Functional Limits and Continuity

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Abstract

Solution of the Exercise set 4.4.12

Example 0.1 (Abbot, 4.4.12). Review exercise 4.4.11 and then determine which of the following statements is true about a continuous function defined on $\mathbf{R}$:

(a) $f^{-1}(B)$ is finite, whenever B is finite.

(b) $f^{-1}(K)$ is compact whenever K is compact.

(c) $f^{-1}(A)$ is bounded whenever A is bounded.

(d) $f^{-1}(F)$ is closed whenever F is closed.

Proof.

(a) This statement is false. Consider $f(x) = 1$, for all $x \in \mathbb{R}$. $B = \{1\}$ is finite, but $f^{-1}(B)$ is uncountable.

(b) This statement is false. Considering the example above, $K = \{1\}$ has no limit points, so it is closed and bounded, hence compact. But, $f^{-1}(K)$ is unbounded and therefore not compact.

(c) This statement is false.

(d) This statement is true. Let c be a limit point of $f^{-1}(F)$. There exists a sequence $(x_n) \subseteq f^{-1}(F)$, such that $(x_n) \to c$, with $x_n \neq c$ for all $n \in \mathbf{N}$.

f is continuous at c. Consequently, f preserves limits and $f(x_n) \to f(c)$. But, $f(x_n) \in F$, so $f(c)$ is a limit point of F. Since, F is closed, $f(c) \in F$. Therefore, $c \in f^{-1}(F)$. Thus, $f^{-1}(F)$ is closed.