Chapter 3. Basic Topology of R

Theorem. The open interval $(0,1) = \{x \in \mathbf{R} : 0 < x < 1\}$ is uncountable.

Proof.
We proceed by contradiction and assume that there exists a function $f : \mathbf{N} \to (0,1)$ that is $1-1$ and onto. $1-1$ implies that distinct elements have distinct images. Onto implies that every element in the co-domain has atleast one pre-image. For each $m \in \mathbf{N}$, $f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\ldots$$

What is meant here is that for each $m, n \in \mathbf{N}$, a_{mn} is the digit from the set $\{0,1,2,3,\ldots,9\}$ that represents the nth digit in the decimal expansion of mth real number, $f(m)$. The $1-1$ correspondence between $\mathbf{N}$ and $(0,1)$ can be summarized in the doubly indexed array:

$$\begin{array}{cccccccccccc}
\mathbf{N} & & (0,1) & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \mathbf{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \mathbf{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \mathbf{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \mathbf{a_{66}} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The key assumption about this correspondence is that **every** real number in $(0,1)$ is assumed to appear somewhere on this list.

Now for the pearl of the argument. Define a real number $x \in (0,1)$ with the decimal expansion $x = .b_1b_2b_3b_4\ldots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

Now, the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$, simply because its first digit b_1 differs from the first digit a_{11} of $f(1)$. Similarly, the second digit b_2 differs from the second digit a_{22} of $f(2)$. In general, the nth digit of x differs from the nth digit of $f(n)$. So, we have constructed a real number x that is not in the set $\{f(1), f(2), f(3), \ldots, f(n)\}$. But, this is a contradiction. Hence, our initial assumption is false. The set of real numbers in $(0,1)$ are uncountable.

Exercise. [Abbott, 1.6.4] Let S be the set consisting of all sequences of 0s and 1s. Observe that S is not a particular sequence, but rather a large set whose elements are sequences, namely:

$$S = \left\{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\right\}$$

As an example, the binary sequence $(1,0,1,0,1,0,1,0,\ldots)$ is an element of S as is the sequence $(1,1,1,1,1,1,\ldots)$. Give a rigorous argument showing that S is uncountable.

Proof.

Suppose that S - the set of all possible binary strings of infinite length is countable. Then, we can define a bijection $f : \mathbf{N} \to S$ between the natural numbers and S. For each $m \in \mathbf{N}$, $f(m)$ is a binary string in S. Let us enlist the first few elements of this correspondence.

$$\begin{array}{cccccccccccc}
\mathbf{N} & & (0,1) & & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \mathbf{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \mathbf{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \mathbf{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \mathbf{a_{66}} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Define a binary sequence $x = (b_1, b_2, b_3, b_4, \ldots)$ such that

$$b_i = \begin{cases} 1 & \text{if } a_{ii} = 0 \\ 0 & \text{if } a_{ii} = 1 \end{cases}$$

Thus, the binary sequence $x = (b_1, b_2, b_3, b_4, \ldots)$ has atleast one bit that differs from all of the elements in S. Consequently, $x \notin S$. This is a contradiction, as S is supposed to contain all binary strings. Hence, S is not countable.

3.1 Discussion: The Cantor Set.

What follows is a fascinating mathematical construction, due to Georg Cantor, which is extremely useful for extending the horizons of our intuition about the nature of subsets of the real line. Cantor's name has already appeared in the first chapter in our discussion of uncountable sets. Indeed, Cantor's proof that $\mathbf{R}$ is uncountable occupies another spot on the short list of the most significant contrubutions towards the understanding of the mathematical infinite. In the words of the mathematician David Hilbert, "No one shall expel us from the paradise that Cantor has created for us."

Let C_0 be the closed interval $[0,1]$ and define C_1 to be the set that results when the open middle one third is removedl that is,

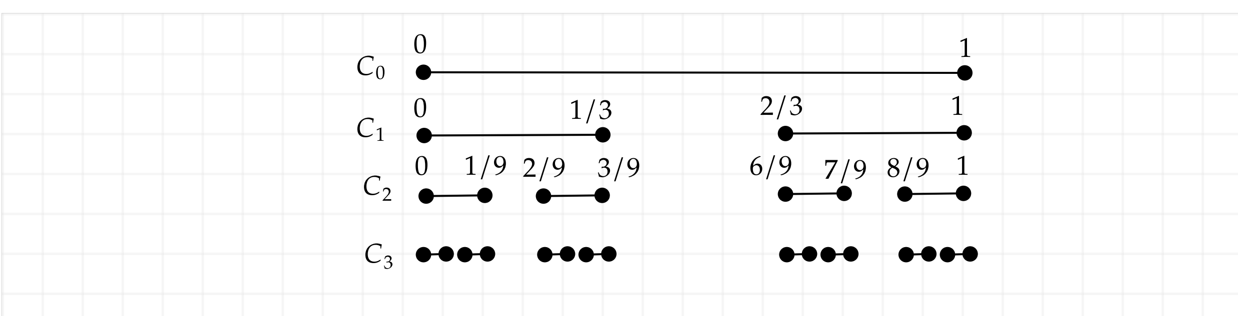
$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Now, construct C_2 in a similar way by removing the open middle third of each of the two components of C_1:

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right]\right) \cup \left(\left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$$

If we continue this process inductively then for each $n = 0, 1, 2, \ldots$ we get a set of C_n consisting of 2^n closed intervals each having length $1/3^n$. Finally, we define the Cantor set C to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$



Defining the Cantor set

It may be useful to understand C as the remainder of the inerval $[0,1]$ after the interative process of removing open middle one thirds is taken to infinity.

$$C = [0,1] - \left[\left(\frac{1}{3}, \frac{2}{3} \right) \cup \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \cup \ldots \right]$$

There is some initial doubt whether anything remains at all, but notice that because we are always removing open middle one thirds, then for every $n \in \mathbf{N}$, $0 \in C_n$ and hence $0 \in C$. The same argument shows that $1 \in C$. In fact, if y is the endpoint of some closed interval of some particular set C_n, then it is also an enpoint of one of the intervals of C_{n+1}. Because at each stage, the endpoints are never removed, it follows that $y \in C_n$ for all n. Thus, C atleast contains the endpoints of all of the intervals that make up each of the sets C_n.

Is there anything else? Is C countable? Does C contain any intervals? Any irrational numbers? These are difficult questions at the moment. All of the endpoints mentioned earlier are rational numbers (they have the form $m/3^n$), which means that if it is true that C consists of only these endponts, then C would be a subset of Q and hence countable. We shall see about this. There is some strong evidence that not much is left in C if we consider the total length of the intervals removed. To form C_1, an open interval of length $1/3$ was taken out. In the second step, we removed two intervals of length $1/9$ and to construct C_n, we removed 2^{n-1} middle theirds of length $1/3^n$. There is some logic, then to defining the length of C to be 1 minus the total

$$\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \ldots + 2^{n-1}\left(\frac{1}{3^n}\right) + \ldots = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1$$

The Cantor set has *zero length*.

To this point, the information we have collected suggests a mental picture of C as relatively small. For these reasons, the set C is often referred to as Cantor dust. But, there are some strong counterarguments that imply a very different picture. First C is actually *uncountable*. with cardinality equal to the cardinality of $\mathbf{R}$. One slightly intuitive but convincing way to see this is to create a $1-1$ correspondence between C and the sequences of the form $(a_n)_{n=1}^{\infty}$, where $a_n = 0$ or 1. For each $c \in C$, set $a_1 = 0$ if c falls in the left-hand component and set $a_1 = 1$ if c falls in the right hand component. Having established where in C_1, the point c is located, there are now two possible components of C_2 that might contain c. This time, we set $a_2 = 0$ or 1 depending on whether c falls in the left of right half of these two components of C_2. Continuing in this way, we come to see that every element $c \in C$ yields a sequence $(a_1, a_2, a_3, \ldots)$ of zeroes and ones (a binary string), that acts as a set of directions for how to locate c within C. Because the set of sequences of zeroes and ones is uncountable, we must conclude that C is uncountable as well.

What does this imply? In the first place, because the end points of the approximating sets C_n form a countable set, we are forced to accept the fact that not only are there other points in C, but there are uncountably many of them, From the point of view of cardinality, C is quite large - as large as $\mathbf{R}$, in fact. This should be contrasted with the fact that from the point of view of length, C measures the same size as a single point. We conclude this discussion with a demonstration that from the point of view of *dimension*, C strangely falls somewhere in between.

There is a sensible agreement that a point has dimension zero, a line segment has dimension one, a square has dimension two, and a cube has dimension three. Without attempting a formal definition of dimension (of which there are several) we can nevertheless get a sense of how one might be defined by observing how the dimension affects the result of magnifying each particular set by a factor of 3. (The reason for the choice of 3 will become clear when we turn our attention back to the Cantor set). A single point undergoes no change at all, whereas a line segment triples in length. For the square, magnifying each length by a factor of 3 results in a larger square that contains 9 copies of the original square. Finally, th magnified cube yields a cube that contains 27 copies of the original cube within its volume. Notice that, in each case, to compute the size of the new set, the dimension appears as the exponent of the magification factor.

	dim	$\times 3$	new copies
point	0	$\rightarrow$	$1 = 3^0$
segment	1	$\rightarrow$	$3 = 3^1$
square	2	$\rightarrow$	$9 = 3^2$
cube	3	$\rightarrow$	$27 = 3^3$

Dimension of C,

Now, apply this transformation to the Cantor set. The set $C_0 = [0, 1]$ becomes the interval $[0, 3]$. Deleting the middle one-third leaves $[0, 1] \cup [2, 3]$, which is where we started in the original construction exept that we now stand to produce an additional copy of C in the interval $[2, 3]$. Magnifying the Cantor set bya factor of 3 yields two copies of the original set. Thus, if x is the dimension of C, we must have $3^x = 2$, or $x = \ln 2 / \ln 3 = 0.631$.

The notion of a non-integer or fractional dimension is the impetus behind the term fractal, coined in 1975 by Benoit Mandlebrot to describe a class of sets whose intricate structures have much in common with the Cantor set. Cantor's construction is over hundred years old and for us represents n invaluable testing ground for upcoming theorems and conjectures about the often ulusive nature of subsets of the real line.

3.2 Open and Closed Sets.

Given $a \in \mathbf{R}$ and $\epsilon > 0$, recall that the ϵ-neighbourhood of a is the set