Understanding Analysis

Solution of exercise problems.

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Abstract

This is a solution manual for Understanding Analysis, 2nd edition, by Stephen Abbott.

Chapter 4. Functional Limits and Continuity

4.1 **Discussion: Examples of Dirichlet and Thomae.**

Although it is common practice in Calculus courses to discuss continuity before differentiation, historically mathematicians' attention to the concept of continuity came long after the derivative was in wide use. Pierre de Fermat (1601-1665) was using tangent lines to solve optimization problems as early as 1629. On the other hand, it was not until around 1820 that Cauchy, Bolzano, Weierstrass, and others began to characterize continuity in terms more rigorous than the prevailing intuitive notions such as "unbroken curves" or "functions which have no jumps or gaps". The basic reason for this two-hundred year waiting period lies in the fact that, for most of this time, the very notion of a *function* did not really permit discontinuities. Functions were entities such as polynomials, sines, cosines, always smooth and continuous over their relevant domains. The gradual liberation of the term function to its modern understanding - a rule associating a unique output with a give input - was simultaneous with the 19th century investigations into the behavior of the infinite series. Extensions of the power of calculus were intimately ties to the ability to represent a function $f(x)$ as a limit of polynomials (called a *power series*) or as a limit of the sums of sines and cosines (called a *trigonometric or Fourier series*). A typical question for Cauchy and his contemporaries was whether the continuity of the limiting polynomials or trigonometric functions necessarily implied that the limit f would also be continuous.

Sequences and series of functions are topics of chapter 6. What is relevant at this moment is that we realize why the issue of finding a rigorous definition of continuity finally made its way to the fore. Any significant progress on the question of whether the limit of continuous functions is continuous (for Cauchy and for us) necessarily depends on a definition of continuity that does not rely on imprecise notions such as "no holes" or "gaps". With a mathematically unambiguous definition for the limit of a sequence in hand, we are well on our towards a rigorous understanding of continuity.

Given a function f with domain $A \subseteq \mathbf{R}$, we want to define continuity at a point $c \in A$ to mean that if $x \in A$ is chosen near c, then $f(x)$ will be near $f(c)$. Symbolically, we will say that f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

The problem is that, at present, we only have the definition for the limit of a sequence, and it is not entirely clear what is meant by $\lim_{x \to c} f(x)$. The subtleties that arise as we try to fashion such a definition are well-illustrated via a family of examples, all based on an idea of the prominent German mathematician, Peter Lejeune Dirichlet. Dirichlet's idea was to define a function g in a piecewise manner based on whether or not the input variable x is rational or irrational. Specifically, let

$$\begin{aligned} g(x) &= 1 \quad \text{if } x \in \mathbf{Q} \\ &= 0 \quad \text{if } x \notin \mathbf{Q} \end{aligned}$$

The intricate way that $\mathbf{Q}$ and $\mathbf{I}$ fit inside of $\mathbf{R}$ makes an accurate graph of g technically impossible to draw, but the below figure gives a rough idea.

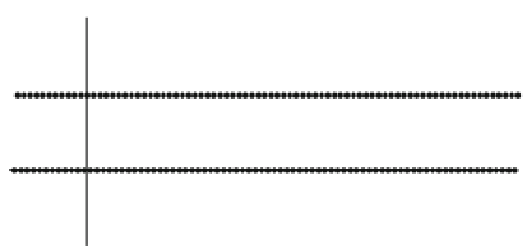


Figure 4.1: Dirichlet's Function, $g(x)$.

Does it make sense to attach a value to the expression $\lim_{x \to 1/2} g(x)$? One idea is to consider a sequence $(x_n) \to 1/2$. Using out notion of the limit of a sequence, we might try to define the $\lim_{x \to 1/2} g(x)$ as simply the limit of the sequence $g(x_n)$. But notice that this limit depends on how the sequence (x_n) is chosen. If each x_n is rational, for instance, when $x_n = \frac{1}{2} + \frac{1}{n}$, then

$$\lim_{n \to \infty} g(x_n) = 1$$

On the other hand, if (x_n) is irrational for each n, for instance when $x_n = \frac{1}{2} + \frac{1}{n + \sqrt{2}}$, then

$$\lim_{n \to \infty} g(x_n) = 0$$

This unacceptable situation demands that we work harder on our definition of functional limits. Generally speaking, we want the value of $\lim_{x \to c} g(x)$ to be independent of how we approach c. In this particular case, the definition of a functional limit that we agree on should lead to the conclusion that

$$\lim_{x \to 1/2} g(x)$$

does not exist.

Postponing the search for formal definitions for the moment, we should nonetheless realise that Dirichlet's function is not continuous at $c = 1/2$. In fact, the real significance of this function is that there is nothing unique about the point $c = 1/2$. Because both $\mathbf{Q}$ and $\mathbf{I}$ (the set of irrationals) are dense in the real line, it follows that for any $z \in \mathbf{R}$, we can find sequences $(x_n) \in \mathbf{Q}$ and $(y_n) \in \mathbf{I}$, such that

$$\lim x_n = \lim y_n = z$$

Because

$$\lim g(x_n) \neq \lim g(y_n)$$

the same line of reasoning reveals that $g(x)$ is not continuous at z. In the jargon of analysis, Dirichlet's function is a *nowhere-continuous* function on $\mathbf{R}$.

What happens if we adjust the definition of $g(x)$ in the following way? Define a new function h on $\mathbf{R}$ by setting

$$h(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

If we take c different from zero, then just as before we can construct sequences $(x_n) \to c$ of rationals and $(y_n) \to c$ of irrationals so that

$$\lim h(x_n) = c \quad \text{and} \quad \lim h(y_n) = 0$$

Thus, h is not contnuous at every point $c \neq 0$.

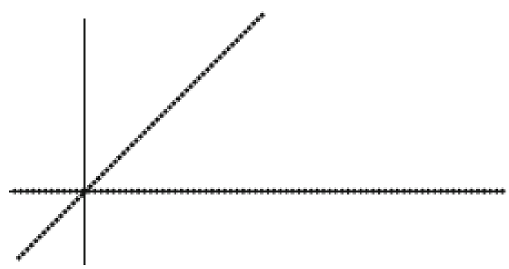


Figure 4.2: Modified Dirichlet Function, $h(x)$.

If $c = 0$, however, then these two limits are both equal to $h(0) = 0$. In fact, it appears as though no matter how we construct a seqeuence (z_n) converging to zero, it will always be the case that $h(z_n) = 0$. This observation goes to the heart of what we want functional limits to entail. To assert that

$$\lim_{x \to c} h(x) = L$$

should imply that

$$h(z_n) \to L \quad \text{for all sequences} \quad (z_n) \to c$$

For reasons not yet apparent, it is beneficial to fashion the definition for functional limits in terms of neighbourhoods constructed around c and L. We will quickly see, however, that this topological formulation is equivalent to the sequential characterization we have arrived at here.

To this point, we have discussing continuity of a function at a particular point in its domain. This is a significant departure from thinking of continuous functions as curves that can be drawn without lifting the pen from the paper, and it leads to some fascinating questions. IN 1875, K.J.Thomae discovered the function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

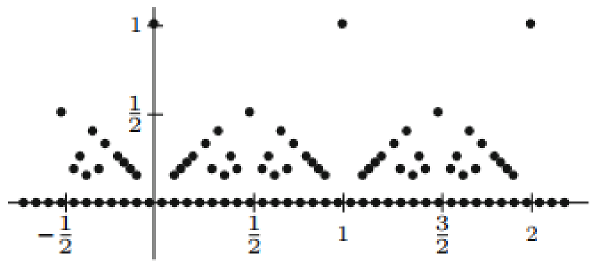


Figure 4.3: Thomae's Function, $t(x)$.

If $c \in \mathbf{Q}$, then $t(c) > 0$. Because the set of irrationals is dense in $\mathbf{R}$, we can find a sequence (y_n) in $\mathbf{I}$ converging to c. The result is that

$$\lim t(y_n) = 0 \neq t(c)$$

and Thomae's function fails to be continuous at any rational point. The twist comes when we try this argument on some irrational point in the domain such as $c = \sqrt{2}$. All irrational values get mapped to get mapped to zero by t, so the natural thing would be to consider a sequence (x_n) of rational numbers that converges to $\sqrt{2}$. Now, $\sqrt{2} \approx 1.414213$, so a good start on a particular sequence of rational approximations for $\sqrt{2}$ might be

$$\left(1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \ldots\right)$$

But, notice that the denominators of these fractions are getting larger. In this case, the sequence $t(x_n)$ begins,

$$\left(1, \frac{1}{5}, \frac{1}{100}, \frac{1}{500}, \frac{1}{5000}, \frac{1}{100000}, \ldots\right)$$

and is fast approaching $0 = t(\sqrt{2})$. We will see that this always happens. The closer a rational number is chosen to a fixed irrational number, the larger its denominator must necessarily be. As a consequence, Thomae's function has the bizarre property of being continuous at every irrational point on $\mathbf{R}$ and discontinuous at every rational point.

Is there an example of a function with the opposite property? In other words, does there exist a function defined on all of $\mathbf{R}$, that is continuous on $\mathbf{Q}$, but fails to be continuous on $\mathbf{I}$? Can the set of discontinuities of a particular function be arbitrary? If we are given some set $A \subseteq \mathbf{R}$, is it always possible to find a function that is continuous only on the set A? In each of the examples in this section, the functions were defined to have erratic oscillations around points in the domain. What conclusions can we draw if we restrict our attention to functions that are somewhat less volatile? One such class is the set of so-called monotone functions, which are either increasing or decreasing on a given domain. What might we be able to say about the set of discontinuities of a montone function on $\mathbf{R}$?

4.2 **Functional Limits.**

Consider a function $f : A \to \mathbf{R}$. Recall that a limit point c of A is a point with the property that every ϵ-neighbourhood $V_\epsilon(c)$ intersects A in some point other than c. Equivalently, c is a limit point of A, if and only if $c = \lim x_n$ for some sequence $(x_n) \subseteq A$ with $x_n \neq c$. It is important to remember that limit points of A do not necessarily belong to the set A unless A is closed.

If c is a limit point of the domain of f, then intuitively, the statement

$$\lim_{x \to c} f(x) = L$$

is intended to convey that the values of $f(x)$ get arbitrarily closed to L as x is chosen closer and closer to c. The issue of what happens when $x = c$ is irrelevant from the point of view of functional limits. In fact, c need not even be in the domain of f.

The structure of the definition of functional limits follows the "challenge-response" pattern established in the definition for the limit of a sequence. Recall that given a sequence (a_n), the assertion $\lim a_n = L$ implies that for every

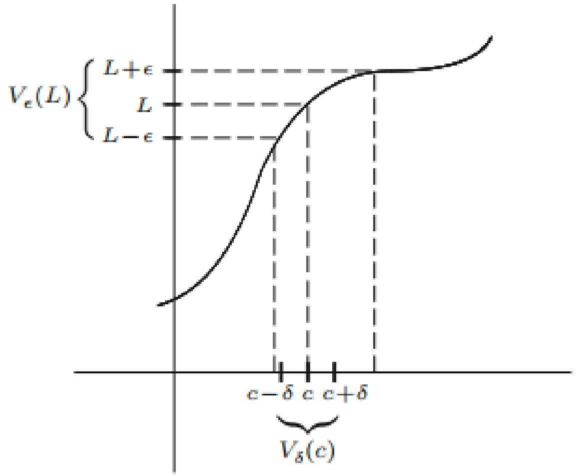


Figure 4.4: Definition of Functional Limit.

for every ϵ-neighbourhood $V_\epsilon(L)$ centered at L, there is a point in the seqeunce - call it a_N - after which all of the terms a_n fall in $V_\epsilon(L)$. Each ϵ-neighbourhood represents a particular challenge, and each N is the respective response. For functional limit statements such as $\lim_{x \to c} f(x) = L$, the challenges are still made in the form of an arbitrary ϵ-neighbourhood around L, ut the response this time is a δ-neighbourhood centered at c.

Definition 4.2.1. (Functional Limit). Let $f : A \to \mathbf{R}$, and let c be a limit point of the domain A. We say that, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x-c| < \delta$ (and $x \in A$), it follows that $|f(x) - L| < \epsilon$.

This is often referred to as the $\epsilon - \delta$ version of the definition for functional limits. Recall that the statement

$$|f(x) - L| < \epsilon \text{ is equivalent to } f(x) \in V_\epsilon(L)$$

Likewise, the statement

$$|x-c| < \delta \text{ is satisfied if and only if } x \in V_\delta(c)$$

The additional restriction $0 > |x-c|$ is just an economical way of saying $x \neq c$. Recasting the definition 4.2.1 in terms of neighbourhoods - just as we did for the definition of convergence of a sequence in section 2.2 - amounts to little more than a change of notation, but it does help emphasize the geometrical nature of what is happening.

Definition 4.2.1B (Functional Limit : Topological Version). Let c be a limit point of the domain of $f : A \to \mathbf{R}$. We say that $\lim_{x \to c} f(x) = L$ provided that, for every ϵ-neighbourhood $V_\epsilon(L)$ of L, there exists a δ-neighbourhood $V_\delta(c)$ around c with the property that for all $x \in V_\delta(c)$ different from c (with $x \in A$) it follows that $f(x) \in V_\epsilon(L)$.

The parenthetical reminder $(x \in A)$ present in both version fo the definition is included to ensure that x is an allowable input for the function in question. When no confusion is likely, we may omit this reminder with the understanding that the appearance of $f(x)$ carries with it the implicit assumption that x is in the domain of f. On a related note, there is nor reason to discuss functional limits at isolated points of the domain. Thus, functional limits will only be considered as x tends toward a limit point of the function's domain.

Example 4.2.2. (i) To familiarize ourselves with the **Definition 4.2.1**, let's prove that if $f(x) = 3x + 1$, then

$$\lim_{x \to 2} f(x) = 7$$

Let $\epsilon > 0$. Definition 4.2.1 equires that we produce a $\delta > 0$ so that $0 < |x-2| < \delta$ leads to the conclusion $|f(x) - 7| < \epsilon$. Notice that

$$|f(x) - 7| = |(3x+1) - 7| = |3x-6| = 3|x-2|$$

Thus, if we choose $\delta = \epsilon/3$, then $0 < |x-2| < \delta$ implies $|f(x) - 7| < 3(\epsilon/3) = \epsilon$.

(ii) Let's show

$$\lim_{x \to 2} g(x) = 4$$

where $g(x) = x^2$. Given an arbitrary $\epsilon > 0$, our goal this time is to make $|g(x) - 4| < \epsilon$ by restricting $|x-2|$ to be smaller than some carefully chosen δ. As in the previous problem, a little algebra reveals

$$|g(x) - 4| = \left|x^2 - 4\right| = |x+2| \cdot |x-2|$$

We can make $|x-2|$ as small as we like, but we need an upper bound on $|x+2|$ in order to choose δ. The presence of the variable x causes some initial confusion, but keep in mind that we are discussing the limit as x approaches 2. If we agree that out δ-nieghbourhood around $c = 2$ must have radius no bigger than $\delta = 1$, then we get the upper bound $|x+2| \le |3+2| = 5$ for all $x \in V_\delta(c)$.

Now, choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x-2| < \delta$, then it follows that

$$\left|x^2 - 4\right| = |x+2||x-2| < (5)\frac{\epsilon}{5} = \epsilon$$

and the limit is proved.

Sequential criterion for Functional Limits.

We worked very hard in Chapter 2 to derive an impressive list of properties enjoyed by sequential limits. In particular, the Algebraic Limit Theorem and the Order Limit Theorem proved invaluable in a large number of arguments that followed. Not surprisingly, we are are going to need analogous statements for functional limits. Although it is not difficult to generate independent proos for these statements, all of them follow quite naturally from their sequential analogs once we derive the sequential criterion for functional limits motivated in the opening discussion of this chapter.

Theorem 4.2.3. **(Sequential Criterion for Functional Limits).** Given a function $f : A \to \mathbf{R}$ and a limit point c of A, the following two statements are equivalent:

(i) $\lim_{x \to c} f(x) = L$

(ii) For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.

Proof.

($\Longrightarrow$) Let's first assume that $\lim_{x \to c} f(x) = L$. To prove (ii), we consider an arbitrary sequence (x_n), which conveges to c and satisfies $x_n \neq c$. Our goal is to show that the image sequence $f(x_n) \in V_\epsilon(L)$. This is most easily seen using the topological formulation of the definition.

Let $\epsilon > 0$. Because we are assuming (i), definition 4.2.1B implies that there exists $V_\delta(c)$ with the property that all $x \in V_\delta(c)$ different from c satisfy $f(x) \in V_\epsilon(L)$. All we need to do then is argue that our particular sequence (x_n) is eventually in $V_\delta(c)$, But we are assuming that $(x_n) \to c$. This implies that there exists a point x_N after which $x_n \in V_\delta(c)$. It follows that $n \geq N$ implies $f(x_n) \in V_\epsilon(L)$ as desired.

($\Longleftarrow$) For this implication, we give a contrapositive proof, which is essentially a proof by contradiction. Thus, we assume that statement (ii) is true and carefully negate statement (i). To say that

$$\lim_{x \to c} f(x) \neq L$$

means that there exists at least one particular $\epsilon_0 > 0$ for which no δ is a suitable response. In other words, no matter what $\delta > 0$ we try, there will always be atleast one point

$$x \in V_\delta(c) \quad \text{with} \quad x \neq c \quad \text{for which} \quad f(x) \notin V_{\epsilon 0}(L)$$

Now, consider $\delta_n = 1/n$. From the preceding discussion, it follows that for each $n \in \mathbf{N}$, we may pick an $x_n \in V_\delta(c)$ with $x_n \neq c$ and $f(x) \notin V_{\epsilon_0}(L)$. But, now notice that the result of this is a sequence $(x_n) \to c$ with $x_n \neq c$, where the image seqeuence $f(x_n)$ certainly does not converge to L.

But, this contradicts (ii), which we are assuming is true for this argument, we may conclude that (i) must also hold.

Theorem 4.2.3 has several useful corollaries. In addition to the previously advertised benefit of granting us some short proofs of statements about how functional limits interact with algebraic combinations of functions, we also get an economical way of establishing that certain limits do not exist.

Corollary 4.2.4 (Algebraic Limit Theorem for Functional Limits). Let f and g be functions defined on a domain $A \subseteq \mathbf{R}$, and assume that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. Then.

(i) $\lim_{x \to c} kf(x) = kL$

(ii) $\lim_{x \to c} [f(x) + g(x)] = L + M$

(iii) $\lim_{x \to c} [f(x)g(x)] = LM$

(iv) $\lim_{x \to c} f(x)/g(x) = L/M$, provided $M \neq 0$.

Proof.

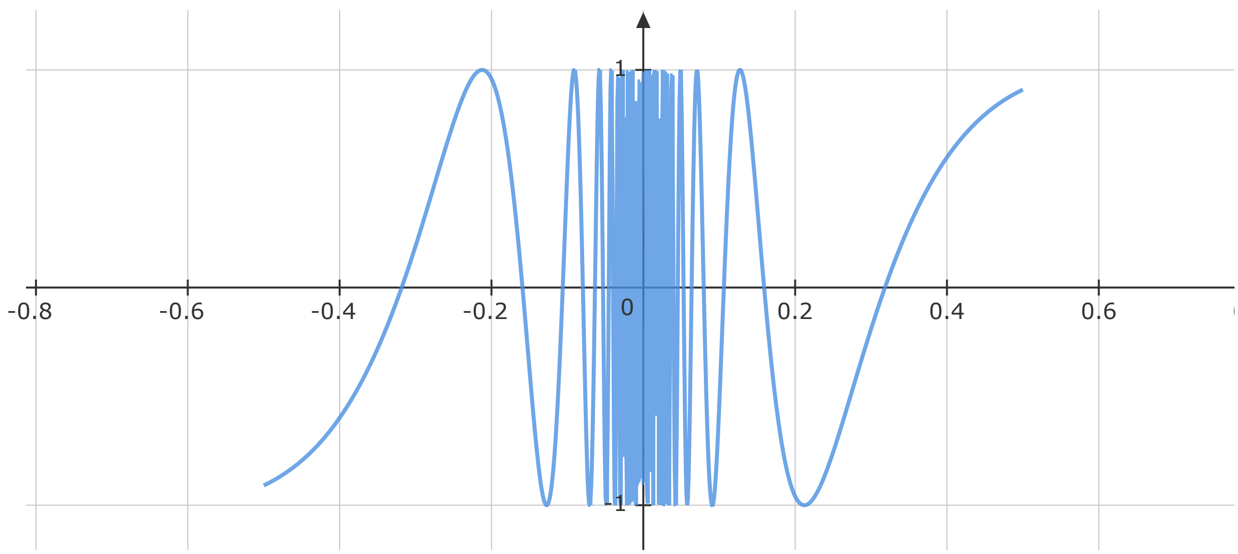
These follow from Theorem 4.2.3 and the Algebraic Limit Theorem for sequences. The details are requested in Exercise 4.2.1.

Corollary 4.2.5 (Divergence Criterion for Functional Limits). Let f be a function defined on A, and let c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and

$$\lim x_n = \lim y_n = c \quad \text{but} \quad \lim f(x_n) \neq \lim f(y_n)$$

then we can conclude that the functional limit $\lim_{x \to c} f(x)$ does not exist.

Example 4.2.6 Assuming the familiar properties of the sine function, let's show that $\lim_{x \to 0} \sin(1/x)$ does not exist.



If $x_n = 1/2n\pi$ and $y_n = 1/(2n\pi + \pi/2)$, then $\lim x_n = \lim y_n = 0$. However, $\sin(1/x_n) = 0$ for all $n \in \mathbf{N}$ while $\sin(1/y_n) = 1$. Thus,

$$\lim \sin(1/x_n) \neq \lim \sin(1/y_n)$$

so by the corollary 4.2.5, $\lim_{x \to 0} \sin(1/x)$ does not exist.

Exercises.

1. [Abbott, 4.2.1] (a) Supply the details for how the Corollary 4.2.4 part (ii) follows from the Sequential Criterion for Functional Limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

Proof.

From the sequential criterion for functional limits, we have that,

(i) For all sequences $(x_n) \subseteq A$, where $x_n \neq c$ such that $(x_n) \to c$, it follows that $f(x_n) \to L$.

(ii) For all sequences $(x_n) \subseteq A$, where $x_n \neq c$ such that $(x_n) \to c$, it follows that $g(x_n) \to M$.

From the Algebraic Limit Theorem, for all sequences $(x_n) \to c$, $x_n \neq c$, we have that:

$$\lim[f(x_n) + g(x_n)] = \lim f(x_n) + \lim g(x_n) = L + M$$

Therefore, from the sequential characterization of functional limits, we see that:

$$\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$$

(b) Now, write another proof of Corollary 4.2.4 part (ii) directly from the definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

Proof.

We are given that, $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.

By definition of functional limits, for all $\epsilon > 0$, there exists a δ-response, such that whenever $0 < |x - c| < \delta$, it follows that $|f(x) - L| < \epsilon$.

There exists $\delta_1 > 0$, such that whenever $0 < |x - c| < \delta_1$, it follows that $|f(x) - L| < \epsilon/2$.
There exists $\delta_2 > 0$, such that whenever $0 < |x - c| < \delta_2$, it follows that $|g(x) - M| < \epsilon/2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Let us explore the expression

$$\begin{aligned}
|f(x)+g(x)-(L+M)| &= |f(x)-L+g(x)-M| \\
&\leq |f(x)-L| + |g(x)-M| \qquad \text{Triangle Inequality} \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus, $f(x)+g(x) \to L+M$.

(c) Repeat (a) and (b) for corollary 4.2.4 part (iii).

(i) For all sequences $(x_n) \subseteq A$, where $x_n \neq c$ such that $(x_n) \to c$, it follows that $f(x_n) \to L$.

(ii) For all sequences $(x_n) \subseteq A$, where $x_n \neq c$ such that $(x_n) \to c$, it follows that $g(x_n) \to M$

From the Algebraic Limit Theorem, for all sequences $(x_n) \to c$, $x_n \neq c$, we have that:

$$\lim[f(x_n) \cdot g(x_n)] = \lim f(x_n) \cdot \lim g(x_n) = LM$$

Therefore, from the sequential characterization of limits, we see that

$$\lim_{x \to c} f(x) \cdot g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = LM$$

Let's now prove this fact using the definition of functional limits.

We are given that, $\lim_{x \to c} f(x) = L$.
By the definition of functional limits, for all $\epsilon > 0$, there exists a δ-response, $\delta > 0$, such that for all x satisfying, $0 < |x-c| < \delta$, we have $|f(x)-L| < \epsilon$..

Consider the expression

$$\begin{aligned}
|f(x)g(x)-LM| &= |f(x)g(x)-f(x)M \ + \ f(x)M \ - \ LM| \\
&\leq |f(x)||g(x)-M| + |M||f(x)-L|
\end{aligned}$$

Firstly, since $f(x) \to L$, there exists $\delta_2 > 0$, such that for all $x \in (c-\delta_2, c+\delta_2)$, we have $|f(x)-L| < \dfrac{\epsilon}{2(|M|+1)}$. We choose to make the distance $|f(x)-L|$ smaller than $\epsilon/2(|M|+1)$ to take care of both $|M| = 0$ and $|M| > 0$ cases.

Sinec, $f(x) \to L$, there exists $\delta_3 > 0$, such that for all $x \in (c-\delta_3, c+\delta_3)$, $|f(x)-L| < 1$. So, $|f(x)| < |L| \ + \ 1$.

Moreover, since $g(x) \to M$, there exists $\delta_2 > 0$, such that for all $x \in (c-\delta_2, c+\delta_2)$, we have $|g(x) - M| < \dfrac{\epsilon}{2(|L|+1)}$.

Now, let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Consequently, for all $x \in (c-\delta, c+\delta)$, we have

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M \ + \ f(x)M \ - \ LM| \\
&\le |f(x)||g(x) - M| + |M||f(x) - L| \\
&< (|L| \ + \ 1) \cdot \frac{\epsilon}{2(|L|+1)} + (|M|+1) \cdot \frac{\epsilon}{2(|M|+1)} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Hence, $\lim_{x\to c} f(x) \cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = LM$.

2. [Abbott, 4.2.2] For each stated limit, find the largest possible δ-neighbourhood that is a proper response to the given ϵ-challenge.

(a) $\lim_{x\to 3}(5x-6) = 9$, where $\epsilon = 1$.

Proof.

We are interested to make the distance $|(5x-6)-9| < 1$. Thus,

$$\begin{aligned}
|(5x-6)-9| &< 1 \\
|5x-15| &< 1 \\
|x-3| &< \frac{1}{5}
\end{aligned}$$

Thus, the largest δ-neighbourhood that is a response to $\epsilon = 1$ is $\left(3 - \dfrac{1}{5}, 3 + \dfrac{1}{5}\right) = \left(\dfrac{14}{5}, \dfrac{16}{5}\right)$.

(b) $\lim_{x\to 4} \sqrt{x} = 2$, where $\epsilon = 1$.

Proof.

We are interested to make the distance $\left|\sqrt{x} - 2\right| < 1$. We have:

$$
\begin{aligned}
\left|\sqrt{x}-2\right| &< 1 \\
\left|\sqrt{x}-2\right| \times \frac{\left|\sqrt{x}+2\right|}{\left|\sqrt{x}+2\right|} &< 1 \\
\frac{|x-4|}{\left|\sqrt{x}+2\right|} &< 1 \\
|x-4| &< \left|\sqrt{x}+2\right|
\end{aligned}
$$

Since $\left|\sqrt{x}-2\right| < 1,$ we have $\sqrt{x} > -1$. Thus, $\sqrt{x}+2 > 1$.

The above inequality will hold
Consequently, the largest δ-neighbourhood that is a response to $\epsilon = 1$, is $(4-5, 4+5) = (-1, 9)$.

(c) $\lim_{x \to \pi} [[x]] = 3$, where $\epsilon = 1$.

We are interested to make the distance $|[[x]]-3| < 1$. This inequality will be satisfied if and only if

$$
\begin{gathered}
|[[x]]-3| < 1 \\
\therefore -1 < [[x]]-3 < 1 \\
\implies 2 < [[x]] < 4 \\
\implies [[x]] = 3 \\
\implies 3 \le x < 4 \\
\implies 3-\pi \le x - \pi < 4-\pi
\end{gathered}
$$

The above inequality is satisfied, if and only if the distance $|x-\pi| < \pi-3$. Thus, the largest δ-neighbourhood that is a response to $\epsilon = 1$, is $(\pi-(\pi-3), \pi+\pi-3) = (3, 2\pi-3)$.

(d) $\lim_{x \to \pi} [[x]] = 3$, where $\epsilon = 0.01$.

We are interested to make the distance $|[[x]]-3| < .01$. This inequality will be satisfied if and only if

$$
\begin{gathered}
|[[x]]-3| < 0.01 \\
\therefore -.01 < [[x]]-3 < .01 \\
\implies 2.99 < [[x]] < 3.01
\end{gathered}
$$

$$\implies [[x]] = 3$$
$$\implies 3 \le x < 4$$
$$\implies 3 - \pi \le x - \pi < 4 - \pi$$

Thus, no matter how small the ϵ-challenge, the δ-response remains the same.

3. [Abbott 4.2.3] Review the definition of Thomae's function $t(x)$ from section 4.1.

(a) Construct three different sequences (x_n), (y_n) and (z_n) each of which converges to 1 without using the number 1 as a term in the sequence.

Let $x_n = 1 + \frac{1}{n}$, $y_n = 1 + \frac{\sqrt{2}}{n}$ and $z_n = \frac{(n+1)^2}{n^2}$.

(b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$ and $\lim t(z_n)$.

$t(x_n) = \frac{1}{n}$, $t(y_n) = 0$, $t(z_n) = \frac{1}{n^2}$. Thus, $\lim t(x_n) = 0$, $\lim t(y_n) = 0$, $\lim t(z_n) = 0$.

(c) ★ TODO.

We proprose that $\lim_{x \to 1} t(x) = 0$.
We are interested to prove that, for every ϵ-neighbourhood $(-\epsilon, \epsilon)$ around 0, there exists a δ-neighbourhood $(1-\delta, 1+\delta)$ around 1, with the property that for all $x \in (1-\delta, 1+\delta)$ different from 1, it follows that $t(x) \in (-\epsilon, \epsilon)$.

Consider the set of points $\{x \in \mathbf{R} : t(x) \ge \epsilon\}$. Consider the open interval $(x-\delta, x+\delta)$.

4. [Abbott, 4.2.4] Consider the reasonable but erroneous claim that

$$\lim_{x \to 10} \frac{1}{[[x]]} = \frac{1}{10}$$

(a) Find the largest δ that represents a proper response to the challenge of $\epsilon = 1/2$.

Proof.

We are interested to make the distance $|1/[[x]]-1/10| < 1/2$. We have:

$$-\frac{1}{2} < \frac{1}{[[x]]} - \frac{1}{10} < \frac{1}{2}$$
$$\frac{1}{10} - \frac{1}{2} < \frac{1}{[[x]]} < \frac{1}{10} + \frac{1}{2}$$
$$-\frac{2}{5} < \frac{1}{[[x]]} < \frac{3}{5}$$

Consider the inequality $-\frac{2}{5} < \frac{1}{[[x]]}$. If $[[x]] > 0$, then $-\frac{5}{2} < [[x]]$. If $[[x]] < 0$, then $-\frac{5}{2} > [[x]]$, so $[[x]] < -\frac{5}{2}$. So, $[[x]] > 0$ or $[[x]] < -\frac{5}{2}$.
Consider the inequality $\frac{1}{[[x]]} < \frac{3}{5}$. If $[[x]] > 0$, then $[[x]] > \frac{5}{3}$. If $[[x]] < 0$, then $[[x]] < \frac{5}{3}$.
So,
$[[x]] < 0$ or $[[x]] > \frac{5}{3}$.

For the inequalities to hold simultaneously, we must have $[[x]] < -\frac{5}{2}$ or $[[x]] > \frac{5}{3}$. Thus, $x < -3$ or $x > 2$. Consequently, $x - 10 < -13$ or $x - 10 > -8$. So, the largest δ-response to $\epsilon = 1/2$ is $\delta = 8$.

(b) Find the largest δ-response that represents a proper response to $\epsilon = 1/50$.

Proof.

We are interested to make the distance $|1/[[x]]-1/10| < 1/50$. We have:

$$-\frac{1}{50} < \frac{1}{[[x]]} - \frac{1}{10} < \frac{1}{50}$$
$$\frac{1}{10} - \frac{1}{50} < \frac{1}{[[x]]} < \frac{1}{10} + \frac{1}{50}$$
$$\frac{2}{25} < \frac{1}{[[x]]} < \frac{3}{25}$$

So, we must have $[[x]] < \frac{25}{2}$ or $[[x]] > \frac{25}{3}$. Therefore, $x < 13$ or $x \geq 9$. Thus, $x - 10 < 3$ or

$x - 10 \geq -1$. Thus, $-1 \leq x - 10 < 3$. So, the largest δ-response is $\delta = 1$.

(c) Find the largest ϵ-challenge for which there is no suitable δ-response possible.

Proof.

★ TODO.

We are interested to make

$$\left| \frac{1}{[[x]]} - \frac{1}{10} \right| > \epsilon$$

for all $\delta > 0$.

5. [Abbott, 4.2.5] Use Definition 4.2.1 to supply a proper proof for the following limit statements.

(a) $\lim_{x \to 2} (3x + 4) = 10$.

Proof.

We are interested to make the distance $|(3x + 4) - 10| < \epsilon$. Let us explore this inequality.

$$\begin{aligned} |(3x+4) - 10| &< \epsilon \\ |3x - 6| &< \epsilon \\ |x - 2| &< \frac{\epsilon}{3} \end{aligned}$$

Pick $\delta = \dfrac{\epsilon}{3}$. Then, for all $\epsilon > 0$, we have found a $\delta > 0$ such that, whenever $|x - 2| < \delta$, it follows that $|(3x + 4) - 10| < \epsilon$. Consequently, $\lim_{x \to 2} (3x + 4) = 10$.

(b) $\lim_{x \to 0} x^3 = 0$.

Proof.

We are interested to make the distance $\left|x^3\right| < \epsilon$. Let us explore this inequality.

$$\begin{aligned} \left|x^3\right| &< \epsilon \\ |x| &< \sqrt[3]{\epsilon} \end{aligned}$$

Pick $\delta = \epsilon^{1/3}$.

(c) $\lim_{x \to 2} \left(x^2 + x - 1\right) = 5$.

Proof.

We are interested to make the distance $\left|\left(x^2 + x - 1\right) - 5\right| < \epsilon$. Let us explore this inequality.

$$\begin{aligned} \left|x^2 + x - 6\right| &< \epsilon \\ |x+3| \cdot |x-2| &< \epsilon \\ |x-2| &< \frac{\epsilon}{|x+3|} \end{aligned} \tag{1}$$

☼ Note. *In mathematical logic, basically "stronger" means more implications. When $P \implies Q$ holds, then P is called* ***stronger*** *than Q*, and *Q is called* ***weaker*** *than P.*

We would like to stengthen the above condition $a < b$, by shrinking the interval. So, we'd like to increase the denominator. We need an upper bound for $|x+3|$.

For simplicity, assume $\delta < 1$. Then, $|x-2| < \delta$ implies $1 < x < 3$, which in turn implies that $4 < x + 3 < 6$. So, 6 is an upper bound for $|x+3|$.

If we prove the stronger condition

$$|x-2| < \epsilon/6 \tag{2}$$

then (1) holds.

Pick $\delta = \min\left\{1, \frac{\epsilon}{6}\right\}$. Then, $|x-2| < \delta$ implies $|x-2| < \frac{\epsilon}{6}$. Therefore, $|(x-2)(x+3)| < \frac{\epsilon}{6} \cdot 6 = \epsilon$. Thus, $\left|\left(x^2 + x - 1\right) - 5\right| < \epsilon$. So, $\lim_{x \to 2} \left(x^2 + x - 1\right) = 5$.

(d) $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

Proof.

We are interested to make the distance $\left|\frac{1}{x}-\frac{1}{3}\right|$ as small as we please. Let us explore the inequality $\left|\frac{1}{x}-\frac{1}{3}\right| < \epsilon$.

$$\frac{|x-3|}{3|x|} < \epsilon$$
$$|x-3| < 3\epsilon|x|$$

We need a lower bound on $|x|$. For simplicity, assume that $\delta < 1/2$. Then, $|x-3| < \delta$ implies that $\frac{5}{2} < x < \frac{7}{2}$, which in turn implies that $|x| > \frac{5}{2}$.

Pick $\delta = \min\left\{\frac{1}{2}, \frac{15}{2}\epsilon\right\}$. Then, $|x-3| < \delta$ implies that

$$\begin{aligned} |x-3| &< \frac{15}{2}\epsilon \\ \implies \frac{|x-3|}{3|x|} &< \frac{15}{2}\epsilon \cdot \frac{1}{3\cdot(5/2)} = \epsilon \\ \implies \left|\frac{1}{x}-\frac{1}{3}\right| &< \epsilon \end{aligned}$$

6. [Abbott, 4.2.6] Decide if the following claims are true or false, and give short justifications for each conclusion.

(a) If a particular δ has been constructed as a suitable response to a particular ϵ challenge, then any small positive δ will also suffice.

This proposition is true.

Justification. By the definition of functional limits, if for every ϵ-neighbourhood $(L-\epsilon, L+\epsilon)$ around L, there exists a δ-neighbourhood $(c-\delta, c+\delta)$ around c, such that for all $x \in (c-\delta, c+\delta)$, we have $f(x) \in (L-\epsilon, L+\epsilon)$, we say that, $\lim_{x\to c} f(x) = L$.

If $\delta' < \delta$, $V_{\delta'}(c) \subseteq V_\delta(c)$. So, if $x \in V_{\delta'}(c)$, it implies that $x \in V_\delta(c)$. Consequently, $f(x) \in V_\epsilon(L)$.

(b) If $\lim_{x \to a} f(x) = L$ and a happens to be in the domain of f, then $L = f(a)$.

This proposition is false.

Consider the function f defined piecewise as follows :

$$f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Then, $\lim_{x \to 0} f(x) = 0$. But, $f(0) = 1$.

(c) If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} 3[f(x) - 2]^2 = 3(L-2)^2$.

This proposition is true.

By the Algebraic Limit Theorem for functional limits,

$$\begin{aligned}
& \lim_{x \to a} 3[f(x) - 2]^2 \\
= \;& \lim_{x \to a} 3(f(x) - 2)) \cdot (f(x) - 2) \\
= \;& 3 \cdot \lim_{x \to a}(f(x) - 2)) \cdot \lim_{x \to a}(f(x) - 2) \\
= \;& 3 \cdot \left(\lim_{x \to a} f(x) - \lim_{x \to a} 2\right)) \cdot \left(\lim_{x \to a} f(x) - \lim_{x \to a} 2\right) \\
= \;& 3 \cdot (L-2)(L-2) \\
= \;& 3(L-2)^2
\end{aligned}$$

(d) If $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$ for any function g (with domain equal to the domain of f).

This proposition is false.

Consider the functions

$$f(x) = \begin{cases} x-a & \text{if } x \neq a \\ 1 & \text{if } x = a \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{1}{x-a} & \text{if } x \neq a \\ 1 & \text{if } x = a \end{cases}$$

Now, $\lim\limits_{x \to a} f(x) = 0$, but $\lim\limits_{x \to a} f(x) \cdot g(x) = \dfrac{x-a}{x-a} = 1$.

7. [Abbott, 4.2.7] Let $g : A \to \mathbf{R}$ and assume that f is a bounded function on A in the sense that there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$. Show that if $\lim\limits_{x \to c} g(x) = 0$, then $\lim\limits_{x \to c} g(x)f(x) = 0$ as well.

Proof.

We are given that $\lim\limits_{x \to c} g(x) = 0$. By the definition of functional limits, for all $\epsilon > 0$, there exists $\delta > 0$, such that whenever $|x - c| < \delta$, we have $|g(x)| \leq \epsilon$.

So, there exists $\delta > 0$, such that if $|x - c| < \delta$, we have $|g(x)| < \epsilon/M$.

Let us explore the expression $|g(x) \cdot f(x)|$.

$$\begin{aligned} |g(x)f(x)| &= |g(x)| \cdot |f(x)| \\ &< \frac{\epsilon}{M} \cdot M = \epsilon \end{aligned}$$

for all $x \in V_\delta(c)$.

Consequently, $\lim\limits_{x \to c} g(x) \cdot f(x) = 0$.

8. [Abbott, 4.2.8] Compute each limit or state that it does not exist. Use the tools developed in this section to justify each conclusion.

(a) $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$

Case I. Let $x_n = 2 + \frac{1}{n}$. Since, $x_n > 2$ for all $n \in \mathbf{N}$, $x_n - 2 > 0$, so $|x_n - 2| = x_n - 2 = 2 + \frac{1}{n} - 2 = \frac{1}{n}$. Consequently, we have:

$$\lim \frac{1/n}{1/n} = 1$$

Case II. Let $y_n = 2 - \frac{1}{n}$. Since, $y_n < 2$, for all $n \in \mathbf{N}$, $y_n - 2 < 0$, so $|y_n - 2| = -(y_n - 2) = -\frac{1}{n}$. Consequently, we have:

$$\lim \frac{-1/n}{1/n} = -1$$

Hence, $\lim x_n = \lim y_n = 2$, but $\lim_{x_n \to 2} f(x_n) \neq \lim_{y_n \to 2} f(y_n)$. Hence, the limit does not exist.

(b) $\lim_{x \to 7/4} \frac{|x-2|}{x-2}$

Since $(x_n) \to 7/4$, for all $\delta > 0$, there exists a point x_N in the sequence, such that for all $n \geq N$, we have $\left| x_n - \frac{7}{4} \right| < \delta$.

For simplicity, pick $\delta < \frac{1}{4}$. Then, $x_n \in \left(\frac{3}{2}, 2 \right)$ for all $n \geq N$. Consequently, $x_n - 2 < 0$ for all $n \geq N$. Hence, $|x_n - 2| = -(x_n - 2)$.

Consider the inequality

$$\begin{aligned} \left| \frac{-(x_n - 2)}{x_n - 2} - (-1) \right| &< \epsilon \\ 0 &< \epsilon \end{aligned}$$

This inequality is always satisfied for all $\epsilon > 0$. Hence, for all $\epsilon > 0$, if we pick $\delta < \frac{1}{4}$, then whenever $\left| x_n - \frac{7}{4} \right| < \delta$, we have $|f(x) - (-1)| < \epsilon$. Consequently, $\lim_{x \to 7/4} f(x) = -1$.

(c) $\lim_{x \to 0} (-1)^{[[1/x]]}$.

Let $x_n = \dfrac{1}{n}$. Then, $\lim_{x_n \to 0} f(x_n) = \lim(-1)^n$. We know that $(-1)^n$ is a divergent sequence. So, this limit does not exist.

(d) $\lim_{x \to 0} \sqrt[3]{x} \cdot (-1)^{[[1/x]]}$.

Let $x_n = \dfrac{1}{n}$. Then,

$$\begin{aligned} \lim_{(x_n) \to 0} f(x_n) &= \lim \sqrt[3]{\frac{1}{n}} \cdot (-1)^n \\ &= \lim \frac{(-1)^n}{n^{(1/3)}} \end{aligned}$$

Pick an arbitrary $\epsilon > 0$. Let us explore the expression

$$\begin{aligned} \left| \frac{(-1)^n}{n^{1/3}} \right| &< \epsilon \\ n^{1/3} &> \frac{1}{\epsilon} \\ n &> \frac{1}{\epsilon^3} \end{aligned}$$

If we let $N > \dfrac{1}{\epsilon^3}$, then for all $n \geq N$, we have

$$\left| \frac{(-1)^n}{n^{1/3}} \right| < \epsilon$$

Consequently, $\lim \dfrac{(-1)^n}{n^{1/3}} = 0$. Therefore, $\lim_{x \to 0} \sqrt[3]{x} \cdot (-1)^{[[1/x]]} = 0$.

9. **[Abbott, 4.2.9] (Infinite Limits).**

The statement $\lim_{x \to 0} 1/x^2 = \infty$ certainly makes intuitive sense. To construcct a rigorous definition in the challenge-response style of definition 4.2.1 for an infinite limit statement of this form, we replace the (arbitrarily small) $\epsilon > 0$ challenge with an (arbitrarily large) $M > 0$ challenge:

Definition: $\lim_{x \to c} f(x) = \infty$ means that for all $M > 0$, we can find a $\delta > 0$ such that whenever $0 < |x - c| < \delta$, it follows that $f(x) > M$.

(a) Show that $\lim_{x \to 0} 1/x^2 = \infty$ in the sense described in the previous definition.

Proof.

Let $M > 0$ be an arbitrary large real number. Let us explore the expression

$$\begin{aligned} \frac{1}{x^2} &> M \\ x^2 &< \frac{1}{M} \\ |x| &< \frac{1}{\sqrt{M}} \end{aligned}$$

Let $\delta = \dfrac{1}{\sqrt{M}}$. Then, for all $M > 0$, we have found a $\delta > 0$, such that whenever $|x| < \delta$, we have $\dfrac{1}{x^2} > M$. Consequently, $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

(b) Now, construct a definition for the statement $\lim_{x \to \infty} f(x) = L$. Show that $\lim_{x \to \infty} 1/x = 0$.

Proof.

For all $\epsilon > 0$, there exists an $M > 0$. such that whenever $|x| > M$, we have $|f(x) - L| < \epsilon$.

Let's prove that $\lim_{x \to \infty} 1/x = 0$.

Pick an arbitrary $\epsilon > 0$. We would like to make the distance $\left| \dfrac{1}{x} - 0 \right| = \left| \dfrac{1}{x} \right|$ as small as we please. Pick $M > \dfrac{1}{\epsilon}$. Then, for all $|x| > M$, we have $\left| \dfrac{1}{x} \right| < \epsilon$. Consequently,

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

(c) What would a rigorous definition for $\lim_{x \to \infty} f(x) = \infty$ look like? Give an example of such a limit.

Proof.

Definition. For all $N > 0$, there exists an $M > 0$, such that whenever $|x| > N$, we have $|f(x)| > M$.

Consider $\lim_{x \to \infty} x^2$. Pick an arbitrary $N > 0$. We would like to make the distance $\left|x^2\right| > M$. This implies $|x| > \sqrt{M}$. Pick $N > \sqrt{M}$. Then for all N, there exists M, such that whenever $|x| > N$, we have $\left|x^2\right| > N^2 > M$.

10. **[Abbott, 4.2.10] (Right and Left Limits).**

Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach a from the right-hand side."

(a) Give a proper definition in the style of the Definition 4.2.1 for the right hand and left-hand limit statements:

$$\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = M$$

Proof.

Definition of right-hand limit of a function. For all $\epsilon > 0$, iff there exists $\delta > 0$, such that whenever $x \in (a, a+\delta)$, we have $|f(x) - L| < \epsilon$, we say that $\lim_{x \to a^+} f(x) = L$.

Definition of left-hand limit of a function. For all $\epsilon > 0$, iff there exists $\delta > 0$, such that whenever $x \in (a-\delta, a)$, we have $|f(x) - M| < \epsilon$, we say that $\lim_{x \to a^-} f(x) = M$.

(b) Prove that $\lim_{x \to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

Proof.
$(\implies)$

We are given that $\lim_{x \to a} f(x) = L$. By definition, for all $\epsilon > 0$, there exists a $\delta > 0$, such that whenever $|x - a| < \delta$, $x \neq a$, we have $|f(x) - L| < \epsilon$. Thus, if $x \in (a - \delta, a)$, we have $f(x) \in (L - \epsilon, L + \epsilon)$. And if $x \in (a, a + \delta)$, it follows that $f(x) \in (L - \epsilon, L + \epsilon)$.

But, this would mean that, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

$(\Longleftarrow)$
Suppose that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

Consider an arbitrary sequence (z_n) in the domain of f, that converges to a, such that $z_n \neq a$.

Then, we have the following cases:
(1) (z_n) eventually lies in $(a - \delta, a)$
(2) (z_n) eventually lies in $(a, a + \delta)$
(3) (z_n) eventually lies in $(a - \delta, a + \delta)$ but neither (1) nor (2) holds.

Case (1).
By the sequential characterization of limits, $f(x)$ eventually lies in $(L - \epsilon, L + \epsilon)$,

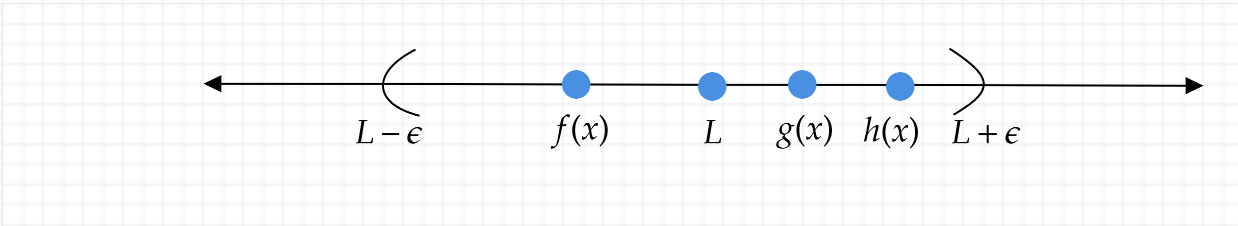
Case (2).
By the sequential characterization of limits, $f(x)$ eventually lies in $(L - \epsilon, L + \epsilon)$,

Case (3).
Suppose we let 0 indicate that the term of the sequence (z_n) belongs to $(a - \delta, a)$ and 1 indicate that a term of the sequence belongs to $(a, a + \delta)$. Then, the entire sequence is a binary string of 0s and 1s, e.g. $0101011110001000111000110011001100\ldots$. Therefore, the whole sequence can be split into two subsequences - one exlusively in the interval $(a - \delta, a)$ and the other exclusively in the interval $(a, a + \delta)$. The subsequences of a convergent sequence converge to the same limit value as the original sequence. Hence, $f(x) \in (L - \epsilon, L + \epsilon)$ for both the subsequences, and consequently for the whole sequence, for $n \geq N$.

11. [Abbott, 4.2.11] **(Squeeze Theorem)**. Let f, g and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point c of A, show that $\lim_{x \to c} g(x) = L$ as well.

Proof.
Pick an arbitrary $\epsilon > 0$. There exists $\delta_1 > 0$, such that for all $|x - c| < \delta_1$, we have

$|f(x)-L|<\epsilon$. There exists $\delta_2>0$, such that for all $|x-c|<\delta_2$, we have $|h(x)-L|<\epsilon$. Let $\delta=\min\{\delta_1,\delta_2\}$. Then, for all $|x-c|<\delta$, we have $|f(x)-L|<\epsilon$ and $|h(x)-L|<\epsilon$.



Since, $f(x)\le g(x)\le h(x)$, the distance $|g(x)-L|$ should be smaller than ϵ. Thus, we have found a $\delta>0$, such that $|g(x)-L|<\epsilon$.

As our choice of ϵ was arbitrary, this holds for all $\epsilon>0$. Consequently, $\lim_{x\to c} g(x)=L$.

4.3 **Continuous Functions.**

We have now come to a significant milestone in our progress towards a rigorous theory of real-valued functions - a proper definition of the seminal concept of continuity that avoids any intuitive appeals to "unbroken curves" or functions without **jumps** or **holes**.

Definition 4.3.1 (Continuity). A function $f: A\to \mathbf{R}$ is continuous at a point $c\in A$, if, for all $\epsilon>0$, there exists a $\delta>0$, such that whenever $|x-c|<\delta$ (and $x\in A$) it follows that $|f(x)-f(c)|<\epsilon$.
If f is continuous at every point in the domain A, then we way that f is continuous on A.

The definition of continuity looks much like the definition for functional limits with a few subtle differences. The most important is that we require the point c to be in the domain of f. The value $f(c)$ then becomes the value of the $\lim_{x\to c} f(x)$. With this observation in mind, it is tempting to shorten Definition 4.3.1 to say that f is continuous at $c\in A$ if:

$$\lim_{x\to c} f(x)=f(c)$$

This is fine as long as c is a limit point of A. If c is an isolated point of A, then $\lim_{x\to c} f(x)$ isn't defined, but definition 4.3.1 can still be applied. An unremarkable but noteworthy consequence of this definition is that functions are continuous at isolated points of their domains.

We saw in the previous section that, in addition to the standard $\epsilon - \delta$ definition, functional limits have a useful formulation in terms of sequences. The same is true of continuity. The next theorem summarizes these various equivalent ways to characterize the continuity of a function at a given point.

Theorem 4.3.2 (Characterizations of Continuity). Let $f : A \to \mathbf{R}$ and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:

(i) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies that $|f(x) - f(c)| < \epsilon$.

(ii) For all $V_\epsilon(f(c))$, there exists a $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $f(x) \in V_\epsilon(f(c))$.

(iii) For all $(x_n) \to c$ (with $x_n \in A$), it follows that $f(x_n) \to f(c)$.

If c is a limit point of A, then the above conditions are equivalent to

(iv) $\lim_{x \to c} f(x) = f(c)$.

Proof.
Statement (i) is just Definition 4.3.1 and statement (ii) standard rewording of (i) using the topilogical neighbourhoods in place of the absolute value notation. Statement (iii) is equivalent to (i) via an argument nearly identical to that of theorem 4.2.3, with some slight modifications for when $x_n = c$. Finally, statement (iv) is seen equivalent to (i) by considering definition 4.2.1 and observing that the case $x = c$ (which is excluded in the definition of functional limits) leads to the requirement that $f(c) \in V_\epsilon(f(c))$, which is trivially true.

The length of this list is somewhat deceiving. Statements (i), (ii) and (iv) are closely related and essentially remind us that functional limits have an $\epsilon - \delta$ formulation as well as a topological description. Statement (iii), however, is qualitatively different from the others. As a general rule, the sequential characterisation of continuity is typically the most useful for demonstrating that a function is not continuous at some point.

Corollary 4.3.3 (Crtierion for Discontinuity). Let $f : A \to \mathbf{R}$ and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$, where $(x_n) \to c$ but such that $f(x_n)$ does not converge to $f(c)$, we may conclude that the function f is not continuous at c.

The sequential characterization of continuity is also important for the other reasons that it was important for functional limits. In particular, it allows us to bring our catalog of results about the behaviour of sequences to bear on the study of continuous functions. The next theorem should be compared to Corollary 4.2.4 as well as to Theorem 2.3.3.

Theorem 4.3.4. (Algebraic Continuity Theorem). Assume that $f : A \to \mathbf{R}$ and $g : A \to \mathbf{R}$ are continuous at a point $c \in A$. Then,

(i) $kf(x)$ is continuous at c for all $k \in \mathbf{R}$.

(ii) $f(x) + g(x)$ is continuous at c.

(iii) $f(x)g(x)$ is continuous at c

(iv) $f(x)/g(x)$ is continuous at c, provided the quotient is defined.

Proof.

(i) Assume that $f(x)$ is continuous at $x = c$. Therefore, the distance $|f(x) - f(c)|$ can be made as small as we please. By the characterization of continuity, for all $\epsilon > 0$, there exists a $\delta > 0$, such that whenever $|x - c| < \delta$, the property $|f(x) - f(c)| < \epsilon$ is satisfied.

So, there exists $\delta > 0$, such that whenever $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon / k$. Consequently,

$$|kf(x) - kf(c)| < \epsilon$$

(ii) $f(x)$, $g(x)$ are continuous at $x = c$. Consequently, $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. Since, these limits exist, we can apply the algebraic limit theorem for functional limits. We have:

$$\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$$

Consequently, $f(x) + g(x)$ is continuous at $x = c$.

(iii) and (iv) are proved similarly.

These results provide us with the tools we need to firm up our arguments in the opening section of this chapter about the behavior of Dirichlet's function and Thomae's function. The details are requested in Exercise 4.3.7. Here are some more exmples of arguments for and against the continuity of some familiar functions.

Example 4.3.5. All polynomials are continuous $\mathbf{R}$. In fact, rational functions (i.e. quotients of pilynomials) are continuous wherever they are defined. To see why this is so, consider the identity function $g(x) = x$. Because, $|g(x) - g(c)| = |x - c|$, we can respond to a given $\epsilon > 0$ by choosing $\delta = \epsilon$, and it follows that g is continuous on all of $\mathbf{R}$. It is even simpler to show that a constant function $f(x) = k$ is continuous. Let $\delta = 1$ regardless of the value of ϵ does the trick. Because an arbitrary polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

consists of sums and products of $g(x)$ with different constant functions, we may conclude from the Algebraic Continuity Theorem that $p(x)$ is continuous.

Likewise, the Algebraic Continuity theorem implies that quotients of polynomials are continuous as long as the denominator is not zero.

Example 4.3.6. In Example 4.2.6, we saw that the oscillations of $\sin(1/x)$ are so rapid near the origin that $\lim_{x \to 0} \sin(1/x)$ does not exist. Now, consider the function

$$g(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

To investigate the continuity of g at $c = 0$, we can estimate

$$|g(x) - g(0)| = |x \sin(1/x) - 0| \leq |x|$$

Given $\epsilon > 0$, set $\delta = \epsilon$, so that whenever $|x - 0| = |x| < \delta$, it follows that $|g(x) - g(0)| < \epsilon$. Thus, g is continuous at the origin.

Example 4.3.7. Throughout the exercises, we have been using the greatest integer function
$h(x) = \lfloor x \rfloor$
which for each $x \in \mathbf{R}$ returns the largest integer n satisfying $n \leq x$. This familiar step function certainly has discontinous jumps at each integer value of its domain, but it is a useful exercise to try and articulate this observation in the language of analysis.

Given $m \in \mathbf{Z}$, define the sequence $(x_n) = m - \dfrac{1}{n}$. It follows that $(x_n) \to m$, $h(x_n) \to m - 1$, which does not equal $h(m) = m$. By Corollary 4.3.3, we see that h fails to be continuous at each $m \in \mathbf{Z}$.

Now let's see why h is continuous at a point $c \neq \mathbf{Z}$. Given $\epsilon > 0$, we must find a δ-

neighbourhood $V_\delta(c)$ such that $x \in V_\delta(c)$ implies that $h(x) \in V_\epsilon(h(c))$. We know that $c \in \mathbf{R}$ falls between consecutive integers $n < c < n+1$ for some $n \in \mathbf{Z}$. If we take $\delta = \min\{c-n, (n+1)-c\}$, then it follows from the definition of h that $h(x) = h(c) = n$ for all $x \in V_\delta(c)$. Thus, we certainly have

$$h(x) \in V_\epsilon(h(c))$$

whenever $x \in V_\delta(c)$.

This latter proof is quite different from the typical situation in that the value of δ does not atually depend on the chice of ϵ. Uusually, a smaller ϵ requires a small δ in response, but here the same value of δ works no matter how small ϵ is chosen.

Example 4.3.8. Consider $f(x) = \sqrt{x}$ defined on $A = \{x \in \mathbf{R} : x \geq 0\}$. Exercise 2.3.1 outlines a sequential proof that f is continuous on A. Here we give an $\epsilon - \delta$ proof of the same face.

Let $\epsilon > 0$. We would like to make the distance $\left|\sqrt{x} - \sqrt{c}\right|$ as small as we please. Let us explore the inequality $\left|\sqrt{x} - \sqrt{c}\right| < \epsilon$.

$$\begin{aligned} \left|\sqrt{x} - \sqrt{c}\right| &< \epsilon \\ \frac{|x-c|}{\sqrt{x}+\sqrt{c}} &< \epsilon \end{aligned}$$

Now, $\sqrt{x} \geq 0$. Replacing $\sqrt{x}$ by a lower bound decreases the value of the denominator thereby increasing the fraction and strengthening the condition. Thus,

$$|x-c| < \sqrt{c} \cdot \epsilon$$

If we pick $\delta = \sqrt{c} \cdot \epsilon$, then $|x-c| < \delta$ implies

$$\left|\sqrt{x} - \sqrt{c}\right| < \frac{\left|\sqrt{x} - \sqrt{c}\right|}{\sqrt{c}} < \frac{\sqrt{c} \cdot \epsilon}{\sqrt{c}} = \epsilon$$

as desired.

Although, we have now shown that

Exercises.

1.[Abbott, 4.3.1]. Let $g(x) = \sqrt[3]{x}$.

(i) Prove that g is continuous at $c = 0$.

Proof.

Pick an arbitrary $\epsilon > 0$. We are interested to make the distance $|g(x) - g(0)|$ as small as we please.

$$\begin{aligned} \left|\sqrt[3]{x} - \sqrt[3]{0}\right| &< \epsilon \\ |x| &< \epsilon^3 \end{aligned}$$

If we pick $\delta = \epsilon^3$, then $|x - 0| < \delta$ implies $|g(x) - g(0)| < \epsilon$. Consequently, g is continuous at $c = 0$.

(ii) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)\left(a^2 + ab - b^2\right)$ will be helpful).

Proof.

We are interested to make the distance $|g(x) - g(c)| = \left|\sqrt[3]{x} - \sqrt[3]{c}\right|$ as small as we please.
Now, this expression can be written as $\left|\sqrt[3]{x} - \sqrt[3]{c}\right| = |x - c| / \left|x^{2/3} - c^{1/3}x^{1/3} + c^{2/3}\right|$.
For simplicity let's assume that $\delta < |c|$. Then, $|x| > 0$. The denominator can be written as

$$\begin{aligned} x^{2/3} - c^{1/3}x^{1/3} + c^{2/3} &= \left(x^{1/3} - \frac{c^{1/3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}c^{1/3}\right)^2 \\ &\geq \left(\frac{\sqrt{3}}{2}c^{1/3}\right)^2 \end{aligned}$$

Replacing the denominator by its lower bound increases the left hand side expression, thereby strengthening what we would like to prove. Therefore, we are interested to show that

$$|x - c| < \left(3c^{2/3}/4\right)\epsilon$$

Pick $\delta = \left(3c^{2/3}/4\right)\epsilon$. Then, $|x-c| < \delta$ implies that

$$\begin{aligned} \left|\sqrt[3]{x}-\sqrt[3]{c}\right| &= \frac{|x-c|}{\left|x^{2/3}-c^{1/3}x^{1/3}+c^{2/3}\right|} \\ &= \frac{|x-c|}{\left|\left(x^{1/3}-\frac{c^{1/3}}{2}\right)^2+\left(\frac{\sqrt{3}}{2}c^{1/3}\right)^2\right|} \\ &\le \frac{|x-c|}{\frac{3}{4}c^{2/3}} < \frac{\left(3c^{2/3}/4\right)\epsilon}{\left(3c^{2/3}/4\right)} = \epsilon \end{aligned}$$

Consequently, $\sqrt[3]{x}$ is continuous at the point $c \neq 0$.

2.[Abbott, 4.3.2]**.** To gain a deeper understanding of the relationship between ϵ and δ in the definition of continuity, let's explore some modest variation of the definition 4.3.1. In all of these, let f be a function defined on all of $\mathbf{R}$.

(a) Let's say f is *onetinuous* at c if for all $\epsilon > 0$, we can choose $\delta = 1$ and it follows that $|f(x)-f(c)| < \epsilon$ whenever $|x-c| < \delta$. Find an example of a function that is onetinuous on all of $\mathbf{R}$.

Consider the constant function $f(x) = k$. For all $\epsilon > 0$, we can pick $\delta = 1$, such that whenever $|x-c| < \delta$, we have $|f(x)-f(c)| = |k-k| = 0 < \epsilon$. So, f is onetinuous on all of $\mathbf{R}$.

(b) Let's say f is *equaltinuous* at c if for all $\epsilon > 0$, we can choose $\delta = \epsilon$ and it follows that $|f(x)-f(c)| < \epsilon$ whenever $|x-c| < \delta$. Find an example of a function that is equaltinuous on $\mathbf{R}$ that is nowhere onetinuous, or explain why there is no such function.

Consider the function $g(x) = x$. This function is equaltinuous on $\mathbf{R}$. Pick an arbitrary $\epsilon > 0$ and let c be an arbitrary point. We are interested to make the distance $|g(x)-g(c)|$ as small as we please. Consider the inequality $|g(x)-g(c)| = |x-c| < \epsilon$. Set $\delta = \epsilon$. Then, $|x-c| < \delta$ implies $|g(x)-g(c)| < \epsilon$. As c was arbitrary, $g(x)$ is equaltinuous on all of $\mathbf{R}$. Moreover, $g(x)$ is nowhere onetinuous, since if we pick $\epsilon < 1$, then we must have $0 < \delta < 1$.

(c) Let's say that f is *lesstinuous* at c, if for all $\epsilon > 0$, we can choose $0 < \delta < \epsilon$ and it follows that $|f(x)-f(c)| < \epsilon$ wheneve $|x-c| < \delta$. Find an example of a function that is lesstinuous on $\mathbf{R}$ that is nowhere equaltinuous, or explain why there is no such function.

Consider $f:\mathbf{R}\to\mathbf{R}$ defined by $f(x) = 2x$. Let $\epsilon > 0$ be arbitrary. We would like to make the distance $|f(x) - f(c)| < \epsilon$. This implies $|2x - 2c| < \epsilon$. So, $|x - c| < \epsilon/2$.
Let $\delta = \epsilon/2$. Thus, $f(x)$ is lesstinuous and it is nowhere equaltinuous.

(d) Is every lesstinuous function continuous? Is every continuous function lesstinuous? Explain.

By definition, a function f is lesstinuous, if for all $\epsilon > 0$ there exists $0 < \delta < \epsilon$, such that whenever the distance $|x - c| < \delta$, the property $|f(x) - f(c)| < \epsilon$. Consequently, lesstinuous functions are continuous.

Continuous functions are lesstinuous. Because, if $\delta \geq \epsilon$ is the largest δ-response suitable for a given ϵ-challenge, any smaller $\delta < \epsilon$ should also guarantee that $f(x) \in V_\epsilon(f(c))$.

3. [Abbott, 4.3.3] (a) Supply a proof for the Theorem 4.3.9 using the $\epsilon - \delta$ characterization of continuity.

Theorem 4.3.9. (Composition of continuous functions). Given $f: A \to \mathbf{R}$ and $g: B \to \mathbf{R}$, assume that the range $f(A) = \{f(x): x \in A\}$ is contained in the domain B so that the composition $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$, and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

Proof.

f is continous at c. By characterization of continuity, for all $\epsilon > 0$, there exists a $\delta > 0$, such that whenever the distance $|x - c| < \delta$, it follows that, $f(x)$ is within ϵ of $f(c)$, that is $|f(x) - f(c)| < \epsilon$.

g is continous at $f(c)$. By characterization of continuity, for all $\xi > 0$, there exists an $\epsilon > 0$, such that whenever the distance $|f(x) - f(c)| < \epsilon$, it follows that, $g(f(x))$ is within ξ of $g(f(c))$, that is $|g(f(x)) - g(f(c))| < \xi$.

Since, $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$ for all $\epsilon > 0$, and $|f(x) - f(c)| < \epsilon \implies |g(f(x)) - g(f(c))| < \xi$ for all $\xi > 0$, it follows that whenever $|x - c| < \delta$, the property $|g(f(x)) - g(f(c))| < \xi$ is satisfied.

(b) Give another proof of this theorem using the sequential characterization of continuity. (from Theorem 4.3.2 (iii)).

f is continuous at c. For all sequences $(x_n) \to c$, it follows that $f(x_n) \to f(c)$. But, for all sequences $f(x_n) \to f(c)$, it follows that $g(f(x_n)) \to g(f(c))$.

Consequently, for all sequences $(x_n) \to c$, it follows that $g(f(x_n)) \to g(f(c))$. Thus, $g(f(\cdot))$ is continuous at c.

4. [Abbott, 4.3.4] Assume that f and g are defined on all of $\mathbf{R}$ and that $\lim_{x \to p} f(x) = q$ and $\lim_{x \to q} g(x) = r$.

(a) Give an example to show that it may not be true that

$$\lim_{x \to p} g(f(x)) = r$$

Proof.
Let $f, g : \mathbf{R} \to \mathbf{R}$ be defined as follows:

$$\begin{aligned} f(x) &= 0 \\ g(x) &= \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \end{aligned}$$

Note that $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 0} g(x) = 0$.
We have:

$$\begin{aligned} \lim_{x \to 0} g(f(x)) &= \lim_{x \to 0} g(0) \\ &= \lim_{x \to 0}(1) \\ &= 1 \end{aligned}$$

(b) Show that the result in (a) does follow if we assume that f and g are continuous?

Assume that f and g are continuous. So, f is continuous at p and g is continuous at q. By the definition of continuity, $\lim_{x \to p} f(x) = f(p) = q$ and $\lim_{x \to q} g(x) = g(q) = r$.

Consider $\lim_{x \to p} g(f(x))$. We have:

$$\begin{aligned}
\lim_{x\to p} g(f(x)) &= g\left(\lim_{x\to p} f(x)\right) \\
&= g\left(f\left(\lim_{x\to p} x\right)\right) \\
&= g(f(p)) = g(q) \\
&= r
\end{aligned}$$

(c) Does the result in (a) hold if we only assume that f is continuous? How about if we only assume that g is continuous?

No, the result in (a) do not hold, if we assume that only f is continuous.

Consider

$$\begin{aligned}
f(x) &= q \\
g(x) &= \begin{cases} r & \text{if } x \neq q \\ r' & \text{if } x = q \end{cases}
\end{aligned}$$

$\lim_{x\to p} f(x) = q$ and $\lim_{x\to q} g(x) = r$. But, $\lim_{x\to p} g(f(x)) = \lim_{x\to p} g(q) = r'$.

★ TODO.
Consider

$$\begin{aligned}
f(x) &= \begin{cases} q & \text{if } x \neq p \\ q' & \text{if } x = p \end{cases} \\
g(x) &= r
\end{aligned}$$

5. [Abbott, 4.3.5] Show using definition 4.3.1 that if c is an isolated point of $A \subseteq \mathbf{R}$, then $f : A \to \mathbf{R}$ is continuous at c.

Proof.

Let c be an isolated point of $A \subseteq \mathbf{R}$. Then, there exists $V_\delta(c)$, such that $V_\delta(c) \cap A = \{c\}$. Thus, for all $\epsilon > 0$, if we pick a δ, with the property $V_\delta(c) \cap A = \{c\}$, then $x \in (c-\delta, c+\delta)$ implies $f(x) = f(c)$ which belongs to $(f(c)-\epsilon, f(c)+\epsilon)$. Consequently, $f(x)$ is continuous at c.

6. [Abbott, 4.3.6] Provide an example of each or explain why the request is impossible.

(a) Two functions f and g, neither of which is continuous at 0, but such that $f(x)g(x)$ and

$f(x)+g(x)$ are continuous at 0.

Solution.

(a) Consider

$$f(x) = \begin{cases} x^2+3 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

$$g(x) = \begin{cases} x+2 & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$$

Then, $\lim\limits_{x \to 0}[f(x)+g(x)] = \lim\limits_{x \to 0}\left(x^2+x+5\right) = 5$. Moreover, $f(0)+g(0) = 5$. Also, $\lim\limits_{x \to 0} f(x) \cdot g(x) = \lim\limits_{x \to 0}\left(x^2+3\right)(x+2) = 6$ and $f(0) \cdot g(0) = 6$.

(b) A function $f(x)$ continuous at 0 and $g(x)$ not continuous at 0 such that $f(x)+g(x)$ is continuous at 0.

Solution.
This request is impossible. Assume that $f(x)+g(x)$ is continuous at 0 and $f(x)$ is continuous at 0.
Therefore, $\lim\limits_{x \to 0} f(x)+g(x) = f(0)+g(0)$ and $\lim\limits_{x \to 0} f(x) = f(0)$. So,
$\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} f(x)+g(x)-f(x) = \lim\limits_{x \to 0}(f(x)+g(x)) - \lim\limits_{x \to 0} g(x) = f(0)+g(0)-f(0) = g(0)$.
Consequently, $g(x)$ is continuous at 0.

(c) A function $f(x)$ is continuous at 0 and $g(x)$ not continuous at 0, such that $f(x) \cdot g(x)$ is continuous at 0.

Consider $f(x) = x^2$ and $g(x) = \dfrac{1}{x}$. Then, $f(x) \cdot g(x) = x$ which is continuous at 0.

(d) A function $f(x)$ not continuous at 0 such that $f(x) + \dfrac{1}{f(x)}$ is continuous at 0.

Consider the equation

$$t + \frac{1}{t} = 4$$

This equation has real solutions t_1, t_2. Solving for the roots of this quadratic equation, we have:

$$\begin{aligned} t^2+1 &= 4t \\ t^2-4t+1 &= 0 \\ (t-2)^2-3 &= 0 \\ \left(t-2-\sqrt{3}\right)\left(t-2+\sqrt{3}\right) &= 0 \end{aligned}$$

$t_1 = 2+\sqrt{3}$, $t_2 = 2-\sqrt{3}$.

Consider

$$f(x) = \begin{cases} 2+\sqrt{3} & \text{if } x \in \mathbf{Q} \\ 2-\sqrt{3} & \text{if } x \notin \mathbf{Q} \end{cases}$$

Consider the sequence $(x_n) = \dfrac{1}{n}$. The sequence $(x_n) \to 0$ and the image sequence $f(x_n) \to 2+\sqrt{3}$. Next consider $(y_n) = \dfrac{1}{\sqrt{n}}$. The sequence $(y_n) \to 0$ and the image sequence $f(y_n) \to 2-\sqrt{3}$. Consequently, f is not continuous at 0.

But, $h(x) = f(x) + \dfrac{1}{f(x)} = 4$ is the constant function which is continuous at 0.

(e) A function $f(x)$ not continuous at 0 such that $[f(x)]^3$ is continuous at 0.

This request is impossible. Assume that $f(x)$ is not continuous at 0. So, there exists $\epsilon > 0$, such that for all $\delta > 0$, there exists $|x| < \delta$, where $|f(x) - f(0)| > \epsilon$.

Consider the distance $\left|f(x)^3 - f(0)^3\right| = |f(x) - f(0)| \cdot \left|f(x)^2 - f(x)\cdot f(0) + f(0)^2\right|$.

$$\begin{aligned}
\left|f(x)^3 - f(0)^3\right| &= |f(x) - f(0)| \cdot \left|f(x)^2 - f(x)\cdot f(0) + f(0)^2\right| \\
&= |f(x) - f(0)| \cdot \left|\left(f(x) - \frac{f(0)}{2}\right)^2 + \frac{3}{4}f(0)^2\right| \\
&\geq \frac{3}{4}f(0)^2 \cdot |f(x) - f(0)| = \frac{3}{4}f(0)^2 \cdot \epsilon
\end{aligned}$$

So, we have found an $\epsilon' = \frac{3}{4}f(0)^2 \cdot \epsilon$ challenge, such that no matter what $\delta > 0$ response, there exists some $|x| < \delta$, where $\left|f(x)^3 - f(0)^3\right| > \epsilon'$.

Consequently, $[f(x)]^3$ is not continuous at 0.

7. [Abbott, 4.3.7] (a) Referring to the proper theorems, give a formal argument that Dirichlet's function from section 4.1 is nowhere-continuous on $\mathbf{R}$.

Proof.
Dirichlet's function based on the idea of the German mathematician Peter Lejune Dirichlet is as follows:

$$g(x) \quad = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

Let $c \in \mathbf{Q}$ be a rational number. Let $x_n = c + \frac{1}{\sqrt{n}}$. Pick $N > \frac{1}{\delta^2}$. Then, $x_n \in V_\delta(c)$ for all $n \geq N$. The image sequence $f(x_n) \to 0$. But, $f(c) = 1$. More formally, there exists $\epsilon = \frac{1}{2} > 0$ such that for all $\delta > 0$, there exists $x_n \in V_\delta(c)$, with $f(x_n) \notin V_\epsilon(f(c))$. So, f is not continuous at $c \in \mathbf{Q}$.

Let $d \in \mathbf{I}$ be an irrational number. Since, $\mathbf{Q}$ is dense in $\mathbf{R}$, for every $y \in \mathbf{R}$, there exists a sequence $(y_n) \in \mathbf{Q}$, such that $(y_n) \to d$. The image sequence $f(y_n) \to 1$, but $f(d) = 0$. Thus, there exists $\epsilon = \frac{1}{2} > 0$ such that for all $\delta > 0$, there exists $y_n \in V_\delta(d)$, with $f(y_n) \notin V_\epsilon(f(d))$.

By the corollary on the criterion for discontinuity, f is nowhere-continuous.

(b) Review the definition of Thomae's function in Section 4.1 and demonstrate that it fails to be continuous at every rational point.

Proof.
The Thomae's function is defined as follows:

$$t(x) \quad = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbf{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

Let $c \in \mathbf{Q}$ be a rational number. Let $x_n = c + \dfrac{1}{\sqrt{n}}$. Pick $N > \dfrac{1}{\delta^2}$. Then, $x_n \in V_\delta(c)$ for all $n \geq N$. The image sequence $t(x_n) \to 0$. But, $t(c) \neq 0$. More formally, there exists $\epsilon = \dfrac{1}{2} > 0$ such that for all $\delta > 0$, there exists $x_n \in V_\delta(c)$, with $t(x_n) \notin V_\epsilon(t(c))$. So, t is not continuous at $c \in \mathbf{Q}$.

(c) Use the characterization of continuity in Theorem 4.3.2 (iii) to show that Thomae's function is continuous at every irrational point in $\mathbf{R}$.

Let's proceed by contradiction. Pick an arbitrary $\epsilon = \dfrac{1}{M} > 0$. Assume that there exists a sequence $(x_n) \to p$ where p is any irrational point and suppose that $t(x_n) > \epsilon$ for infinitely many n. Then, because any fraction having a denominator larger than M is strictly less ϵ, we conclude that the denominator of $t(x_n)$ must be in $\{M-1, M-2, \ldots, 1\}$, that is $t(x_n) \in \left\{\dfrac{1}{M-1}, \dfrac{1}{M-2}, \ldots, 1\right\}$. This is a finite set. Consequently, the subsequence of (x_n) such that $t(x_n) > \epsilon$ is composed of terms of the form $x_n = \dfrac{k}{l}$, where $k \in \{l, l-1, \ldots, 0\}$ and $l \in \{M-1, M-2, \ldots, 1\}$. Thus, the set $\{x_n : |t(x_n)| > \epsilon\}$ is finite.

Pick $\delta = \min\{|x_i - p| : t(x_i) > \epsilon\}$. No matter what $N > 0$ we begin at, we find that there are no terms of the sequence closer to p, than the distance δ. Consequently, (x_n) does not converge to p. This is a contradiction.

Therefore, we conclude, that for all sequences $(x_n) \to p$ where p is an irrational point, the distance $|t(x_n) - t(p)| < \epsilon$, that is $t(x_n) \to 0$.

8. [Abott, 4.3.8] Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous

on all of $\mathbf{R}$.

Proof.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

This proposition is true.

Justification. Let (x_n) be an arbitrary sequence such that $(x_n) \to 1$, $x_n < 1$ for all n. Since, g is continuous at 1, $g(x_n) \to g(1)$. Now, $g(x_n) \geq 0$ for all $n \in \mathbf{N}$. Since, $g(x_n)$ is a convergent sequence, we can apply the Algebraic Order Limit theorem, and take limits on both sides.

Therefore,

$$\begin{aligned} \lim \ g(x_n) &\geq 0 \\ g(1) &\geq 0 \end{aligned}$$

(b) If $g(r) = 0$ for all $r \in \mathbf{Q}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.

This proposition is true.

Justification. Every $x \in \mathbf{R}$ is a limit point of $\mathbf{Q}$. So, for any real number $x \in \mathbf{R}$, there exists a sequence $(r_n) \subseteq \mathbf{Q}$, such that $r_n \neq x$ and $(r_n) \to x$. As g is continuous at x, $g(r_n) \to g(x)$.

Since $g(r_n)$ is the constant sequence $(0, 0, 0, \ldots)$, $g(x) = 0$.

(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbf{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

This proposition is true. The domain of g is the whole of $\mathbf{R}$ and there are no isolated points.

Justification. g is continuous at x_0. By definition of continuity of functions, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x \in (x_0 - \delta, x_0 + \delta)$, we have $g(x) \in (g(x_0) - \epsilon, g(x_0) + \epsilon)$.

Choose $\epsilon = \dfrac{g(x_0)}{2}$. So, there exists $V_\delta(x_0)$ such that $g(x) \in \left(\dfrac{g(x_0)}{2}, \dfrac{3g(x_0)}{2}\right)$.

9. [Abbott, 4.3.9] Assume $h : \mathbf{R} \to \mathbf{R}$ is continous on $\mathbf{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Proof.

Let x be an arbitrary limit point of K. We are interested to prove that $x \in K$.

Since, x is a limit point of K, there exists a sequence $(x_n) \subseteq K$, with $x_n \neq x$, such that $(x_n) \to x$.

As h is continuous on $\mathbf{R}$. for all sequences $(x_n) \to x$, it follows that $h(x_n) \to h(x)$.

But, $x_n \in K$, so $h(x_n) = 0$ for all $n \in \mathbf{N}$. Thus, $h(x_n)$ is the constant sequence $(0, 0, 0, \dots)$ which converges to 0. Therefore, $h(x) = 0$. Consequently, $x \in K$. Therefore, K is closed.

10. [Abbott, 4.3.10] Observe that if a and b are real numbers, then

$$\max\{a, b\} = \frac{1}{2}[(a + b) + |a - b|]$$

(a) Show that if $f_1, f_2, \dots, f_n$ are continuous functions then

$$g(x) = \max\{f_1(x), f_2(x), \dots, f_n(x)\}$$

is a continuous function.

Proof.

Assume that $g_k(x) = \max\{f_1(x), f_2(x), \dots, f_k(x)\}$ is continuous. We are interested to prove that $g_{k+1}(x) = \max\{g_k(x), f_{k+1}(x)\}$ is also continuous.

We have:

$$g_{k+1}(x) = \frac{1}{2}[(g_k(x) + f_{k+1}(x)) + |g_k(x) - f_{k+1}(x)|$$

We would like to prove that $g_{k+1}(x)$ is continuous.

By the Algebraic continuity theorem, since $g_k(x)$ and $f_{k+1}(x)$ are continuous functions, the sum and difference $g_k(x) + f_{k+1}(x)$ and $g_k(x) - f_{k+1}(x)$ are continuous functions.

Also, let $f(x)$ be an arbitrary continuous function. We are interested to prove that $|f(x)|$ is also continuous.

We are interested to make the distance $||f(x)| - |f(c)||$ as small as we please. Let $\epsilon > 0$ be arbitrary. Let us explore the condition $||f(x)| - |f(c)|| < \epsilon$. Since $||f(x)| - |f(c)|| \leq |f(x) - f(c)|$, replacing $||f(x)| - |f(c)||$ by $|f(x) - f(c)|$ strengthens the condition we are interested to prove.

We want to show that $|f(x) - f(c)| < \epsilon$. But, by the definition of continuity of f, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$.

Consequently, there exists $\delta > 0$, such that for all $|x - c| < \delta$, the condition $||f(x)| - |f(c)|| < \epsilon$ is satisfied. So, $\lim_{x \to c} |f(x)| = |f(c)|$. $|f(x)|$ is a continuous function.

We infer that, both $g_k(x) + f_{k+1}(x)$ and $|g_k(x) - f_{k+1}(x)|$ are continuous. Again by algebraic continuity theorem, $g_{k+1}(x) = \frac{1}{2}[(g_k(x) + f_{k+1}(x)) + |g_k(x) - f_{k+1}(x)|]$ is continuous.

(b) Let's explore whether the result in (a) extends to the infinite case. For each $n \in \mathbf{N}$, define f_n on $\mathbf{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq 1/n \\ n|x| & \text{if } |x| < 1/n \end{cases}$$

Now, explicitly compute $h(x) = \sup\{f_1(x), f_2(x), f_3(x), \ldots\}$.

Proof.

Consider the sequence $x_k = \frac{1}{k+1}$. We have $x_k \neq 0$ for all k and $(x_k) \to 0$. Enumerating the terms of this sequence,

$$(x_k) = \left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k+1}, \ldots\right)$$

The set $\{f_1(x_1), f_2(x_1), \ldots,\}$ is

$$\left\{\frac{1}{2}, 1, 1, 1, \dots\right\}$$

So, $h(x_1) = \sup_{n\in\mathbf{N}} f_n(x_1) = 1$.
The set $\{f_1(x_2), f_2(x_2), \dots,\}$ is

$$\left\{\frac{1}{2}, \frac{2}{3}, 1, 1, \dots\right\}$$

So, $h(x_2) = \sup_{n\in\mathbf{N}} f_n(x_2) = 1$.

The set $\{f_1(x_k), f_2(x_k), \dots, f_k(x_k), f_{k+1}(x_k), \dots\}$ is

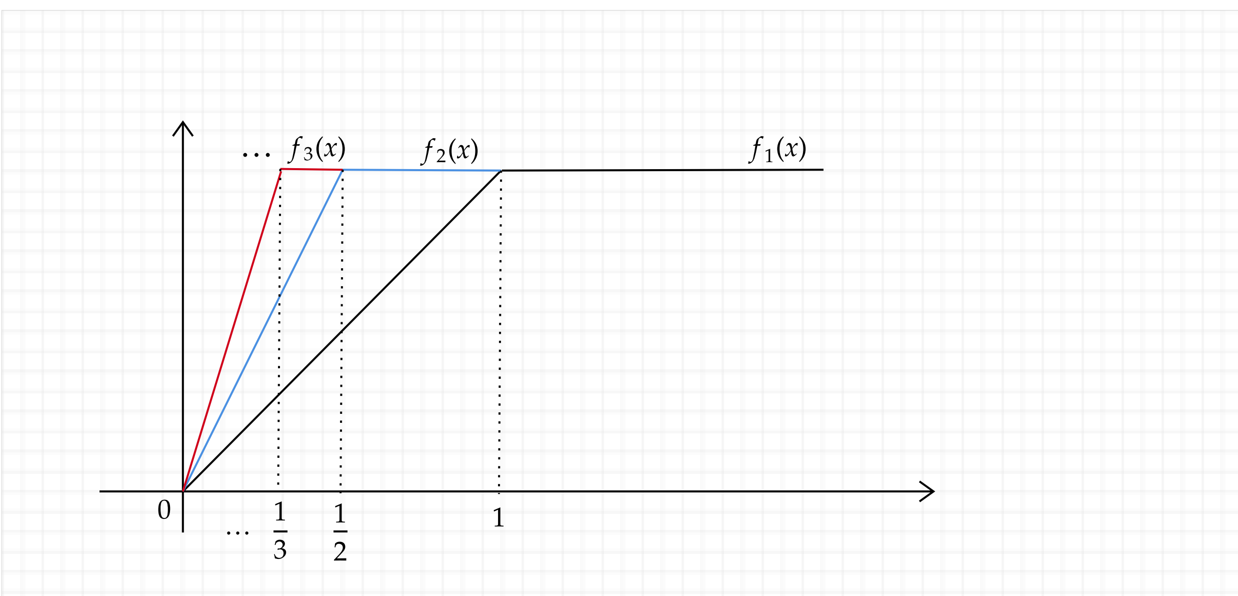
$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{k}{k+1}, 1, 1, \dots\right\}$$

So, $h(x_k) = \sup_{n\in\mathbf{N}} f_n(x_k) = 1$.

As $(x_k) \to 0$, the image sequence $h(x_k) \to 1$.

But, $h(0) = 0$.

Consequently, $h(x)$ is discontinuous at 0.



11. [Abbott, 4.3.11] **Contraction Mapping Theorem.** Let f be a function defined on all of $\mathbf{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

(a) Show that f is continuous on $\mathbf{R}$.

Let $x_0 \in \mathbf{R}$ and let $(x_n) \to x_0$ be an arbitrary sequence. By definition, for all $\delta > 0$, there exists a $N \in \mathbf{N}$, such that $|x_n - x_0| < \delta$ for all $n \geq N$.

We are interested to make the distance $|f(x) - f(x_0)| < \epsilon$. Replacing $|f(x) - f(x_0)|$ by $c|x - x_0|$ strengthens the condition we want to prove. So, we have $|x - x_0| < \dfrac{\epsilon}{c}$.

Pick $\delta = \dfrac{\epsilon}{c}$. Thus, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$.

Consequently, $f(x)$ is continuous on $\mathbf{R}$.

(b) Pick some point $y_1 \in \mathbf{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots)$$

In general, $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence, we may let $y = \lim y_n$.

Proof.

We are interested to make the distance $|y_n - y_m|$ as small as we please. Pick an arbitrary $\epsilon > 0$. Let's explore the expression $|y_n - y_m|$.

$$\begin{aligned}
|y_n - y_m| &= |y_n - y_{n-1} + y_{n-1} - y_{n-2} + \ldots + y_{m+1} - y_m| \\
&\leq |y_n - y_{n-1}| + |y_{n-1} - y_{n-2}| + \ldots + |y_{m+1} - y_m| && \{\text{Triangle Inequality}\} \\
&\leq c|y_{n-1} - y_{n-2}| + c|y_{n-2} - y_{n-3}| + \ldots \\
&+ c|y_m - y_{m-1}| && \{f \text{ is a contraction map}\} \\
&\leq \left(c^{n-2} + \ldots + c^{m-1}\right)|y_2 - y_1| \\
&= c^{m-1}\left(c^{n-m-1} + \ldots + 1\right)|y_2 - y_1| \\
&= c^{m-1}\left(\frac{1 - c^{n-m}}{1-c}\right)|y_2 - y_1| && \{\text{Assuming } n > m\} \\
&< c^{m-1}\left(\frac{1}{1-c}\right)|y_2 - y_1|
\end{aligned}$$

If $0 < b < 1$, we know that the sequence $(b^n) \to 0$. Therefore, for all $\epsilon > 0$, if we pick $N > \dfrac{\log \epsilon}{\log b}$, then $b^n < \epsilon$ for all $n \geq N$.

Consequently, if we pick a large N such that

$$c^N < \epsilon \left(\frac{1-c}{|y_2 - y_1|}\right)$$

then for all $n > m > N$, we have:

$$|y_n - y_m| < c^{m-1}\left(\frac{1}{1-c}\right)|y_2 - y_1| < \epsilon \left(\frac{1-c}{|y_2 - y_1|}\right) \cdot \left(\frac{1}{1-c}\right)|y_2 - y_1| = \epsilon$$

Thus, (y_n) is Cauchy. Cauchy sequences are convergent, so let $\lim y_n = y$.

(c) Prove that y is a fixed point of f (i.e. $f(y) = y$) and that it is unique in this regard.

y is a limit point of domain of f, since there exists a sequence (y_n) in $\mathbf{R}$, such that $(y_n) \to y$, with $y_n \neq y$ for all $n \in \mathbf{N}$.

By the definition of functional continuity, since f is continous, if x is a limit point of the domain of f, for all sequences $(x_n) \to x$, it follows that $f(x_n) \to f(x)$.

Consequently, as $(y_n) \to y$, $\lim f(y_n) = f(y)$. But, $\lim f(y_n) = \lim y_{n+1} = y$. Thus, $f(y) = y$. y is a fixed point of f.

Assume that y and y' are points such that $f(y) = y$ and $f(y') = y'$. We have that, $|f(y) - f(y')| = |y - y'|$. But, f is a contraction mapping. So, $|f(y) - f(y')| \leq c|y - y'|$. Thus,
$(1-c)|y - y'| \leq 0$. But, $1 > c \implies 1 - c > 0$. So, the only possibility is $|y - y'| = 0$. It follows that $y = y'$.

(d) As seen earlier, the sequence $(x, f(x), f(f(x)), \dots)$ converges to some x' such that $f(x') = x'$. But, the fixed points of f are unique. So, $x' = y$.

4.4 Continuous Functions on Compact Sets.

Given a function $f : A \to \mathbf{R}$, and a subset $B \subseteq A$, the notation $f(B)$ refers to the range of f over the set B; that is

$$f(B) = \{f(x) : x \in B\}$$

The adjectives open, closed, bounded, compact, perfect and connected are all used to describe the subsets of the real line. An interesting question is to sort out which, if any of these properties are preserved when a particular set B is mapped to $f(B)$ via a continuous function. For instance, if B is open and f is continuous, is $f(B)$ necessarily open? The answer to this question is no. If $f(x) = x^2$ and B is the open interval $(-1, 1)$, then $f(B)$ is the interval $[0, 1)$, which is not open.

The corresponding conjecture for closed sets also turns out to be false, although constructing a counterexample requires a little more thought. Consider the function

$$g(x) = \frac{1}{1 + x^2}$$

and the closed set $B = [0, \infty) = \{x : x \geq 0\}$. Because $g(B) = (0, 1]$ is not closed, we must conclude that continuous functions do not, in general, map closed sets to closed sets. Notice, however, that our particular counterexample required using an unbounded closed set B. This is not incidental. Sets that are closed and bounded - that is, compact sets - always get mapped to closed and bounded subsets by continuous functions.

Theorem 4.4.1 (Preservation of Compact Sets). Let $f : A \to \mathbf{R}$ be continuous on A. If

$K \subseteq A$ is compact, then $f(K)$ is compact as well.

Proof.

Let (y_n) be an arbitrary sequence contained in the range set $f(K)$. To prove this result, we must find a subsequence (y_{n_k}) that converges to a limit also in $f(K)$. The strategy is to take advantage of the assumption that the domain set K is compact by translating the sequence (y_n) - which is in the range of f - back to a sequence in the domain K.

To assert that $(y_n) \subseteq f(K)$ means that, for each $n \in \mathbf{N}$, we can find atleast one $x_n \in K$ with $f(x_n) = y_n$. This yields a sequence $(x_n) \subseteq K$. As K is compact, there exists a subsequence (x_{n_k}) that converges to a limit that is also in K. So, we can write $\lim x_{n_k} = x \in K$. Finally, we make use of the fact that, f is assumed to be continuous on A and so is continous at x in particular. Given that $(x_{nk}) \to x$, we conclude that $y_{nk} = f(x_{nk}) \to f(x)$. Because, $x \in K$, $f(x) \in f(K)$. So, $f(K)$ is compact.

An extremely important corollary is obtained by combining this result with the observation that compacts sets are bounded and contain their supremums and infimums.

Theorem 4.4.2 (Extreme Value Theorem). If $f : K \to \mathbf{R}$ is continuous on a compact set $K \subseteq \mathbf{R}$, then f attains maximum and minimum value. In other words, there exist x_0, x_1 such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Proof.

Because $f(K)$ is compact, we can set $\alpha = \sup f(K)$ and know $\alpha \in f(K)$. It follows that there exist $x_1 \in K$ with $\alpha = f(x_1)$. The argument for the minimum value is similar.

Uniform Continuity.

Although we have prove that polynomials are always continuous on $\mathbf{R}$, there is an important lesson to be learned by constructing direct proofs that the functions $f(x) = 3x + 1$ and $g(x) = x^2$ are everywhere continuous.

Example 4.4.3. (i) To show directly that $f(x) = 3x + 1$ is continuous at an arbitrary point $c \in \mathbf{R}$, we must argue that $|f(x) - f(c)|$ can be arbitrarily small for values of x near c. Now,

$$|f(x) - f(c)| = |(3x + 1) - (3c + 1)| = 3|x - c|$$

So, given $\epsilon > 0$, if we pick $\delta = \epsilon/3$. Then, $|x-c| < \delta$ implies that

$$|f(x) - f(c)| = 3|x-c| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

Of particular importance for this discussion is the fact that the choice of δ is the same regardless of which point $c \in \mathbf{R}$, we are consider.

(ii) Let's contrast this with what happens when we prove $g(x) = x^2$ is continuous on $\mathbf{R}$. Given $c \in \mathbf{R}$, we have

$$|g(x) - g(c)| = \left|x^2 - c^2\right| = |x+c||x-c|$$

As discussed, we are interested to show that $|x+c||x-c| < \epsilon$. Replacing $|x+c|$ by its upper bound will stengthen the condition we are interested to prove. This can be obtained by insisting that our choice of δ not exceed 1. This guarantees that all values of x under consideration will necessarily fall in the interval $(c-1, c+1)$. It follows that:

$$|x+c| \le |x| + |c| \le (|c| \ + \ 1) + |c| = 2|c| + 1$$

Now, let $\epsilon > 0$. If we choose $\delta = \min\left\{1, \dfrac{\epsilon}{2|c|+1}\right\}$, then $|x-c| < \delta$ implies

$$|g(x) - g(c)| = |x+c||x-c| < (2|c|+1) \cdot \frac{\epsilon}{(2|c|+1)} = \epsilon$$

Now, there is nothing deficient about this argument, but it is important to notice that, in the second prove the response δ depends on the value of c. The statement

$$\delta = \frac{\epsilon}{2|c|+1}$$

means that larger values of c are going to require smaller values of δ, a fact that should be evident from a consideration of the graph of $g(x) = x^2$. Given say, $\epsilon = 1$, a response of $\delta = 1/3$ is sufficient for $c = 1$, because $2/3 < x < 4/3$ certainly implies $0 < x^2 < 2$. However, if $c = 10$, then the steepness of the graph of $g(x)$ means that a much smaller δ is required -

$\delta = 1/21$ by our rule to force $99 < x^2 < 101$.

The next definition is meant to distinguish between these two examples.

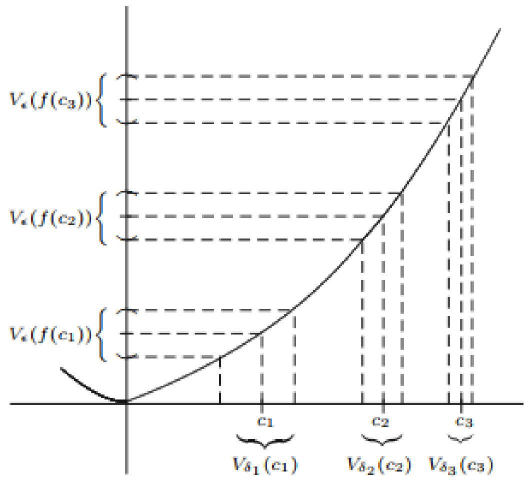


Figure 1: $g(x) = x^2$; a larger c requires a smaller δ

Definition 4.4.4. Uniform Continuity. A function $f : A \to \mathbf{R}$ is uniformly continuous on A if for every $\epsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Recall that to say that "f is continous on A" means to say that f is continuous at each individual point $c \in A$. In other words, given $\epsilon > 0$ and $c \in A$, we can find a $\delta > 0$ perhaps depending on c suh that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Uniform continuity is a strictly stronger property. The key distinction betwen asserting that f is **uniformly continuous on** A, versus simply **continuous on** A is that given an $\epsilon > 0$, a single $\delta > 0$ can be chosen that works simultaneously for all points $c \in A$. To say that a function is not uniformly continuous on a set A, then does not necessarily mean it is not continuous at some point. Rather, it means that there is some $\epsilon_0 > 0$ for which no single $\delta > 0$ is a suitable response for all $c \in A$.

Theorem 4.4.5. (Sequential Criterion for the Absence of Uniform Continuity). A function $f : A \to \mathbf{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0$$

Proof.
The negation of the definition 4.4.4 states that f is not uniformly continuous on A if and only if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, we can find two points x and y satisfying $|x - y| < \delta$ but with $|f(x) - f(y)| \geq \epsilon_0$.

Thus, if we set $\delta_1 = 1$, then there exist two points x_1 and y_1 where $|x_1 - y_1| < 1$ but $|f(x) - f(y)| \geq \epsilon_0$.

In a similar way, if we set $\delta_n = \dfrac{1}{n}$ where $n \in \mathbf{N}$, it follows that there exist points x_n and y_n with $|x_n - y_n| < \dfrac{1}{n}$, but where $|f(x_n) - f(y_n)| \geq \epsilon_0$. The resulting sequences (x_n) and (y_n) satisfy the requirements described in the theorem.

Conversely, if ϵ_0, (x_n) and (y_n) exist as described, it is straightfoward to see that no $\delta > 0$ is a suitable response for ϵ_0.

Example 4.4.6. The function $h(x) = \sin(1/x)$ is continuous at every point in the open interval $(0, 1)$ but is not uniformly continuous on this interval. The problem arises near zero, where the increasingly rapid oscillations take domain values that are quite close together to range values a distance 2 apart. To illustrate Theorem 4.4.5 $\epsilon_0 = 2$ and set

$$x_n = \frac{1}{\pi/2 + 2n\pi} \quad \text{and} \quad y_n = \frac{1}{3\pi/2 + 2n\pi}$$

Because each of these sequences tends to zero, we have $|x_n - y_n| \to 0$ and a short calculation reveals that $|h(x_n) - h(y_n)| = 2$ for all $n \in \mathbf{N}$.

Whereas continuity is defined at a single point, uniform continuity is always discussed in reference to a particular domain. In Example 4.4.3, we were not able to prove that $g(x) = x^2$ is uniformly continuous on $\mathbf{R}$, because larger values of x require smaller and smaller values of δ.

As another illustration of theorem 4.4.5, pick $\epsilon_0 = 2$, $x_n = n$ and $y_n = n + \dfrac{1}{n}$. Then $|x_n - y_n| = \dfrac{1}{n} \to 0$, and

$$|g(x_n) - g(y_n)| = \left|\left(n + \frac{1}{n}\right)^2 - n^2\right| = \left|n^2 + 2 + \frac{1}{n^2} - n^2\right| = \left|2 + \frac{1}{n^2}\right| \geq 2 \ .$$

It is true however, that $g(x)$ is uniformly continuous on the bounded set $[-10, 10]$. Returning to the agument set forth in Example 4.4.3 (ii), notice that if we restrict our attention to the domain $[-10, 10]$, then $|x+y| \leq 20$ for all x and y. Given $\epsilon > 0$, we can now choose $\delta = \dfrac{\epsilon}{20}$ and verify that if $x, y \in [-10, 10]$ satisfying $|x-y| < \delta$, then

$$|f(x) - f(y)| = \left|x^2 - y^2\right| = |x+y||x-y| < 20 \cdot \frac{\epsilon}{20} = \epsilon$$

In fact, it is not difficult to see how to modify this argument to show that $g(x)$ is uniformly continuous on any bounded set A in $\mathbf{R}$.

Now, Example 4.4.6 is included to keep us from jumping to the erroneous conclusion that functions that are contious on bounded domains are necessarily uniformly continuous. A general result does follow however, if we assume that the domain is compact.

Theorem 4.4.7. (Uniform Continuity on Compact Sets). A function that is continous on a compact set K is uniformly continuous on K.

Proof.
Assume $f : K \to \mathbf{R}$ is continuous at every point of a compact set $K \subseteq \mathbf{R}$. To prove that f is uniformly continuous on K, we argue by contradiction.

By the criterion in Theorem 4.4.5, if f is not uniformly continuous on K, then there exist two sequences (x_n) and (y_n) in K such that

$$\lim|x_n - y_n| = 0 \quad \text{while} \quad |f(x_n) - f(y_n)| \geq \epsilon_0$$

for some particular $\epsilon_0 > 0$.

Because K is compact, the sequence (x_n) has a convergent subsequence (x_{nk}) with $x = \lim x_{nk}$ also in K.

We could use the compactness of K again to produce a convergent subsequence of (y_n), but notice what happens when we consider the particular subsequence (y_{nk}) consisting of those terms in (y_n) that correspond to the terms in the convergent subsequence (x_{n_k}). By the Algebraic Limit Theorem,

$$\lim(y_{n_k}) = \lim((y_{n_k} - x_{n_k}) + x_{n_k}) = 0 + x$$

The conclusion is that both (x_{n_k}) and (y_{n_k}) converge to $x \in K$. Because f is assumed to be continuous at x, we have

$$\lim(f(x_{n_k}) - f(y_{n_k})) = f(x) - f(x) = 0$$

A contradiction arises when we recall that (x_n) and (y_n) were chosen to satisfy

$$|f(x_n) - f(y_n)| \geq \epsilon_0$$

for all $n \in \mathbf{N}$. We conclude, then, that f is indeed uniformly continuous on K.

Exercise Problems.

1. [Abbott, 4.4.1] (a) Show that $f(x) = x^3$ is continuous on all of $\mathbf{R}$.

Proof.

Let $c \in \mathbf{R}$ be an arbitrary point, and pick some $\epsilon > 0$. We are interested to make the distance $|f(x) - f(c)|$ as small as we please. Let's explore this inequality. We have:

$$|f(x) - f(c)| = \left|x^3 - c^3\right| = |x - c|\left|x^2 + c^2 + cx\right| = |x - c|\left|\left(x + \frac{c}{2}\right)^2 + \frac{3c^2}{4}\right|$$

Assume that $\delta < \dfrac{c}{2}$. Then, $\dfrac{c}{2} < x < \dfrac{3c}{2}$. Then

$$\left|\left(x + \frac{c}{2}\right)^2 + \frac{3c^2}{4}\right| < 4c^2 + \frac{3c^2}{4} = \frac{19c^2}{4}$$

We are interested to make the distance $|f(x) - f(c)| < \epsilon$. Replacing $|f(x) - f(c)|$ by its upper bound strengthens the condition we want to prove.

$$|x - c| < \frac{\epsilon}{(19c^2/4)}$$

Let $\delta = \min\left\{\dfrac{c}{2}, \dfrac{\epsilon}{(19c^2/4)}\right\}$. To check that this δ-response is indeed suitable for the given ϵ-challenge, we observe that:

$$|f(x) - f(c)| = |x - c|\left|\left(x + \frac{c}{2}\right)^2 + \frac{3c^2}{4}\right| < \frac{\epsilon}{(19c^2/4)} \cdot (19c^2/4) = \epsilon$$

Hence, for all $\epsilon > 0$, there exists $\delta > 0$, such that whenever $0 < |x - c| < \delta$, we have

$|f(x) - f(c)| < \epsilon$. Consequently, f is continuous at $c \in \mathbf{R}$. As c was arbitrary, f is everywhere continuous.

(b) Argue, using Theorem 4.4.5 that f is not uniformly continuous on $\mathbf{R}$.

Pick $x_n = n$ and $y_n = n + \dfrac{1}{n}$. Then, $|x_n - y_n| = \dfrac{1}{n}$. So, $\lim |x_n - y_n| = 0$. But,

$$\begin{aligned}
|f(x_n) - f(y_n)| &= \left| n^3 - \left(n + \frac{1}{n}\right)^3 \right| \\
&= \left| n^3 - \left(n^3 + 3 \cdot n^2 \cdot \frac{1}{n} + 3 \cdot n \cdot \frac{1}{n^2} + \frac{1}{n^3}\right) \right| \\
&= \left| n^3 - \left(n^3 + 3n + \frac{3}{n} + \frac{1}{n^3}\right) \right| \\
&= 3n + \frac{3}{n} + \frac{1}{n^3} \geq 3n
\end{aligned}$$

By the Archimedean property, there exists $N \in \mathbf{N}$, such that $N > \epsilon_0/3$.

Thus, given any arbitrary ϵ_0, if we look at the terms $n \geq N$, where $N > \epsilon_0/3$, the distance $|f(x_n) - f(y_n)| \geq 3n > 3(\epsilon_0/3) = \epsilon_0$. Consequently, f is not uniformly continuous on $\mathbf{R}$.

(c) Show that f is uniformly continuous on any bounded subset of $\mathbf{R}$.

Let $A \subseteq \mathbf{R}$ be a bounded subset of $\mathbf{R}$. Pick an arbitrary $\epsilon > 0$. Let $x, y \in A$. We are interested to make the distance $|f(x) - f(y)|$ as small as we please.

The expression $|f(x) - f(y)|$ can be simplified as:

$$\begin{aligned}
|f(x) - f(y)| &= \left|x^3 - y^3\right| \\
&= |x - y| \left|x^2 + xy + y^2\right| \\
&\leq |x - y| \left(\left|x^2 + y^2\right| + |x||y| \right) && \left\{\text{Triangle Inequality}\right\} \\
&\leq |x - y| \left(2M^2 + M^2\right) && \left\{\because A \text{ is bounded}\right\} \\
&= |x - y| 3M^2
\end{aligned}$$

If we replace $|f(x)-f(y)|$ by its upper bound in the inequality $|f(x)-f(y)| < \epsilon$, this strengthens the condition that we want to prove. So, we must prove that:

$$|x-y| < \frac{\epsilon}{3M^2}$$

Pick $\delta = \dfrac{\epsilon}{3M^2}$. To see that, this choice of δ indeed works, we observe:

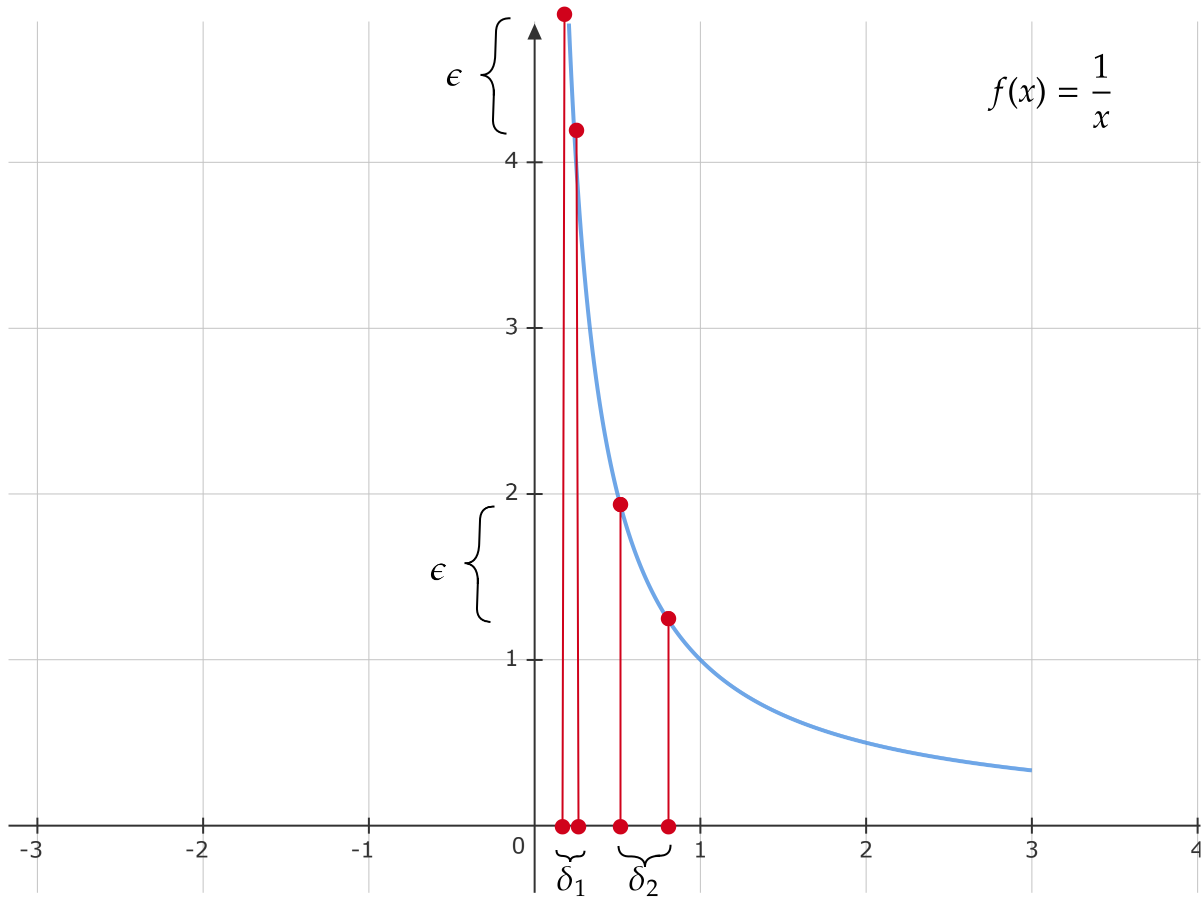
$$\begin{aligned} |f(x)-f(y)| &= |x-y|\left|x^2+xy+y^2\right| \\ &< \frac{\epsilon}{3M^2}\cdot 3M^2 \\ &= \epsilon \end{aligned}$$

As x, y were arbitrary to begin with, this δ-response works for all $x, y \in A$.

Consequently, for all $\epsilon > 0$, there exists a $\delta > 0$, such that for all points $x, y \in A$, that are no more than a distance δ apart, that is $|x-y| < \delta$, the range values are no more than ϵ apart, i.e. $|f(x)-f(y)| < \epsilon$. We conclude that, f is uniformly continuous.

2. [Abbott, 4.4.2] (a) Is $f(x) = 1/x$ uniformly continuous on $(0, 1)$?

Proof.



Intuitively, I don't think $f(x) = 1/x$ is uniformly continuous on $(0,1)$. Given an $\epsilon > 0$ and a point x, no matter what δ-response you pick, its always possible to find a point x_0 to its left, which requires a still smaller δ-response.

Let $c \in (0,1)$ and suppose we try to make the distance $|f(x) - f(c)| < \epsilon$. Then,

$$\begin{aligned} |f(x) - f(c)| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\ &= \frac{|x-c|}{|c||x|} \end{aligned}$$

Assume $\delta < 1$. Then, $|x| > |c| - 1$.

Therefore,

$$
\begin{aligned}
|f(x) - f(c)| &< \frac{|x-c|}{|c|(|c|-1)} \\
&< \frac{|x-c|}{(|c|-1)^2}
\end{aligned}
$$

If we replace $|f(x) - f(c)|$ by its upper bound in the inequality $|f(x) - f(c)| < \epsilon$, it strengthens the condition we are interested to prove. So, let us prove that:

$$|x - c| < \epsilon \cdot (|c| - 1)^2$$

Pick $\delta = \min\left\{1, \epsilon \cdot (|c|-1)^2\right\}$. To show that, this choice of δ indeed works, we observe:

$$|f(x) - f(c)| < \frac{|x-c|}{(|c|-1)^2} < \frac{1}{(|c|-1)^2} \cdot \epsilon \cdot (|c|-1)^2 = \epsilon$$

So, f is continous over $(0, 1)$. However, the algorithm for choosing the δ-response depends upon the choice of c.

Formal proof.

Let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{n+1}$. Then, $|x_n - y_n| = \left|\dfrac{1}{n} - \dfrac{1}{n+1}\right| = \dfrac{1}{n(n+1)}$.
So, $\lim |x_n - y_n| = 0$. But,

$$|f(x_n) - f(y_n)| = |n - (n+1)| = 1 > \frac{1}{2}$$

So, f is not uniformly continuous on $(0, 1)$.

(b) Is $g(x) = \sqrt{x^2 + 1}$ uniformly continuous on $(0, 1)$?

Proof.
Let $\epsilon > 0$ be arbitrary and pick $x, y \in (0, 1)$. We are interested to make the distance $|g(x) - g(y)|$ as small as we please.

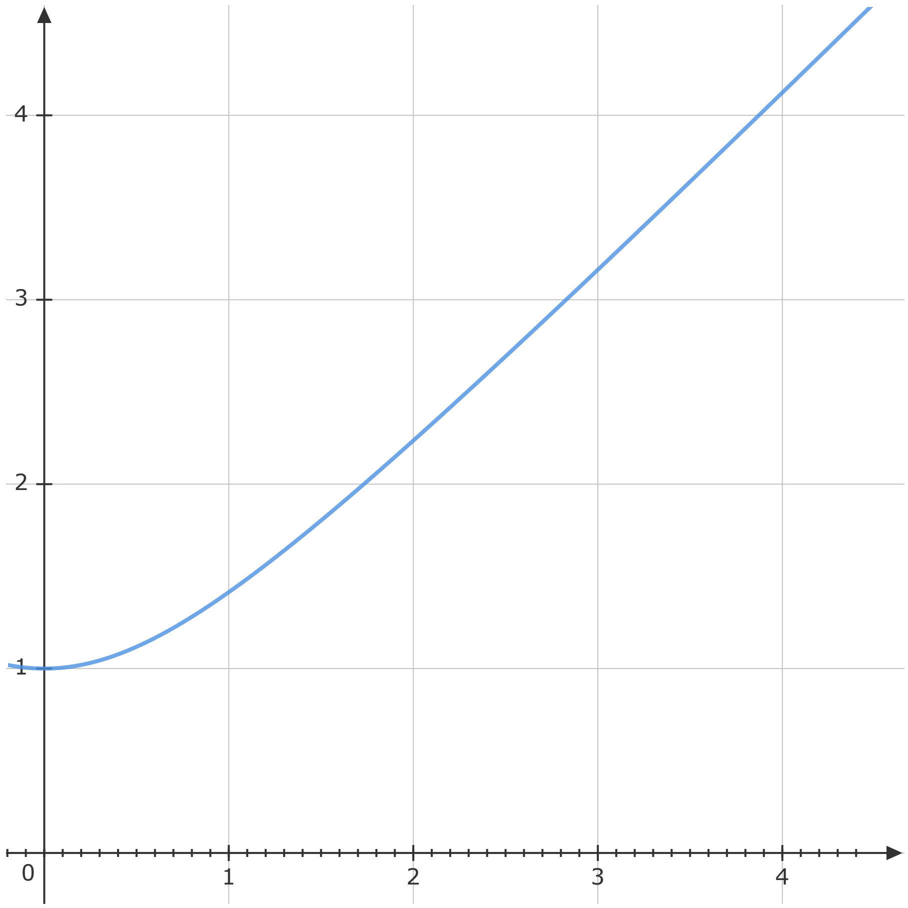
$$
\begin{aligned}
|g(x)-g(y)| &= \left|\sqrt{x^2+1}-\sqrt{y^2+1}\right| \\
&= \frac{\left|\left(\sqrt{x^2+1}\right)^2-\left(\sqrt{y^2+1}\right)^2\right|}{\left|\sqrt{x^2+1}+\sqrt{y^2+1}\right|} \\
&= \frac{\left|x^2-y^2\right|}{\left|\sqrt{x^2+1}+\sqrt{y^2+1}\right|} \\
&< \frac{|x-y|\,|x+y|}{\left|\sqrt{x^2}+\sqrt{y^2}\right|} = \frac{|x-y|\,|x+y|}{|x+y|} \\
&= |x-y|
\end{aligned}
$$

If we replace the expression $|g(x)-g(y)|$ by its upper bound in the inequality $|g(x)-g(y)| < \epsilon$, we are strengthening the condition we want to prove. So, let's prove that

$$|x-y| < \epsilon$$

Pick $\delta = \epsilon$. Then, $|x-y| < \delta$ implies $|g(x)-g(y)| < \epsilon$. As x, y were arbitrary points in $(0,1)$, this holds for all $x, y \in A$. Consequently, g is uniformly continuous on $(0,1)$.

We haven't defined the derivative of a function. But, $g'(x) = \dfrac{x}{\sqrt{x^2+1}}$. So, g is asymptotic to the line $y = x$. So, over the bounded interval (0,1), the same δ-ball would work for all points x, y.



(c) Is $h(x) = x\sin(1/x)$ uniformly continuous on $(0, 1)$?

Let's extend $h(x)$ to $[0, 1]$ as follows:

$$h(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since $K = [0, 1]$ is compact, and $h(x) = x\sin(1/x)$ is continuous over K, h is uniformly continuous on $[0, 1]$. Consequently, it is uniformly continuous over any subset of $[0, 1]$ including $(0, 1)$.

3. [Abbott, 4.4.3] Show that $f(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Proof.

Let's explore the expression $|f(x) - f(y)|$. We have:

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \\
&= \frac{\left|y^2 - x^2\right|}{y^2x^2} \\
&= \frac{|y-x|\,|y+x|}{x^2y^2} \\
&\leq |y-x|\left(\frac{|y|}{x^2y^2} + \frac{|x|}{x^2y^2}\right) && \left\{\text{Triangle Inequality}\right\} \\
&\leq |y-x|\left(\frac{1}{x^2|y|} + \frac{1}{|x|y^2}\right) \\
&\leq 2|y-x| && \{x \geq 1, y \geq 1\}
\end{aligned}$$

If we replace $|f(x) - f(y)|$ by its upper bound $|x-y|$, then we are strengthening the condition we want to prove. So, we must prove that

$$|x-y| \leq \epsilon/2$$

Pick $\delta = \epsilon/2$. Then, $|x-y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. Consequently, f is uniformly continuous over $[1, \infty)$.

To show that f is not uniformly continuous over $(0, 1]$, pick $x_n = \frac{1}{n+1}$, $y_n = \frac{1}{n}$. Then, $|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} \to 0$. But,

$$|f(x_n) - f(y_n)| = \left|(n+1)^2 - n^2\right| = |2n+1| = 2n+1 \geq \epsilon_0$$

4. [Abbott, 4.4.4] Decide whether each of the following statements is true or false, justifying each conclusion.

(a) If f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$ (meaning $1/f$ has bounded range).

Proof.

This proposition is true.

Justification. Since f is continuous on $[a,b]$, with $f(x) > 0$ for $x \in [a,b]$, and $[a,b]$ is compact, by the extreme value theorem, f attains a maximum and a minimum. There exists m and M such that $m \leq f(x) \leq M$ for all $x \in [a,b]$.

By the Algebraic continuity theorem, $\dfrac{1}{f}$ is continuous.
Since $f(x) > 0$ for all $x \in [a,b]$, we have $\dfrac{1}{M} \leq \dfrac{1}{f(x)} \leq \dfrac{1}{m}$.

(b) If f is uniformly continuous on a bounded set A, then $f(A)$ is bounded.

Let $\overline{A}$ be the closure of A. We are interested in extending f to $\overline{A}$.

f is uniformly continuous on the set A. Pick an arbitrary $\epsilon_0 > 0$. Then, there exists $\delta > 0$, such that for all $x, y \in A$, where $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon_0$.

Let c be a limit point in A. Then, there exists a sequence $(x_n) \subseteq A$, such that $(x_n) \to c$ and $x_n \neq c$ for all n. Since, (x_n) is Cauchy, there exists $N \in \mathbf{N}$, such that the distance $|x_n - x_m|$ is smaller than δ, for all $n > m \geq N$.

Because f is uniformly continuous, the distance $|f(x_n) - f(x_m)| < \epsilon_0$ for all $n > m \geq N$. Since, ϵ_0 was arbitrary, we can make the terms of the image sequence arbitrarily close. $f(x_n)$ is Cauchy and hence convergent. Let $f(c) = \lim_{n \to \infty} f(x_n)$.

Further, we would like to show that, for all sequences $(x_n) \to c$, the image sequence $f(x_n) \to f(c)$.
Let $(y_n) \to c$ and $(z_n) \to c$ be two sequences. By Algebraic limit theorem, $\lim(y_n - z_n) = 0$, so they can be made δ-close.

Since f is uniformly continuous on A, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $|y_n - z_n| < \delta$, we have $|f(y_n) - f(z_n)| < \epsilon$.

If we make the distance $|y_n - z_n|$ smaller than the prescribed δ above, $|f(y_n) - f(z_n)|$ are ϵ-close.

Consequently, $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n)$.

Define

$$g(x) = \begin{cases} \lim f(x_n) & ,\forall (x_n) \text{ such that } (x_n) \to c, \text{ if } x = c, c \in \overline{A} \setminus A \\ f(x) & x \in A \end{cases}$$

g is continuous over $\overline{A}$.

Since, $\overline{A}$ is compact, and g is continous over $\overline{A}$, g attains a maximum and minimum value over $\overline{A}$. That is $g(\overline{A})$ is bounded. Therefore, $g(A) = f(A)$ is bounded.

(c)

5. [Abbott, 4.4.5] Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

Proof.

Case I. $x, y \in (a, b]$.
There exists $\delta_1 > 0$, such that for all $x, y \in (a, b]$, $|x - y| < \delta_1$ implies that $|f(x) - f(y)| < \epsilon/2$.

Case II. $x, y \in [b, c)$.
There exists $\delta_2 > 0$, such that for all $x, y \in [b, c)$, $|x - y| < \delta_2$ implies that $|f(x) - f(y)| < \epsilon/2$.

Case III. Let $a < x < b < y < c$.
Then,

$$\begin{aligned} |g(x) - g(y)| &= |g(x) - g(b) + g(b) - g(y)| \\ &\leq |g(x) - g(b)| + |g(b) - g(y)| \end{aligned}$$

We have:
(1) $|x - b| < \delta_1 \implies |g(x) - g(b)|$
(2) $|b - y| < \delta_2 \implies |g(b) - g(y)|$

Pick $\delta = \min\{\delta_1, \delta_2\}$.

Then, for all $|x-y| < \delta$,

$$\begin{aligned} |g(x)-g(y)| &\le |g(x)-g(b)| + |g(b)-g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Consequently, g is uniformly continuous.

6. [Abbott, 4.4.6] Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation of why this is the case.

(a) A continuous function $f : (0,1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Proof.

Consider $f(x) = \dfrac{1}{x}$ defined and continuous on $(0,1)$.

Let $(x_n) = \dfrac{1}{n}$ be a Cauchy sequence in $(0,1)$, where $(x_n) \to 0$.

$f(x_n)$ is an unbounded sequence, and so it's not Cauchy.

(b) A uniformly continuous function $f : (0,1) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not Cauchy.

This request is impossible.

Suppose (x_n) is a Cauchy sequence in $(0,1)$. Then, for all $\delta > 0$, there exists $N \in \mathbf{N}$, $|x_n - x_m| < \delta$ for all $n > m \ge N$.

Since f is uniformly continuous, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all points x, y satisfying $|x-y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

Consequently, for all $\epsilon > 0$, there exists $N \in \mathbf{N}$, such that $|f(x_n) - f(x_m)| < \epsilon$ for all $n > m \ge N$.

So, $f(x_n)$ is Cauchy.

(c) A continuous function $f : [0, \infty) \to \mathbf{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence.

Proof.

This request is impossible.

$[0, \infty)$ is a closed set. For all Cauchy sequences $(x_n) \subseteq [0, \infty)$ such that $(x_n) \to c$, the $f(x_n) \to f(c)$. So, $f(x_n)$ is Cauchy.

7. [Abbott, 4.4.7] Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof.
We are interested to make $|f(x) - f(y)|$ as small as we please.

$$\begin{aligned} |f(x) - f(y)| &= \left|\sqrt{x} - \sqrt{y}\right| \\ \left|\sqrt{x} - \sqrt{y}\right|^2 &\leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| \end{aligned}$$

So, if we pick $\delta = \epsilon^2$, then $|x - y| < \delta$ implies that
$|f(x) - f(y)| = \sqrt{\left|\sqrt{x} - \sqrt{y}\right|^2} < \sqrt{|x - y|} = \epsilon.$

Consequently, f is uniformly continuous.

8. [Abbott, 4.4.8] Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.

Proof.

This request is impossible.

By the result on the preservation of compact sets, if a function $f : A \to \mathbf{R}$ is continuous and $K \subseteq A$ is compact, then $f(K)$ is compact as well. So, $f([0, 1])$ must be compact.

(b) A continuous function on $(0,1)$ with range $[0,1]$.

Proof.

Consider $f(x)$ defined as

$$f(x)=\begin{cases}0 & \text{if } 0<x\le\frac{1}{3}\\ 3x-1 & \text{if } \frac{1}{3}<x\le\frac{2}{3}\\ 1 & \text{if } \frac{2}{3}<x<1\end{cases}$$

The range of this function is $[0,1]$.

(c) A continuous function on $(0,1]$ with range $[0,1]$.

Proof.

Consider $|\sin(1/x)|$ restricted to $(0,1)$. The range of this function is $[0,1]$.

9. [Abbott, 4.4.9] (**Lipschitz Functions**) A function $f : A \to \mathbf{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left|\frac{f(x)-f(y)}{x-y}\right|\le M$$

for all $x \ne y \in A$. Geometrically speaking, a function f is Lipschitz, if there is a uniform bound on the magnitude of the slopes of the lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbf{R}$ is Lipschitz, then it is uniformly continuous on A.

Proof.

If $f : A \to \mathbf{R}$ is Lipschitz, then there exists M, for all $x, y \in A$ such that

$$\begin{aligned} \left| \frac{f(x) - f(y)}{x - y} \right| &\leq M \\ |f(x) - f(y)| &\leq M|x - y| \end{aligned}$$

Replacing $|f(x) - f(y)|$ by $M|x - y|$ strengthens the inequality we wish to prove. So,

$$|x - y| \leq \frac{\epsilon}{M}$$

Choose $\delta = \epsilon/M$. Then, for all $x, y \in A$ satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$. Consequently, f is uniformly continuous.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Proof.

The converse is not true. Consider $y = x^2$.

$$\begin{aligned} \frac{|f(x) - f(y)|}{|x - y|} &= \frac{\left|x^2 - y^2\right|}{|x - y|} \\ &= |x + y| \end{aligned}$$

But, $|x + y|$ is unbounded. So, $f(x)$ is not Lipschitz.

10. [Abbott, 4.4.10] Assume that f and g are uniformly continuous functions defined on a common domain A. Which of the following combinations are necessarily uniformly continuous on A?

$$f(x) + g(x), \quad f(x)g(x), \quad \frac{f(x)}{g(x)}, \quad f(g(x))$$

(Assume that the quotient and the composition are properly defined and thus atleast continuous).

Proof.

Let us explore the expression

$$\begin{aligned}|f(x)+g(x)-(f(y)+g(y))| &= |f(x)-f(y)+g(x)-g(y) \\ &\leq |f(x)-f(y)|+|g(x)-g(y)| \quad \left\{\text{Triangle Inequality}\right\}\end{aligned}$$

Pick an arbitrary $\epsilon > 0$.

There exists $\delta_1 > 0$, such that for all $x, y \in A$ satisfying $|x-y| < \delta_1$, we have $|f(x)-f(y)| < \epsilon/2$.
There exists $\delta_2 > 0$, such that for all $x, y \in A$ satisfying $|x-y| < \delta_2$, we have $|g(x)-g(y)| < \epsilon/2$.

Pick $\delta = \min\{\delta_1, \delta_2\}$.

For all $|x-y| < \delta$, we have

$$\begin{aligned}|f(x)+g(x)-(f(y)+g(y))| &\leq |f(x)-f(y)|+|g(x)-g(y)| \\ &\leq \epsilon/2+\epsilon/2 = \epsilon\end{aligned}$$

Consider $A = [1, \infty)$. Pick $f(x) = x$ and $g(x) = x$. f, g are uniformly continous on A. But, $f(x) \cdot g(x) = x^2$ is not uniformly continuous.

Consider $A = (0, 1)$. Let $f(x) = 1$, $g(x) = x$. Both f, g are uniformly continous on A. Moreover, $\dfrac{f(x)}{g(x)} = \dfrac{1}{x}$ is well-defined and continuous on $(0, 1)$. But, $\dfrac{1}{x}$ is not uniformly continous on A.

For $f(g(x))$ to be well-defined, the range of g must be a subset of the domain of f. So, $g(A) \subseteq A$. Since $f(x)$ is uniformly continuous over A, it is uniformly continuous over any subset of A including $g(A)$. Consequently, $f(g(x))$ is uniformly continuous on A.

11. [Abbott, 4.4.11] (**Topological Characterization of Continuity**). Let g be defined on all of $\mathbf{R}$. If B is a subset of $\mathbf{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbf{R}$ is an open set.

Proof.

4.5 **The Intermediate Value Theorem.**

The Intermediate Value Theorem (IVT) is the name given to the very intuitive observation that a continuuous function f on a closed interval $[a, b]$ attains every value that falls between the range values $f(a)$ and $f(b)$.

Here is this observation in the language of analysis.

Theorem 4.5.1 (**Intermediate Value Theorem**). Let $f : [a, b] \to \mathbf{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.

This theorem was freely used by the mathematicians of the 18th century (including Euler and Gauss) without any consideration of its validity. In fact, the first analytical proof was not offered until 1817 by Bolzano in a paper that also contains the first appearance of somewhat modern definition of continuity. This emphasizes the significance of this result. As discussed, Bolzano and his contemporaries had arrived at a point in the evolution of mathematics where it was become increasingly important to firm up the foundations of the subject. Doing so, however, was not simply a matter of going back and supplying the missing proofs. The real battle lay in first obtaining a thorough and mutually

upon understanding of the relevant concepts. The importance of the Intermediate Value Theorem for us is similar in that our understanding of continuity and the nature of the real line is now mature enough for a proof to be possible. Indeed there are several satisying arguments for this simple result, each one isolating, in a slightly different way, the interplay between continuity and completeness.

Preservation of Connected Sets

The most potentially useful way to understand the Intermediate Value Theorem (IVT) is as a special case of the fact that continuous functions map connected sets to connected sets. In Theorem 4.4.1, we saw that if f is continuous function on a compact set K, then the range set $f(K)$ is also compact. The analogoous observation holds for connected sets.

Theorem 4.5.2. (**Preservation of Connected Sets**). Let $f : G \to \mathbf{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected as well.

Proof.

Intending to use the characterization of connected sets in Theorem 3.4.6, let $f(E) = A \cup B$ where A and B are disjoint and non-empty. Our goal is to produce a sequence contained in one of these sets that converges to a limit in the other.

Let

$$C = \{x \in E : f(x) \in A\} \quad \text{and} \quad D = \{x \in E : f(x) \in B\}$$

The sets C and D are the pre-images of A and B, respectively. Using the properties of A and B, it is straight-forward to check that C and D are non-empty and disjoint and satisfy $E = C \cup D$.

Since E is connected, there exists a convergent sequence $(x_n) \to x$, with the sequqnce terms (x_n) belonging exactly one of the sets C, D and the limit x in the other. As f is continuous over E, the image sequence $f(x_n) \to f(x)$ with $f(x_n)$ in one of the sets A, B and $f(x)$ in the other. Therefore, with another nod to theorem 3.4.5, $f(E)$ is connected.

In $\mathbf{R}$, a set is connected if and only if it is a (possibly unbounded) interval. This fact together with theorem 4.5.2 leads to a short proof of the Intermediate Value Theorem (IVT). Since $f([a, b])$ is a connected set and $f(a) < L < f(b)$ with $f(a), f(b) \in f([a, b])$, then $L \in f([a, b])$. Thus, there exists $c \in (a, b)$ such that $f(c) = L$.

We should point out that the proof in theorem 4.5.2 does not make use of the equivalence between connected sets and intervals in $\mathbf{R}$, but relies only on the general definitions. The previous comment that this is the most useful way to approach IVT stems from the fact that, although it is not discussed here, the definitions of continuity and connectedness can be easily adapted to higher dimensional settings. Theorem 4.5.2 then, remains a valid conclusion in higher dimensions, whereas Intermediate Value Theorem is essentially a one-dimensional result.

Completeness.

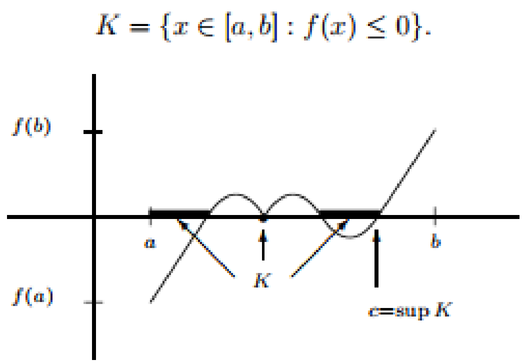
A typical way the Intermediate Value Theorem is applied is to prove the existence of roots. Given $f(x) = x^2 - 2$, for instance, we see that $f(1) = -1$ and $f(2) = 2$. Therefore, there exists a point $c \in (1, 2)$ such that $f(c) = 0$.

In this case, we can easily compute $c = \sqrt{2}$, meaning we really did not need IVT to show that f has a root. We spent a good deal of time in Chapter 1 proving that $\sqrt{2}$ exists, which was only possible once we insisted on the Axiom of Completeness as part of our assumptions about the real numbers. The fact that the Intermediate Value Theorem has just asserted that $\sqrt{2}$ exists suggests that another way to understand this result is in terms of the relationship between the continuity of f and the completeness of $\mathbf{R}$.

The Axiom of Completeness (AoC) from the first chapter states that non-empty subsets of $\mathbf{R}$, that are bounded above have a least upper bound. Later we saw that the Nested Interval Property (NIP) is an equivalent way to assert that the real numbers have no gaps. Either of these characterizations of completeness can be used as the cornerstone for an alternative proof of theorem 4.5.1.

Proof. **I.** (First approach using AoC).
To simplify matters a bit, let's consider the special case where f is a continuous function satisfying $f(a) < 0 < f(b)$ and show that $f(c) = 0$ for some $c \in (a, b)$. First let

$$K = \{x \in [a, b] : f(x) \leq 0\}$$



Notice that K is bounded above by b, and $a \in K$ so K is not empty. Thus, we may appeal to the axiom of completeness to assert that $c = \sup K$ exists.

There are three cases to consider:

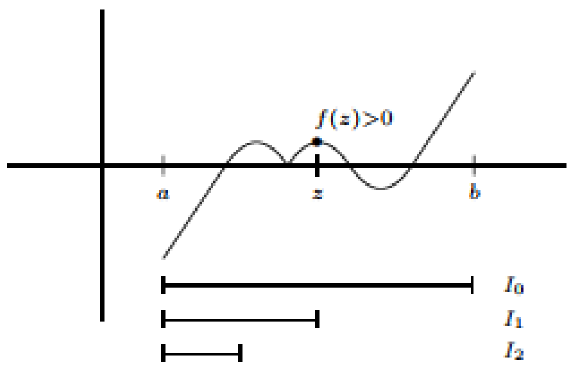
$$f(c) > 0, \ f(c) < 0 \quad \text{and} \quad f(c) = 0$$

The fact that c is the least upper bound of K can be used to rule out the first two cases. Consider $x_n \in \left(c - \frac{1}{n}, c\right]$. Then the sequence $(x_n) \to c$, and since $x_n \in K$, $f(x_n) \leq 0$. As f is continuous over $[a, b]$, $f(x_n) \to f(c)$. By the order limit theorem, $f(c) \leq 0$. Consider

$y_n \in \left(c, c + \dfrac{1}{n}\right)$. Then, the sequence $(y_n) \to c$, and since $y_n \notin K$, we must have that $f(y_n) > 0$. As f is continuous over $[a,b] - K$, $f(y_n) \to f(c)$. By the order limit theorem, $f(c) \geq 0$. So, the only possibility is $f(c) = 0$.

II. (Second approach using NIP) Again, consider the special case where $L = 0$ and $f(a) < 0 < f(b)$. Let $I_0 = [a,b]$ and consider the midpoint

$$z = (a+b)/2$$



If $f(z) \geq 0$, then set $a_1 = a$ and $b_1 = z$. If $f(z) < 0$, then set $a_1 = z$ and $b_1 = b$. In either case, the interval, $I_1 = [a_1, b_1]$ has the property that f is negative at the left endpoint and nonnegative at the right.

This procedure can be inductively repeated, setting the stage for the Nested Interval Property. The remainder of the the argument is left as exercise 4.5.5(b).

The Intermediate Value Property.

Does the Intermediate Value Theorem have a converse?

Definition 4.5.3. A function f has the intermediate value property on an interval $[a,b]$ if for all $x < y$ in $[a,b]$ and all L between $f(x)$ and $f(y)$, it is always possible to find a point $c \in (x,y)$ where $f(c) = L$.

Another way to summarize the Intermediate Value Theorem is to say that every continuous function on $[a,b]$ has the Intermediate Value Property. There is an understandable temptation to suspect that any function that has the intermediate value property must necessarily be continuous, but that is not the case. We have seen that

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at zero, but it does have the intermediate value property on $[0, 1]$.

The intermediate value property does imply continuity if we insist that our function is monotone.

Exercises.

1. [Abbott, 4.5.1] Show how the Intermediate Value Theorem follows as a corollary to the Theorem 4.5.2.

Proof.
Since $f : [a, b] \to \mathbf{R}$ is continuous, and the set $[a, b]$ is connected, by the theorem on the preservation of connected sets, $f([a, b])$ is connected. Consequently, if $f(a) < L < f(b)$ or $f(b) > L > f(a)$, we must have that, $L \in f((a, b))$. Consequently, there exists c such that $f(c) = L$.

2. [Abbott, 4.5.2] Provide an example of each of the following, or explain why the request is impossible:

(a) A continuous function defined on an open interval with a range equal to a closed interval.

Consider $f(x) = \left|\sin \frac{1}{x}\right|$ where $0 < x < 1$. $f\left(\frac{1}{n\pi}\right) = 0$ and $f\left(\frac{1}{2n\pi + \pi/2}\right) = 1$. Further, $f((0, 1)) = [0, 1]$.

(b) A continuous function defined on a closed interval with range equal to an open interval.

This request is impossible.

Justification. If $f : [a, b] \to \mathbf{R}$ is a continuous function, then f preserves compact sets. The image of $[a, b]$ under f is compact. $f[a, b]$ must be compact.

(c) A continuous function defined on an open interval with range equal to an unbounded closed set different from $\mathbf{R}$.

Consider the function f defined piecewise as follows:

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \le \dfrac{1}{4} \\ 8\left(x - \dfrac{1}{4}\right) & \text{if } \dfrac{1}{4} < x \le \dfrac{1}{2} \\ \dfrac{1}{1-x} & \text{if } \dfrac{1}{2} < x < 1 \end{cases}$$

f is well-defined and continuous on the open interval $(0,1)$ and maps this set onto $[0,\infty)$.

(d) A continuous function defined on all of $\mathbf{R}$ with range equal to $\mathbf{Q}$.

This request is impossible.

Justification.

$\mathbf{R}$ is connected. Consider any two disjoint non-empty subsets of $\mathbf{R}$ such as $A = \left(-\infty, \sqrt{2}\right]$ and $B = \left(\sqrt{2}, \infty\right)$. We see that $\mathbf{R} = A \cup B$ and $A \cap \overline{B} = \left\{\sqrt{2}\right\}$. This is true for all non-empty disjoint subsets A, B of $\mathbf{R}$.
As f is a continuous function, it preserves connected sets. So, the image of $\mathbf{R}$ under f must be connected.

But, $\mathbf{Q}$ is disconnected.

3. [Abbott, 4.5.3] A function f is increasing on A if $f(x) \le f(y)$ for all $x < y$ in A. Show that if f is increasing on $[a,b]$ and satisfies the intermediate value property, then f is continuous on $[a,b]$.

Proof.