, 11.
(ii) One or more sixes are rolled.

$AB^c :=$ The event that the sum of faces is odd, but no six is rolled.

$$AB = \{(6,1), (1,6), (6,3), (3,6), (6,5), (5,6)\} = \frac{6}{6^2} = \frac{1}{6}.$$

$$P\{AB\} = \frac{6}{6^2} = \frac{1}{6}.$$

$$A\cup B = \{(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5)\}$$
$$\cup \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)\}$$

$$P(A\cup B) = \frac{24}{36} = \frac{2}{3}.$$

$$P(AB^c) = \frac{12}{36} = \frac{1}{3}.$$

10. In example (2.g) discuss the meaning of the following events: (a) ABC (b) A − AB (c) AB^cC.

Solution.
A := Husband is older than 40
B := Husband is older than is wife
C := Wife is older than 40.

(a) ABC
The event that the husband is older 40, husband is older to ~~is~~ his wife and wife is older than 40 simultaneously.

(b) $A - AB = A \cap (AB)^c = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = AB^c$
$= A - B$

The event that husband is older than 40, but not older than his wife.

(c) AB^cC

The event that the
(i) husband is older than 40.
(ii) husband is younger or the same age as his wife
(iii) wife is older than 40
simultaneously.

11. In example (2.g), verify that $AC^c \subset B$.

Solution:

AC^c = The event that husband is older than 40, but the wife's age ≤ 40.
$= \{(x,y) : x > 40, y \leq 40\}$
B = The event that the husband is older than the wife
$= \{(x,y) : x > y\}$.

$\therefore AC^c \subset B$.

12. Bridge. For $k = 1, 2, 3, 4$, let N_k be the event that North has atleast k aces. Let S_k, E_k, W_k be the analogous events for South, East, West. What can be said about the number x of aces in West's possession in the events

(a) W_1^c (b) $N_2 S_2$ (c) $N_1^c S_1^c E_1^c$ (d) $W_2 - W_3$ (e) $N_1 S_1 E_1 W_1$
(f) $N_3 W_1$ (g) $(N_2 \cup S_2) E_2$

Solution:
(a) $x = 0$ aces.
(b) $x = 0$ aces.
(c) $x = 4$ aces.
(d) $x = 2$ aces.
(e) $x = 1$ ace.
(f) $x = 1$ ace
(g) $x = 0$ aces.

13. In the preceding problem, verify that
(a) $S_3 \subset S_2$
(b) $S_3 W_2 = 0$
(c) $N_2 S_1 E_1 W_1 = 0$
(d) $N_2 S_2 \subset W_1^c$
(e) $(N_2 \cup S_2) W_3 = 0$
(f) $W_4 = N_1^c S_1^c E_1^c$

Solution:
(a) S_3 is the event that South gets atleast 3 aces. If south has atleast 3 aces, the event S_2: south has atleast 2 aces is also realized. so, $S_3 \subset S_2$.

(b) $S_3 W_2$ — The event that South has atleast 3 aces, and West has atleast 2 aces simultaneously is an impossible event, as a standard deck of 52 cards has ~~a~~ 4 ~~suits~~ aces.

$\therefore S_3 W_2 = \emptyset$.

(c) $N_2 S_1 E_1 W_1 = 0$.
Again this is an impossible event.

(d) $N_2 S_2 \subset W_1^c$.
The event W_1^c implies West receives 0 aces. ~~It implies that~~ Hence, $N_2 S_2 \subset W_1^c$.

(e) $(N_2 \cup S_2) W_3 = 0$.
Impossible event.

(f) $W_4 = N_1^c S_1^c E_1^c$.
$N_1^c S_1^c E_1^c$ is the simultaneous realization of ~~Als~~
(i) North receives 0 aces.
(ii) South receives 0 aces.
(iii) East receives 0 aces.
Hence, West must receive 4 aces and vice versa.
so, $W_4 = N_1^c S_1^c E_1^c$.

14. Verify the following relations.
(a) $(A \cup B)^c = A^c B^c$

Proof.
Let $x \in (A \cup B)^c$.
$\Leftrightarrow \quad x \notin (A \cup B)$
$\Leftrightarrow \quad (x \notin A) \wedge (x \notin B)$
$\Leftrightarrow \quad (x \in A^c) \wedge (x \in B^c)$
$\Leftrightarrow \quad x \in A^c B^c$

so, $(A \cup B)^c \subseteq A^c B^c$ (1)

Also, let $x \in A^c \cap B^c$.
$\Leftrightarrow \quad (x \in A^c) \wedge (x \in B^c)$
$\Leftrightarrow \quad (x \notin A) \wedge (x \notin B)$
$\Leftrightarrow \quad x \notin \{y : (y \in A) \vee (y \in B)\}$
$\Leftrightarrow \quad x \notin A \cup B$
$\Leftrightarrow \quad x \in (A \cup B)^c$

so, $A^c \cap B^c \subseteq (A \cup B)^c$ (2)

From (1) and (2), $(A \cup B)^c = A^c \cap B^c$.

(b) $(A \cup B) - B = A - AB = AB^c$
Proof
Let $x \in (A \cup B) - B$.
Then, $x \in A \cup B$, but not B.
$\Leftrightarrow \quad x \in A$, but not B
$\Leftrightarrow \quad x \in AB^c$.

so $(A \cup B) - B \subseteq AB^c$

$\Leftarrow$ direction.

If $x \in A \cap B^c$, then $x \in A$, $x \notin B$.
Since, $A \subseteq A \cup B$, $x \in A \cup B$, $x \notin B$.
So, $x \in (A \cup B) - B$.
Thus, $AB^c \subseteq (A \cup B) - B$.

Consequently, $(A \cup B) - B = AB^c$.

(c) $AA = A \cup A = A$.
Proof
$\Rightarrow$ direction Let $x \in A \cap A$.
Then, $(x \in A) \wedge (x \in A)$.
$\Rightarrow$ $x \in A$.

$\Leftarrow$ direction.
Let $x \in A$.
Then, $(x \in A) \wedge (x \in A)$ holds.
So, $x \in (A \cap A)$.

Consequently, $A = AA$.

Also, $A \cup A = A$.

(d) $(A - AB) \cup B = A \cup B$.
Proof.
Let $x \in (A - AB) \cup B$.
Then, x belongs to atleast one of $A - AB$ and B.
If $x \in B$, then since $B \subseteq A \cup B$, $x \in A \cup B$ and we are done.
If $x \in A - AB$, then $x \in A$, but not AB.
$\Rightarrow$ $x \in AB^c$.
Since, $AB^c \subseteq A \cup B$, ~~then~~ $x \in A \cup B$.

$\Leftarrow$ direction.
Let $x \in A \cup B$.
Then x belongs to atleast one of A, B.
If x belongs to B, then since $B \subseteq (A - AB) \cup B$, we have that $x \in (A - AB) \cup B$.
If x belongs to A, then $x \in A - AB$ or $x \in AB$.
{ Since $A = (A - AB) \cup (AB)$ and these are disjoint }

If $x \in A - AB \Rightarrow x \in (A - AB) \cup B$.
If $x \in AB \Rightarrow x \in B \Rightarrow x \in (A - AB) \cup B$.

In both cases, $(A - AB) \cup B = A \cup B$.

(e) $(A \cup B) - AB = AB^c \cup A^cB$.

Proof.

$\Rightarrow$ direction.

Let $x \in (A \cup B) - AB$. Then, $x \in (A \cup B)$, but not AB.
Therefore, x belongs to exactly one of A, B but not both.
Thus, ($x \in A$, but not B) $\vee$ ($x \in B$, but not A).
so, $x \in (A^cB \cup AB^c)$

$\Leftarrow$ direction.

Suppose $x \in AB^c \cup A^cB$.
~~Let $x \in$~~ Then, x belongs to atleast one of AB^c, A^cB.
If x belongs to AB^c, then $(x \in A, \wedge x \notin B)$. ~~Since,~~
Thus, $(x \in A, x \notin AB)$. It implies that ($x \in A \cup B$, with $x \notin AB$).

(f) TODO
(g) TODO

15. Find simple expressions for
(a) $(A \cup B)(A \cup B^c)$

$(A \cup B)(A \cup B^c)$.
$= (A \cup B) \cap (A \cup B^c)$
$= ((A \cup B) \cap A) \cup ((A \cup B) \cap B^c)$
$= A \cup AB^c$
$= AB^c. = A - B$

(b) $(A \cup B) \cap (A^c \cup B)$.

$(A \cup B) \cap (A^c \cup B)$
$= ((A \cup B) \cap A^c) \cup ((A \cup B) \cap B)$
$= (A^cB) \cup (B)$
$= A^cB = B - A$.

(c) $(A \cup B) \cap (B \cup C)$
$= (A \cap (B \cup C)) \cup (B \cap (B \cup C))$
$= (A \cap (B \cup C)) \cup B$
$=$

18. The union $A \cup B$ of two events can be expressed as the union of two mutually exclusive events, thus:

$$A \cup B = A \cup (B - AB).$$

Express in a similar way, the union of three events A, B, C.

Solution

$$A \cup B \cup C = A \cup (B - AB) \cup (C - (AC \cup BC))$$

~~$A \cup B \cup C \cup D$~~

$$A_1 \cup A_2 \cup \dots \cup A_n = A_1 \cup (A_2 - A_1A_2) \cup (A_3 - (A_1A_3 \cup A_2A_3)) \cup (A_4 - (A_1A_4 \cup A_2A_4 \cup A_3A_4)) \cup \dots$$

19. Using the result of problem 18, prove that

$$P\{A \cup B \cup C\} = P\{A\} + P\{B\} + P\{C\} - P\{AB\} - P\{BC\} - P\{CA\} + P\{ABC\}.$$

Proof.

If A_1 and A_2 are disjoint events,

$$P\{A_1 \cup A_2\} = P(A_1) + P(A_2)$$

so,

$$\begin{aligned}
P\{A \cup B \cup C\} &= P\{A\} + P\{B - AB\} + P\{C - (AC \cup BC)\} \\
&= P\{A\} + P\{B\} - P\{AB\} + P\{C\} - P\{AC \cup BC\} \\
&= P\{A\} + P\{B\} - P\{AB\} + P\{C\} \\
&\quad - (P\{AC\} + P\{BC\} - P\{ABC\})
\end{aligned}$$

$$= P\{A\} + P\{B\} + P\{C\} - P\{AB\} - P\{BC\} - P\{CA\} + P\{ABC\}$$