

# Are derivatives continuous

Quasar Chunawala

## Abstract

An investigation into whether the derivatives of functions continuous.

---

The geometric motivation for the derivative is most likely familiar territory. Given a function  $g(x)$ , the derivative  $g'(x)$  is understood to be the slope of the graph of  $g$  at each point  $x$  in the domain. A graphical picture reveals the impetus behind the mathematical definition

$$g'(x) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c}$$

The difference quotient (ratio)  $(g(x) - g(c))/(x - c)$  represents the slope of the line through the two points  $(x, g(x))$  and  $(c, g(c))$ . By taking the limit as  $x$  approaches  $c$ , we arrive at a well-defined mathematical meaning for the slope of the tangent line at  $x = c$ .

The myriad applications of the derivative function are the topic of much of the calculus sequence, as well as several other upper-level courses in mathematics. None of these applied questions are pursued here in any length, but it should be pointed out that the rigorous underpinnings of differentiation worked out in this chapter are an essential foundation for any applied study. Eventually, as the derivative is subjected to more and more complex manipulations, it becomes crucial to know precisely how differentiation is defined and how it interacts with other mathematical operations.

Although physical applications are not explicitly discussed, we will encounter several questions of a more abstract quality as we develop the theory. Many of these are concerned with the relationship between differentiation and continuity. Are continuous functions always differentiable? If not, how nondifferentiable can a continuous function be? Are differentiable functions continuous?

Given that a function  $f$  has a derivative at every point in its domain, what can we say about the function  $f'$ ? Is  $f'$  continuous? How accurately can we describe the set of all possible derivatives, or are there no restrictions? Put another way, if we are given an arbitrary function  $g$ , is it always possible to find a differentiable function  $f$  such that  $f' = g$ , or are there some properties that  $g$  must possess for this to occur? In our study of continuity, we saw that restricting our attention to monotone functions had a significant impact on the answers to questions about sets of discontinuity. What effect, if any, does this same restriction have on our questions about potential sets of non-differentiable points? Some of these issues are harder to resolve than the others, and some remain unanswered in any satisfactory way.

A particularly useful class of examples for this discussion are functions of the form

$$g(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

When  $n = 0$ , we have seen that the oscillations of  $\sin(1/x)$  prevent  $g_0(x)$  from being continuous at  $x = 0$ . When  $n = 1$ , these oscillations are squeezed between  $|x|$  and  $-|x|$ , the result being that  $g_1$  is continuous at  $x = 0$ . Is  $g'_1(0)$  defined? Using the preceding definition, we get

$$g'_1(0) = \lim_{x \rightarrow 0} \frac{g_1(x) - g_1(0)}{x - 0} = \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

which, as we now know, does not exist. Thus,  $g_1$  is not differentiable at  $x = 0$ . On the other hand, the same calculation shows that  $g_2$  is differentiable at zero. In fact, we have

$$g'_2(0) = \lim_{x \rightarrow 0} x \sin \left( \frac{1}{x} \right) = 0$$

At points different from zero, we can use the familiar rules of differentiation (soon to be justified) to conclude that  $g_2$  is differentiable everywhere in  $\mathbf{R}$  with

$$\begin{aligned} g_2'(x) &= (x^2 \sin(1/x))' \\ &= 2x \sin(1/x) + x^2 \cdot \cos(1/x) \cdot \frac{-1}{x^2} \\ &= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) \end{aligned}$$

when  $x \neq 0$  and  $g_2'(x) = 0$  when  $x = 0$ .

But, now consider

$$\lim_{x \rightarrow 0} g_2'(x)$$

Because the  $\cos(1/x)$  term is not preceded by a factor of  $x$ , we must conclude that this limit does not exist and that, consequently, the derivative function is not continuous. To summarize, the function  $g_2(x)$  is continuous and differentiable everywhere on  $\mathbf{R}$ , the derivative function  $g_2'$  is thus defined everywhere on  $\mathbf{R}$ , but  $g_2'$  has a discontinuity at zero. The conclusion that derivatives need not, in general, be continuous.

The discontinuity in  $g_2'$  is *essential*, meaning  $\lim_{x \rightarrow 0} g_2'(x)$  does not exist as a one-sided limit. But, what about a function with a simple jump discontinuity? For example, does there exist a function  $h$  such that

$$h'(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

A first impression may bring to mind the absolute value function, which has slopes of  $-1$  at the points to the left of zero and  $1$  to the right. However, the absolute value function is not differentiable at zero. We are seeking a function that is differentiable everywhere, including the point zero, where we are insisting that the slope of the graph be  $-1$ . The degree of difficulty of this request should start to become apparent. Without sacrificing differentiability at any point, we are demanding that the slopes jump from  $-1$  to  $1$  and not attain any value in between.

Although we have seen that continuity is not a required property of derivatives, the intermediate value property will prove a more stubborn quality to ignore.