Understanding Analysis

Solution of exercise problems.

Quasar Chunawala

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Abstract

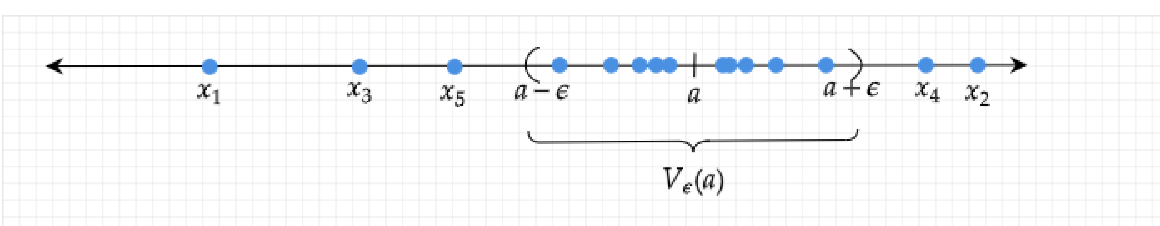
This is a solution manual for Understanding Analysis, 2nd edition, by Stephen Abbott.

Chapter 2. Sequences and Series.

The Limit of a Sequence.

Definition. A sequence is a function whose domain is $\mathbf{N}$.

Definition. (Convergence of a sequence). A sequence (a_n) converges to a real number a, if for every positive real number $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that whenever $n \geq N$, it follows that the distance $|a_n - a| < \epsilon$.



Definition. Given a real number $a \in \mathbf{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbf{R} : |x - a| < \epsilon\}$$

is called the ϵ-neighbourhood of a.

Notice that, $V_\epsilon(a)$ consists of all the points whose distance from a is less than ϵ. Said another way, $V_\epsilon(a)$ is an open interval centered at a, with radius ϵ.

Recasting the definition of convergence in terms of ϵ-neighbourhoods gives a more geometric impression of what is being described.

Definition(Convergence of a sequence - Topological version). A sequence (a_n) converges to a, if, given any ϵ-neighbourhood $V_\epsilon(a)$ of a, there exists a point in the sequence, after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighbourhood contains all but a finite number of the terms of the sequence (a_n).

The above definitions say precisely the same thing: the natural number N in the original version of the definition is the point where the seqeuence (a_n) enters $V_\epsilon(a)$, never to leave. It should be apparent that the value of N depends on the choice of ϵ. The smaller the ϵ-neighbourhood, the large N may have to be.

The Algebraic and the Order Limit Theorems.

The real purpose of creating a rigorous definition of a sequence is <i>not</i> to have a tool to verify computational statements such as $\lim 2n/(n+2) = 2$. Historically, a definition of the limit came 150 years after the founders of Calculus began working with intuitive notions of convergence. The point of having such a logically tight description of convergence is so that we can confidently <i>prove statements about convergent sequences in general.</i> We are ultimately trying to resolve arguments about what is and what is not true regarding the behavior of limits with respect to the mathematical manipulations we intend to inflict on them.

As an example, let us prove that convergent sequences are bounded. The term "bounded" has a rather familiar connotation but, like everything else, we need to be explicit about what it means in this context.

Definition. A sequence (x_n) is bounded if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$.

<p>
Geometrically, this means that we can find an interval $[-M, M]$ that contains every term in the sequence (x_n).

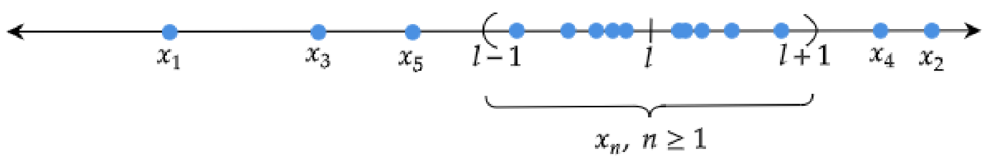
Theorem. Every convergent sequence is bounded.

Proof.
Assume that the seqeuence (x_n) converges to a limit l. This means that given a particular value of ϵ, say

$\epsilon = 1$, we know that there must exist an $N \in \mathbf{N}$ such that if $n \geq N$, then x_n is in the interval $(l-1, l+1)$. Not knowing whether l is positive or negative, we can certainly conclude that

$$|x_n| < |l| + 1$$

for all $n \geq N$.



We still need to worry (slightly) about the terms in the sequence, that come before the Nth term. Because there are only a finite number of these, we let

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l| + 1\}$$

It follows that $|x_n| \leq M$ for all $n \in \mathbf{N}$, as desired.

This post began with a demonstration of how applying familiar algebraic properties (commutativity of addition) to infinite objects (series) can lead to paradoxical results. These examples are meant to instill in us a sense of caution and justify the extreme care we are taking in drawing our conclusions. The following theorems illustrate that sequences behave extremely well with respect to the operations of addition, subtraction, multiplication, division and order.

<p>Theorem (Algebraic Limit Theorem). Let $\lim a_n = a$ and $\lim b_n = b$. Then,

(i) $\lim(ca_n) = ca$ for all $c \in \mathbf{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n / b_n) = a/b$ provided $b \neq 0$

<i>Proof</i>.

(i) Consider the case where $c \neq 0$. We want to show that the sequence (ca_n) converges to ca, so the structure of the proof follows the template I described earlier. First, we let ϵ be some arbitrary positive number. Our goal is to find some point in the sequence (ca_n) after which we can have

$$| - pairca_n - ca < \epsilon$$

Now,

$$| - pairca_n - ca = | - pairc| - paira_n - a$$

We are given that $(a_n) \to a$, so we know that we can make the distance $| - paira_n - a$ as small as we like. In particular, we can choose an N such that

$$| - paira_n - a < \epsilon / | - pairc$$

whenever $n \geq N$. To see that this N indeed works, observe that, for all $n \geq N$,

$$| - pairca_n - ca = | - pairc| - paira_n - a < | - pairc\frac{\epsilon}{| - pairc} = \epsilon$$

The case $c = 0$ reduces to showing that the constant sequence $(0, 0, 0, \ldots)$ converges to 0, which is easily verified.

Before continuing with, parts (ii), (iii) and (iv), I should point out that the proof of (i), while somewhat short, is extremely typical for a convergence proof. Before embarking on a formal argument, it is a good idea to take an inventory of what we <i>want</i> to make less than ϵ, and what we are given can be made small for suitable choices for n. For the previous proof, we wanted to make $| - pairca_n - ca < \epsilon$, and we were given $| - paira_n - a <$ <i>anything we like</i> (for large values of n). Notice that in (i) and all of the ensuing arguments, the strategy each time is to bound the quantity we want to be less than ϵ, which is in this case is

$$| - pair(\text{terms of the sequence}) - (\text{proposed limit})$$

with some algebraic combination of quantities over which we have control.

(ii) To prove this statement, we need to argue that the quantity

$$| - pair(a_n + b_n) - (a + b)$$

can be made less than an arbitrary ϵ using the assumptions that $|-pair a_n - a$ and $|-pair b_n - b$ can be made as small as we like for large n. The first step is to use the triangle inequality, to say

$$|-pair(a_n + b_n) - (a+b) = |-pair(a_n - a) - (b_n - b) \le |-pair a_n - a + |-pair b_n - b$$

Again, we let $\epsilon > 0$ be arbitrary. The technique this time is to divide the ϵ between the two expressions on the right-hand side in the preceding inequality. Using the hypothesis that $(a_n) \to a$, we know that there exists an N_1 such that

$$|-pair a_n - a < \frac{\epsilon}{2}$$

whenever $n \ge N_1$. Likewise, the assumption that $(b_n) \to b$ means that we can choose N_2 so that

$$|-pair b_n - b < \frac{\epsilon}{2}$$

whenever $n \ge N_2$.

The question now arises as to which N_1 or N_2 we should take to be our choice of N. By choosing $N = \max\{N_1, N_2\}$, we ensure that if $n \ge N$, then $n \ge N_1$ and $n \ge N_2$. This allows us to conclude that

$$\begin{aligned} |-pair a_n + b_n - (a+b) &\le |-pair a_n - a + |-pair b_n - b \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \ge N$, as desired.

<p>
(iii) To show that $(a_n b_n) \to ab$, we begin by observing that

$$\begin{aligned} |-pair a_n b_n - ab &= |-pair a_n b_n - ab_n + ab_n - ab \\ &= |-pair b_n(a_n - a) + a(b_n - b) \\ &\le |-pair b_n |-pair(a_n - a) + |-pair a |-pair b_n - b \end{aligned}$$

In the initial step, we subtracted and then added ab_n, which created an opportunity to use the triangle inequality. Essentially, we have broken up the distance from $a_n b_n$ to ab with a midway point and are using the sum of two distances to overestimate the original distance. This clever trick will become a familiar technique in arguments to come.

Letting $\epsilon > 0$ be arbitrary, we again proceed with the strategy of making each piece in the preceding inequality less than $\epsilon/2$. For the piece on the right hand $(|-paira|-pairb_n - b)$, if $a \neq 0$, we can choose an N_1 so that $n \geq N_1$ implies that

$$|-pairb_n - b < \frac{1}{|-paira}\frac{\epsilon}{2}$$

Getting the term on the left-hand side of $(|-pairb_n|-paira_n - a)$ to be less than $\epsilon/2$ is complicated by the fact that we have variable quantity $|-pairb_n$ to contend with as opposed to the constant $|-paira$ we encountered in the right-hand term. The idea is to replace $|-pairb_n$ with a worst-case estimate. Using the fact that convergent sequences are bounded, we know there exists a bound $M > 0$ satisfying $|-pairb_n \leq M$ for all $n \in \mathbf{N}$. Now, we can choose N_2 so that, whenever $n \geq N_2$, it implies that,

$$|-paira_n - a < \frac{1}{M}\frac{\epsilon}{2}$$

To finish the argument, pick $N = \max\{N_1, N_2\}$ and observe that if $n \geq N$, then

$$\begin{aligned}
|-paira_n b_n - ab &= |-paira_n b_n - ab_n + ab_n - ab \\
&\leq |-pairb_n|-paira_n - a + |-paira|-pairb_n - b \\
&< M \cdot \frac{1}{M}\frac{\epsilon}{2} + |-paira\frac{1}{|-paira}\frac{\epsilon}{2} = \epsilon
\end{aligned}$$

(iv) This final statement will follow from (iii) if we can prove that $(b_n) \to b$ implies that

$$\left(\frac{1}{b_n}\right) \to \frac{1}{b}$$

whenever $b \neq 0$. We begin by observing that

$$|-pair\frac{1}{b_n} - \frac{1}{b} = \frac{|-pairb - b_n}{|-pairb|-pairb_n}$$

Because $(b_n) \to b$, we can make the preceding numerator as small as we like by choosing large n. The problem comes in that we need a worst-case estimate on the size of $1/(|-pairb|-pairb_n)$. Because the (b_n) terms are in the denominator, we are no longer interested in an upper bound on $|-pairb_n$ but rather in an equality of the form $|-pairb_n \geq \delta > 0$. This will then lead to a bound on the size of $1/(|-pairb|-pairb_n)$.

The trick is to look far enough out into the sequence (b_n) so that the terms are closer to b than they are to 0. Consider the particular value $\epsilon_0 = \frac{|-pairb}{2}$. Because, $(b_n) \to b$, there exists an N_1 such that $|-pairb_n - b < |-pairb/2$. Because, $(b_n) \to b$, there exists a N_1 such that $|-pairb_n - b < |-pairb/2$ for all $n \geq N_1$. This implies $|-pairb_n > |-pairb/2$.

Next, choose N_2 so that $n \geq N_2$ implies

$$|-pairb_n - b < \epsilon \frac{|-pairb^2}{2}$$

Finally, if we let $N = \max\{N_1, N_2\}$, then $n \geq N$ implies

$$\begin{aligned} |-pair\frac{1}{b_n} - \frac{1}{b} &= \frac{|-pairb_n - b}{|-pairb_n|-pairb} \\ &< \epsilon \frac{|-pairb^2}{2} \cdot \frac{2}{|-pairb} \cdot \frac{1}{|-pairb} = \epsilon \end{aligned}$$

as desired.

<h5>Limits and Order.</h5>

Although there are few dangers to avoid, the Algebraic Limit Theorem verifies that the relationship between algebraic combinations of sequences and the limiting process is as trouble-free as we could hope for. Limits can be computed from the individual component sequences provided that each component limit exists. The limiting process is also well-behaved with respect to the order operation.

Theorem (Order Limit Theorem). Assume $\lim a_n = a$ and $\lim b_n = b$.

(i) If $a_n \geq 0$ for all $n \in \mathbf{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbf{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbf{R}$ for which $c \leq b_n$ for all $n \in \mathbf{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbf{N}$, then $a \leq c$.

<i>Proof.</i>

We will prove this by contradiction; thus let's assume that $a < 0$. The idea is to produce a term in the sequence (a_n) that is also less than zero. To do this, we consider the particular value $\epsilon = |-paira$. The definition of convergence guarantees that we can find an N such that $|-paira_n - a < |-paira$ for all $n \geq N$. In particular, this would mean that $|-paira_N - a < |-paira$, which implies $a_N < 0$. This contradicts our hypothesis that $a_N \geq 0$. We therefore conclude that $a \geq 0$.

(ii) The Algebraic Limit theorem ensures that the sequence $(b_n - a_n)$ converges to $(b-a)$. Because, $b_n - a_n \geq 0$, we can apply part (i) to get that $(b-a) \geq 0$.

(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbf{N}$ and apply (ii).

A word about the idea of tails is in order. Loosely speaking, limits and their properties do not depend at all on what happens at the beginning of the sequence but are strictly determined by what happens when n gets larger. We are more interested in the behavior of the infinite tail. Changing the value of the first ten - or ten thousand - terms in a particular sequence has no effect on the limit. The Order Limit Theorem part(i) assumes, for instance that $a_n \geq 0$ <i>for all</i> $n \in \mathbf{N}$. However, the hypothesis could be weakened by assuming only that there exists some point N_1 where $a_n \geq 0$ for all $n \geq N_1$. The theorem remains true, and in fact, the same proof is valid with the provision that when N is chosen, it be atleast as large as N_1.

In the language of analysis, when a property (such as non-negativity) is not necessarily possessed by some finite number of initial terms but is possessed by all terms in the sequence after some point N, we say that the sequence *eventually* has this property. The order limit theorem part (i), could be restated, "Convergent sequences that are eventually nonnegative converge to nonnegative limits." Parts (ii) and (iii) have similar modifications, as will many other upcoming results.

Exercises.

1. [Abbott, 2.3.1] Let $x_n \geq 0$ for all $n \in \mathbf{N}$.

(a) If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$.

(b) If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$.

Proof.

(a) The sequence $(x_n) \to 0$. From the definition of convergence, for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$, it implies that $|-pair x_n < \epsilon$. In particular, there is an N_1 such that for all $n \geq N_1$, $|x_n| < \epsilon^2$. So, $\sqrt{|x_n|} < \epsilon$. Since $x_n \geq 0$ for all $n \in \mathbf{N}$, $\sqrt{|x_n|} = \sqrt{x_n}$. As we are taking the positive square root, strictly speaking we write, $|\sqrt{x_n}| < \epsilon$. This closes the proof.

(b) The sequence $(x_n) \to x$. From the definition of convergence, for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that whenever $n \geq N$, it implies that $|-pair x_n - x < \epsilon$. Let us explore the inequality $|-pair\sqrt{x_n} - \sqrt{x} < \epsilon$.

$$\begin{aligned} |-pair\sqrt{x_n} - \sqrt{x} &< \epsilon \\ |-pair\sqrt{x_n} - \sqrt{x} \times \frac{|-pair\sqrt{x_n} + \sqrt{x}}{|-pair\sqrt{x_n} + \sqrt{x}} &< \epsilon \\ \frac{|-pair x_n - x}{|-pair\sqrt{x_n} + \sqrt{x}} &< \epsilon \\ |-pair x_n - x &< \epsilon \cdot |-pair\sqrt{x_n} + \sqrt{x} \end{aligned}$$

Shrinking the open interval (a, b) is strengthening $a < b$. Since, $\sqrt{x_n} \geq 0$, the expression on the right hand side is atleast as big as $|-pair\sqrt{x}$. So, if we can show that $|-pair x_n - x < \epsilon|-pair\sqrt{x}$, the above condition will hold. Clearly, we can find an $N \in \mathbf{N}$, such that for all $n \geq N$, we have $|-pair x_n - x < \epsilon|-pair\sqrt{x}$. This closes the proof.

<p>
2. [Abbott, 2.3.2] Using only the definition of the convergence of a sequence, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$

(b) $\left(\frac{1}{x_n}\right) \to \frac{1}{2}$

<i>Proof.</i>

(a) Let us explore the condition $|-pair\frac{2x_n-1}{3} - 1 < \epsilon$. We have,

$$
\begin{aligned}
|-pair\frac{2x_n-1}{3}-1 &< \epsilon \\
|-pair\frac{2x_n-1-3}{3} &< \epsilon \\
|-pair\frac{2x_n-4}{3} &< \epsilon \\
|-pair x_n-2 &< \frac{3}{2}\epsilon
\end{aligned}
$$

Since, $(x_n)\to 2$, there exists an $N\in\mathbf{N}$, such that for all $n\geq N$, $|-pair x_n-2<\frac{3}{2}\epsilon$. This closes the proof.

<p>
(b) Let us explore the condition $|-pair\frac{1}{x_n}-\frac{1}{2}<\epsilon$. We have,

$$
\begin{aligned}
|-pair\frac{1}{x_n}-\frac{1}{2} &< \epsilon \\
\frac{|-pair x_n-2}{|-pair 2x_n} &< \epsilon \\
|-pair x_n-2 &< 2\epsilon|-pair x_n
\end{aligned}
$$

We would like to find a lower bound on $|-pair x_n$, so that we can shrink the interval and strengthen the condition. For simplicity, suppose $\epsilon<1$, then $|-pair x_n>1$. So, it suffices to show that $|-pair x_n-2<2\epsilon$. But, since the sequence $(x_n)\to 2$, there exists an $N\in\mathbf{N}$, such that $n\geq N$, $|-pair x_n-2<\epsilon$.

<p>
3. [Abbott, 2.3.3] Squeeze Theorem. Show that if $x_n\leq y_n\leq z_n$ for all $n\in\mathbf{N}$, and if $\lim x_n=\lim z_n=l$, then $\lim y_n=l$ as well.

<i>Proof.</i>

Pick an arbitrary $\epsilon>0$. Consider a small neighbourhood around the point l, $V_\epsilon(l)=(l-\epsilon,l+\epsilon)$. Since, (x_n) is a convergent sequence, there exists N_1 such that for all $n\geq N_1$, $x_n\in V_\epsilon(l)$. Since, (y_n) is a convergent sequence, there exists N_2 such that for all $n\geq N_2$, $y_n\in V_\epsilon(l)$. Let $N=\max\{N_1,N_2\}$. Then, for all $n\geq N$, we have,

$$l-\epsilon<x_n\leq y_n\leq z_n<l+\epsilon$$

so, $z_n \in V_\epsilon(l)$. Consequently, (z_n) is a convergent sequence and $(z_n) \to l$.

4. [Abbott, 2.3.4] Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$

(b) $\lim\left(\frac{(a_n+2)^2-4}{a_n}\right)$

(c) $\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right)$

<i>Proof.</i>

(a) Applying the algebraic limit theorem several times, we obtain,

$$\begin{aligned}\lim\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right) &= \frac{\lim(1+2a_n)}{\lim(1+3a_n-4a_n^2)}\\ &= \frac{1+2\cdot 0}{1+3\cdot 0-4\cdot 0^2}\\ &= 1\end{aligned}$$

(b) Applying the algebraic limit theorem several times, we obtain,

$$\begin{aligned}\lim\left(\frac{(a_n+2)^2-4}{a_n}\right) &= \lim\left(\frac{a_n^2+4a_n+4-4}{a_n}\right)\\ &= \lim\left(\frac{a_n(a_n+4)}{a_n}\right)\\ &= \lim(a_n+4)\\ &= 4\end{aligned}$$

(c) Applying the algebraic limit theorem several times, we obtain,

$$\lim\left(\frac{\frac{2}{a_n}+3}{\frac{1}{a_n}+5}\right) = \lim\left(\frac{2+3a_n}{1+5a_n}\right)$$

$$= \frac{\lim(2+3a_n)}{\lim(1+5a_n)}$$

$$= 2$$

<p>
5. [Abbott, 2.3.5] Let (x_n) and (y_n) be given, and define(z_n) to be the "shuffled" sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots, x_n, y_n, \ldots)$. Prove that (z_n) is convergent if and only if (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$.

<i>Proof.</i>

$(\implies)$ direction.

Assume that the shuffled sequence $(z_n) := (x_1, y_1, x_2, y_2, x_3, y_3, \ldots,)$ is convergent and that it's limiting value is l. From the definition of convergence, we know that, for all $\epsilon > 0$, there exists $N \in \mathbf{N}$, such that for all $n \geq N$, $|-pairz_n - l < \epsilon$. Pick an arbitrary $\epsilon > 0$. If N is even, then $z_N = z_{2K} = y_K$. Then, $|-pairy_k - l < \epsilon$ for all $k \geq K$. And $|-pairx_k - l < \epsilon$ for all $k \geq K+1$. We can similarly argue for N odd. So, $(x_k) \to l$ and $(y_k) \to l$.

<p>
$(\impliedby)$ direction.

Assume that both the sequences (x_n) and (y_n) converge to l. Pick an $\epsilon > 0$. Since (x_n) is convergent, there exists an N_1 such that, for all $n \geq N_1$, the distance $|-pairx_n - l < \epsilon$. Since, (y_n) is convergent, there exists an N_2 such that, for all $n \geq N_2$, the distance $|-pairy_n - l < \epsilon$. Let $N = 2\max\{N_1, N_2\}$. Then, for all $n \geq N$, the distance $|-pairz_n - l < \epsilon$ as desired. Consequently, $(z_n) \to l$. This closes the proof.

<p>
6. [Abbott, 2.3.6] Consider the sequence given by $b_n = n - \sqrt{n^2 + 2n}$. Taking $(1/n) \to 0$ as given, and using both the Algebraic Limit theorem and the result in exercise 1, show that $\lim b_n$ exists and find the value of the limit.

<i>Proof.</i>

The expression $n - \sqrt{n^2 + 2n}$ can be written as,

$$
\begin{aligned}
n-\sqrt{n^2+2n} &= n-\sqrt{n^2+2n}\times\frac{n+\sqrt{n^2+2n}}{n+\sqrt{n^2+2n}} && \text{[Multiply and divide by conjugate]}\\
&= \frac{n^2-(n^2+2n)}{n+\sqrt{n^2+2n}}\\
&= -\frac{2n}{n+\sqrt{n^2+2n}}\\
&= -\frac{2}{1+\sqrt{1+\frac{2}{n}}} && \text{[Dividing throughout by n^2]}
\end{aligned}
$$

Applying the Algebraic limit theorem and the result in exercise 1, we have, $\lim b_n = -2$.

<p>
7. [Abbott, 2.3.7] Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;

(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges.

(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n, such that $(1/b_n)$ diverges.

(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;

(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge, but (b_n) does not.

<i>Proof.</i>

(a) Let $(x_n) = \sqrt{n+1}$ and $y_n = -\sqrt{n}$. Both the sequences (x_n) and (y_n) are divergent, but their sum $(x_n + y_n)$ is converges to 0.

(b) This request is impossible. Let $(x_n + y_n) \to l$ and $x_n \to m$. Then, we have,

$$
\begin{aligned}
\lim(x_n+y_n)-\lim(y_n) &= \lim(x_n)+\lim(y_n)-\lim(y_n) && \text{[by Algebraic Limit theorem]}\\
l-m &= \lim x_n
\end{aligned}
$$

Thus, the sequence (x_n) is convergent.

(c) Consider the sequence (b_n) where $b_n := \frac{1}{n}$. Then, (b_n) converges to 0, but $\left(\frac{1}{b_n}\right)$ is a divergent seqeuence.

(d) This request is impossible. We are told that the sequence $(a_n - b_n)$ is bounded. Then, for all $n \in \mathbf{N}$, there exists a $K > 0$, such that $| - paira_n - b_n < K$. Also, the sequence (b_n) is convergent. As convergent sequences are bounded, for all $n \in \mathbf{N}$, there exists an $M > 0$, such that $| - pairb_n < M$. It follows that, for all $n \in \mathbf{N}$,

$$\begin{aligned} | - paira_n &= | - paira_n - b_n + b_n \\ &\leq | - paira_n - b_n + | - pairb_n \\ &< K + M \end{aligned}$$

Hence, the sequence (a_n) is bounded.

(e) Consider the sequence $a_n := \frac{1}{n}$ and $b_n := \sin(n)$. Then, $(a_n b_n)$ and (a_n) are convergent, but (b_n) does not converge.

8. [Abbott, 2.3.8] Let $(x_n) \to x$ and let $p(x)$ be a polynomial.

(a) Show that $p(x_n) \to p(x)$.

(b) Find an example of a function $f(x)$ and a convergent sequence $(x_n) \to x$ where the sequence $f(x_n)$ converges but not to $f(x)$.

<i>Proof.</i>

Let us first prove that if $(x_n) \to x$, then $(x_n^k) \to x^k$ for any $k \in \mathbf{N}$.

Let us explore the expression $| - pairx_n^k - x^k$. The polynomial expression $x_n^k - x^k$ always has a factor $x_n - x$. Therefore, we can write,

<p>

$$\begin{aligned} |x_n^k - x^k| &= |x_n - x| \left| x_n^{k-1} + (x_n^{k-2})(x) + (x_n^{k-3})(x^2) + \ldots + (x_n)(x^{k-2}) + x^{k-1} \right| \\ &\leq | - pairx_n - x(| - pairx_n^{k-1} + | - pairx_n^{k-2}| - pairx + | - pairx_n^{k-3}| - pairx^2 + \ldots + | - pairx^{k-1}) \\ &\leq | - pairx_n - x \cdot C \end{aligned}$$

The quantity C is such that,

$$|-pair x_n^{k-1} + |-pair x_n^{k-2}|-pair x + |-pair x_n^{k-3}|-pair x^2 + \ldots + |-pair x^{k-1} \leq C$$

Shrinking the interval (a, b) strengthens the inequality $a < b$. So, we have a stronger condition:

$$|-pair x_n - x \cdot C < \epsilon$$

<p>
Since $(x_n) \to x$, clearly, there exists an $N \in \mathbf{N}$, such that for all $n \geq N$,

$$|-pair x_n - x < \frac{\epsilon}{C}$$

The value of C remains to be established. (x_n) is a convergent seqeuence and convergent sequences are bounded. So, $|-pair x_n \leq M$ for all $n \in \mathbf{N}$. Applying this fact in the expression below, we get:

$$|-pair x_n^{k-1} + |-pair x_n^{k-2}|-pair x + |-pair x_n^{k-3}|-pair x^2 + \ldots + |-pair x^{k-1} \leq (M^{k-1} + M^{k-2}|-pair x + M^k$$

So, we can set $C = M^{k-1} + M^{k-2}|-pair x + \ldots + |-pair x^{k-1}$.

Further, let $p(t)$ be a polynomial of degree m. Suppose, $p(t) = a_m t^m + \ldots + a_1 t + a_0$. By applying the Algebraic Limit Theorem, we have

<p>
$$\begin{aligned}
\lim p(x_n) &= \lim[a_m(x_n)^{m-1} + \ldots + a_1(x_n) + a_0] \\
&= \lim[a_m(x_n)^{m-1}] + \ldots + \lim[a_1(x_n)] + \lim[a_0] && [\lim(a_n + b_n) = \lim a_n + \lim b_n] \\
&= a_m \lim(x_n)^{m-1} + \ldots + a_1 \lim(x_n) + a_0 \lim(1) && [\lim ca_n = c \lim a_n] \\
&= a_m x^{m-1} + \ldots + a_1 x + a_0 \\
&= p(x)
\end{aligned}$$

<p>This closes the proof.

(b) Let (x_n) be the sequence defined by $x_n := \frac{1}{n}$. We know, that $(x_n) \to 0$. Consider the piecewise function $f(x)$ defined as,

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{otherwise} \end{cases}$$

Thus, $f(x_n) \to 0$, but $f(x) = 1$.

9. [Abbott, 2.3.9] (a) Let (a_n) be bounded (not necessarily convergent) sequence, and assume $\lim b_n = 0$. Show that $\lim(a_n b_n) = 0$. Why are we not allowed to use the Algebraic Limit Theorem to prove this?

(b) Can we conclude anything about the convergence of $(a_n b_n)$ if we assume that (b_n) converges to some non-zero limit b?

<i>Proof.</i>

Let us explore the condition $| - pair a_n b_n < \epsilon$. Since, (a_n) is a bounded sequence, we know that, for all $n \in \mathbf{N}$, $| - pair a_n \leq M$. Thus, we can strengthen the condition by writing,

$$\begin{aligned} | - pair M | - pair b_n &< \epsilon \\ | - pair b_n & \frac{\epsilon}{M} \end{aligned}$$

Since (b_n) is a convergent sequence, there exists an $N \in \mathbf{N}$, such that $| - pair b_n < \epsilon / M$ for all $n \geq N$, as desired.

In order to apply the Algebraic Limit Theorem, both sequences must be convergent.

<p>
(b) No. For example, let $a_n := (-1)^n$ and $b_n := \frac{n+1}{n}$. Then, (a_n) is bounded, but divergent and $(b_n) \to 1$. Yet, $(a_n b_n) = ((-1)^n \frac{n+1}{n})$ which diverges.

10. [Abbott, 2.3.10] Consider the following list of conjectures. Provide a short proof for those that are true and a counterexample for any that are false.

(a) If $\lim(a_n - b_n) = 0$, then $\lim a_n = \lim b_n$.

(b) If $(b_n) \to b$, then $(| - pair b_n) \to | - pair b$.

(c) If $(a_n) \to a$ and $(b_n - a_n) \to 0$, then $(b_n) \to a$.

(d) If $(a_n) \to 0$ and $| - pair b_n - b \leq a_n$ for all $n \in \mathbf{N}$, then $(b_n) \to b$.

<i>Proof</i>.

(a) This is false. The Algebraic limit theorem can be applied, if and only, both the sequences (a_n) and (b_n) are convergent.

For example, consider $a_n := \sqrt{n+1}$ and $b_n := -\sqrt{n}$. Now, $\lim(a_n - b_n) = 0$, but (a_n) and (b_n) are divergent sequences.

(b) I first prove the inequality $|-pair| - paira - |-pairb \leq |-paira - b$.

We may write, $a = a - b + b$. Therefore,

$$\begin{aligned} |-paira &= |-paira - b + b \\ &\leq |-paira - b + |-pairb \\ |-paira - |-pairb &\leq |-paira - b \end{aligned}$$

Also, we may write $b = b - a + a$. Therefore,

$$\begin{aligned} |-pairb &= |-pairb - a + a \\ &\leq |-pairb - a + |-paira \\ |-pairb - |-paira &\leq |-paira - b \\ -(|-paira - |-pairb) &\leq |-paira - b \end{aligned}$$

So, it turns out that $|-pair| - paira - |-pairb \leq |-paira - b$.

Let us explore the expression $|-pair| - pairb_n - |-pairb$. We have,

$$|-pair| - pairb_n - |-pairb \leq |-pairb_n - b$$

We know that, since the sequence $(b_n) \to b$, for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that $|-pairb_n - b < \epsilon$. Consequently, $|-pair| - pairb_n - |-pairb < \epsilon$ as desired.

<p>
(c) By Algebraic Limit Theorem,

$$\begin{aligned} \lim(b_n) &= \lim(b_n - a_n + a_n) \\ &= \lim(b_n - a_n) + \lim(a_n) \\ &= 0 + a \\ &= a \end{aligned}$$

<p>
(d) This is true. Since, the sequence (a_n) converges to 0, for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that for all $n \geq N$, we have:

$$|-pair a_n < \epsilon$$
$$\therefore |-pair b_n - b \leq |-pair a_n < \epsilon$$
$$\therefore |-pair b_n - b < \epsilon$$

Consequently, $(b_n) \to b$.

<p>
11. [Abbott, 2.3.11] Cesaro Means. (a) Show that if (x_n) is a convergent sequence, then the sequence given by the averages

$$y_n = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

also converges to the same limit.

(b) Give an example to show that it is possible for the sequence (y_n) of averages to converge even if (x_n) does not.

<i>Proof.</i>

(a) Let us explore the expression $|-pair \frac{x_1+x_2+\ldots+x_n}{n} - x$.

$$|-pair \frac{x_1 + x_2 + \ldots + x_n}{n} - x = |-pair \frac{x_1 + x_2 + \ldots + x_n - nx}{n}$$
$$= |-pair \frac{(x_1 - x) + (x_2 - x) + (x_3 - x) + \ldots + (x_n - x)}{n}$$

As the sequence (x_n) is convergent, if we pick an arbitrary $\epsilon/2 > 0$, there exists an $N_1 \in \mathbf{N}$, $0 \leq N_1 \leq n$, such that whenever $n \geq N_1$, $|-pair x_n - x < (\epsilon/2)$.

We can therefore write out the above expression as,

$$|-pair\frac{(x_1-x)+(x_2-x)+(x_3-x)+\ldots+(x_n-x)}{n} = |-pair\frac{(x_1-x)+(x_2-x)+(x_3-x)+\ldots+(x_{N_1-1}-x)\cdot}{n}$$

$$\leq \frac{|-pairx_1-x+|-pairx_2-x+|-pairx_3-x+\ldots+|-}{n}$$

[Applying Triangle Inequality]

$$< \frac{C}{n}+\frac{n-(N_1-1)}{n}\cdot\frac{\epsilon}{2}$$

[Restricting n to $n \geq N_1$]

$$< \frac{C}{n}+\frac{\epsilon}{2}$$

Shrinking the interval (a,b) strengthens the inequality $a<b$. So, we have a stronger condition:

<p>

$$\frac{C}{n}+\frac{\epsilon}{2}<\epsilon$$

$$\frac{C}{n}<\frac{\epsilon}{2}$$

$$n>\frac{2C}{\epsilon}$$

So, we can pick $N_2 > \frac{2C}{\epsilon}$. Then, for all $n \geq N_2$, the above inequality holds.

This closes the proof.

<p>
12. [Abbott, 2.3.12] A typical task in Analysis is to decipher whether a property possessed by every term in a convergent sequence is necessarily inherited by the limit. Assume $(a_n) \to a$, and determine the validity of each claim. Try to produce a counterexample for any that are false.

(a) If every a_n is an upper bound for the set B, then a is also an upper bound for B.

(b) If every a_n is in the complement of the interval $(0,1)$, then a is also in the complement of $(0,1)$.

(c) If every a_n is rational, then a is rational.

<i>Proof.</i>

(a) If $b \leq a_n$ for all elements $b \in B$ and $n \in \mathbf{N}$, then by the Order Limit theorem $b \leq a$.

(b) This is true. We proceed by contradiction. Assume that $0 < a < 1$ and $a_n < 0$ or $a_n > 1$ for all $n \in \mathbf{N}$. What we're after is, a term of the sequence that lies in the open interval $(0,1)$. The sequence (a_n) converges to a, so the terms of the sequence must come arbitrarily close to a; therefore if we can produce a teeny weeny ϵ-neighbourhood around a, that guarantees a term of the sequence will lie in $(0,1)$.

Let $\epsilon = \min\{\frac{a}{2}, \frac{|-pair1-a}{2}\}$. Applying the definition of convergence, we know that, there exists an $N(\epsilon) \in \mathbf{N}$ such that the $a_n \in (a-\epsilon, a+\epsilon)$ for all $n \geq N$. In particular, $0 < a_{N(\epsilon)} < 1$. This contradicts our hypothesis. Hence, the limiting value of the sequence $a \in \mathbf{R} - (0,1)$.

<p>
(c) This is false. Consider the sequence of rationals in the set $\{x \in \mathbf{Q} : x^2 < 2, x > 0\}$ given by

$$\left(1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \ldots, \right)$$

that converges to $\sqrt{2}$. Every a_n is rational, but a is irrational.

13. [Abbott, 2.3.13] Given a doubly indexed array a_{mn} where $m, n \in \mathbf{N}$, what should $\lim_{m,n \to \infty} a_{mn}$ represent?

(a) Let $a_{mn} = \frac{m}{m+n}$ and compute the <i>iterated</i> limits.

$$\lim_{n \to \infty} \left(\lim_{m \to \infty} a_{mn} \right) \quad \text{and} \quad \lim_{m \to \infty} \left(\lim_{n \to \infty} a_{mn} \right)$$

Define $\lim_{m,n \to \infty} a_{mn} = a$ to mean that for all $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that if both $m, n \geq N$, then $|-paira_{mn} - a < \epsilon$.

<p>
(b) Let $a_{mn} = \frac{1}{m+n}$. Does the limit $\lim_{m,n \to \infty} a_{mn} = a$ exist in this case? Do the two iterated limits exist? How do these three values compare? Answer these same questions for $a_{mn} = \frac{mn}{m^2+n^2}$.

<p>
<i>Proof.</i>
(a) We have:

$$
\begin{aligned}
\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) &= \lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{m}{m+n}\right)\\
&= \lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{1}{1+\frac{n}{m}}\right)\\
&= \lim_{n\to\infty}(1)\\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) &= \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right)\\
&= \lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{\frac{m}{n}}{\frac{m}{n}+1}\right)\\
&= \lim_{n\to\infty}(0)\\
&= 0
\end{aligned}
$$

(b) Consider $a_{mn} = \frac{1}{m+n}$.

$$
\begin{bmatrix}
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots\\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots\\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

<p>
We find that, $\lim_{n\to\infty}\frac{1}{m+n} = 0$ and $\lim_{m\to\infty}\frac{1}{m+n} = 0$. So, the iterated limits are 0. The double limit is also 0. Assume that there exists an $N \in \mathbf{N}$, such that for all $m, n \geq N$, $\frac{1}{m+n} < \epsilon$. $\frac{1}{m+n} < \frac{1}{2N}$. Thus, if we set $N > \frac{1}{2\epsilon}$, then for all $m, n \geq N$, we have $|-pair\frac{1}{m+n} < \epsilon$.

<p>
Let's look at the 2-dimensional matrix corresponding the sequence (a_{mn}). If $a_{mn} = \frac{mn}{m^2+n^2}$ the iterated limits are zero. This is easy to see visually, all sequences down the columns converge to 0, all sequences across the rows converge to 0. The double limit exists, if and only if, regardless of what path is taken, there is a unique limiting value. If we look at the diagonal of the matrix, this is a constant sequence

$1/2, 1/2, 1/2, \ldots$. Hence, the double limit does not exist.

<p>
$$\begin{bmatrix} \frac{1}{2} & \frac{2}{5} & \frac{3}{10} & \frac{4}{17} & \cdots \\ \frac{2}{5} & \frac{4}{8} & \frac{6}{13} & \frac{8}{20} & \cdots \\ \frac{3}{10} & \frac{6}{13} & \frac{9}{18} & \frac{6}{37} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

<h4>The Monotone Convergence Theorem and a First Look at Infinite Series.</h4>

We showed that, convergent sequences are bounded. The converse statement is certainly not true. It is not too difficult to produce an example of a bounded sequence that does not converge. On the other hand, if a bounded sequence is <i>monotone</i>, then in fact, it does converge.

Definition. A sequence (a_n) is <i>increasing</i> if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$ and <i>decreasing</i> if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$. A sequence is monotone if it is either increasing or decreasing.

<p>
Theorem (Monotone Convergence Theorem). If a sequence is monotone and bounded, then it converges.

<i>Proof.</i>

Let (a_n) be monotone and bounded. To prove that (a_n) converges using the definition of convergence, we are going to need a candidate for the limit. Let's assume the sequence is increasing (the decreasing case is handled similarly), and consider the set of points $s = \sup\{a_n : n \in \mathbf{N}\}$. It is reasonable to claim that $\lim a_n = s$.

To prove this, let $\epsilon > 0$. Because s is the least upper bound for $\{a_n : n \in \mathbf{N}\}$, $s - \epsilon$ is not an upper bound, so there exists a point in the sequence a_N such that $s - \epsilon < a_N$. Now the fact, that (a_n) is increasing implies that if $n \geq N$, then $a_n \geq a_N$. Hence,

$$s - \epsilon < a_N \leq a_n \leq s < s + \epsilon$$

which implies that $| - paira_n - s < \epsilon$ for all $n \geq N$ as desired.

The Monotone Convergence Theorem is extremely useful for the study of infinite series, largely because it asserts the convergence of a sequence without explicit mention of the actual limit. This is a good moment to do some preliminary investigations, so it is time to formalize the relationship between sequences and series.

<p>
 Definition (Convergence of a series). Let (b_n) be a sequence. An <i>infinite series</i> is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + b_4 + b_5 + \ldots$$

We define the corresponding <i>sequence of partial sums</i> (s_m) by

$$s_m = b_1 + b_2 + b_3 + \ldots + b_m$$

and say that the series $\sum_{n=1}^{\infty} b_n$ <i>converges</i> to B if the sequence (s_m) converges to B. In this case, we write $\sum_{n=1}^{\infty} b_n = B$.

Example. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Because the terms in the sum are all positive, the sequence of partial sums given by:

$$s_m = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{m^2}$$

is monotonically increasing. The question is whether or not we can find some upper bound on (s_m). To this end, we observe:

$$\begin{aligned}
s_m &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \ldots + \frac{1}{m\cdot m} \\
&\le 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(m-1)\cdot m} \\
&\le 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
&= 2 - \frac{1}{m} \\
&< 2
\end{aligned}$$

<p> Thus, 2 is an upper bound for the sequence of partial sums, so by the Monotone Convergence Theorem (MCT), $\sum_{n=1}^{\infty} 1/n^2$ converges to some (for the moment) unknown limit less than 2.

<p>
Example (Harmonic Series). This time, consider the so-called <i>harmonic series</i>

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Again, we have an increasing seqeuence of partial sums

$$s_m = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{m}$$

that upon naive inspection appears as though it may be bounded. However, 2 is no longer an upper bound because,

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 2$$

A similar calculation shows that $s_8 > 2\frac{1}{2}$, and we can see that in general

$$
\begin{aligned}
s_{2^k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{k-1}} + \ldots + \frac{1}{2^k}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^k} + \ldots + \frac{1}{2^k}\right) \\
&= 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \ldots + 2^{k-1} \cdot \frac{1}{2^k} \\
&= 1 + k\left(\frac{1}{2}\right)
\end{aligned}
$$

which is unbounded. Thus, despite the incredibly slow pace, the sequence of partial sums of $\sum_{n=1}^{\infty} 1/n$ eventually surpasses every number on the positive real line. Because, convergent sequences are bounded, the harmonic series diverges. The previous example is a special case of a general argument that can be used to determine the convergence or divergence of a large class of infinite series.

<p>
 Theorem (Cauchy Condensation Test). Suppose (b_n) is decreasing and satisfies $b_n \geq 0$ for all $n \in \mathbf{N}$. Then, the infinite series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series

$$
\sum_{n=0}^{\infty} 2^n b_{2^n} = b_1 + 2b_2 + 4b_4 + 8b_8 + 16b_{16} + \ldots
$$

converges.

<i>Proof.</i> First, assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Since, convergent sequences are bounded, we conclude that the partial sums

$$
t_k = b_1 + 2b_2 + 4b_4 + \ldots + 2^k b_{2^k}
$$

are bounded; that is there exists an $M > 0$ such that $t_k \leq M$ for all $k \in \mathbf{N}$. We want to prove that $\sum_{n=1}^{\infty} b_n$ converges. Because, $b_n \geq 0$ we know that the partial sums are increasing, so we only need to show that

$$
s_m = b_1 + b_2 + b_3 + b_4 + \ldots + b_m
$$

is bounded.

Fix m and let k be large enough to ensure that $m \leq 2^{k+1} - 1$. Then, $s_m \leq s_{2^{k+1}-1}$ and

$$
\begin{aligned}
s_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \ldots + (b_{2^k} + b_{2^k+1} + \ldots + b_{2^{k+1}-1}) \\
&\leq b_1 + (b_2 + b_2) + (b_4 + b_4 + b_4 + b_4) + \ldots + (b_{2^k} + b_{2^k} + \ldots + b_{2^k}) \\
&= b_1 + 2b_2 + 4b_4 + \ldots + 2^k b_{2^k} = t_k
\end{aligned}
$$

Thus, $s_m \leq t_k \leq M$, and the sequence of partial sums (s_m) is bounded. By the Monotone Convergence Theorem (MCT), we can conclude that the sequence of partial sums (s_m) and hence the infinite series $\sum_{n=1}^{\infty} b_n$ converges.

1. [Abbott 2.4.1] (a) Prove that the sequence defined by $x_1 = 3$ and

$$
x_{n+1} = \frac{1}{4 - x_n}
$$

converges.

(b) Now that we know that $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part(a) to explicitly compute $\lim x_n$.

<i>Proof.</i>

(a) By direct computation, we find that the first few values of the sequence are:

$$
\left(1, \frac{1}{3}, \frac{3}{11}, \frac{11}{41}, \ldots\right)
$$

It appears that this is a decreasing sequence. Let's use mathematical induction to prove that $x_{n+1} \leq x_n$ for all $n \in \mathbf{N}$.

Let's prove the double inequality $\frac{1}{4} \leq x_n \leq 3$ for all $n \in \mathbf{N}$.

For $n = 1$, $\frac{1}{4} \leq x_1 \leq 3$. Our inductive hypotheses is to assume: $\frac{1}{4} \leq x_n \leq 3$ for all $n \in \mathbf{N}$.

We can write,

$$
\begin{aligned}
-\frac{1}{4} &\geq -x_n \geq -3 \\
\frac{15}{4} &\geq 4 - x_n \geq 1 \\
\frac{4}{15} &\leq \frac{1}{4-x_n} \leq 1 \qquad [\because x_n < 4] \\
\therefore \frac{1}{4} < \frac{4}{15} &\leq x_{n+1} \leq 1 < 3
\end{aligned}
$$

Thus, (x_n) is bounded below.

<p>
Let us prove rigorously, (x_n) is a monotonically decreasing sequence.

For $n = 1$, we have $x_2 \leq x_1$. Our inductive hypotheses is to assume: $x_{n+1} \leq x_n$. Then,

$$
\begin{aligned}
4 - x_{n+1} &\geq 4 - x_n \\
\frac{1}{4-x_{n+1}} &\leq \frac{1}{4-x_n} \qquad [\because x_n \leq 3 \ \ \forall n \in \mathbf{N}] \\
x_{n+2} &\leq x_{n+1}
\end{aligned}
$$

Consequently, (x_n) is monotonically decreasing and bounded. By the Monotone Convergence Thorem, (x_n) is a convergent sequence.

(b) Since (x_n) is a convergent sequence, for all $\epsilon > 0$, there exists an $N \geq 2$, such that whenever $n \geq N$, it follows that the distance $\backslash absval x_n - x < \epsilon$. Since, (x_{n+1}) is the subsequence of (x_n) containing all but the first term of the original sequence, the behavior of its infinite tail is identical to (x_n). More rigorously, as $x_{n+1} \leq x_n$, we have:

<p>
$x - \epsilon < x_{n+1} \leq x_n < x + \epsilon$

for all $n \geq N$. Consequently, the sequence $(x_{n+1}) \to x$.

<p>(c) We have:

<p>

$$
\begin{aligned}
\lim x_{n+1} &= \frac{1}{4-\lim x_n} \\
x &= \frac{1}{4-x} \\
4x - x^2 &= 1 \\
x^2 - 4x + 1 &= 0 \\
(x-2)^2 - 3 &= 0 \\
(x-2+\sqrt{3})(x-2-\sqrt{3}) &= 0 \\
\therefore x &= 2-\sqrt{3}
\end{aligned}
$$

2. [Abbott, 2.4.2] (a) Consider the recursively defined sequence $y_1 = 1$,

$y_{n+1} = 3 - y_n$

and set $y = \lim y_n$. Because (y_n) and (y_{n+1}) have the same limit, taking the limit across the recursive equation gives $y = 3 - y$. Solving for y we conclude that $\lim y_n = 3/2$. What is wrong with this argument?

(b) This time $y_1 = 1$ and $y_{n+1} = 3 - \frac{1}{y_n}$. Can the strategy in (a) be applied to compute the limit of this sequence?

Proof.

(a) By direct computation, we find that the first few terms of the series are:

$$(y_n) = (1, 2, 1, 2, 1, 2, \ldots)$$

This sequence oscillates and does not converge. Hence, $\lim(y_n)$ does not exist.

(b) By direct computation, we find that the first few terms of the series are:

$$(y_n) = \left(1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \frac{89}{34}, \ldots\right)$$

Let us first prove that (y_n) is a bounded sequence. Our claim is, $1 \leq y_n \leq 3$ for all $n \in \mathbf{N}$.

For $n = 1$, $y_1 = 1$. Our inductive hypotheses is : Assume $1 \leq y_n \leq 3$. We have,

$$\begin{aligned}
1 \le y_n &\le 3 \\
1 \ge \frac{1}{y_n} &\ge \frac{1}{3} \qquad [\because y_n > 0] \\
-1 \le -\frac{1}{y_n} &\le -\frac{1}{3} \\
2 \le 3 - \frac{1}{y_n} &\le \frac{8}{3} \\
\therefore 1 < y_{n+1} &< 3
\end{aligned}$$

Let us prove rigorously that (y_n) is an increasing sequence. For $n = 1$, we have $y_2 > y_1$. Our inductive hypotheses is to assume that $y_{n+1} \ge y_n$. We are interested to prove that $y_{n+2} \ge y_{n+1}$. We have,

$$\begin{aligned}
3 - \frac{1}{y_{n+1}} &\ge 3 - \frac{1}{y_n} \qquad [\because y_n > 0 \text{ for all } n] \\
y_{n+2} &\ge y_{n+1}
\end{aligned}$$

Thus, (y_n) is monotonically increasing and a bounded sequence. By the Monotone Convergence Theorem, (y_n) is a convergent sequence.

Taking limits on both sides,

<p>

$$\begin{aligned}
\lim y_{n+1} &= 3 - \frac{1}{\lim y_n} \\
y &= 3 - \frac{1}{y} \\
y^2 &= 3y - 1 \\
y^2 - 3y + 1 &= 0 \\
y^2 - 2 \cdot y \cdot (3/2) + (3/2)^2 - (5/4) &= 0 \\
(y - (3/2))^2 - (\sqrt{5}/2)^2 &= 0 \\
y &= \frac{3 - \sqrt{5}}{2}
\end{aligned}$$

3. [Abbott, 2.4.3] (a) Show that

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

converges and find the limit.

(b) Does the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converge? If so, find the limit.

Proof.
(a) The sequence (x_n) given by the recursive relation $x_{n+1} = \sqrt{2+x_n}$ is monotonically increasing. We use mathematical induction to prove it rigorously. For $n = 1$, $x_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = x_1$. Our inductive hypotheses is: assume $x_{n+1} \geq x_n$. It follows that, $\sqrt{2+x_{n+1}} \geq \sqrt{2+x_n}$. Consequently, $x_{n+2} \geq x_{n+1}$.

The sequence (x_n) is bounded. For $n = 1$, we have $x_1 = \sqrt{2} < 2$. Our inductive hypotheses is: assume $x_n < 2$. It follows that, $\sqrt{2+x_n} < \sqrt{2+2} = 2$. Thus, $x_{n+1} < 2$. Hence, $x_n < 2$ for all $n \in \mathbf{N}$.

By the Monotone Convergence Theorem, (x_n) is a convergent sequence.

Taking limits on both sides, we have:

<p>

$$\begin{aligned}
\lim x_{n+1} &= \sqrt{2+\lim x_n} \\
x &= \sqrt{2+x} \\
x^2 &= 2+x \\
x^2 - x - 2 &= 0 \\
x^2 - 2x + x - 2 &= 0 \\
x(x-2) + (x-2) &= 0 \\
(x+1)(x-2) &= 0 \\
\therefore x &= 2
\end{aligned}$$

(b) The sequence (x_n) given by the recursive relation $x_{n+1} = \sqrt{2x_n}$ is monotonically increasing. We use mathematical induction to prove it rigorously. For $n = 1$, $x_2 = \sqrt{2\sqrt{2}} > \sqrt{2} = x_1$. Our inductive hypotheses is: assume $x_{n+1} \geq x_n$. It follows that, $\sqrt{2x_{n+1}} \geq \sqrt{2x_n}$. Consequently, $x_{n+2} \geq x_{n+1}$.

The sequence (x_n) is bounded. For $n = 1$, we have $x_1 = \sqrt{2} < 2$. Our inductive hypotheses is: assume $x_n < 2$. It follows that, $\sqrt{2x_n} < \sqrt{2 \cdot 2} = 2$. Thus, $x_{n+1} < 2$. Hence, $x_n < 2$ for all $n \in \mathbf{N}$.

By the Monotone Convergence Theorem, (x_n) is a convergent seqeuence.

Taking limits on both sides, we have:

<p>

$$\begin{aligned}
\lim x_{n+1} &= \sqrt{2 \lim x_n} \\
x &= \sqrt{2x} \\
x^2 &= 2x \\
x^2 - 2x &= 0 \\
x(x-2) &= 0 \\
\therefore x &= 2
\end{aligned}$$

4. [Abbott, 2.4.4] (a) Earlier we used the Axiom of Completeness (AoC) to prove the Archimedean Property of $\mathbf{R}$. Show that the Monotone Convergence Theorem can also be used to prove the Archimedean Property without making any use of AoC.

(b) Use the Monotone Convergence Theorem to supply a proof for the Nested Interval Property that doesn't make use of the AoC.

These two results suggest that we could have used the Monotone Convergence Theorem in place of AoC as our starting axiom for building a proper theory of real numbers.

Proof.

(a) Consider the sequence of natural numbers (a_n), $a_n := n$. Assume, for contradiction that natural numbers are bounded above. That is, (a_n) is bounded above. Since, $n+1 > n$ for all $n \in \mathbf{N}$, (a_n) is a monotonically increasing sequence. By the Montone Convergence Theorem(MCT), $(a_n) = n$ is a convergent sequence, and we can set $\alpha = \lim \ \mathbf{N}$. By definition of limits, for all ϵ, there exists $N \in \mathbf{N}$ such that, $n \in V_\epsilon(\alpha)$ for all $n \geq N$. Pick $\epsilon = 1$.We have:

$$\alpha - 1 < N < \alpha + 1$$

Because, $N+2 \in \mathbf{N}$, and $N+2 \ > \ \alpha+1$, we have a contradiction to the fact that $n \in (\alpha-1, \alpha+1)$ for all $n \geq N$. Hence, given any real number $x \in \mathbf{R}$, there exists an $n \in \mathbf{N}$, such that $n > x$.

(b) The nested interval property states that the real number line $\mathbf{R}$ contains no gaps. I reproduce the statement of NIP for completeness.

For each $n \in \mathbf{N}$, we assume that we are given a closed interval $[a_n, b_n] = \{x \in \mathbf{R} : a_n \leq x \leq b_n\}$.

Assume also that each I_n contains I_{n+1}. Then the resulting sequence of closed intervals

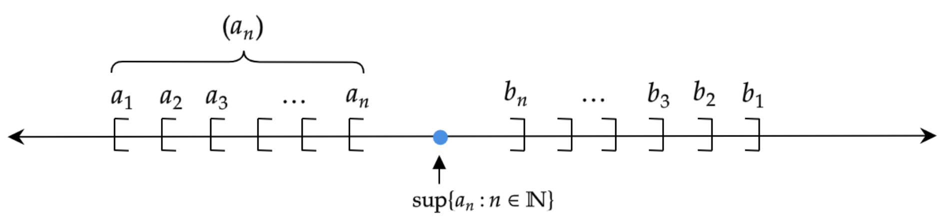
$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$$

have a non-empty intersection, that is

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing$$

Proof.

Consider a sequence of nested intervals $I_1 \supseteq I_2 \supseteq I_3 \ldots$ on the real number line $\mathbf{R}$, where $I_n = [a_n, b_n]$.



We define the sequence (a_n), as consisting of all the left-hand endpoints $a_1, a_2, a_3, \ldots,$ of the intervals $I_n, n \in \mathbf{N}$. The sequence (b_n) is defined, as consisting of all the right-hand endpoints of the $b_1, b_2, b_3, \ldots$ of the nested intervals. Since, the intervals are nested, every b_n is an upper bound for the sequence a_n. Likewise, every a_n is a lower bound for the sequence (b_n). Moreover, since

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq b_n \leq \ldots \leq b_2 \leq b_1$$

(a_n) is a monotonically increasing sequence and bounded above. (b_n) is a monotonically decreasing sequence and bounded below. By the Monotone convergence theorem, the sequences (a_n) and (b_n) are convergent. We would like to produce a single real number x satisfying $x \in I_n$ for all $n \in \mathbf{N}$. We are justified in setting $x = \lim(a_n)$. Pick an arbitrary interval $I_n = [a_n, b_n]$. Consider the constant sequence $(a_n, a_n, a_n, \ldots)$ and the subsequence $(a_n, a_{n+1}, a_{n+2}, \ldots)$. Since, $a_n \leq a_{n+r}$ for all r, by the Order Limit Theorem, $a_n \leq \lim a_n = x$. Similarly, we can argue that $x \leq b_n$. Altogether, then we have $a_n \leq x \leq b_n$, which means $x \in I_n$ for every choice of $n \in \mathbf{N}$. Hence, $x \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is not empty.

5. [Abbott, 2.4.5] **(Calculating the Square roots)**. Let $x_1 = 2$ and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

(a) Show that x_n^2 is always greater than or equal to 2, and then use this to prove that $x_n - x_{n+1} \geq 0$. Conclude that the $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Proof.

(a) Let us prove rigorously that $x_n^2 \geq 2$. We can write the recursive relation as,

<p>

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ 2x_n x_{n+1} &= (x^2 + 2) \\ x_n^2 - 2x_n x_{n+1} + 2 &= 0 \end{aligned}$$

Since the above equation has real roots, the discriminant must be non-negative. So, $4x_{n+1}^2 - 4(2) \geq 0$. Consequently, $x_{n+1}^2 \geq 2$. Further, let's prove that $1 < x_n \leq 2$ for all $n \in \mathbf{N}$. For $n = 1$, we have $x_1 = 2$. For the inductive hypotheses, assume $1 < x_n \leq 2$. We have:

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\ &\geq \frac{1}{2}\left(1 + \frac{2}{2}\right) \qquad [\because x_n \geq 1, \frac{1}{x_n} \geq \frac{1}{2}] \\ &= 1 \end{aligned}$$

Moreover,

$$
\begin{aligned}
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\
&\leq \frac{1}{2}\left(2 + \frac{2}{1}\right) \qquad [\because x_n \leq 2, \frac{1}{x_n} \leq 1] \\
&= 2
\end{aligned}
$$

Consequently, $1 < x_n \leq 2$ for all $n \in \mathbf{N}$.

We are interested to show that $x_n - x_{n+1} \geq 0$.
<p>

$$
\begin{aligned}
x_n - x_{n+1} &= x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\
&= x_n - \frac{x_n}{2} - \frac{1}{x_n} \\
&= \frac{x_n}{2} - \frac{1}{x_n} \\
&= \frac{x_n^2 - 2}{2x_n} \\
&\geq 0 \qquad [\because x_n^2 \geq 2 \text{ and } 1 < x_n \leq 2]
\end{aligned}
$$

Thus, the above sequence is monotonically decreasing and bounded. Hence, (x_n) is convergent. Taking limits on both sides, we have:

$$
\begin{aligned}
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \\
\lim x_{n+1} &= \frac{1}{2}\left(\lim x_n + \frac{2}{\lim x_n}\right) \\
x &= \frac{1}{2}\left(x + \frac{2}{x}\right) \\
2x &= x + \frac{2}{x} \\
x &= \frac{2}{x} \\
x^2 &= 2 \\
x &= \sqrt{2}
\end{aligned}
$$

(b) The sequence (x_n) given by the recursive relation $x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$ converges to $\sqrt{c}$.

6. [Abbott, 2.4.6] **(Arithmetic-Geometric Mean)**. (a) Explain why $\sqrt{xy} \leq \frac{(x+y)}{2}$ for any two positive real numbers x and y. (The Geomtric mean is always less than or equal to the arithmetic mean.)

(b) Now, let $0 \leq x_1 \leq y_1$ and define

$$x_{n+1} = \sqrt{x_n y_n} \quad \text{and} \quad y_{n+1} = \frac{x_n + y_n}{2}$$

Show that $\lim x_n$ and $\lim y_n$ both exist and are equal.

<i>Proof</i>.

(a) We have,

$$\begin{aligned} (x_n + y_n)^2 - 4x_n y_n &= (x_n - y_n)^2 \geq 0 \\ \left(\frac{x_n + y_n}{2}\right)^2 &\geq x_n y_n \\ \left(\frac{x_n + y_n}{2}\right) &\geq \sqrt{x_n y_n} \qquad\qquad [x_n, y_n \geq 0] \end{aligned}$$

(b) Our claim is, $0 \leq x_n \leq y_n$ for all $n \in \mathbf{N}$. For $n = 1$, we have $0 \leq x_1 \leq y_1$. Our inductive hypotheses is : assume $0 \leq x_n \leq y_n$. Since, x_n and y_n are positive reals, their geometric mean is less than or equal to the arithmetic mean. So, $\sqrt{x_n y_n} \leq \frac{x_n + y_n}{2}$. Therefore, $x_{n+1} \leq y_{n+1}$. Consequently, by mathematical induction, $0 \leq x_n \leq y_n$.

Let us explore the equation $x_{n+1} = \sqrt{x_n y_n}$. We have:

$$\begin{aligned} x_{n+1} &= \sqrt{x_n \cdot y_n} \\ &\geq \sqrt{x_n \cdot x_n} = x_n \end{aligned}$$

Thus, (x_n) is a monotonically increasing sequence. Also, let us explore the equation, $y_{n+1} = \frac{x_n + y_n}{2}$. We have:

$$
\begin{aligned}
y_{n+1} &= \frac{x_n + y_n}{2} \\
&\leq \frac{y_n + y_n}{2} \\
&= y_n
\end{aligned}
$$

Thus, (y_n) is a monotonically decreasing sequence. Thus, every x_n is a lower bound for the sequence (y_n). Also, every y_n is an upper bound for the sequence (x_n). Consequently, by the Monotone Convergence Theorem, both (x_n) and (y_n) are convergent sequences. Assume that $\lim x_n = x$ and $\lim y_n = y$. Then,

$$
\begin{aligned}
\lim x_{n+1} &= \sqrt{\lim x_n \cdot \lim y_n} \\
x &= \sqrt{xy} \\
x^2 &= xy \\
x(x-y) &= 0 \\
\therefore x &= y
\end{aligned}
$$

7. [Abbott, 2.4.7] **(Limit Superior)** Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The *limit superior* of (a_n), or $\lim\sup a_n$ is defined by

$$\lim\sup a_n = \lim y_n$$

where (y_n) is the sequence from part (a) of the exercise. Provide a reasonable definition for $\lim\inf a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\lim\inf a_n \leq \lim\sup a_n$ for every bounded sequence, and give an example for which the inequality is strict.

(d) Show that $\lim\inf a_n = \lim\sup a_n$, if and only if $\lim a_n$ exists. In this case, all three share the same value.

<i>Proof.</i>

(a) Consider the sequence $(y_n) = \{\sup a_k : k \geq n\}$. Our claim is that (y_n) is a monotonically decreasing sequence. For $n = 1$, $y_1 = \sup\{a_1, a_2, \ldots\}$ and $y_2 = \sup\{a_2, \ldots\}$. Clearly, $y_1 \geq y_2$. Our inductive hypotheses is to assume that $y_n \geq y_{n+1}$. We are interested to prove that $y_{n+1} \geq y_{n+2}$. We have,

$$
\begin{aligned}
y_{n+1} &= \sup\{a_k : k \geq n+1\} \\
&= \sup\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\} \\
&= \sup\{a_{n+1}, \sup\{a_{n+2}, a_{n+3}, \ldots\}\} \\
&= \sup\{a_{n+1}, y_{n+2}\} \\
&\geq y_{n+2}
\end{aligned}
$$

Consequently, by mathematical induction, we have $y_n \geq y_{n+1}$ for all $n \in \mathbf{N}$.

Since the sequence (a_n) is bounded, $|a_n| \leq M$ for all $n \in \mathbf{N}$. Therefore, $\backslash absvaly_n \leq M$ for all $n \in N$. Consequently, (y_n) is bounded. By the Montone Convergence Theorem, (y_n) is a convergent seqeuence.

(b) The limit infimum of the sequence (a_n) is the limit of the infimum of the k-tails of the sequence (a_n). Mathematically,

$$
\lim\inf a_n = \lim\inf\{a_k : k \geq n\}
$$

Consider the sequence (z_n) defined as, $z_n = \inf\{a_k : k \geq n\}$. Our claim is that (z_n) is a monotonically increasing sequence. For $n = 1$, $z_1 = \inf\{a_1, a_2, \ldots\}$ and $z_2 = \sup\{a_2, \ldots\}$. Clearly, $z_1 \leq z_2$. Our inductive hypotheses is to assume that $z_n \geq z_{n+1}$. We are interested to prove that $z_{n+1} \leq z_{n+2}$. We have,

$$
\begin{aligned}
z_{n+1} &= \inf\{a_k : k \geq n+1\} \\
&= \inf\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\} \\
&= \inf\{a_{n+1}, \inf\{a_{n+2}, a_{n+3}, \ldots\}\} \\
&= \inf\{a_{n+1}, z_{n+2}\} \\
&\leq z_{n+2}
\end{aligned}
$$

Consequently, by mathematical induction, we have $z_n \leq z_{n+1}$ for all $n \in \mathbf{N}$.

Since the sequence (a_n) is bounded, $|a_n| \leq M$ for all $n \in \mathbf{N}$. Therefore, $\backslash absvalz_n \leq M$ for all $n \in N$. Consequently, (z_n) is bounded. By the Montone Convergence Theorem, (z_n) is a convergent seqeuence.

(c) Consider the terms of the sequence (y_n) and (z_n). We have, $y_n = \sup\{a_k : k \geq n\}$ and $z_n = \inf\{a_k : k \geq n\}$. Since, $\inf A \leq \sup A$ for any set A, we have $y_n \geq z_n$ for all $n \in \mathbf{N}$. Since, (y_n) and (z_n) are convergent sequences, their limits exist. By the Order Limit Theorem, $\lim y_n \geq \lim z_n$, that is $\lim\sup a_n \geq \lim\inf a_n$.

Consider the sequence (a_n) defined to be, $a_n = \frac{n+1}{n} \sin\left(\frac{n\pi}{4}\right)$. Here, the inequality is strict.

(d) Let us compare the terms of the sequence (a_n), (y_n) and (z_n).

$$\begin{aligned} y_n &= \sup\{a_k : k \geq n\} \geq a_n \\ z_n &= \inf\{a_k : k \geq n\} \leq a_n \end{aligned}$$

Since the elements of any set lie between its infimum and supremum, $z_n \leq a_n \leq y_n$ for all $n \in \mathbf{N}$.

Define

$$\begin{aligned} a_* = \liminf a_n &= \sup_n \inf_{k \geq n}\{a_k\} \\ a^* = \limsup a_n &= \inf_n \sup_{k \geq n}\{a_k\} \end{aligned}$$

Pick an arbitrary $\epsilon > 0$. Since, a^* is an infimum for (y_n), there exists $N_1 \in \mathbf{N}$, such that $y_{N_1} < a^* + \epsilon$. Since, (y_n) is monotonically decreasing, $y_n \leq y_{N_1} < a^* + \epsilon$. Similarly, since a_* is a supremum for (z_n), there exists $N_2 \in \mathbf{N}$ such that $a_* - \epsilon < z_{N_2}$. Since, (z_n) is monotonically increasing, $a_* - \epsilon < z_{N_2} \leq z_n$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$, we have:

$$a_* - \epsilon < z_n \leq a_n \leq y_n < a^* + \epsilon$$

Now, suppose for some sequence, $a_* = a^* = a$. Then, the above argument shows that the sequence $a_n \in V_\epsilon(a)$. In other words, the sequence (a_n) converges to a.

For the converse, pick an arbitrary $\epsilon > 0$. Since, $(a_n) \to a$, we know that, there exists an $N_0 \in \mathbf{N}$, such that

$$a - \epsilon < a_n < a + \epsilon$$

for all $n \geq N_0$. Now, what can we say about (y_n) and (z_n). Since, $y_n = \sup_{n \geq k}\{a_k\}$, we must have $y_n < a + \epsilon$ for all $n \geq N_0$. Similarly, as $z_n = \inf_{n \geq k}\{a_k\}$, we must have $a - \epsilon < y_n$. Therefore,

$$a - \epsilon < z_n \leq a_* \leq a_n \leq a^* \leq\leq y_n < a + \epsilon$$

Therefore, the distance between a_* and a^* can be made less than 2ϵ. Hence, $a_* = a^*$ since they can be made arbitrarily close.

8. [Abbott, 2.4.8] For each series, find an explicit formula for the sequence of the partial sums and determine if the series converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(c) $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$

Proof.
(a) The kth term of the sequence of partial sums s_k is given by,

$$\begin{aligned} s_k &= \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^k} \\ &= \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^{k-1}}\right] \\ &= \frac{1}{2}\left[\frac{\left(1 - \frac{1}{2^k}\right)}{1 - \frac{1}{2}}\right] \\ &= 1 - \frac{1}{2^k} \end{aligned}$$

Clearly, (s_k) is a monotonically increasing sequence and $s_k \leq 1$ for all $k \in \mathbf{N}$. By the Monotone Convergence Theorem(MCT), (s_k) is a convergent sequence.

(b) The kth term of the sequence of partial sums (s_k) is given by,

$$\begin{aligned} s_k &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{k\cdot(k+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 1 - \frac{1}{k+1} \end{aligned}$$

Clearly, (s_k) is a monotonically increasing sequence and $s_k \leq 1$ for all $k \in \mathbf{N}$. By the Monotone

Convergence Theorem (MCT), (s_k) is a convergent sequence.

(c) The kth term of the sequence of partial sums (s_k) is given by,

$$
\begin{aligned}
s_k &= \sum_{n=1}^{k} \log n + 1 - \log n \\
&= \log 2 - \log 1 + \log 3 - \log 2 + \log 4 - \log 3 + \ldots + log(k+1) - \log k \\
&= \log(k+1) - \log 1
\end{aligned}
$$

The sequence of partial sums (s_k) is not bounded. Thus, this infinite series is divergent.

9. [Abbott, 2.4.9] Complete the proof of the Cauchy Condensation Test by showing that if the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges, then so does $\sum_{n=1}^{\infty} b_n$. Exercise 2.4.5 may be a useful reference.

<i>Proof.</i>

Consider the kth term of the subsequence $s_{2^{k+1}}$ of partial sums.

$$
\begin{aligned}
s_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + b_6 + b_7 + b_8) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k}) \\
&\geq b_1 + b_2 + (b_4 + b_4) + (b_8 + b_8 + b_8 + b_8) + \ldots + 2^{k-1} b_{2^k} && [\because b_n \geq b_{n+1}] \\
&= \frac{1}{2}\left(b_1 + 2b_2 + 4b_4 + 8b_8 + \ldots + 2^k b_2^k\right) + \frac{b_1}{2} \\
&= \frac{t_k}{2} + \frac{b_1}{2}
\end{aligned}
$$

(t_k) is a divergent sequence. As it monotonically increasing, it must be unbounded. Therefore, s_{2^k} is unbounded and divergent. Consequently, the sequence of partial sums (s_m) is divergent.

Subsequences and the Bolzanno-Weirstrass Theorem.

We have seen earlier that the sequence of partial sums (s_m) of the harmonic series does not converge by focussing our attention on a particular <i>subsequence</i> (s_{2^k}) of the original sequence. For the moment, we will put the topic of infinite series aside and more fully develop the important concept of subsequences.

Definition. Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence

$$(a_{n1}, a_{n2}, a_{n3}, a_{n4}, a_{n5}, \ldots)$$

is called a *subsequence* of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbf{N}$ indexes the subsequence.

Notice that the order of the terms in a subsequence is the same as in the original sequence, and repetitions are not allowed. Thus, if

$$(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right)$$

then

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right) \quad \text{and} \quad \left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots\right)$$

are examples of legitimate subsequences, whereas

$$\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \frac{1}{500}, \ldots\right) \quad \text{and} \quad \left(1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \ldots\right)$$

are not.

Theorem. Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof.

Assume that $(a_n) \to a$ and let (a_{n_k}) be a subsequence. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ whenever $n \geq N$. Because $n_k \geq k$ for all k, the same N will suffice for the subsequence; that is $|a_{n_k} - a| < \epsilon$ whenever $k \geq N$. This closes the proof.

This not too surprising result has somewhat surprising applications. It is the key ingredient for understanding when infinite sums are associative. We can also use it in the following clever way to compute values of some familiar limits.

Example. Let $0 < b < 1$. Because

$$b > b^2 > b^3 > b^4 \ldots 0$$

the sequence (b^n) is decreasing and bounded below. The Monotone Convergence Theorem allows us to conclude that (b^n) converges to some $l$satisfying $b > l \geq 0$. To compute l, notice that (b^{2n}) is a subsequence, so $(b^{2n}) \to l$ using the result that subsequences of a convergent sequence converge to the same limit as the original sequence. But, $b^{2n} = b^n \cdot b^n$. So, by the Algebraic Limit theorem, $b^{2n} \to l \cdot l = l^2$. Because limits are unique, $l^2 = l$ and thus $l = 0$.

Without much trouble, we can generalize this example to conclude $(b^n) \to 0$ if and only if $-1 < b < 1$.

Example (Divergence Criterion).The above theorem is also useful for providing economical proofs for divergence. Consider the sequence

$$\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \ldots\right)$$

did not converge to any proposed limit. Notice that

$$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots\right)$$

is a subsequence that converges to $\frac{1}{5}$. Also,

$$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \ldots\right)$$

is a different subsequence of the original sequence that converges to $-1/5$. Because we have two subsequences converging to two different limits, we can rigorously conclude that the original sequence diverges.

Exercise 2.5

2.5.1 (d) A sequence that contains subsequences converging to every point in the infinite set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ and no subsequences converging to points outside this set.

This request is impossible.

Exercises.

1. [Abott, 2.5.1] Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \dots\}$, and no subsequences converging to points outside of this set.

Proof.

(a) This request is impossible. Since (a_{n_k}) is a convergent sequence, by the Bolzanno-Weierstrass theorem, it must have atleast one convergent subsequence.

(b) Consider the sequence

$$\left(2, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{4}{3}, \frac{1}{4}, \frac{5}{4}, \frac{1}{5}, \frac{6}{5}, \dots\right)$$

The subsequence $(a_{2n}) \to 0$ and the subsequence $(a_{2n-1}) \to 1$.

(c) Consider the sequence

$$\begin{bmatrix} 2 & \frac{3}{2} & \frac{4}{3} & \frac{5}{4} & \dots \\ \frac{3}{2} & 1 & \frac{5}{6} & \frac{3}{4} & \dots \\ \frac{4}{3} & \frac{5}{6} & \frac{2}{3} & \frac{7}{12} & \\ \frac{5}{4} & \frac{3}{4} & \frac{7}{12} & \frac{1}{2} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The (i, j) element of the matrix is $\frac{1}{i} + \frac{1}{j}$. Thus, each row subsequence converges to $\frac{1}{i}$. We traverse the matrix diagonally to get successive elements. So, $n = i(i-1)/2 + j$.

A much simpler example could be

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ \ldots\right)$$

(d) This request is impossible. Consider

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ \ldots\right)$$

The subsequence $1, 1/2, 1/3, 1/4, \ \ldots$ converges to 0, a point strictly outside of this set.

2. [Abbott, 2.5.2] Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

This proposition is true.

Justification: Consider the subsequence $(x_{n+1}) = (x_2, x_3, \ \ldots)$. Since this subsequence converges, for all $\epsilon > 0$, there exists an $N > 1$ such that for all $n_K \geq N$, the terms of the subsequence (x_{n+1}) belong to $V_\epsilon(L)$. This implies that, whenever $n \geq \ N$, the terms of the parent sequence (x_n) belong to $V_\epsilon(L)$.

(b) If (x_n) contains a divergent sequence, then (x_n) diverges.

This proposition is true. If (x_n) is a convergent sequence, every subsequence of (x_n) converges and converges to the same limit as the original sequence. The contrapositive of this result is, if (x_n) contains a divergent sub-seqeuence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, there exist two subsequences of (x_n) that converge to different limits.

If (x_n) is a bounded sequence, it has a limit inferior and limit superior. Recall the definition:

$$\alpha = \lim\sup(x_n) = \lim \alpha_n = \lim \sup\{x_k : k \geq n\}$$
$$\beta = \lim\inf(x_n) = \lim \beta_n = \lim \inf\{x_k : k \geq n\}$$

Pick $\epsilon = 1$.

By definition, there exists an N_1, such that :

$$\alpha - 1 < \alpha_{N_1} < \alpha + 1$$

Let's give ourselves a δ of room.

Let $\delta = \min\left\{\frac{|\alpha_{N_1} - (\alpha+1)|}{2}, \frac{|\alpha_{N_1} - (\alpha-1)|}{2}\right\}$. Then,

$$\alpha - 1 < \alpha_{N_1} - \delta < \alpha_{N_1} < \alpha_{N_1} + \delta < \alpha + 1$$

Since $\alpha_{N_1} - \delta$ is not an upper bound for the set $\{x_k : k \geq N_1\}$, there exists $x_{n_1} \in \{x_k : k \geq N_1\}$ such that

$$\alpha - 1 < \alpha_{N_1} - \delta < x_{n_1} < \alpha_{N_1} < \alpha_{N_1} + \delta < \alpha + 1$$

Next, we pick $\epsilon = \frac{1}{2}$.
By definition, there exists $N_2 > N_1$, such that:

$$\alpha - (1/2) < \alpha_{N_2} < \alpha + (1/2)$$

Let $\delta = \min\left\{\frac{|\alpha_{N_2} - (\alpha+1/2)|}{2}, \frac{|\alpha_{N_2} - (\alpha-1/2)|}{2}\right\}$.

Then,

$$\alpha - (1/2) < \alpha_{N_2} - \delta < \alpha_{N_2} < \alpha_{N_2} + \delta < \alpha + (1/2)$$

Since $\alpha_{N_2} - \delta$ is not an upper bound for the set $\{x_k : k \geq N_2\}$, there exists $x_{n2} \in \{x_k : k \geq N_2\}$ such that

$$\alpha - (1/2) < \alpha_{N_2} - \delta < x_{n2} < \alpha_{N_2} < \alpha_{N_2} + \delta < \alpha + (1/2)$$

In general, we can find x_{n_t} such that $x_{n_t} \in V_{1/t}(\alpha)$. Thus, for any arbitrary $\epsilon > 0$, if we pick $T > \frac{1}{\epsilon}$, then for all $t \geq T$, the terms of the subsequence $x_{n_t} \in V_\epsilon(\alpha)$. Consequently, $(x_{n_t}) \to \alpha$.

Similarly, we can produce a subsequence that converges to β. Therefore, if a sequence is bounded and divergent, there exist atleast two subsequences that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Let (x_{n_k}) be a convergent subsequence of (x_n). Assume that $(x_{n_k}) \to L$. Pick an arbitrary $\epsilon > 0$. Since

(x_{n_k}) is convergent there exists a $K \in \mathbf{N}$, such that for all $n_k \geq n_K$, we have $x_{n_k} \in (L-\epsilon, L+\epsilon)$. Because the sequence (x_n) is monotone, every $x_m \in (x_n)$ lies between x_{n_k} and $x_{n_{k+1}}$, for $n_k \leq m \leq n_{k+1}$.

Therefore,

$$|x_m - L| < \epsilon$$

for all $m \geq n_K$. Thus, $(x_n) \to L$.

3. [Abbott, 2.5.3] (a) Prove that if an infinite series converges, then the associative property holds. Assume $a_1 + a_2 + a_3 + a_4 + a_5 + \ldots$ converges to a limit L (that is, the sequence of partial sums $(s_n) \to L$. Show that any regrouping of the terms

$$(a_1 + a_2 + \ldots + a_{n1}) + (a_{n1+1} + \ldots + a_{n2}) + (a_{n2+1} + \ldots + a_{n3}) + \ldots$$

leads to a series that also converges to L.

(b) Compare this result to the example discussed at the beginning of this post where infinite addition was shown not to be associative. Why doesn't our proof in (a) apply to this example?

Proof.

(a) We are given that the infinite series

$$\sum_{n=1}^{\infty} a_n$$

converges to finite limit L. Consider the sequence of partial sums (s_m), where $s_k = \sum_{n=1}^{k} a_n$. The sequence of partial sums converges to L. If (s_m) is a convergent sequence, all subsequences of (s_m) converge to the same limiting value as the original sequence. Therefore, the subsequences $(s_{2m}) = (s_2, s_4, s_6, s_8, \ldots)$ and $(s_{3m}) = (s_3, s_6, s_9, s_{12}, \ldots)$ converge to the same limiting value. In general, let $n_1 < n_2 < \ldots < n_k < \ldots$ be an increasing sequence of natural numbers. Then, the subsequence $(s_{n_1}, s_{n_2}, \ldots, s_{n_k}, \ldots)$ converges to the same limit value for any choice of $n_1, n_2, \ldots, n_k$. This subsequence (extraction) of the sequence of partial sums represents the following grouping of the infinite series:

$$(a_1 + a_2 + \ldots + a_{n_1}) + (a_{n_1+1} + \ldots + a_{n_2}) + (a_{n_2+1} + \ldots + a_{n_3}) + \ldots$$

(b) The infinite series $\sum_{n=1}^{\infty}(-1)^n$ is divergent. The associative property of infinite addition does not apply to a divergent series.

4. [Abbott, 2.5.4] The Bolzanno-Weierstrass theorem is extremely important, and so is the strategy employed in the proof. To gain some more experience with this technique, assume that the Nested Interval Property is true and use it to provide a proof of the Axiom of Completeness. To prevent the argument from being circular, assume also that $(\frac{1}{2^n}) \to 0$. Why precisely is this last assumption needed to avoid circularity?

Proof.

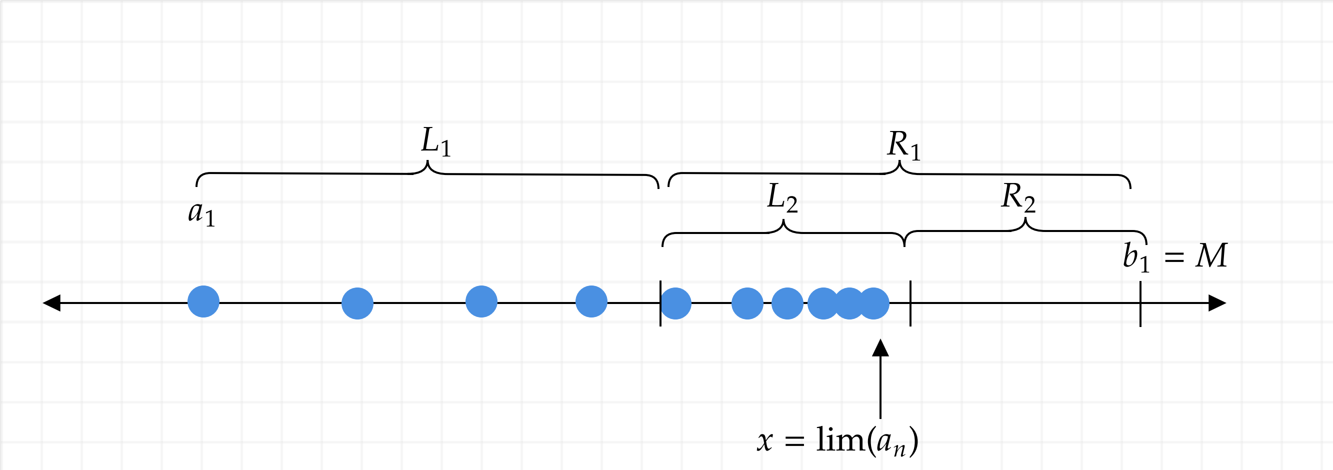
Claim: Every non-empty subset of real numbers that is bounded above has a least-upper bound in $\mathbf{R}$.

Since, A is bounded above, assume that M is an upper bound for A. I would like to construct a sequence of nested closed intervals, containing the supremum of the subset A. Let $I_1 = [a_1, b_1]$ where $a_1 \in A$ and $b_1 = M$. We bisect the interval I_1 into two halves, L_1 and R_1. We define I_2 as follows:

$$I_2 = \begin{cases} R_1 & \text{if } A \cap R_1 \neq \emptyset \\ L_1 & \text{otherwise} \end{cases}$$

If $I_2 = L_2$, then $a_2 = a_1$ and $b_2 = (a_1 + b_1)/2$. If $I_2 = R_2$, then $a_2 = (a_1 + b_1)/2$ and $b_2 = b_1$. In general, we construct the closed interval I_{k+1} by bisecting the closed interval I_k and selecting the left or right half-interval, based on:

$$I_{k+1} = \begin{cases} R_k & \text{if } A \cap R_k \neq \emptyset \\ L_k & \text{otherwise} \end{cases}$$



Since $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \dots$, by the Nested Interval Property (NIP), the intersection of the nested closed intervals is non-empty.

$$\bigcap_{k=1}^{\infty} I_k \neq \varnothing$$

and contains atleast one element x.

Indeed, every b_n is an upper bound for a_n. And (a_n) is a monotonically increasing sequence. Therefore, by the Monotone convergence Theorem (a_n) is convergent.

Our claim is $x = \lim(a_n) = \sup(a_n)$. Pick any arbitrary $\epsilon > 0$, then there exists an interval I_k of length $(b_k - a_k)/2^{k-1}$, such that

$$\left| \frac{b_k - a_k}{2^{k-1}} \right| < \epsilon$$

Since, every interval I_k contains x, we find that $a_k \in V_\epsilon(x)$. Since $a_k \leq x$, for all k and there exists $a_m \in \{a_n\}$, such that $x - \xi < a_m \leq x$ for all $\xi > 0$, $\sup(a_n) = x$.

We can do more than this.

(1) Let's show that x is an upper bound for A.

By construction, every I_k is guranteed to contain elements of A. Also, every b_k is an upper bound for the elements of the set A.

Assume that x is not an upper bound for A. Then there exists an element $\tilde{a} \in A$, such that $\tilde{a} > x$. Take $\tilde{a} - x = \delta$. Then, there exists an interval $I_m = [a_m, b_m]$ of length less than δ. A consequence of this is:

$$x \leq b_m < x + \delta = \tilde{a}$$

This contradicts the fact that every b_k is an upper bound for the elements of the set A. Thus, x is an upper bound for A.

(2) Moreover, for all $\epsilon > 0$, there exists an element $\tilde{a}$ of the set A, such that

$$x - \epsilon < a_k \leq \tilde{a} \leq b_k < x + \epsilon$$

From (1) and (2), x is an upper bound for A, such that for all $\epsilon > 0$, there exists an element a_ϵ in A, such that $x - \epsilon < a_\epsilon$. By definition, $x = \sup A$.

5. [Abbott, 2.5.5] Assume that (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbf{R}$. Show that (a_n) must converge to a.
Proof.

If a sequence (a_n) is bounded, there exists two distinct subsequences that converge to $\lim\inf(a_n)$ and $\lim\sup(a_n)$. We are given that all convergent subsequences converge to the same limit a. Therefore, $\lim\inf(a_n) = \lim\sup(a_n) = a$.

Moreover, since (α_n) is monotonically decreasing and $\inf\{\alpha_n\} = a$, if $\epsilon > 0$, then $a + \epsilon$ should not be a lower bound for the set $\{\alpha_n\}$. Thus, there exists N_1 such that

$$\alpha_n < a + \epsilon$$

for all $n \geq N_1$.

Since (β_n) is monotonically increasing and $\sup\{\alpha_n\} = a$, if $\epsilon > 0$, then $a - \epsilon$ should not be an upper bound for the set $\{\beta_n\}$. Thus, there exists N_2 such that

$$a - \epsilon < \beta_n$$

for all $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have:

$$a-\epsilon < \beta_n \leq a_n \leq \alpha_n < a+\epsilon$$

Consequently, for all $\epsilon > 0$, we have found an N, such that whenever $n \geq N$, $a_n \in V_\epsilon(a)$. Thus, $(a_n) \to a$.

6. [Abbott, 2.5.6] Use a similar strategy to the one in example 2.5.3 to show that $\lim(b^{1/n})$ exists for all $b \geq 0$ and find the value of the limit. (The results in exercise 2.3.1 may be assumed.)

Proof.

Case I. $b \geq 1$

We have:

$$b^1 \geq b^{1/2} \geq b^{1/3} \geq \ \ldots \ \geq b^{1/n} \geq b^{1/(n+1)} \geq \ \ldots$$

Moreover, $b^{1/n} \geq 1$ if $b \geq 1$. So, $(b^{1/n})$ is a monotonically decreasing and bounded sequence. Consequently, the Monotone Convergence Theorem allows is to conclude that, $(b^{1/n})$ converges to some limit l where, $1 \leq l \leq b^{1/n}$.

Now, $(b^{1/2n})$ is a subsequence of $(b^{1/n})$. Every subsequence of convergent sequence converges to the same limiting value as the original sequence. So, $\lim b^{1/2n} = l$. But, $b^{1/2n} = \sqrt{b^{1/n}}$. Taking limits on both sides, $l = \sqrt{l}$. Therefore, $\sqrt{l} = 1$ and it follows $l = 1$.

Case II. $0 \leq b < 1$

We have:

$$b^1 \leq b^{1/2} \leq b^{1/3} \leq \ \ldots \ \leq b^{1/n} \leq b^{1/(n+1)} \geq \ \ldots$$

Moreover, $b^{1/n} < 1$ if $b < 1$. So, $(b^{1/n})$ is a monotonically increasing and bounded sequence. Consequently, the Monotone Convergence Theorem allows is to conclude that, $(b^{1/n})$ converges to some limit l where, $b^{1/n} < l \leq 1$. Similar to the case above, $l = 1$.

7. [Abbott, 2.5.7] Extend the result proved in the example 2.5.3 to the case $| - pairb < 1$, that is show that $\lim(b^n) = 0$ if and only if $-1 < b < 1$.
Proof.

If $-1 < b < 0$, the subsequence of even terms $(b^2, b^4, b^6, \ \ldots)$ is positive and $b^2 > b^4 > b^6 > \ \ldots \ > 0$. So, it is a monotonically decreasing and bounded seqeuence. The subsequence of odd terms $(b, b^3, b^5, \ldots)$ is negative and $b < b^3 < b^5 < \ldots < 0$. So, it is a monotonically increasing and bounded sequence. By the Monotone convergence theorem, both (b_{2n}) and (b_{2n+1}) are convergent.

Again, $b_{4n} = b_{2n}^2$, so taking limits on both sides, $l = l^2$. Consequently, the subsequence (b_{2n}) converges to $l = 0$. Likewise, the subsequence of odd terms (b_{2n+1}) converges to 0. Since, both convergent sequences converge to the same limiting value, the sequence (b_n) is convergent, when $-1 < b < 0$.

8. [Abbott, 2.5.8] Another way to prove the Bolzano-Weierstrass theorem is to show that every sequence contains a monotone subsequence. A useful device in this endeavor is the notion of a *peak term*. Given a sequence (x_n), a particular term (x_m) is a peak term, if no later term in the sequence exceeds it; i.e., if $x_m \geq x_n$ for all $n \geq m$.

(a) Find examples of sequences with zero, one and two peak terms. Find an example of a sequence with infinitely many peak terms that is not montone.

(b) Show that every sequence contains a monotone subsequence and explain how this furnishes a new proof of the Bolzanno-Weierstrass Theorem.

Proof.

(a) An monotonically increasing sequence such as

$$(0, 1, 2, 3, \ldots)$$

has zero peak terms. The sequence

$$(1, 0, 1, 2, 3, 4, 5, \ldots)$$

also has zero peak terms.

The sequence

$$2, 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \ldots, 2 - \frac{1}{2^{n-2}}, \ldots$$

has one peak term. So, also the subsequence

$$2, 1, -\frac{3}{2}, \frac{7}{4}, -\frac{15}{8}, \frac{31}{16}, -\frac{63}{32}, \ldots, (-1)^n\left(2 - \frac{1}{2^{n-2}}\right), \ldots$$

has one peak term.

The sequence

$$3, 2, 1, \frac{3}{2}, \frac{14}{8}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \ldots, 2 - \frac{1}{2^{n-3}}, \ldots$$

has two peak terms.

The sequence $(0, 1, 0, 1, 0, 1, \ldots)$ forming an alternating sequence of zeros and ones has infinitely many peak terms and is not monotone.

(b) We have a dichotomy. A sequence could have either a finite number of peak terms or a countably infinite number of peak terms. If a sequence has a finite number of peak terms, there is a tail of the sequence containing a monotonic subsequence. If a sequence has an infinite number of peak terms, the subsequence consisting of the peak terms is montonic decreasing.

Therefore, every sequence (x_n) has atleast one monotonic subsequence. This subsequence is bounded, if the parent (x_n) is bounded. Consequently, by the Monotone Convergence Theorem(MCT), the subsequence is convergent.

The Cauchy Criterion.

The following definition bears resemblance to the definition of convergence for a sequence.

Definition. A sequence (a_n) is called a **Cauchy sequence**, if, for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that whenever $m, n \geq N$, it follows that $|a_n - a_m| < \epsilon$.

In other words, if a sequence is Cauchy, its terms come arbitrarily close to each other as we go further out into the sequence. To make the comparison easier, let's restate the definition of convergence.

Definition. A sequence (a_n) converges to a real number a, if for every $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n \geq N$, it follows that $|a_n - a| < \epsilon$.

As we have discussed, the definition of convergence asserts that, given an arbitrary positive $\epsilon > 0$, it is possible to find a point in the sequence after which the terms of the sequence are all closer to the limit a than the given ϵ. On the other hand, a sequence is a Cauchy sequence if, for every $\epsilon > 0$, there is a point in the sequence after which the terms are all closer <i>to each other </i>than the given ϵ. To spoil the surprise, we will argue in this section that in fact these two definitions are equivalent:

Convergent sequences are Cauchy sequences, and Cauchy sequences are converge. The significance of the definition of a Cauchy sequence is that there is no mention of a limit. This is somewhat like the situation with the Monotone Convergence Theorem in that we will have another way of proving that sequences converge without having any explicit knowledge of what the limit might be.

Theorem. Every convergent sequence is a Cauchy sequence.

Proof.
Assume that (x_n) converges to x. To prove that (x_n) is Cauchy, we must find a point in the sequence after which we have $|x_n - x_m| < \epsilon$. This can be done using an application of the triangle inequality.

Since, $(x_n) \to x$, for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that $|x_n - x| < \epsilon$ for all $n \geq N$. Consider a term x_m of the sequence, where $n > m \geq N$. We have:

$$\begin{aligned} |x_n - x_m &= |(x_n - x) - (x_m - x) \\ &\leq |x_n - x + |x_m - x \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

We conclude that, every convergent sequence is a Cauchy sequence.

The converse is a bit more difficult to prove, mainly because, in order to prove that a sequence converges, we must have a proposed limit for the sequence to approach. We have been in this situation in the proofs of the Monotone Convergence Theorem and the Bolzanno-Weierstrass theorem. Our strategy here will be to use the Bolzanno-Weierstrass Theorem. This is the reason for the next lemma.

Lemma. Cauchy sequences are bounded.

Proof.
Given $\epsilon = 1$, there exists an N such that $|x_n - x_m| < 1$ for all $m, n \geq N$. Thus,we must have $|x_n| < |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1,|, |x_2,|, |x_3|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for the sequence (x_n).

Theorem (Cauchy Criterion). A sequence converges if and only if it is a Cauchy sequence.

Proof.
($\implies$) This direction is the theorem proved earlier.

($\Longleftarrow$) For this direction, we start with a Cauchy sequence (x_n). The Lemma guarantees that (x_n) is bounded, so we may use the Bolzanno-Weierstrass Theorem to produce a convergent subsequence (x_{n_k}). Set

$$x = \lim x_{n_k}$$

The idea is to show that the original sequence (x_n) converges to this same limit. Once again, we will use a triangle inequality argument. We know that the terms in the subsequence are getting close to the limit x, and the assumption that (x_n) is Cauchy implies that the terms in the tail of the sequence are close to each other. Thus, we want to make each of these distances less than half of the prescribed ϵ.

Let $\epsilon > 0$. Because (x_n) is Cauchy, there exists N such that

$$|x_n - x_m| < \frac{\epsilon}{2}$$

whenever $m, n \geq N$. Now, we also know that $(x_{n_k}) \to x$, so choose a term in this subsequence, call it n_K, with $n_K \geq N$ and

$$|x_{n_K} - x| \ < \frac{\epsilon}{2}$$

To see that N has the desired property (for the original sequence (x_n)), observe that if $n \geq N$, then

$$\begin{aligned} |x_n - x| &= |x_n - x_{nK} + x_{nK} - x| \\ &\leq |x_n - x_{nK}| + |x_{nK} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

This closes the proof.

The Cauchy criterion is named after the French Mathematician Augustin Louis Cauchy. Cauchy is a major figure in the history of many branches of mathematics - number theory and the theory of finite groups - to name a few - but he is most widely recognised for his enormous contributions in analysis, especially complex analysis. He is deservedly credited with inventing the ϵ-based definitions of limits we use today, although it is probably better to view him as a pioneer of analysis in the sense that his work did not attain the level of refinement that modern mathematicians have come to expect. The Cauchy criterion, for instance, was devised and used by Cauchy to study infinite series, but he never actually proved it in both directions. The fact that there were gaps in Cauchy's work should not diminish his brilliance in any way. The issues of the day were both difficult and subtle, and Cauchy was far and away the most influential in laying the groundwork for modern standards of rigor. Karl Weierstrass played a major role in sharpening Cauchy's arguments. We will hear a good deal more from Weierstrass most notably in a later chapter, when we take up uniform convergence. Bernhard Bolzanno was working in Prague and was writing and thinking about many of these same issues surrounding limits and continuity. Because, his work was not widely available to the rest of the mathematical community, his historical reputation never achieved the distinction that his impressive accomplishments would seem to merit.

Exercises.

1. [Abbott, 2.6.2] Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.

Consider $a_n := \dfrac{(-1)^n}{n}$. The sequence $(a_n) \to 0$. A sequence is convergent $\iff$ A sequence is Cauchy. And (a_n) is not monotone.

(b) A Cauchy sequence with an unbounded subsequence.

This request is impossible.

If a subsequence of (a_n) is unbounded, the parent (a_n) is also unbounded. Since, cauchy sequences are bounded, the contrapositive of this statement implies, unbounded sequences are not Cauchy.

(c) A divergent monotone sequence with a Cauchy subsequence.

This request is impossible.

By the Montone Convergence Theorem, if a sequence (a_n) is monotone and bounded, it is convergent. By contrapositive, a divergent monotone sequence is unbounded. Let $(a_{n_k}) \subset (a_n)$ be a proper subsequence of (a_n). Then, (a_{n_k}) is unbounded. Cauchy sequences are bounded. Consequently, (a_{n_k}) is not Cauchy.

(d) An unbounded sequence containing subsequence that is Cauchy.

Consider the sequence $\{a_n\} := \left(n \sin \dfrac{n\pi}{2}, n \in \mathbf{N}\right)$. Then,

$$(a_n) = (0, 1, 0, -3, 0, 5, 0, -7, \ldots)$$

contains the constant subsequence

$$(a_{n_k}) = (0, 0, 0, \ldots)$$

which is Cauchy.

2. [Abbott 2.6.3] If (x_n) and (y_n) are Cauchy sequences, then one easy-way to prove that $(x_n + y_n)$ is Cauchy is to use the Cauchy criterion. Using the result that convergent sequences are Cauchy and vice-versa, (x_n) and (y_n) are convergent and the Algebraic Limit Theorem then implies $(x_n + y_n)$ is convergent and hence Cauchy.

(a) Give a direct argument that $(x_n + y_n)$ is a Cauchy sequence that does not use the Cauchy criterion or the algebraic limit theorem.

Proof.

(x_n) and (y_n) are Cauchy sequences. Therefore, for all $\epsilon > 0$, there exists N_1, N_2 such that $|x_q - x_p| < \epsilon/2$ and $|y_s - y_r| < \epsilon/2$ for $q \geq p \geq N_1$, $s \geq r \geq N_2$.

Let $N = \max\{N_1, N_2\}$ and suppose $n \geq m \geq N$. Consider the sequence $(x_n + y_n)$. Let us explore the expression $|(x_n + y_n) - (x_m + y_m)|$.

$$
\begin{aligned}
|(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\
&\leq |(x_n - x_m)| + |(y_n - y_m)| \quad \Big[\text{Triangle Inequality}\Big] \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

Hence, $(x_n + y_n)$ is Cauchy.

(b) Do the same for the product $(x_n y_n)$.

Proof.
Since (x_n) and (y_n) are Cauchy, and Cauchy sequences are bounded, there is an $M > 0$, such that $|x_n| \leq M$ for all $n \in \mathbf{N}$. Also, there exists N_1, such that $|x_n - x_m| < \epsilon/2M$ for all $n \geq N_1$. There exists N_2, such that $|y_n - y_m| < \epsilon/2K$.

Let us explore the expression $|x_n y_n - x_m y_m|$.

$$
\begin{aligned}
|x_n y_n - x_m y_m| &= |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \\
&\leq |y_n||x_n - x_m| + |x_m||y_n - y_m| && \Big[\text{Triangle Inequality}\Big] \\
&\leq K|x_n - x_m| + M|y_n - y_m| && [|x_n| \leq M, |y_n| \leq K, \\
& && \text{since these are bounded sequences}] \\
&< K \cdot (\epsilon/2K) + M \cdot (\epsilon/2M) \; = \; \epsilon
\end{aligned}
$$

3. [Abbott 2.6.4] Let (a_n) and (b_n) be Cauchy sequences. Decide whether each of the following

sequences is a Cauchy sequence justifying each conclusion.

(a) $c_n = |a_n - b_n|$

Since (a_n) is a Cauchy sequence, if we pick an abitrarily small positive number $\epsilon/2$, there exists N_1 such that $|a_q - a_p| < \epsilon/2$ for all $q > p \geq N_1$.

Since (b_n) is a Cauchy sequence, there exists N_2 such that $|b_s - b_r| < \epsilon/2$ for all $s \geq r \geq N_2$. Let $N = \max\{N_1, N_2\}$ and let $n > m \geq N$.

For any two real numbers $a, b \in \mathbf{R}$, we know that: $||a| - |b|| \leq |a - b|$.

Let us explore the expression $|c_n - c_m|$. We have:

$$\begin{aligned} |c_n - c_m| &= ||a_n - b_n| - |a_m - b_m|| \\ &\leq |(a_n - b_n) - (a_m - b_m)| \\ &= |(a_n - a_m) - (b_n - b_m)| \\ &\leq |a_n - a_m| + |b_n - b_m| \qquad \left[\text{Triangle inequality}\right] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence, (c_n) is a Cauchy sequence.

(b) $c_n = (-1)^n a_n$

Consider the constant sequence $(a_n) = (1, 1, \ldots,)$. (a_n) is a Cauchy sequence, but clearly

$$(c_n) = (1, -1, 1 - 1, \ldots)$$

is not Cauchy.

for all $n > m \geq N$.

Hence, (c_n) is a Cauchy sequence.

(c) $c_n = [[a_n]]$ where $[[x]]$ refers to the greatest integer less than or equal to x.

Consider the sequence $(a_n) = (-1)^n/n$. Then $[[a_n]]$ is:

$$(-1, 0, -1, 0, -1, 0, \ldots)$$

Hence, $[[a_n]]$ is not a Cauchy sequence.

4. [Abbott 2.6.5] Consider the following invented definition: A sequence (s_n) is pseudo-Cauchy if, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \epsilon$.

Decide which of the following two propositions are actually true. Supply a proof for the valid statement and a counterexample for the other.

(i) Pseudo-Cauchy sequences are bounded.

This proposition is false.

Consider the infinite harmonic series. The sequence of partial sums (s_n) of the harmonic series are defined as

$$s_n := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$$

The distance between any two successive terms of (s_n) can be made as small as possible as we go further out into the sequence.

$$|s_{n+1} - s_n| = \left|\frac{1}{n+1} - \frac{1}{n}\right| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n^2}$$

For all $\epsilon > 0$, if we pick $N > \dfrac{1}{\sqrt{\epsilon}}$, then for all $n \geq N$, we have:

$$|s_{n+1} - s_n| < \epsilon$$

However, we find that:

$$
\begin{aligned}
s_{2^n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots+\frac{1}{2^n} \\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\ldots+\underbrace{\left(\frac{1}{2^n}+\ldots+\frac{1}{2^n}\right)}_{2^{n-1}\text{ times}} \\
&= 1+\frac{1}{2}+2\cdot\frac{1}{4}+4\cdot\frac{1}{8}+\ldots+2^{n-1}\frac{1}{2^n} \\
&= 1+\underbrace{\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}}_{n\text{ times}} \\
&= 1+\frac{n}{2}
\end{aligned}
$$

Thus, (s_n) is an unbounded sequence.

(ii) If (x_n) and (y_n) are pseudo-Cauchy, then (x_n+y_n) is pseudo-Cauchy as well.

This proposition is true.

Since, (x_n) is pseudo-cauchy, there exists N_1 such that for all $n \geq N_1$, $|x_{n+1}-x_n| < \epsilon/2$.

Since, (y_n) is pseudo-cauchy, there exists N_2 such that for all $n \geq N_2$, $|y_{n+1}-y_n| < \epsilon/2$.

Let $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we have:

$$
\begin{aligned}
|(x_{n+1}+y_{n+1})-(x_n+y_n)| &= |(x_{n+1}-x_n)+(y_{n+1}-y_n)| \\
&\leq |x_{n+1}-x_n|+|y_{n+1}-y_n| \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

Consequently, (x_n+y_n) is pseudo-cauchy.

5. [Abbott 2.6.6] Let's call a sequence (a_n) quasi-increasing if for all $\epsilon > 0$ there exists an N such that whenever $n > m \geq N$, it follows that $a_n > a_m - \epsilon$.

(a) Give an example of a sequence that is quasi-increasing but not monotone or eventually monotone.

Consider the sequence (a_n) defined by $(-1)^n/n$. We are interested to make the difference $a_m - a_n < \epsilon$. We have:

$$
\begin{aligned}
a_m - a_n &= (-1)^m \frac{1}{m} - (-1)^n \frac{1}{n} \\
&= (-1)^m \frac{1}{m} + (-1)^{n+1} \frac{1}{n} \\
&\leq \frac{1}{m} + \frac{1}{n} < \frac{1}{N} + \frac{1}{N} = \frac{2}{N}
\end{aligned}
$$

Thus, if we choose $N > \frac{2}{\epsilon}$, then we make the difference smaller than ϵ.

(b) Give an example of a quasi-increasing sequence that is divergent and not monotone or eventually monotone.

Properties of Infinite Series.

Given an infinite series $\sum_{k=1}^{\infty} a_k$, it is important to keep a lear distinction between

(i) the sequence of terms: $(a_1, a_2, a_3, \ldots)$ and

(ii) the sequence of *partial sums* $(s_1, s_2, \ldots)$ where $s_n = a_1 + a_2 + \ldots a_n$.

The convergence of the series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the sequence (s_n). Specifically, the statement

$$
\sum_{k=1}^{\infty} a_k = A \quad \text{means that} \quad \lim s_n = A
$$

It is for this reason that we can immediately translate many of our results from the study of sequences into statments about the behavior of infinite series.

Theorem. (Algebraic Limit Theorem for Series). If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbf{R}$ and

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof.

(i) In order to show that $\sum_{k=1}^{\infty} ca_k = cA$, we must argue that the sequence of partial sums

$$t_m = ca_1 + ca_2 + \ldots + ca_m$$

converges to cA. But, we are given that $\sum_{k=1}^{\infty} a_k$ converges to A, meaning the sequence of partial sums

$$s_m = a_1 + a_2 + \ldots a_m$$

converge to A. Because, $t_m = cs_m$, applying the Algebraic Limit Theorem, for sequences yields

$$\lim t_m = \lim cs_m = cA$$

(ii) We are given that the infinite series $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Thus, the sequence of partial sums

$$s_m = a_1 + a_2 + \ldots a_m$$

converge to A, and

$$t_m = b_1 + b_2 + \ldots + b_m$$

converge to B. We must argue that the sequence of partial sums

$$\begin{aligned} u_m &= (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_m + b_m) \\ &= (a_1 + a_2 + \ldots + a_m) + (b_1 + b_2 + \ldots + b_m) \\ &= s_m + t_m \end{aligned}$$

converge. Applying the Algebraic Limit theorem for sequences yields

$$\lim(s_m + t_m) = \lim s_m + \lim t_m \;=\; A + B$$

Hence, $\lim u_m = A + B$. Consequently, $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

One way to summarize part (i) of the above theorem. is to say that infinite addition still satisfies distibutive property. Part (ii) verifies that infinite series can be added the usual way. Missing from this theorem is any statement about the *product* of two infinite series. At the heart of this question is the issue of commutativity which requires a more delicate analyis and so is postponed until the next section.

Theorem. (Cauchy Criterion for Series.) The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, given an $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that whenever $n > m \geq N$ it follows that:

$$|a_{m+1}+a_{m+2}+\ldots+a_n| < \epsilon$$

Proof.

Observe that

$$|s_n - s_m| = |a_{m+1}+a_{m+2}+\ldots+a_n|$$

By the Cauchy Criterion for sequences, the sequence of partial sums (s_n) converges if and only if (s_n) is Cauchy. That is, for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that

$$|s_n - s_m| < \epsilon$$

for all $n > m \geq N$.

The Cauchy Criterion leads to economical proofs of several basic facts about the series.

Theorem. (nth term test). If the infinite series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof.

Consider the special case $n = m+1$ in the Cauchy Criterion for infinite series. Since $\sum_{k=1}^{\infty} a_k$ is convergent, by the Cauchy Criterion for infinite series, we must have, for all $\epsilon > 0$, an $N \in \mathbf{N}$, such that

$$|a_{m+1}| < \epsilon$$

for all $m \geq N$. Consequently, $(a_k) \to 0$.

Every statement of this result should be accompanied with a reminder to look at the harmonic series and to erase any misconceptipon, that the converse of the statement is true. Knowing that (a_k) tends to 0, does not imply that the series converges.

Theorem. (Comparison Test). Assume that (a_k) and (b_k) are seqeuences satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbf{N}$.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof.

Observe that:

$$|a_{m+1} + a_{m+2} + \ldots + a_n| \leq |b_{m+1} + b_{m+2} + \ldots + b_n|$$

Assume that $\sum_{k=1}^{\infty} b_k$ converges. Then, by the Cauchy criterion for series, we know that for all $\epsilon > 0$, there exists an N such that

$$|b_{m+1} + b_{m+2} + \ldots + b_n| < \epsilon$$

whenever $n > m \geq N$.

Consequently, we have, for any arbitary $\epsilon > 0$,

$$|a_{m+1} + a_{m+2} + \ldots a_n| < \epsilon$$

whenever $n > m \geq N$.

Similarly, suppose that $\sum_{k=0}^{\infty} a_k$ diverges. Then, there exists an $\epsilon > 0$ for all N, such that

$$|a_{m+1} + a_{m+2} + \ldots + a_n| > \epsilon$$

for some pair n, m such that $n > m \geq N$.

Consequently, we have an $\epsilon > 0$ for all N, such that

$$|b_{m+1} + b_{m+2} + \ldots + b_n| > \epsilon$$

for some pair n, m such that $n > m \geq N$.

This is a good point to remind ourselves again that the statements about the convergence of sequences and series are immune to changes in some finite number of initial terms. In the Comparison test, the requirement that $0 \leq a_k \leq b_k$ does not really need to hold for all $k \in \mathbf{N}$, but just needs to be eventually true. A weaker, but sufficient hypothesis would be to assume that there exists some point $M \in \mathbf{N}$ such that the inequality $a_k \leq b_k$ is true for all $k \geq M$.

The Comparision Test is used to deduce he convergence or divergence of one series based on the behavior of another. Thus, for this test to be of any great use, we need a catalog of series we can use as measuring sticks. Earlier, we proved the Cauchy Condensation Test, which led to the general statement that the series $\sum_{n=1}^{p} \left(1/n^p\right)$ converges if and only if $p > 1$. The next example summarizes the situation for another important class of seris.

Example. (Geometric Series). A series is called geometric if it is of the form:

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \ldots$$

If $r = 1$ and $a \neq 0$, the series evidently diverges. For $r \neq 1$, the algebraic identity

$$(1-r)\left(1 + r + r^2 + r^3 + \ldots + r^{m-1}\right) = 1 - r^m$$

enables us to rewrite the partial sum

$$s_m = a + ar + ar^2 + \ldots + ar^{m-1} = a\frac{\left(1-r^m\right)}{(1-r)}$$

Now, applying the Algebraic Limit Theorem (for sequences), we have :

$$\begin{aligned}
\lim s_m &= \lim \frac{a\left(1-r^m\right)}{(1-r)} \\
&= \frac{\lim\ a\left(1-r^m\right)}{\lim\ (1-r)} \\
&= \frac{a - a \cdot \lim\ r^m}{(1-r)} \\
&= \frac{a - a \cdot 0}{(1-r)} && \left[\lim\ b^n = 0,\ |b| < 1\right] \\
&= \frac{a}{1-r} && \left[\text{if } |r| < 1\right]
\end{aligned}$$

Although the comparision test requires that the terms of the series be positive, it is often used in conjunction with the next theorem to handle series that contain some negative terms.

Theorem. (Absolute Convergence Test). If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof.

This proof makes use of both the necessity (the *if* direction) and the sufficient (the *only if* direction) of the Cauchy Criterion for Series. Because, $\sum_{n=1}^{\infty} |a_n|$ converges, we know that, given an $\epsilon > 0$, there exists an $N \in \mathbf{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \ldots + |a_n| \ < \epsilon$$

for all $n > m \geq N$. By the triangle inequality,

$$|a_{m+1} + a_{m+2} + \ldots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \ldots + |a_n| < \epsilon$$

so the sufficiency condition of the Cauchy criterion guarantees that $\sum_{n=1}^{\infty} a_n$ also converges.

The converse of this theorem is false. In the opening discussion of this chapter, we considered the alternating harmonic series.

Taking absolute values of the terms gives us the harmonic series $\sum_{n=1}^{\infty} (1/n)$, which we have seen diverges. However, it is not too difficult to prove that with the alternating negative signs the series indeed converges. This is a special case of the alternating series test.

Theorem. (Alternating Series Test). Let (a_n) be a sequence satisfying

(i) $a_1 \geq a_2 \geq a_3 \geq \ \ldots \ \geq a_n \geq a_{n+1} \geq \ \ldots$

(ii) $(a_n) \to 0$

Then, the alternating sequence $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof.
A consequence of the conditions (i) and (ii) is that $a_n \geq 0$. Several proofs of this theorem are outline in the exercise 2.7.1.

Definition. If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges absolutely*. If on the other hand, the series $\sum_{n}^{\infty} a_n$ converges, but the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ does not converge, then we say that the original series $\sum_{n=1}^{\infty} a_n$ *converges conditionally*.

In terms of this newly defined jargon, we have shown by the Alternating Series test that, the Alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges conditionally, whereas

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{2^n}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n},$$

converges absolutely.

In particular, any convergent series with (all but finitely many) positive terms must converge absolutely.

The Alternating Series Test is the most accessible test for conditional convergence, but several others are explored in the exercises. In particular, Abel's test useful in our investigations of the power series.

Rearrangements.

Informally speaking, a rearrangement of a series is obtained by permuting the terms in the sum into some other order. It is important that all of the original terms eventually appear in the new ordering and that no term gets repeated. In an earlier discussion, we formed a rearrangement of the alternating harmonic series by taking two positive terms for each negative term.

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$$

There are clearly an infinite number of rearrangements of any sum; however, it is helpful to see why neither

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

nor

$$1+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{11}-\frac{1}{12}$$

is considered a rearrangement of the original alternating harmonic series.

Definition. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a rearrangement of $\sum_{k}^{\infty} a_k$ if there exists a one-to-one, onto function $f : \mathbf{N} \to \mathbf{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbf{N}$.

We now have all the tools and notation in place to resolve an issue raised at the beginning of the chapter. Previously, we constructed a particular rearrangement of the alternating harmonic series that converges to a limit different from that of the original seris. This happens because the convergence is conditional.

Theorem. If a series converges absolutely, then any rearrangement of this series converges to the same limit.
Proof.
Assume that $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$. Let's use

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

for the partial sums of the original series and use

$$t_m = \sum_{k=1}^{m} b_k = b_1 + b_2 + \ldots + b_m$$

for the partial sums of the rearranged series. Thus, we want to show that $(t_m) \to A$.
Let $\epsilon > 0$. By the hypotheses, the series converges absolutely to A. Therefore, from the absolute convergence test it follows, $(s_n) \to A$, so choose N_1 such that:

$$|s_n - A| < \frac{\epsilon}{2}$$

for $n \geq N_1$.

Because the convergence is absolute, we can choose N_2, such that

$$|a_{s+1}| + |a_{s+2}| + \ldots + |a_t| < \frac{\epsilon}{2}$$

for all $s > t \geq N_2$.

Now take $N = \max\{N_1, N_2\}$. We know that the finite set of terms $\{a_1, a_2, \ldots, a_N\}$ must all appear in the rearranged series, and we want to move far enough out into the series $\sum_{n=1}^{\infty} b_n$ so that we have included all these terms. Thus, choose

$$M = \max\{f(k) : 1 \leq k \leq N\}$$

It should be now be evident that if $m \geq M$, then $(t_m - s_N)$ consists of a finite set of terms, the absolute values of which appear in the tail $\sum_{k=N+1}^{m} |a_k|$. Our choice of N_2 then guarantees that $|t_m - s_N| < \epsilon/2$ and so

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \qquad \Big[\text{Triangle Inequality}\Big] \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $m \geq M$.

Theorem. Let (a_n) be a sequence such that the following conditions are met:

(1) $a_1 \geq a_2 \geq a_3 \ldots \geq a_n \geq a_{n+1} \geq \ldots$

(2) $(a_n) \to l$

Then, intuitively, we know that:

$$l = \inf\{a_n : n \in \mathbf{N}\}$$

Proof.

Claim I. l is a lower bound for (a_n).

Let $A := \{a_n : n \in \mathbf{N}\}$. Suppose there exists an $N_1 \in \mathbf{N}$, such that $a_{N_1} < l$. Put $l - a_{N_1} = \epsilon$. Since $(a_n) \to l$, there exists an $N_2 \in \mathbf{N}$, such that for all $n \geq N_2$,

$$|a_n - l| < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Then,

$$|a_N - l| < \frac{\epsilon}{2} = \frac{a_{N_1} - l}{2} \tag{1}$$

Since $N \geq N_1$, we have $a_N \leq a_{N_1} < l$. Therefore,

$$\left|\frac{a_{N_1} - l}{2}\right| \leq \left|\frac{a_N - l}{2}\right| \tag{2}$$

From (1) and (2),

$$|a_N - l| < \left|\frac{a_N - l}{2}\right|$$

This is a contradiction. Consequently, $l \leq a_n$ for all $n \in \mathbf{N}$.

Claim 2. l is the greatest lower bound.

Pick an arbitrary $\epsilon > 0$. Let's give ourselves an ϵ of room. Consider $(l, l+\epsilon)$. Because $(a_n) \to l$, there exists N such that for all $n \geq N$, $l < a_n < l + \epsilon$. Since, this is true for all $\epsilon > 0$, l is the greatest lower bound.

Exercises.
1. [Kaczor and Nowak, 3.1.2] Find the sum of the series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

Solution.
The nth term can be simplified as:

$$\begin{aligned} a_n &= \frac{2n+1}{n^2(n+1)^2} \\ &= \frac{(n+1)^2 - n^2}{n^2(n+1)^2} \\ &= \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \end{aligned}$$

Therefore, the partial sum $s_n = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots + \frac{1}{n^2} - \frac{1}{(n+1)^2} = 1 - \frac{1}{(n+1)^2}$. The sequence of partial sums (s_n) is monotonically increasing and bounded above by 1. By the Monotone Convergence Theorem (MCT), (s_n) is a convergent sequence. Applying the Algeraic Limit theorem, we have

$$\lim s_n = 1 - \lim \frac{1}{(n+1)^2} = 1 - 0 = 1$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(2n+1)^2}$

Solution.
The term a_n of the infinite series can be simplified as:

$$\begin{aligned} a_n &= \frac{1}{8} \cdot \frac{(2n+1)^2 - (2n-1)^2}{(2n-1)^2(2n+1)^2} \\ &= \frac{1}{8} \cdot \left(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right) \end{aligned}$$

Therefore, the partial sum s_n equals

$$\begin{aligned} s_n &= \frac{1}{8}\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{3^2} - \frac{1}{5^2} + \ldots + \frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2}\right) \\ &= \frac{1}{8}\left(1 - \frac{1}{(2n+1)^2}\right) \end{aligned}$$

The sequence of partial sums is monotone increasing and bounded above by $1/8$. Consequently, by the Montone Convergence Theorem, (s_n) is convergent. Applying the Algebraic Limit Theorem, we have:

$$\lim s_n = \frac{1}{8} \cdot \left(1 - \lim \frac{1}{(2n+1)^2}\right) = \frac{1}{8}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n - \sqrt{n^2 - 1}}{\sqrt{n(n+1)}}$

We have,

$$a_n = \frac{\sqrt{n}}{\sqrt{n+1}} - \frac{\sqrt{n-1}}{\sqrt{n}}$$

So, the partial sum s_n can be expressed as:

$$\begin{aligned}
s_n &= \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{\sqrt{2}}{\sqrt{3}} - \frac{\sqrt{1}}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{\sqrt{4}} - \frac{\sqrt{2}}{\sqrt{3}}\right) + \ldots + \left(\frac{\sqrt{n}}{\sqrt{n+1}} - \frac{\sqrt{n-1}}{\sqrt{n}}\right) \\
&= \left(\cancel{\frac{1}{\sqrt{2}}} - 0\right) + \left(\cancel{\frac{\sqrt{2}}{\sqrt{3}}} - \cancel{\frac{\sqrt{1}}{\sqrt{2}}}\right) + \left(\cancel{\frac{\sqrt{3}}{\sqrt{4}}} - \cancel{\frac{\sqrt{2}}{\sqrt{3}}}\right) + \ldots + \left(\frac{\sqrt{n}}{\sqrt{n+1}} - \cancel{\frac{\sqrt{n-1}}{\sqrt{n}}}\right) \\
&= \frac{\sqrt{n}}{\sqrt{n+1}}
\end{aligned}$$

Thus, (s_n) is a montononically increasing sequence and bounded above by 1. By the Monotone Convergence Theorem, (s_n) is convergent. Applying the Algebraic Limit Theorem to s_n, we have $\lim s_n = 1$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

We have,

$$\begin{aligned}
a_n &= \frac{1}{\left(4n^2 - 1\right)} \\
&= \frac{1}{(2n+1)(2n-1)} \\
&= \frac{1}{2} \cdot \left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)
\end{aligned}$$

Consequently, the partial sum (s_n) is given by:

$$s_n = \frac{1}{2} \cdot \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \ldots + \frac{1}{2n-1} - \frac{1}{2n+1} \right) = \frac{1}{2} \cdot \left(1 - \frac{1}{2n+1} \right)$$

Thus, the sequence of partial sums (s_n) is monotonic increasing and bounded above by $1/2$. By the Monotone Convergence Theorem, (s_n) converges. Applying the Algebraic limit theorem, we find $\lim s_n = \frac{1}{2}$.

(e) $\sum_{n=1}^{\infty} \frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\left(\sqrt{n(n+1)}\right)}$

Solution.

Since $\left(\sqrt{n+1}\right)^2 - \left(\sqrt{n}\right)^2 = 1$, we have $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n}+\sqrt{n+1}}$.

Thus,

$$\begin{aligned} a_n &= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}} \\ &= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \end{aligned}$$

The partial sum s_n would be,

$$s_n = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}}$$

Consequently, the sequence of partial sums (s_n) is monotonic increasing and bounded above by 1. Applying the Algebraic Limit theorem, we get $\lim s_n = 1$.

2. [Kaczor and Nowak, 3.1.3] Compute the following sums:

(a) $\ln \frac{1}{4} + \sum_{n=1}^{\infty} \ln \frac{(n+1)(3n+1)}{n(3n+4)}$.

Solution.

(a) We have

$$a_n \quad = \log(n+1) + \log(3n+1) - \log n - \log(3n+4)$$

The partial sum s_n can be expressed as,

$$\begin{aligned}
s_n &= (\log 2 + \log 4 - \log 1 - \log 7) + (\log 3 + \log 7 - \log 2 - \log 10) \\
&+(\log 4 - \log 10 - \log 3 - \log 13) + \ldots (\log(n+1) + \log(3n+1) - \log n - \log(3n+4)) \\
&= \left(\cancel{\log 2} + \log 4 - \log 1 - \cancel{\log 7}\right) + \left(\log 3 + \cancel{\log 7} - \cancel{\log 2} - \log 10\right) \\
&+ \ldots + \left(\cancel{\log(n)} + \log(3n-2) - \log(n-1) - \cancel{\log(3n+1)}\right) \\
&+\left(\log(n+1) + \cancel{\log(3n+1)} - \cancel{\log n} - \log(3n+4)\right)
\end{aligned}$$

$$\begin{aligned}
s_n \quad &= (\log 2 + \log 4 - \log 1 - \log 7) + (\log 3 + \log 7 - \log 2 - \log 10) \\
&+(\log 4 - \log 10 - \log 3 - \log 13) + \ldots (\log(n+1) + \log(3n+1) - \log n - \log(3n+4)) \\
&= \left(\cancel{\log 2} + \log 4 - \log 1 - \cancel{\log 7}\right) + \left(\log 3 + \cancel{\log 7} - \cancel{\log 2} - \log 10\right) \\
&+ \ldots + \left(\cancel{\log(n)} + \log(3n-2) - \log(n-1) - \cancel{\log(3n+1)}\right) \\
&+\left(\log(n+1) + \cancel{\log(3n+1)} - \cancel{\log n} - \log(3n+4)\right) \\
&= \log 4 + \log(n+1) - \log(3n+4)
\end{aligned}$$

If we add the first term $\log \frac{1}{4}$, this becomes

$$s_n = \log \frac{(n+1)}{(3n+4)}$$

The sequence $t_n = (n+1)/(3n+4)$ is monotone increasing and has an upper bound $1/3$, since

$$t_n = \frac{n+1}{3n+4} = \frac{1}{3} \cdot \frac{3n+3}{3n+4} = \frac{1}{3} \cdot \frac{(3n+4)-1}{(3n+4)} = \frac{1}{3} \cdot \left(1 - \frac{1}{3n+4}\right)$$

Since $\log x$ is also a monotone increasing function of x, the partial sum (s_n) is monotone increasing and bounded above by $\log(1/3)$. By the Algebraic Limit Theorem, $\lim s_n = \log(1/3)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)n}{(n+1)(2n-1)}$.

Solution.

3. [Abbott, 2.7.1] Proving the Alternating Series Test amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - a_4 + \ldots \pm a_n$$

converges. Different characterizations of completeness lead to different proofs.

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy Sequence.

Consider the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where the following conditions are satisfied:

(1) $a_1 \geq a_2 \geq \ \ldots \ \geq a_n \geq a_{n+1} \geq \ \ldots$

(2) The sequence $(a_n) \to 0$.

We are given that (a_n) is a monotonic decreasing sequence and $(a_n) \to 0$. Then, $\inf\{a_n : n \in \mathbf{N}\} = 0$. Therefore, $a_n \ \geq 0$ for all $n \in \mathbf{N}$.

Let us explore the expression $s_n - s_m$.

$$\begin{aligned} s_{12} - s_5 \quad &= a_6 - a_7 + a_8 - a_9 + a_{10} - a_{11} + a_{12} \\ &= a_6 - (a_7 - a_8) - (a_9 - a_{10}) - (a_{11} - a_{12}) \\ &\leq a_6 \leq a_5 \end{aligned}$$

And, $s_{12} - s_5 \geq 0$.

$$\begin{aligned} s_{12} - s_6 \quad &= -a_7 + a_8 - a_9 + a_{10} - a_{11} + a_{12} \\ &= -a_7 + (a_8 - a_9) \ + \ (a_{10} - a_{11}) + a_{12} \\ &\geq -a_7 \\ &\geq -a_6 && \left[\text{since } -a_6 \leq \ -a_7\right] \end{aligned}$$

And $s_{12} - s_6 \leq 0$.

In general, $|s_n - s_m| \ \leq \ |a_m|$.

Since $(a_n) \to 0$, we know that for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that

$$|a_m| < \epsilon$$

for all $m \geq N$.

Consequently, for all ϵ, there exists N such that,

$$|s_n - s_m| \ \leq \ |a_m| < \epsilon$$

for all $n > m \geq N$.

Therefore, by the Cauchy criterion, (s_n) is a convergent sequence.

(b) Supply another proof for this result using the Nested Interval Property.

Consider the sequence of partial sums $(s_n) = (s_1, s_2, s_3, s_4, s_5, s_6, \ldots)$. We have:

$$\begin{aligned}
s_1 &= a_1 \\
s_3 &= a_1 - a_2 + a_3 = s_1 - (a_2 - a_3) \leq s_1 \\
s_5 &= a_1 - a_2 + a_3 - a_4 + a_5 = s_3 - (a_4 - a_5) \leq s_3
\end{aligned}$$

And

$$\begin{aligned}
s_2 &= a_1 - a_2 \\
s_4 &= a_1 - a_2 + a_3 - a_4 = s_1 + (a_3 - a_4) \geq s_2 \\
s_6 &= a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = s_4 + (a_5 - a_6) \geq s_4
\end{aligned}$$

And,

$$\begin{aligned}
s_2 &= a_1 - a_2 \leq a_1 = s_1 \\
s_2 &\leq (a_1 - a_2) + a_3 = s_3 \\
s_2 &\leq (a_1 - a_2) + (a_3 - a_4) + a_5 \leq s_5
\end{aligned}$$

Also, consider the terms s_{2m} and s_{2n+1}, for arbitrary $m, n \in \mathbf{N}$.

Case I. $2m < 2n + 1$.

$$\begin{aligned}
s_{2m} &= (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2m-1} - a_{2m}) \\
&\leq (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2m-1} - a_{2m}) + \ldots + a_{2n+1} \\
&= s_{2n+1}
\end{aligned}$$

Case II. $2m > 2n + 1$.

$$\begin{aligned}
s_{2m} &= a_1 - a_2 + a_3 - a_4 + \ldots - a_{2n} + a_{2n+1} - a_{2n+2} + a_{2n+3} - \ldots - a_{2m} \\
&= a_1 - a_2 + a_3 - a_4 + \ldots - a_{2n} + a_{2n+1} - (a_{2n+2} - a_{2n+3}) - \ldots - a_{2m} \\
&= s_{2n+1} - \ldots - a_{2m} \leq s_{2n+1}
\end{aligned}$$

Thus, every s_{2m} less than each s_{2n+1}.

Consequently,

$$s_2 \leq s_4 \leq s_6 \leq s_8 \leq \ldots \leq s_7 \leq s_5 \leq s_3 \leq s_1$$

Define I_1 as the closed interval $[s_2, s_1]$, $I_2 := [s_4, s_3]$, $\ldots$. In general, define $I_k = [s_{2k}, s_{2k-1}]$. We have a sequence of closed nested intervals.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_k \supseteq I_{k+1} \supseteq \ldots$$

By the Nested Interval Property, there exists atleast one element $x \in \mathbf{R}$ such that

$$\bigcap_{k=1}^{\infty} I_k = x$$

Since, $(a_n) \to 0$, $(a_{2n}) \to 0$. Therefore, for all $\epsilon > 0$, there exists an N, such that $|a_{2n}| < \epsilon$ for all $2n \geq N$.

But, $s_{2n} = s_{2n-1} - a_{2n}$. So, $s_{2n-1} - s_{2n} = a_{2n}$. Thus, for all $\epsilon > 0$, there exists an interval I_N such that the length of the interval:

$$|s_{2n-1} - s_{2n}| = |a_{2n}| < \epsilon$$

Thus, $(s_{2n-1} - s_{2n}) \to 0$.

Pick any arbitrary $\epsilon > 0$. Since $I_N \subset (x - \epsilon/2, x + \epsilon/2)$, we have $s_k \in (x - \epsilon/2, x + \epsilon/2)$ for all $k \geq N$. Hence, (s_n) is a convrgent sequence.

(c) Consider the subsequences (s_{2n}) and (s_{2n+1}) and show how the Montone Convergence Theorem leads to a third proof for the Alternating Series Test.

Proof.

Let $(p_n) = (s_{2n}) = (s_2, s_4, s_6, s_8, \ldots)$ and $(q_n) = (s_{2n-1}) = (s_1, s_3, s_5, s_7, \ldots)$. Every q_n is an upper bound for each p_n and every p_n is a lower bound for each q_n. (p_n) is a monotonic increasing sequence, whilst (q_n) is a monotonic decreasing sequence. By the Monotone Convergence Theorem, the sequences (p_n) and (q_n) are convergent.

(p_n) is convergent sequence. Since (p_n) is monotonic increasing, $\lim(p_n) = \sup\{p_n : n \in \mathbf{N}\}$.
(q_n) is convergent sequence. Since (q_n) is monotonic increasing, $\lim(q_n) = \inf\{q_n : n \in \mathbf{N}\}$.

We are interested to prove that $\lim(p_n) = \sup\{p_n : n \in \mathbf{N}\} = \inf\{q_n : n \in \mathbf{N}\} = \lim(q_n)$.

Let $l = \inf\{q_n : n \in \mathbf{N}\}$.

Claim I. l is an upper bound for $\{p_n : n \in \mathbf{N}\}$.
By contradiction, suppose l is not an upper bound for $\{p_n : n \in \mathbf{N}\}$. Then, there exists an p_N such that $l < p_N$. Let $p_N - l = \epsilon > 0$. Therefore, $p_N = l + \epsilon$. Since, $l = \inf\{q_n : n \in \mathbf{N}\}$, there exists q_M such that $l < q_M < l + \epsilon$. So, $q_M < p_N$. But, this is a contradiction, since every q_k is an upper bound for each p_k.

Therefore, l must be an upper bound for $\{p_n : n \in \mathbf{N}\}$.

Claim II. l is the supremum of $\{p_n : n \in \mathbf{N}\}$.

If $s = \sup A$, then we know that for all $\epsilon > 0$, there exists an $a \in A$, such that $s - \epsilon < a$. The negation of this statement is : there exists an $\epsilon > 0$, for all $a \in A$, such that $a < s - \epsilon$.

By contradiction, suppose l is not the least upper bound for $\{p_n : n \in \mathbf{N}\}$. Then, there exists an $\epsilon > 0$ for all p_n, such that $p_n < l - \epsilon$.

We know that $q_n - p_n = a_{2n}$. Since $(a_{2n}) \to 0$, this distance can be made smaller than any ϵ we please. Therefore, $-\epsilon < q_n - p_n < \epsilon$.

Altogether, $(q_n - p_n) + p_n < (l - \epsilon) + \epsilon = l$. Therefore, $q_n < l$. But, this contradicts the fact, that l is a lower bound for q_n.

Hence, our initial assumption is false. $l = \sup\{p_n : n \in \mathbf{N}\}$.

We have $(p_n) \to l$ and $(q_n) \to l$. Therefore, the shuffled sequence $(p_1, q_1, p_2, q_2, \ldots, p_n, q_n, \ldots)$ converges to l.

4. [Abbott, 2.7.5] Now that we have proved the basic facts about the geometric series, supply a proof for the corollary 2.4.7

Proof.

We are interested to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Let (s_n) be the sequence of partial sums of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$. We have,

$$
\begin{aligned}
s_{2^n} &= 1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+\ldots+\frac{1}{15^p}\right)+\ldots \\
&+\left(\frac{1}{\left(2^{n-1}\right)^p}+\ldots+\frac{1}{\left(2^n-1\right)^p}\right) \\
&\le 1+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\left(\frac{1}{8^p}+\ldots+\frac{1}{8^p}\right)+\ldots \\
&+\left(\frac{1}{\left(2^{n-1}\right)^p}+\ldots+\frac{1}{\left(2^{n-1}\right)^p}\right) \\
&= 1+2\cdot\frac{1}{2^p}+4\cdot\frac{1}{4^p}+8\cdot\frac{1}{8^p}+\ldots+2^{n-1}\cdot\frac{1}{\left(2^{n-1}\right)^p} \\
&= 1+\frac{1}{2^{p-1}}+\frac{1}{\left(2^{p-1}\right)^2}+\frac{1}{\left(2^{p-1}\right)^3}+\ldots+\frac{1}{\left(2^{p-1}\right)^{n-1}} \\
&\le \textstyle\sum_{n=0}^{\infty}\frac{1}{\left(2^{p-1}\right)^n}=\frac{1}{1-\left(1/2^{p-1}\right)} && \left[\text{iff } p>1\right]
\end{aligned}
$$

Thus, the sequence of partial sums is monotonic increasing and bounded if and only $p > 1$. By the Monotone Convergence Theorem, $\sum_{n=1}^{\infty}\left(1/n^p\right)$ converges if and only if $p > 1$.

5. [Abbott, 2.7.2] Decide whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty}\frac{1}{2^n+n}$

Solution.

We know that the geometric series $\sum_{n=1}^{\infty}ar^{n-1}$ converges for $|r| < 1$. Consequently, $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a convergent series. Since, $0 \le \frac{1}{2^n+n} \le \frac{1}{2^n}$, by the comparision test, the given infinite series is convergent.

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Solution.

We know that $-1 \le \sin(n) \le 1$. Therefore, $0 \le |\sin(n)| \le 1$.Thus, $0 \le |\sin(n)| / n^2 \le 1/n^2$.
$\sum_{n=1}^{\infty} (1/n^p)$ is a convergent series if and only if $p > 1$. Therefore, $\sum_{n=1}^{\infty} (1/n^2)$ is a convergent series. By the comparision test, $\sum_{n=1}^{\infty} |\sin(n)| / n^2$ is a convergent series.

By the absolute convergence test, $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ is a convergent series.

(c) By the nth term test, we have:

$$\lim(a_n) = \lim(-1)^n \cdot \frac{n+1}{2n} = \lim(-1)^n \cdot \frac{1}{2}$$

This limit does not exist. Consequently, since (a_n) does not approach 0, $\sum_{n=1}^{\infty} a_n$ is divergent.

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \ldots$

We have:

$$\begin{aligned} s_n &= 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \pm \frac{1}{n} \\ &\ge 1 + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} + \frac{1}{6} - \frac{1}{6} + \ldots \pm \frac{1}{n} \\ &= 1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \ldots \pm \frac{1}{n} \end{aligned}$$

But, the infinite series $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \ldots$ is unbounded and divergent. Consequently, (s_n) is unbounded. Hence, it is divergent.

(e) This can be written as the sum of two infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Whereas $\sum_{n=1}^{\infty} \left(1/n^2\right)$ is a convergent series, the infinite series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is divergent. This can be proven as follows:

Let (s_n) be the sequence of partial sums of the infinite series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

We have:

$$\begin{aligned}
s_5 &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \\
&\geq 1 + \frac{1}{3} + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) = 1 + \frac{1}{3} + \frac{1}{3} = 1 + 2 \cdot \frac{1}{3} \\
s_{(27+1)/2} &= 1 + \frac{1}{3} + \left(\frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right) + \left(\frac{1}{11} + \ldots + \frac{1}{27}\right) \\
&\geq 1 + \frac{1}{3} + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + \left(\frac{1}{27} + \ldots + \frac{1}{27}\right) \\
&= 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 + 3 \cdot \frac{1}{3} \\
s_{(3^4+1)/2} &\geq 1 + 4 \cdot \frac{1}{3} \\
s_{(3^n+1)/2} &\geq 1 + n \cdot \frac{1}{3}
\end{aligned}$$

Hence, the sequence of partial sums (s_n) is unbounded. Consequently, (s_n) is divergent. Hence, the original infinite series is also divergent.

6. [Abbott, 2.7.3] (a) Provide the details for the proof of the Comparison Test (Theorem 2.7.4) using the Cauchy criterion for series.

(b) Give another proof of the Comparison Test, this time using the Monotone Convergence Theorem.

Proof.
We are given that (a_k) and (b_k) are two sequences satisfying

$$0 \leq a_k \leq b_k$$

Let (s_n) and (t_n) be the sequence of partial sums of the infinite series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$. Thus, (s_n) and (t_n) are monotonic increasing sequences. Further, $0 \leq s_n \leq t_n$.

(i) If (t_n) is a convergent sequence, then $\lim(t_n)$ exists. So, $0 \leq s_n \leq \lim(t_n)$.Therefore, (s_n) is a bounded sequence. By the Monotone Convergence Theorem, (s_n) convergence.

(ii) If (s_n) is a divergent monotone sequence, (s_n) is unbounded. If (s_n) is unbounded, then it follows (t_n) is unbounded, since $t_n \geq s_n$. Therefore, (t_n) is divergent.

7. [Abbott 2.7.4] Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.

Consider the infinite series $\sum (1/n)$ and $\sum \left(1/\sqrt{n}\right)$. Both theses series are divergent. But, $\sum \left(1/n\sqrt{n}\right)$ is convergent.

(b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

Let $\sum x_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and $\sum y_n = \sum_{n=1}^{\infty} (-1)^{n+1}$. (y_n) is a bounded sequence, $|y_n| \leq 1$. However, $\sum x_n y_n = \sum \frac{1}{n}$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum (x_n + y_n)$ both converge but $\sum y_n$ diverges.

This request is impossible. Since, both $\sum (x_n + y_n)$ and $\sum x_n$ converge, by the Algebraic Limit theorem for infinite series,

$$\begin{aligned} \sum y_n &= \sum (x_n + y_n - x_n) \\ &= \sum (x_n + y_n) - \sum x_n \end{aligned}$$

and therefore $\sum y_n$ is convergent.

(d) A sequence (x_n) satisfying $0 \le x_n \le (1/n)$ where $\sum (-1)^n x_n$ diverges.

To do.

8. [Abbot, 2.7.6] Let's say that a series subverges if the sequence of partial sums contains a subsequence that converges. Consider this (invented) definition for a moment, and then decide which of the following statements are valid propositions about subvergent series:

(a) If (a_n) is bounded, then $\sum a_n$ subverges.

This proposition is false.

Counterexample.

Let $(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)$.

(a_n) is a bounded sequence, since $0 < a_n \le 1$. However, $\sum a_n$ does not subverge.

(b) All convergent series are subvergent.

This proposition is true.

Justification.
The sequence of partial sums (s_n) is convergent. Let $(s_n) \to L$. Thus, for all $\epsilon > 0$ there exists an $N \in \mathbf{N}$, such that $|s_n - L| < \epsilon$ for all $n \ge N$. Let (s_{nk}) be a proper subsequence of (s_n). Then, for all $n_k \ge N$, $|s_{n_k} - L| < \epsilon$. Consequently, $(s_{n_k}) \to L$. Thus, all convergent series are subvergent.

(c) If the $\sum |a_n|$ subverges, then $\sum a_n$ subverges as well.

This proposition is true.

Let (s_n) be the sequence of partial sums of the infinite series $\sum |a_n|$.

Let (s_{n_k}) be any subsequence of the sequence of partial sums of the series $\sum |a_n|$.

Let (t_{n_k}) be the corresponding subsequence of the sequence of partial sums of the series $\sum a_n$.

Since (s_n) is subvergent, by the Cauchy criterion, there exists an $K \in \mathbf{N}$, such that for all $c > b \ge K$, we have:

$$|s_{n_c} - s_{n_b}| < \epsilon$$

Therefore,

$$|a_{n_b+1}| + |a_{n_b+2}| + \ldots + |a_{n_c}| < \epsilon$$

But, we know that:

$$|a_{n_b+1} + a_{n_b+2} + \ldots + a_{n_c}| \leq |a_{n_b+1}| + |a_{n_b+2}| + \ldots + |a_{n_c}| < \epsilon$$

for all $n_c > n_b \geq n_K$.

Thus,

$$|t_{n_c} - t_{n_b}| < \epsilon$$

for all $c > b \geq K$. Therefore, (t_{nk}) is convergent. Thus, $\sum a_n$ is subvergent.

(d) If $\sum a_n$ subverges, then (a_n) has a convergent subsequence.

This proposition is false.

Counterexample.

Let $a_{2n} = -n$ and $a_{2n+1} = n + \dfrac{1}{n^2}$. Now, (s_{2n+1}) is a convergent subsequence. So, $\sum a_n$ subverges. However, (a_n) has not convergent subsequence.

9. [Abbott, 2.7.7] (a) Show that if $a_n > 0$ and $\lim na_n = l$ with $l \neq 0$, then the series $\sum a_n$ diverges.

Proof.

Since $\lim na_n = l$, for all $\epsilon > 0$, there exists an N such that

$$|na_n - l| < \epsilon$$

Consequently,

$$|na_n| > |l| - \epsilon$$

Since $a_n > 0$,

$$na_n > |l| - \epsilon$$

Pick $\epsilon = |l|/2$. Then, we have:

$$na_n > \frac{|l|}{2}$$

So,

$$a_n > \frac{|l|}{2n}$$

for $n > N$.

By the comparison test, the series $\sum \frac{1}{2n}$ is divergent. Consequently, $\sum a_n$ is a divergent series.

(b) Since $\lim n^2 a_n = l$, we have:

$$n^2 a_n \quad < |l| + \epsilon \qquad \left[\text{for all } n > N\right]$$

If we pick $\epsilon = |l|$, we get:

$$n^2 a_n < 2|l|$$

for all $n > N$.

$$a_n < 2\frac{|l|}{n^2}$$

We know that $\sum \frac{2|l|}{n^2}$ is a convergent series. By the comparision test, consequently, $\sum a_n$ is a convergent series.

10. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

This proposition is true.

Proof.

Assume that $\sum a_n$ converges absolutely. Therefore, the series of absolute terms $\sum |a_n|$ is convergent. By the Cauchy criterion for series, there exists N such that

$$|a_{m+1}| + |a_{m+2}| + \ldots + |a_n| < \sqrt{\epsilon}$$

for all $n > m \geq N$.

Consider the infinite series $\sum \left|a_n^2\right| = \sum |a_n|^2$. We have:

$$|a_{m+1}|^2 + \ldots + |a_n|^2 \leq (|a_{m+1}| + \ldots + |a_n|)^2 < \epsilon$$

for all $n > m \geq N$.

Therefore the series $\sum a_n^2$ is absolutely convergent.

(b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.

This proposition is false.

Counterexample.

Let $a_n = \dfrac{1}{\sqrt{n}}$. Since $a_1 \geq a_2 \geq a_3 \geq \ \ldots \ \geq a_n \geq a_{n+1} \geq \ \ldots$, the sequence (a_n) is monotonic decreasing. Moreover, $(a_n) \to 0$. Therefore, by the alternating series test $\sum (-1)^{n+1} \dfrac{1}{\sqrt{n}}$ is a convergent series. Moreover, let $b_n = \dfrac{(-1)^{n+1}}{\sqrt{n}}$. (b_n) is a convergent sequence. We have,

$$a_n b_n = \frac{(-1)^{2n+2}}{\sqrt{n} \cdot \sqrt{n}} = \frac{1}{n}$$

The harmonic series $\sum a_n b_n$ is a divergent series.

(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

This proposition is true.

Proof (By contrapositive).

Assume that $\sum n^2 a_n$ converges. Therefore, $\left(n^2 a_n\right)$ is a convergent sequence and approaches 0. Since, $\left(n^2 a_n\right)$ is convergent, it is bounded. Therefore, there exists $M > 0$ for all $n \in N$, such that

$$\begin{aligned} n^2 |a_n| &\leq M \\ |a_n| &\leq \frac{M}{n^2} \end{aligned}$$

By the comparision test, we know that $\sum M/n^2$ is a convergent series. Therefore, the series of absolute values, $\sum |a_n|$ is convergent. So, $\sum a_n$ converges absolutely. The contrapositive of this statement is:

If $\sum a_n$ is conditionally convergent, then $\sum n^2 a_n$ diverges.

11. [Abbott, 2.7.9] **(Ratio Test.)** Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the ratio test states that, if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1$$

then the series converges absolutely.

(a) Let r' satisfy $r \; < \; r' < 1$. Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$

Proof.

We are given, $\lim \left| \frac{a_{n+1}}{a_n} \right| = r$. By definition of the convergence of a sequence, for all $\epsilon > 0$, there exists N, such that $|a_{n+1}/a_n| \in V_\epsilon(r)$ for all $n \geq N$. Pick $\epsilon = r' - r$. Then, there exists N such that $n \geq N$, implies

$$\begin{aligned} -\epsilon &< \left| \frac{a_{n+1}}{a_n} \right| - r < \epsilon \\ -(r' - r) &< \left| \frac{a_{n+1}}{a_n} \right| - r < (r' - r) \end{aligned}$$

$$-(r'-r)+r < \left|\frac{a_{n+1}}{a_n}\right| < (r'-r)+r$$

$$\therefore \left|\frac{a_{n+1}}{a_n}\right| < r'$$

$$|a_{n+1}| < r'|a_n|$$

(b) Why does $|a_N| \sum (r')^n$ converge?

Proof.

Since, $a_n \neq 0$, we must have$|a_{n+1}/a_n| > 0$ for all n. Therefore, by the order limit theorem $\lim |a_{n+1}/a_n| = r > 0$ by the order limit theorem.

Since $r < r' < 1$, altogether we have $0 < r < r' < 1$.

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent, provided $|r| < 1$.

Therefore, the infinite series $|a_N| \sum (r')^n$ is convergent.

(c) Now, show that $\sum |a_n|$ converges, and conclude that $\sum a_n$ converges.

Proof.

We have:

$$\begin{aligned} |a_n| &\leq r'|a_{n-1}| \\ &\leq (r')^2|a_{n-2}| \\ &\vdots \\ &\leq (r')^{n-N}|a_N| \end{aligned}$$

Since $|a_n| > 0$, altogether we have:

$$0 < |a_n| \leq (r')^{n-N}|a_N|$$

From (b), $|a_N| \sum (r')^n$ is a convergent series, given $|r'| < 1$. Consequently, by the comparison test, $\sum |a_n|$ is a convergent series.

By the absolute convergence test, if $\sum |a_n|$ converges, then $\sum a_n$ converges as well.

12. [Abbot, 2.4.10] **(Infinite Products.)** A close relative of the infinite series is the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \ldots$$

which is understood in terms of its sequence of *partial products*

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \ldots$$

where $a_n \geq 0$.

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = 1/n$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = 1/n^2$ and make a conjecture about the convergence of this sequence.

We have:

$$\begin{aligned} p_n &= \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \ldots \left(1 + \frac{1}{n}\right) \\ &= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n} \\ &= \frac{\cancel{2}}{1} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdots \frac{n+1}{\cancel{n}} \\ &= n + 1 \end{aligned}$$

We know that the sequence $(p_n) = (n + 1)$ is a divergent sequence.

Let $a_n = \dfrac{1}{n^2}$. We have:

$$
\begin{aligned}
p_1 &= \left(1+\frac{1}{1^2}\right) = \frac{2}{1}\\
p_2 &= \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right) = \frac{2}{1}\cdot\frac{5}{4}\\
p_3 &= \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right) = \frac{2}{1}\cdot\frac{5}{4}\cdot\frac{10}{9}\\
p_4 &= \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right) = \frac{2}{1}\cdot\frac{5}{4}\cdot\frac{10}{9}\cdot\frac{17}{16}
\end{aligned}
$$

The ratio of the partial products is $\left|\dfrac{p_n}{p_{n-1}}\right| = \dfrac{n^2+1}{n^2} > 1$. The ratio sequence approaches 1. In fact,

$$
\begin{aligned}
\left|\frac{p_n}{p_m}\right| &= \frac{1+\frac{1}{n^2}}{1+\frac{1}{m^2}} = \frac{n^2+1}{n^2}\cdot\frac{m^2}{m^2+1} < \frac{n^2+1}{n^2}\cdot\frac{m^2+1}{m^2+1} = \frac{n^2+1}{n^2}\\
\left|\frac{p_n}{p_m}\right| &< 1+\frac{1}{n^2}\\
\frac{|p_n|}{|p_m|}-1 &< \frac{1}{n^2}\\
\frac{|p_n|-|p_m|}{|p_m|} &< \frac{1}{n^2}\\
|p_n|-|p_m| &< \frac{|p_m|}{n^2} && [|p_n|>|p_m|,\ n>m]\\
p_n-p_m &< \frac{p_m}{n^2} && \left[\text{since } p_n, p_m > 0\right]\\
&< \frac{2}{n^2} && \left[\text{ since } p_m \geq 2\right]\\
\therefore |p_n-p_m| &< \frac{2}{n^2}
\end{aligned}
$$

Thus, if we pick $N > \sqrt{\dfrac{2}{\epsilon}}$, then for all $n \geq N$, the above inequality holds. Consequently, by the Cauchy criterion for sequences, (p_n) is a convergent sequence.

(b) Show, in general, that the sequence of partial products converges if and only if $\sum\limits_{n=1}^{\infty} a_n$ converges.

(The inequality $1+x \leq 3^x$ for positive x will be useful in one direction.)

Proof.

($\Longleftarrow$) direction.

Assume that the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Let p_n be the partial product of the terms of the infinite product series $\displaystyle\prod_{n=1}^{\infty}(1+a_n)$.

$$\begin{aligned} p_n &= (1+a_1)(1+a_2)\,\cdots\,(1+a_n) \\ &\leq (1+2)^{a_1}(1+2)^{a_2}\,\cdots\,(1+2)^{a_n} \\ &= 3^{\sum_{n=1}^{\infty} a_n} \end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent series, (p_n) is bounded above and monotonic increasing. By the Monotone Convergence theorem, (p_n) is a convergent sequence.

($\Longrightarrow$) direction.

Assume that the sequence of partial products (p_n) is convergent. By the Cauchy criterion, for all $\epsilon > 0$, there exists an N such that:

$$|p_n - p_m| < \epsilon$$

13. [Abbott, 2.7.10] **(Infinite Products Contd.)** Review the exercise 2.4.10 about infinite products and then answer the following questions:

(a) Does $\dfrac{2}{1}\cdot\dfrac{3}{2}\cdot\dfrac{5}{4}\cdot\dfrac{9}{8}\cdot\dfrac{17}{16}\,\cdots$ converge?

Solution.

This infinite product converges since,

$$p_n = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\left(1+\frac{1}{16}\right)\cdots\left(1+\frac{1}{2^n}\right)$$

Since $\sum a_n = 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}+\ldots$ is a convergent series, the above infinite product converges.

(b) The infinite product $\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}\cdots$ certainly converges (Why?). Does it converge to 0?

Solution.

We have:

$$\begin{aligned} p_n &= p_{n-1}\left(1-\frac{1}{2n}\right) \\ &\leq p_{n-1} \end{aligned}$$

for all $n \in \mathbf{N}$. Consequently, $p_1 \geq p_2 \geq \ \ldots \ \geq p_n \geq p_{n+1} \geq \ \ldots$, and (p_n) is a monotonic decreasing sequence. Moreover

$$\begin{aligned} p_n &= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2n}\right) \\ &\geq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\cdots\left(1-\frac{1}{2}\right) \\ &\geq \left(1-\frac{1}{2}\right)^n = \frac{1}{2^n} > 0 \end{aligned}$$

So, sequence of partial products is bounded below. By the Monotone Convergence Theorem, (p_n) is a convergent sequence.

We claim that, the infinite product series $(p_n) \to 0$. We have:

$$\begin{aligned} p_n &\leq \left(1-\frac{1}{2n}\right)^n \\ &= (b_n)^n \end{aligned}$$

Thus,

$$0 < p_n \leq (b_n)^n$$

where $0 < b_n < 1$. We know that, the sequence $\left(b^n\right)$, where $0 < b < 1$ converges to 0. Thus, the constant sequence $(0, 0, 0, \ldots)$ and $\left(b_1^1, b_2^2, \ldots\right)$ have limits 0. By the squeeze theorem, $\lim(p_n)$ exists and $\lim p_n = 0$.

(c) In 1655, John Wallis famously derived the formula

$$\left(\frac{2 \cdot 2}{1 \cdot 3}\right)\left(\frac{4 \cdot 4}{3 \cdot 5}\right)\left(\frac{6 \cdot 6}{5 \cdot 7}\right)\left(\frac{8 \cdot 8}{7 \cdot 9}\right) \cdots = \frac{\pi}{2}$$

Show that the left side of this identity at least onverges to something. (A complete proof of this result is taken up in section 8.3).

Proof.
The Wallis formula can be expressed as:

$$\begin{aligned}
p_n &= \frac{2}{1} \cdot \left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 + \frac{1}{5}\right) \cdots \left(1 - \frac{1}{2n+1}\right)\left(1 + \frac{1}{2n+1}\right) \\
&= \frac{2}{1} \cdot \left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right) \cdots \left(1 - \frac{1}{(2n+1)^2}\right) \\
&\geq \frac{2}{1} \cdot \underbrace{\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{3^2}\right)}_{n \text{ terms}} \\
&= \frac{2}{1} \cdot \left(\frac{8}{9}\right)^n \geq 0
\end{aligned}$$

So, the sequence (p_n) is bounded below by 0. Moreover,

$$\begin{aligned}
p_n &= p_{n-1}\left(1 - \frac{1}{(2n+1)^2}\right) \\
&\leq p_{n-1}
\end{aligned}$$

for all $n \in \mathbf{N}$.

Consequently, (p_n) is a monotonic decreasing sequence and bounded below. By the Monotone Convergence Theorem (p_n) is a convergent sequence.

14. [Abbott, 2.7.11] Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge, but for which $\sum \min\{a_n, b_n\}$ converges. To make it more challenging, produce examples where (a_n) and (b_n) are

strictly positive and decreasing.

Solution.

Consider the sequences

$$a_{2n} = \frac{1}{n} \quad b_{2n} = \frac{1}{n^2}$$
$$a_{2n-1} = \frac{1}{n^2} \quad b_{2n-1} = \frac{1}{n}$$

The sequences (a_n) and (b_n) are strictly decreasing and positive. Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent series. We have:

$$\begin{aligned}
(a_n) &= \left(\frac{1}{1^2}, \frac{1}{1}, \frac{1}{2^2}, \frac{1}{2}, \frac{1}{3^2}, \frac{1}{3}, \dots\right) \\
(b_n) &= \left(\frac{1}{1}, \frac{1}{1^2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{3}, \frac{1}{3^2}, \dots\right) \\
\min\{a_n, b_n\} &= \left(\frac{1}{1^2}, \frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{3^2}, \dots\right)
\end{aligned}$$

We know, that the p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.

15. [Abott, 2.7.12] **(Summation by parts)**. Let (x_n) and (y_n) be sequences, let $s_n = x_1 + x_2 + \ldots + x_n$ and set $s_0 = 0$. Use the observation that $x_j = s_j - s_{j-1}$ to verify the formula

$$\sum_{j=m}^{n} x_j y_j = s_n y_{n+1} - s_{m-1} y_m + \sum_{j=m}^{n} s_j (y_j - y_{j+1})$$

Proof.