Derivatives and the Intermediate Value Property

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Abstract

A short note on Derivatives and the Intermediate Value Property.

1 Derivatives and the Intermediate Value Property.

Although the definition would technically make sense for more complicated domains, all of the interesting results about the relationship between a function and its derivative require that the domain of the given function be an interval. Thinking geometrically of the derivative as the rate of change, it should not be too surprising that we would want to confine the independent variable to move about a connected domain.

The theory of functional limits from section 4.2 is all that is need to supply a rigorous definition for the derivative.

Definition 1.1. (**Differentiability.**) Let $g : A \to \mathbf{R}$ be a function defined on an interval A. Given $c \in A$, the derivative of g at c is defined by

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

provided this limit exists. In this case, we say that g is differentiable at c. If g' exists for all points $c \in A$, we say that g is differentiable on A.

Example 1.1. (i) Consider $f(x) = x^n$, where $n \in \mathbf{N}$, and let c be any arbitrary point in $\mathbf{R}$. Using the algebraic identity

$$x^n - c^n = (x - c)(x^{n-1} + cx^{n-2} + c^2x^{n-3} + \ldots + c^{n-1}$$

we can calculate the familiar formula

$$\begin{aligned} f'(c) &= \lim_{x \to c}(x^{n-1} + cx^{n-2} + \ldots + c^{n-1}) \\ &= c^{n-1} + c^{n-1} + \ldots + c^{n-1} = nc^{n-1} \end{aligned}$$

(ii) If $g(x) = |x|$, then attempting to compute the derivative at $c = 0$ produces the limit

$$g'(0) = \lim_{x \to 0} \frac{|x|}{x}$$

which is $+1$ or -1 depending on whether you choose a sequence $(x_n) \to 0$ with positive terms or negative terms. Consequently, this limit does not exist, and we conclude that g is not differentiable at zero.

The example (ii) above is a reminder that the continuity of g does not imply that g is necessarily differentiable. On the other hand, if g is differentiable at a point, then it is true that g must be continuous at this point.

Theorem 1.1. If $g : A \to \mathbf{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.

Proof. We are assuming that

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

exists, and we want to prove that $\lim_{x \to c} g(x) = g(c)$. But notice, that $\lim_{x \to c}(x - c)$ exists and therefore, by the Algebraic Limit theorem for functional limits, we are allowed to take a product of these two functions and write,

$$\begin{aligned} g'(c) \lim_{x \to c}(x - c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \cdot \lim_{x \to c}(x - c) \\ 0 &= \lim_{x \to c}(g(x) - g(c)) \\ \lim_{x \to c} g(x) = g(c) & \end{aligned}$$

Consequently, $g(x)$ is continuous at c.

1.1 Combinations of Differentiable functions.