Understanding Analysis

Solution of exercise problems.

Quasar Chunawala

Version 1.0.0, last revised on 2021-05-13.

Abstract

This is a solution manual for Understanding Analysis, 2nd edition, by Stephen Abbott.

Chapter 3. Basic Topology of R

Theorem. The open interval $(0,1) = \{x \in \mathbf{R} : 0 < x < 1\}$ is uncountable.

Proof.
We proceed by contradiction and assume that there exists a function $f : \mathbf{N} \to (0,1)$ that is $1-1$ and onto. $1-1$ implies that distinct elements have distinct images. Onto implies that every element in the co-domain has atleast one pre-image. For each $m \in \mathbf{N}$, $f(m)$ is a real number between 0 and 1, and we represent it using the decimal notation

$$f(m) = .\,a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}\ldots$$

What is meant here is that for each $m, n \in \mathbf{N}$, a_{mn} is the digit from the set $\{0,1,2,3,\ldots,9\}$ that represents the nth digit in the decimal expansion of mth real number, $f(m)$. The $1-1$ correspondence between $\mathbf{N}$ and $(0,1)$ can be summarized in the doubly indexed array:

$$
\begin{array}{ccccccccccc}
\mathbf{N} & & (0,1) & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \mathbf{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \mathbf{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \mathbf{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \mathbf{a_{66}} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

The key assumption about this correspondence is that **every** real number in $(0,1)$ is assumed to appear somewhere on this list.

Now for the pearl of the argument. Define a real number $x \in (0,1)$ with the decimal expansion $x = .b_1b_2b_3b_4\ldots$ using the rule

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

Now, the real number $x = .b_1b_2b_3b_4\ldots$ cannot be $f(1)$, simply because its first digit b_1 differs from the first digit a_{11} of $f(1)$. Similarly, the second digit b_2 differs from the second digit a_{22} of $f(2)$. In general, the nth digit of x differs from the nth digit of $f(n)$. So, we have constructed a real number x that is not in the set $\{f(1), f(2), f(3), \ldots, f(n)\}$. But, this is a contradiction. Hence, our initial assumption is false. The set of real numbers in $(0,1)$ are uncountable.

Exercise. [Abbott, 1.6.4] Let S be the set consisting of all sequences of 0s and 1s. Observe that S is not a particular sequence, but rather a large set whose elements are sequences, namely:

$$S = \left\{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\right\}$$

As an example, the binary sequence $(1,0,1,0,1,0,1,0,\ldots)$ is an element of S as is the sequence $(1,1,1,1,1,1,\ldots)$. Give a rigorous argument showing that S is uncountable.

Proof.

Suppose that S - the set of all possible binary strings of infinite length is countable. Then, we can define a bijection $f : \mathbf{N} \to S$ between the natural numbers and S. For each $m \in \mathbf{N}$, $f(m)$ is a binary string in S. Let us enlist the first few elements of this correspondence.

$$\begin{array}{ccccccccccccc}
\mathbf{N} & & (0,1) & & & & & & & & \\
\hline
1 & \longleftrightarrow & f(1) & = & .\mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \longleftrightarrow & f(2) & = & .a_{21} & \mathbf{a_{22}} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \longleftrightarrow & f(3) & = & .a_{31} & a_{32} & \mathbf{a_{33}} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \longleftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & \mathbf{a_{44}} & a_{45} & a_{46} & \cdots \\
5 & \longleftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a_{55}} & a_{56} & \cdots \\
6 & \longleftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & \mathbf{a_{66}} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Define a binary sequence $x = (b_1, b_2, b_3, b_4, \ldots)$ such that

$$b_i = \begin{cases} 1 & \text{if } a_{ii} = 0 \\ 0 & \text{if } a_{ii} = 1 \end{cases}$$

Thus, the binary sequence $x = (b_1, b_2, b_3, b_4, \ldots)$ has atleast one bit that differs from all of the elements in S. Consequently, $x \notin S$. This is a contradiction, as S is supposed to contain all binary strings. Hence, S is not countable.

3.1 Discussion: The Cantor Set.

What follows is a fascinating mathematical construction, due to Georg Cantor, which is extremely useful for extending the horizons of our intuition about the nature of subsets of the real line. Cantor's name has already appeared in the first chapter in our discussion of uncountable sets. Indeed, Cantor's proof that $\mathbf{R}$ is uncountable occupies another spot on the short list of the most significant contrubutions towards the understanding of the mathematical infinite. In the words of the mathematician David Hilbert, "No one shall expel us from the paradise that Cantor has created for us."

Let C_0 be the closed interval $[0, 1]$ and define C_1 to be the set that results when the open middle one third is removedl that is,

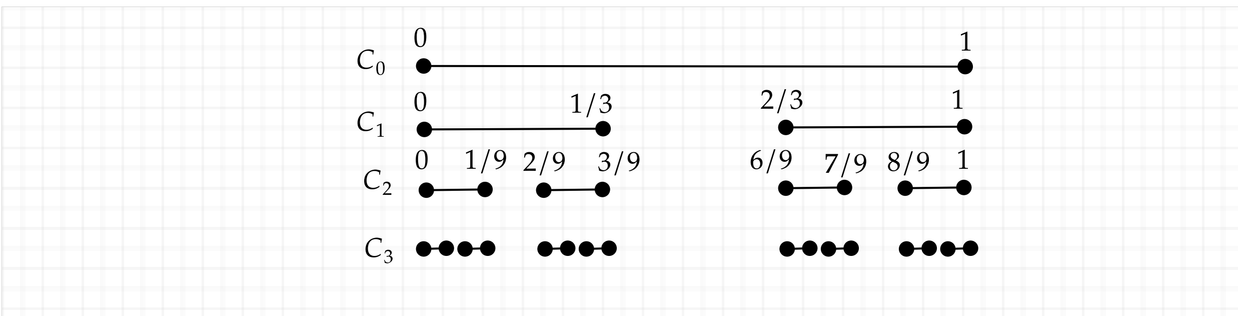
$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Now, construct C_2 in a similar way by removing the open middle third of each of the two components of C_1:

$$C_2 = \left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right]\right) \cup \left(\left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$$

If we continue this process inductively then for each $n = 0, 1, 2, \ldots$ we get a set of C_n consisting of 2^n closed intervals each having length $1/3^n$. Finally, we define the Cantor set C to be the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$



Defining the Cantor set

It may be useful to understand C as the remainder of the ineral $[0, 1]$ after the interative process of removing open middle one thirds is taken to infinity.

$$C = [0, 1] - \left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots\right]$$

There is some initial doubt whether anything remains at all, but notice that because we are always removing open middle one thirds, then for every $n \in \mathbf{N}$, $0 \in C_n$ and hence $0 \in C$. The same argument shows that $1 \in C$. In fact, if y is the endpoint of some closed interval of some particular set C_n, then it is also an enpoint of one of the intervals of C_{n+1}. Because at each stage, the endpoints are never removed, it follows that $y \in C_n$ for all n. Thus, C atleast contains the endpoints of all of the intervals that make up each of the sets C_n.

Is there anything else? Is C countable? Does C contain any intervals? Any irrational numbers? These are difficult questions at the moment. All of the endpoints mentioned earlier are rational numbers (they have the form $m/3^n$), which means that if it is true that C consists of only these endponts, then C would be a subset of Q and hence countable. We shall see about this. There is some strong evidence that not much is left in C if we consider the total length of the intervals removed. To form C_1, an open interval of length $1/3$ was taken out. In the second step, we removed two intervals of length $1/9$ and to construct C_n, we removed 2^{n-1} middle theirds of length $1/3^n$. There is some logic, then to defining the length of C to be 1 minus the total

$$\frac{1}{3}+2\left(\frac{1}{9}\right)+4\left(\frac{1}{27}\right)+\ldots+2^{n-1}\left(\frac{1}{3^n}\right)+\ldots = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$$

The Cantor set has *zero length*.

To this point, the information we have collected suggests a mental picture of C as relatively small. For these reasons, the set C is often referred to as Cantor dust. But, there are some strong counterarguments that imply a very different picture. First C is actually *uncountable*. with cardinality equal to the cardinality of $\mathbf{R}$. One slightly intuitive but convincing way to see this is to create a $1-1$ correspondence between C and the sequences of the form $(a_n)_{n=1}^{\infty}$, where $a_n = 0$ or 1. For each $c \in C$, set $a_1 = 0$ if c falls in the left-hand component and set $a_1 = 1$ if c falls in the right hand component. Having established where in C_1, the point c is located, there are now two possible components of C_2 that might contain c. This time, we set $a_2 = 0$ or 1 depending on whether c falls in the left of right half of these two components of C_2. Continuing in this way, we come to see that every element $c \in C$ yields a sequence $(a_1, a_2, a_3, \ldots)$ of zeroes and ones (a binary string), that acts as a set of directions for how to locate c within C. Because the set of sequences of zeroes and ones is uncountable, we must conclude that C is uncountable as well.

What does this imply? In the first place, because the end points of the approximating sets C_n form a countable set, we are forced to accept the fact that not only are there other points in C, but there are uncountably many of them, From the point of view of cardinality, C is quite large - as large as $\mathbf{R}$, in fact. This should be contrasted with the fact that from the point of view of length, C measures the same size as a single point. We conclude this discussion with a demonstration that from the point of view of *dimension*, C strangely falls somewhere in between.

There is a sensible agreement that a point has dimension zero, a line segment has dimension one, a square has dimension two, and a cube has dimension three. Without attempting a formal definition of dimension (of which there are several) we can nevertheless get a sense of how one might be defined by observing how the dimension affects the result of magnifying each particular set by a factor of 3. (The reason for the choice of 3 will become clear when we turn our attention back to the Cantor set). A single point undergoes no change at all, whereas a line segment triples in length. For the square, magnifying each length by a factor of 3 results in a larger square that contains 9 copies of the original square. Finally, th magnified cube yields a cube that contains 27 copies of the original cube within its volume. Notice that, in each case, to compute the size of the new set, the dimension appears as the exponent of the magification factor.

	dim	$\times 3$	new copies
point	0	$\rightarrow$	$1 = 3^0$
segment	1	$\rightarrow$	$3 = 3^1$
square	2	$\rightarrow$	$9 = 3^2$
cube	3	$\rightarrow$	$27 = 3^3$

Dimension of C,

Now, apply this transformation to the Cantor set. The set $C_0 = [0, 1]$ becomes the interval $[0, 3]$. Deleting the middle one-third leaves $[0, 1] \cup [2, 3]$, which is where we started in the original construction exept that we now stand to produce an additional copy of C in the interval $[2, 3]$. Magnifying the Cantor set bya factor of 3 yields two copies of the original set. Thus, if x is the dimension of C, we must have $3^x = 2$, or $x = \ln 2 / \ln 3 = 0.631$.

The notion of a non-integer or fractional dimension is the impetus behind the term fractal, coined in 1975 by Benoit Mandlebrot to describe a class of sets whose intricate structures have much in common with the Cantor set. Cantor's construction is over hundred years old and for us represents n invaluable testing ground for upcoming theorems and conjectures about the often ulusive nature of subsets of the real line.

3.2 Open and Closed Sets.

Given $a \in \mathbf{R}$ and $\epsilon > 0$, recall that the ϵ-neighbourhood of a is the set

$$V_\epsilon(a) = \{x \in \mathbf{R} : |x - a| < \epsilon\}$$

In other words, $V_\epsilon(a)$ is the open interval $(a - \epsilon, a + \epsilon)$ centered at a with radius ϵ.

Definition. A set $O \subseteq \mathbf{R}$ is open if for all points $a \in O$ there exists an ϵ-neighbourhood $V_\epsilon(a) \subseteq O$.

Example.
(i) Perhaps, the simplest example of an open set is $\mathbf{R}$ itself. Given an arbitrary element $a \in \mathbf{R}$, we are free to pick any ϵ-neighbourhood we like and it will always be true that $V\epsilon(a) \subseteq \mathbf{R}$. It is also the case that the logical structure of the definition requires us to classify the empty set $\emptyset$ as an open subset of the real line.

(ii) For a more useful collection of exampls, consider the open interval

$$(c, d) = \{x \in \mathbf{R} : c < x < d\}$$

To see that (c, d) is open in the same sense just defined, let $x \in (c, d)$ be arbitrary. If we take $\epsilon = \min\{x - c, d - x\}$, then it follows that $V_\epsilon(x) \subseteq (c, d)$. It is important to see where this argument breaks down if the interval includes either one of its endpoints.

The union of open intervals is another example of an open set. This observation leads to the next result.

Theorem. (i) The union of an arbitrary collection of open sets is open.
(ii) The intersection of a countably finite collection of open sets is open.

Proof.

To prove (i), we let $\{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets and $O = \bigcup_{\lambda \in \Lambda} O_\lambda$. Here, Λ could be a countably infinite or an uncountable set. Let a be an arbitrary element of O. In order to show that O is open, the definition insists that we produce an ϵ-neighbourhood of a completely contained in O. But, $a \in O$ implies that a is an element of atleast one particular $O_{\lambda'}$. Because we are assuming that $O_{\lambda'}$ is open, by definition we can assert, that there exists $V_\epsilon(a) \subseteq O_{\lambda'}$. The fact that $O_{\lambda'} \subseteq O$ allows us to conclude that $V_\epsilon(a) \subseteq O$ for all $a \in O$. This completes the proof of (i).

For (ii), let $\{O_1, O_2, \ldots, O_N\}$ be a finite collection of open sets. Now, if $a \in \bigcap_{k=1}^{N} O_k$, then a is an element of each of the open sets. By the definition of an open set, we know that, for each, $1 \leq k \leq N$, there exists a $V_{\epsilon_k}(a) \in O_k$. We are in search of a single neighbourhood of a that is contained in every O_k, so the trick is to take the smallest one. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$, it follows that $V_\epsilon(a) \subseteq V_{\epsilon_k}(a)$ for all k, and hence $V_\epsilon(a) \subseteq \bigcap_{k=1}^{N} O_k$ as desired.

Closed Sets.

Definition. A point x is the **limit point** of a set A, if every ϵ-neighbourhood $V_\epsilon(x)$ intersects the set A at some point other than x.

Limit points are also referred to as cluster points or accumulation points, but the phrase, "x is a limit point of A" has the advantage of explicitly reminding us that x is quite literally the limit of the sequence in A.

Theorem. A point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A, satisfying $a_n \neq x$ for all $n \in \mathbf{N}$.

Proof.
($\Longrightarrow$) Assume that x is a limit point of A. In order to produce a sequence (a_n) converging to x, we are going to consider the particular ϵ-neighbourhoods obtained using $\epsilon = 1/n$. By definition, every neighborhood of x intersects A in some point other than x. This means that, for each $n \in \mathbf{N}$, we are justified in picking a point

$$a_n \in V_{1/n}(x) \cap A$$

with the stipulation that $a_n \neq x$. It should not be too difficult to see why $(a_n) \rightarrow x$. Given an arbitrary $\epsilon > 0$, choose N such that $1/N < \epsilon$. Then for all $n \geq N$, we have $|a_n - x| < \epsilon$.

($\Longleftarrow$)For the reverse implication, we assume that $\lim a_n = x$ where $a_n \in A$ but $a_n \neq x$ and let $V_\epsilon(x)$ be an arbitrary neighbourhood of x. The definition of convergence assures us that there exists a term a_N in the sequence satisfying $a_N \in V_\epsilon(x)$. So, every ϵ-neighbourhood, $V_\epsilon(x)$ intersects A in some element other than x. Hence, $x = \lim a_n$. QED.

The restriction that $a_n \neq x$ in the above theorem deserves a ccmment. Given a point $a \in A$, it is always the case that a is the limit of a sequence in A, if we are allowed to consider the constant sequence $(a, a, a, \ldots)$. There will be occasions where will want to avoid this somewhat uninteresting situation, so it is important to have vocabulary that can distinguish limit points of a set fom isolated points.

Definition. A point $a \in A$ is an isolated point of A if it is not a limit point of A.

As a word of caution, we need to be little careful about how we understand the relationship between these concepts. Whereas an isolated point is always an element of the relevant set A, it is quite possible for a limit point of A not to belong to A. As an example, consider the endpoint of an open interval. This situation is the subject of the next important definition.

Definition. A set $F \subseteq \mathbf{R}$ is closed if it contains its limit points.

The adjective "closed" appears in several other mathematical contexts and is usually employed to mean that an operation on the elements of a given set does not take us outside of the set. In linear algebra, for example, a vector space is a set that is closed under vector addition and scalar multiplication. In analysis, the operation we are concerned with is the limiting operation. Topologically speaking, a closed set is the one where convergent sequences within the set have limits that are also in the set.

Theorem. A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Proof. Exercise 3.2.5.

Example. (i) Consider

$$A = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$$

Let's show that each point of A is isolated. Given $1/n \in A$, choose $\epsilon = \frac{1}{n} - \frac{1}{n+1}$. (Note that, $\frac{1}{n-1} - \frac{1}{n}$ is larger than $\frac{1}{n} - \frac{1}{n+1}$). Then,

$$V_\epsilon(1/n) \cap A = \frac{1}{n}$$

It follows from the definition that $1/n$ is not a limit point and so is isolated. Although all of the points of A are isolated, the set does have one limit point, namely 0. This is because every ϵ-neighbourhood of centered at zero, no matter how small, is going to contain points of A. Because $0 \notin A$, A is not closed. The set $F = A \cup \{0\}$ is an example of a closed set and is called the closure of A. (The closure of a set is discussed in a moment).

(ii) Let's prove that a closed interval

$$[c, d] = \{x \in \mathbf{R} : c \leq x \leq d\}$$

is a closed set using the definition. If x is a limit point of $[c, d]$, then by the previous theorem there exists a sequence (x_n) contained in $[c, d]$ such that $(x_n) \to x$ and $x_n \neq x$. We need to prove that x belongs to $[c, d]$.

The key to this argument is contained in the Order Limit Theorem, which sumarrizes the relationship between inequalities and the corresponding limiting process. Because, $c \leq x_n \leq d$, it follows from the the Order Limit Theorem, that $c \leq x \leq d$. Thus, $[c, d]$ is closed.

(iii) Consider the set $\mathbf{Q} \subseteq \mathbf{R}$ of rational numbers. An extremely important property of $\mathbf{Q}$ is that the set of all limit points of $\mathbf{Q}$ is actually all of $\mathbf{R}$. To see why this is so, recall the theorem, which is referred to as the density property of $\mathbf{Q}$ in $\mathbf{R}$. $\mathbf{Q}$ is dense in $\mathbf{R}$ implies that $\mathbf{Q}$ sits inside of $\mathbf{R}$. Between any two real numbers $a, b \in \mathbf{R}$, you can always find a rational number r satisfying, $a < r < b$.

Let $y \in \mathbf{R}$ be arbitrary and consider any neighbourhood $(y - \epsilon, y + \epsilon)$. The density theorem allows us to conclude that there exists a rational number $r \neq y$ that falls in this neighbourhood. Thus, y is a limit point of $\mathbf{Q}$.

The density property of $\mathbf{Q}$ can now be reformulated in the following way.

Theorem.(Density of $\mathbf{Q}$ in $\mathbf{R}$). For every $y \in \mathbf{R}$, there exists a sequence of the rational numbers that converges to y.

Proof.
From the above discussion, we know that for an arbitary $y \in \mathbf{R}$, every ϵ-neighbourhood of y intersects $\mathbf{Q}$ in some point other than y. y is a limit point of $\mathbf{Q}$. By the theorem on limit points of sets, there exists a sequence $(x_n) \subseteq \mathbf{Q}$, such that $(x_n) \to y$, and $x_n \neq y$ for all n.

The same argument can also be used to show that every real number is the limit of a sequence of irrational numbers. Although interesting, part of the allure of the rational numbers is that, in addition to being dense in $\mathbf{R}$, they are countable. As we will see, this tangible aspect of $\mathbf{Q}$ makes it an extremely useful set, both for proving theorems and for producing interesting counterexamples.

Closure.

Definition. Given a set $A \subseteq \mathbf{R}$, let L be the set of all limit points of A. The closure A is defined to $cl(A) = A \cup L$.

In example 3.2.9 (i), we saw that if $A = \{1/n : n \in \mathbf{N}\}$, then the closure of $A = cl(A) = A \cup \{0\}$. Example 3.2.9 (iii) verifies that $cl(\mathbf{Q}) = \mathbf{R}$. If A is an open interval, then $cl(A) = [a, b]$. If A is a closd interval, then $cl(A) = A$. It is not for lack of imagination that in each of these examples $cl(A)$ is always a closed set.

Theorem. For any $A \subseteq \mathbf{R}$, the closure $cl(A)$ is a closed set and is the smallest closed set containing A.

Proof.
If L is the set of limit points of A, then it is immediately clear that $cl(A)$ contains the limit points of A. There is still something more to prove, however because taking the union of L with A could potentially produce some new limit points of $cl(A)$. In exercise 3.2.7, we outline the argument that this does not happen.

Now, *any* closed set containing A must contain L as well. This shows that $cl(A) = A \cup L$ is the smalled such closed set containing A.

Complements.

The mathematical notions of open and closed are not antonyms the way they are in standard English. If a set is not open, that does not imply it must be closed. Many sets such as the half-open interval $(c, d] = \{x \in \mathbf{R} : c < x \leq d\}$ are neither open nor closed. The sets $\mathbf{R}$ and $\emptyset$ are simultaneously open and closed, although, thankfully, these are he only ones with this disorientating property. (Exercises 3.2.13). There is, however, an important relationship between open and closed sets. Recall that the complement of a set $A \subseteq \mathbf{R}$ is defined to be the set:

$$A^C = \{x \in \mathbf{R} : x \notin A\}$$

Theorem. A set O is open if and only if O^c is closed. Likewise, a set F is closed if and only F^C is open.

Proof.
($\Longrightarrow$) Given an open set $O \subseteq \mathbf{R}$, let's first prove that O^C is a closed set. To prove that O^C is a closed, we need to show that it contains all of its limit points. If x is a limit point of O^C, then every ϵ-neighbourhood of x contains some point of O^C other than x. But that is enough to conclude that x cannot be in O, because if $x \in O$, then every open interval $(x-\epsilon, x+\epsilon)$ is contained in O. Consequently, $x \in O^C$. Thus, O^C is a closed set.

($\Longleftarrow$) For the converse statement, we assume that O^C is closed and argue that O is open. Thus, given an arbitrary point $x \in O$, we must produce an ϵ-neighbourhood $V_\epsilon(x) \subseteq O$. Because, $x \notin O^C$, we can be sure that x is not a limit point of O^C. If x is not a limit point of O^C, there exists an $\epsilon > 0$, such that $(x-\epsilon, x+\epsilon) \nsubseteq O^C$, or equivalently $(x-\epsilon, x+\epsilon) \subseteq O$. This is precisely what we needed to show.

Theorem. (i) The union of a countably finite collection of closed sets is closed.

(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof.
De Morgan's Laws state that for any collection of sets $\{E_\lambda : \lambda \in \Lambda\}$ it is true that

$$\left(\bigcup_{\lambda \in \Lambda} E_\lambda\right)^C = \bigcap_{\lambda \in \Lambda} E_\lambda^C \quad \text{and} \quad \left(\bigcap_{\lambda \in \Lambda} E_\lambda\right)^C = \bigcup_{\lambda \in \Lambda} E_\lambda^C$$

The result follows directly from these statements and Theorem 3.2.3. The details are requested in Exercise 3.2.9.

Exercises.

1. [Abbot, 3.2.1] (a) Where in the proof of Theorem 3.2.3 part (ii) does the assumption that the collection of open sets be *finite* get used?

Solution. a is contained in each of the open sets $O_1, O_2, \ldots, O_N$. Since, there are open sets, there exists an ϵ_k-neighbour of a, such that $V_{\epsilon k}(x) \subseteq O_k$, $k = 1, \ldots N$.

We would like to select an ϵ-neighbourhood that is contained in $\bigcap_{k=1}^{n} O_k$. To do so, we choose ϵ to be the smallest distance in the collection $\{\epsilon_1, \ldots, \epsilon_N\}$. This is where the assumption that the collection of open sets be finite gets used.

For if, the collection is countably infinite or uncountable, we risk $\inf\{\epsilon_1, \epsilon_2, \ldots\} = 0$. We simply might not be able to find an ϵ-neighbourhood contained in every open set O_k.

(b) Give an example of a countable collection of open sets $\{O_1, O_2, O_3, \ldots\}$ whose intersection $\bigcap_{n=1}^{\infty} O_n$ is closed, not empty and not all of $\mathbf{R}$.

Proof.

Consider $O_n = \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$. O_n is an open set. $\bigcap_{n=1}^{\infty} O_n = [-1, 1]$ and this is a closed set.

2. [Abbott, 3.2.2] Let

$$A = \left\{(-1)^n + \frac{2}{n} : n = 1, 2, 3, \dots\right\} \quad \text{and} \quad B = \{x \in \mathbf{Q} : 0 < x < 1\}$$

Answer the following questions for each set.

(a) What are the limit points?

The Set A:
Enlisting the first few elements of A:

$$A = \left\{1, 2, -\frac{1}{3}, \frac{3}{2}, -\frac{3}{5}, \frac{4}{3}, \dots\right\}$$

The subsequence (a_{2n}) converges to 1, whilst the subsequence (a_{2n-1}) converges to (-1). Hence, the limit points of A are $L = \{1, -1\}$.

The Set B:

Note that $\mathbf{Q}$ is dense in $\mathbf{R}$. Consider an arbitrary real number $y \in [0, 1]$. Pick $\epsilon = \min\{y - 0, 1 - y\}$. Every open interval $(y - \epsilon, y + \epsilon)$ intersects B in some point other than y, because we can find a rational x such that $y - \epsilon < x < y$. Consequently, y is a limit point of B.

So, the limit points of B is the closed interval $[0, 1]$.

(b) Is the set open? Closed?

The Set A:

Pick an arbitrary point $x \in A$, for example $x = \dfrac{3}{2}$. Let $\epsilon > 0$. Clearly, for all $\epsilon > 0$, $\left(\dfrac{3}{2} - \epsilon, \dfrac{3}{2} + \epsilon\right)$ is not contained in A. Thus, there exists an $x \in A$, such that for all $\epsilon > 0$, $(x - \epsilon, x + \epsilon)$ is not in A. So, A is not open.

The limit point -1 does not belong to A. So, A is not closed.

The Set B:
Between any two rational numbers, there is an irrational. For example, consider $a, b \in B$, where $a < b$. Then,

$$a < a + \frac{b-a}{\sqrt{2}} < b$$

So, every open interval $(b-\epsilon, b+\epsilon)$ will contain irrational points, and is therefore not contained in B.
We have shown that $\forall\epsilon > 0$, we have $(b-\epsilon, b+\epsilon) \not\subseteq B$ for atleast one $b \in B$. Consequently, B is not open.

The set B does not contain its limit points. So, B is not closed.

(c) Does the set contain any isolated points?

The Set A:
Consider the distance $\left|\left((-1)^n + \frac{2}{n}\right) - \left((-1)^{n+2} + \frac{2}{n+2}\right)\right| = \frac{4}{n(n+2)}$. Since, $\frac{4}{(n+2)^2} < \frac{4}{n(n+2)}$, pick $\epsilon = \frac{4}{(n+2)^2}$. For this choice of ϵ, the open interval $(a_n - \epsilon, a_n + \epsilon) \cap A = a_n$, for $n > 1$. Hence, all of the points except 1, are isolated points of A.

The Set B:

Let $x = \frac{p}{q}$ be any rational such that $0 < x < 1$. The sequence $x_n = \frac{p}{q} + \frac{1}{n}$ converges to $\frac{p}{q}$. So every rational number $x \in B$ is a limit point. B has no isolated points.

(d) Find the closure of the set.

The set A:
$cl(A) = A \cup \{-1\}$

The set B:
$cl(B) = [0, 1]$

3. [Abbott, 3.2.3] Decide whether the following sets are open, closed or neither. If a set is not open, find a point in the set for which there is no ϵ-neighbourhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) $\mathbf{Q}$.

Given a rational number $q \in \mathbf{Q}$, there exists an irrational number $x \in \mathbf{I}$, such that $|x - q| < \epsilon$ for all $\epsilon > 0$. For example, if $q = \frac{m}{n}$ and $\epsilon = \frac{1}{n}$, then $\frac{m}{n} < \frac{m+1}{\sqrt{2}n} < \frac{m}{n} + \frac{1}{n}$. So, $(x - \epsilon, x + \epsilon)$ is not contained in $\mathbf{Q}$. Therefore, $\mathbf{Q}$ is not open.

Also, let $y \in \mathbf{R}$. Every ϵ-neighbourhood of y intersects $\mathbf{Q}$, since $\mathbf{Q}$ is dense in $\mathbf{R}$. So, we can always find a rational q such that $y - \epsilon < q < y$. Consequently, the limit points of $\mathbf{Q}$ is all of $\mathbf{R}$. So, $\mathbf{Q}$ is not closed.

(b) $\mathbf{N}$.

For all $\epsilon > 0$, $(n - \epsilon, n + \epsilon)$ is not contained in $\mathbf{N}$. To see this, consider $n + \epsilon/2 \in (n - \epsilon, n + \epsilon)$. Clearly, $(n + \epsilon/2) \notin \mathbf{N}$. So, $\mathbf{N}$ is not open.

$\mathbf{N}$ is unbounded and has no limit points. Therefore, $\mathbf{N}$ is closed.

(c) $\{x \in \mathbf{R} : x \neq 0\}$

Let $x \in \mathbf{R} - \{0\}$. There exists an ϵ-neighbourhoood $V_\epsilon(x) = (x - \epsilon, x + \epsilon)$, of x, that is contained entirely in the set $\{x \in \mathbf{R} : x \neq 0\}$ for all x. Hence, it is an open set.

Consider $x_n = \frac{1}{n}$. Clearly, $x_n \in \{x : \mathbf{R} : x \neq 0\}$ and $\lim(x_n) = 0$. Thus, 0 is a limit point of the given set. Since, 0 is not in the given set, it is not a closed set.

(d) $\left\{1 + 1/4 + 1/9 + \ldots + 1/n^2 : n \in \mathbf{N}\right\}$.

Let (s_n) be the partial sums of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. So, the elements of the set are:

$$\left\{s_n : s_n = \sum_{k=1}^{n} \frac{1}{k^2}\right\}$$

For all $\epsilon > 0$, $(s_n - \epsilon, s_n + \epsilon)$ is not contained in the given set. So, it is not an open set.

Moreover, (s_n) is a convergent sequence. So, $(s_n) \rightarrow s$ with $s_n \neq s$ for all n. Thus, the limit point s is not an element of the given set. Consequently, $\left\{s_n : s_n = \sum_{k=1}^{n} \frac{1}{k^2}\right\}$ is not closed.

(e) $\{1+1/2+1/3+\ldots+1/n : n \in \mathbf{N}\}$.

For all $\epsilon > 0$, open interval $(s_n - \epsilon, s_n + \epsilon)$ is not contained in the given set. So, it is not an open set. Further, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so the set has no limit points. Hence, it is closed.

4. [Abbott, 3.2.4] Let A be non-empty and bounded above so that $s = \sup A$ exists.

(a) Show that $s \in cl(A)$.

By the definition of the supremum, for all $\epsilon > 0$, there exists an $a_n \in A$, such that $s - \epsilon < a_n < s$. If we successively choose, $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}$, we produce a sequence $(a_n) \subseteq A$ such that $\lim(a_n) = s$ and $a_n \neq s$. Thus, s is the limit point of A. Consequently, $s \in cl(A)$.

(b) Can an open set contain its supremum?

Let O be an open set and suppose $s \in O$. Then, by definition of open sets, there exists an $\epsilon > 0$, such that the open interval $(s - \epsilon, s + \epsilon) \subseteq O$. So, $s + \epsilon/2 \in O$. But this implies, s is not the supremum of O. This is a contradiction. Hence, our initial assumption is false, and $s \notin O$.

5. [Abbott, 3.2.5] Prove theorem 3.2.8

Theorem. A set $F \subseteq \mathbf{R}$ is closed if and only if every Cauchy sequence contained in F has a limit point that is also an element of F.

p: A set F is closed.
q: **Every** Cauchy sequence contained in F converges to some point in F.

($p \implies q$) Assume that the set F is closed. Let (x_n) be an arbitrary Cauchy sequence in F. Cauchy sequences are convergent, so (x_n) is a convergent sequence. Suppose $\lim x_n = x$. By definition of convergence, for all $\epsilon > 0$, there exists an $N \in \mathbf{N}$, such that $x_n \in (x - \epsilon, x + \epsilon)$ for all $n \geq N$.

Thus, for all $\epsilon > 0$, $(x-\epsilon, x+\epsilon)$ intersects F in some point other than x. Consequently, x is a limit point of F. Since, F is closed, $x \in F$.

Thus, every Cauchy sequence contained in F has a limit point in F.

$(p \Longleftarrow q)$ Assume that every Cauchy sequence contained in F converges to some point in F. We must prove that F contains all its limit points. Let x be an arbitrary limit point of F.

By definition of a limit point, there exists a sequence $(x_n) \subseteq F$, such that $\lim x_n = x$, with $x_n \neq x$. Convergent sequences are Cauchy. So, (x_n) is a Cauchy sequence. Therefore, $x \in F$. Consequently, F contains all its limit points.

6. [Abbott, 3.2.6] Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbf{R}$.

This proposition is false.

Counterexample. Consider the set $\mathbf{R}-\{0\} = \mathbf{R}_+ \cup \mathbf{R}_-$. There exists an ϵ-neighbourhood, $V_\epsilon(x)$, of x, such that $V_\epsilon(x) \subseteq \mathbf{R}-\{0\}$ for all x beonging the set. So, $\mathbf{R}-\{0\}$ is open, but it does not contain all of the real numbers.

(b) The nested interval property (NIP) remains true if the term **closed interval** is replaced by closed set.

This proposition is false.

Counterexample.

Consider $\displaystyle\bigcap_{n=1}^{\infty} [n, \infty)$.

(c) Every non-empty open set contains a rational number.

This proposition is true.

Proof. Let O be an open set and suppose $x \in O$. By definition, there exists an $\epsilon > 0$, such $(x-\epsilon, x+\epsilon) \subseteq O$. As $\mathbf{Q}$ is dense in $\mathbf{R}$, there exists a rational number q such that

$x - \epsilon < q < x$. So, every non-empty open set contains a rational number.

(d) Every bounded infinite closed set contains a rational number.

This proposition is false.

Counterexample.

Consider the set $A = \left\{\pi + \dfrac{1}{n} : n \in \mathbf{N}\right\} \cup \{\pi\}$. π is a limit point of the set and it belongs to A. So, A is a closed set. A is a bounded infinite set. A does not contain rational numbers.

(e) The Cantor set is closed.

C_n is a closed set for all all $n \in \mathbf{N}$. The intersection of an arbitrary collection of closed sets is closed, so $\bigcap_{n=1}^{\infty} C_n$ is also a closed set.

7. [Abbott, 3.2.7] Given $A \subseteq \mathbf{R}$, let L be the set of all limit points of A.

(a) Show that the set L is closed.

Let x be a limit point of L. We are interested to prove that $x \in L$.

Since x is a limit point of L, every ϵ-neighbourbood $V_\epsilon(x)$, of x intersects L in some point y other than x.

So, for all $\epsilon > 0$, there exists a $y \in L$, such that

$$|y - x| < \frac{\epsilon}{2}$$

But, y is a limit point of A. So, for all $\epsilon > 0$, there exists a $z \in A$, such that

$$|z - y| < \frac{\epsilon}{2}$$

The distance $|z - x|$ can bounded as follows:

$$\begin{aligned} |z-x| &= |z-y+y-x| \\ &\leq |z-y| + |y-x| \\ &< \epsilon/2+\epsilon/2 = \epsilon \end{aligned}$$

So, every ϵ-neighbourhood of x intersects A in some point z other than x. Consequently, x is a limit point of A. Thus, $x \in L$. So, L is closed.

(b) Argue that if x is a limit point of $A \cup L$, then x is a limit point of A.

Suppose x is a limit point of $A \cup L$. Thus, either x is a limit point of A or x is a limit point of L. If x is a limit point of A, then $x \in L$. If x is a limit point of L, then since L is closed, $x \in L$. Also, $A \cup L$ is the smallest closed set containing A. So, the above two possibilities are exhaustive.

8. [Abbott, 3.2.8] Assume that A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both or neither.

(a) $cl(A \cup B)$.

The closure of any set S is closed. So, $cl(A \cup B)$ is a closed set.

(b) $A \setminus B = \{x : x \in A, x \notin B\}$

$A \setminus B = A \cap B^C$. A is an open set. B^C is an open set. The intersection of countably finite open sets is open. So, $A \setminus B$ is open.

(c) $\left(A^C \cup B\right)^C$

A^C is closed, so $A^C \cup B$ is a closed set. Therefore, $\left(A^C \cup B\right)^C$ is open.

(d) $(A \cap B) \cup \left(A^C \cap B\right)$

By De Morgan's laws,

$$(A \cap B) \cup \left(A^c \cap B\right) = \left(A \cup A^C\right) \cap B = B$$

So, this set is closed.

(e) $(cl(A))^C \cap cl\left(A^C\right)$.

Note that $cl\left(A^C\right) = A^C$. Thus, $(cl(A))^C \cap A^C = (cl(A) \cup A)^C = (cl(A))^C$. So, this is an open set.

9. [Abbott, 3.2.9] De-Morgan's laws.

(i) Given a collection of sets $\{E_\lambda : \lambda \in \Lambda\}$, show that

$$\left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^C = \bigcap_{\lambda\in\Lambda} E_\lambda^C \quad \text{and} \quad \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^C = \bigcup_{\lambda\in\Lambda} E_\lambda^C$$

Proof.

The complement of the union of an arbitrary collection of sets is the intersection of their complements.

($\Longrightarrow$) direction.

Let $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^C$. Therefore, for all $\lambda \in \Lambda$, $x \notin E_\lambda$. Consequently, $x \in E_\lambda^C$ for all $\lambda \in \Lambda$.

Thus, $x \in \bigcap_{\lambda\in\Lambda} E_\lambda$.

($\Longleftarrow$) direction.

Suppose $x \in \bigcap_{\lambda\in\Lambda} E_\lambda^C$. Thus, x is the common element of all sets E_λ^C, such that $\lambda \in \Lambda$. Thus, x is not in E_λ for all $\lambda \in \Lambda$. This is the negation of x belonging to atleast one set E_λ for some $\lambda \in \Lambda$. So, $x \notin \bigcup_{\lambda\in\Lambda} E_\lambda$. Therefore, $x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^C$.

The complement of the intersection of an arbitrary collection of sets is the union of their complements.

($\Longrightarrow$) direction.

Let $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^C$. Then, there exists atleast one $\lambda \in \Lambda$ such that $x \notin E_\lambda$. That is, there is atleast one set E_λ, such that $x \in E_\lambda^C$. Consequently, $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^C$.

($\Longleftarrow$) direction.

Let $x \in \bigcup_{\lambda\in\Lambda} E_\lambda^C$. Then, there exists atleast one $\lambda \in \Lambda$ such that $x \notin E_\lambda$. Consequently, x does not belong to $\bigcap_{\lambda\in\Lambda} E_\lambda$. Therefore, $x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^C$.

(ii) Now, provide the details for the proof of theorem 3.2.14

Proof.

Let $O_1, O_2, \ldots, O_N$ be a finite collection of open sets. Then, the finite intersection

$$\bigcap_{k=1}^{N} O_k$$

is open. Applying De-Morgan's law, we have

$$\left(\bigcap_{k=1}^{N} O_k\right)^C = \bigcup_{k=1}^{N} O_k^C$$

The complementation of an open set is a closed set. Let $E_k = O_k^C$. Then, E_k is a closed set. Further, $\left(\bigcap_{k=1}^{N} O_k\right)^C$ is closed. So, altogether we have,

$$\bigcup_{k=1}^{N} E_k = \left(\bigcap_{k=1}^{N} E_k^C\right)^C$$

Consequently, the finite union of closed sets is closed.

Now, let $\{O_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of open sets. We know that, the union of an arbitrary collection of open sets is open. Thus, $\bigcup_{\lambda \in \Lambda} O_\lambda$ is open. Taking the complement and applying De-Morgan's laws, we have:

$$\left(\bigcup_{\lambda \in \Lambda} O_\lambda\right)^C = \bigcap_{\lambda \in \Lambda} O_\lambda^C$$

Let $E_\lambda = O_\lambda^C$. E_λ is closed. Moreover, $\left(\bigcup_{\lambda \in \Lambda} O_\lambda\right)^C$ is closed. Consequently, the intersection of arbitrary collection of closed sets is closed.

$$\bigcap_{\lambda \in \Lambda} E_\lambda = \left(\bigcup_{\lambda \in \Lambda} E_\lambda^C\right)^C$$

This closes the proof. $\square$

10. [Abbott, 3.2.10] Only one of the following three descriptions can be realized. Provide an example that illustrates the viable description, and explain why the other two cannot exist.

(i) A countable set contained in $[0, 1]$ with no limit points.

This is not viable.
Let (a_n) be any sequence such that $0 \leq a_n \leq 1$. The sequence (a_n) is bounded. By the Bolzanno-Weierstrass theorem, every bounded sequence has atleast one convergent subsequence. Thus, any countable set contained in $[0, 1]$ must have atleast one limit point.

(ii) A countable set contained in $[0, 1]$ with no isolated points.
This is plausible.
Consider $\mathbf{Q} \cap [0, 1]$. Let $q \in \mathbf{Q} \cap [0, 1]$. Then, $\forall \epsilon > 0$, the open interval $(q - \epsilon, q + \epsilon)$ intersects $\mathbf{Q} \cap [0, 1]$ in some point other than q. Consequently, $0 \leq q \leq 1$ is a limit point of the set, for all q. Hence, $\mathbf{Q} \cap [0, 1]$ has no isolated points.

(iii) A set with an uncountable number of isolated points.
This is not viable. Let x be any isolated point in the set. Then, there exists an $\epsilon > 0$, such that $(x - \epsilon, x + \epsilon)$ does not intersect the set in any other point than x.

11. [Abbott, 3.2.11] (a) Prove that $cl(A \cup B) = cl(A) \cup cl(B)$.

Solution.

Suppose $x \in \overline{A \cup B}$. Let L be the set of limit points $A \cup B$. Then, either $x \in A \cup B$ or $x \in L$. If $x \in A \cup B$, then $x \in A$ or $x \in B$ or x belongs to both. If $x \in A$, then $x \in \overline{A}$ and the same for B. In either case, $x \in \overline{A} \cup \overline{B}$.

Suppose that, $x \notin A \cup B$, but $x \in L$, then by definition:

$$(\forall \epsilon > 0, \exists y : (y \neq x) \wedge (y \in (A \cup B) \cap V_\epsilon(x)))$$

This implies that

$$(\forall \epsilon > 0, \exists y : (y \neq x) \wedge ((y \in (A \cap V_\epsilon(x))) \ \vee \ (y \in (B \cap V_\epsilon(x)))))$$

That is, it intersects $A \cup B$ all the time.

What we need to prove is,

$$\begin{gathered}(\forall \epsilon > 0, \exists y : (y \neq x) \wedge (y \in (A \cap V_\epsilon(x)))) \\ \vee (\forall \epsilon > 0, \exists y : (y \neq x) \wedge (y \in (B \cap V_\epsilon(x))))\end{gathered}$$

that is, it intersects A all the time, or it intersects B all the time.

We still need to explain, why we cannot have that for some $\epsilon > 0$, $V_\epsilon(x)$ intersects A and for some ϵ, $V_\epsilon(x)$ intersects B.

★ To do.

Thus, $V_\epsilon(x)$ intersects atleast one of A, B in some point other than x. So, x is a limit point of atleast one of the sets, A or B. Thus, $x \in \overline{A} \cup \overline{B}$.

Altogether, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

In the opposite direction, suppose $x \in \overline{A} \cup \overline{B}$. Then, $x \in \overline{A}$ or $x \in \overline{B}$ or x belongs to both. Since $A \subseteq A \cup B$, $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$. Consequently, $x \in \overline{A \cup B}$.

(b) Does this result about closures extend to the infinite unions of sets?

★ To do.

12. [Abbott, 3.2.12] Let A be an uncountable set and B be the set of real numbers that divides A into two uncountable sets; that is, $s \in B$ if both $\{x : x \in A \ \wedge x < s\}$ and $\{x : x \in A \wedge x > s\}$ are uncountable. Show that B is nonempty and open.

Solution.

Suppose $A = [0, 1) \cup (2, 3]$. If $s = 0.5$, then you have $[0, 0.5)$ on one side and $(0.5, 1) \cup (2, 3]$ on the other. So, $s \in B$. If $s = 0$, then we don't get two disjoint uncountable sets. If $s = 1.5$, then you have $[0, 1)$ on one side and $(2, 3]$ on the other. If $s = 3$, again we don't get disjoint sets. So, clearly, $B = (0, 3)$,

3.3 Compact Sets.

The central challenge in analysis is to exploit the power of the mathematical infinite - via limits, series, deriviatives and intergral, integrals - without falling victim to erroneous logic or faulty intuitition. A major tool for maintaining a rigorous footing in this endeavor is the concept of compact sets. In ways that will become clear, especially in our upcoming study of continuous functions, employing compact sets in a proof has the effect of bringing a finite quality to the argument, thereby making it much more tractable.

Definition (Compactness). A set $K \subseteq \mathbf{R}$ is compact, if every sequence in K has a subsequence that converges to a limit that is also in K.

Example. The most basic example of a compact set is a closed interval. To see this notice that, if (a_n) is contained in an interval $[c, d]$, then the Bolzanno Weierstrass Theorem guarantees that we can find a convergent subsequence (a_{n_k}). Because a closed interval is a closed set, we know that the limit of a subsequence is also in $[c, d]$.

What are the properties of closed intervals that we used in the preceding argument? The Bolzanno-Weierstrass theorem requires boundedness, and we used the fact that closed sets contain their limit points. As we are about to see, these two properties completely characterize compact sets in $\mathbf{R}$. The term bounded has so far been used to only describe sequences but an analogous statement can also be made about sets.

Definition. A set $A \subseteq \mathbf{R}$, is bounded if there exists $M > 0$ such that $|a| \leq M$ for all $a \in A$.

Theorem. (Characterisation of Compactness in $\mathbf{R}$). A set $K \subseteq \mathbf{R}$, is compact if and only if it is closed and bounded.

Proof.
Let K be compact. We will first prove K must be bounded, so assume for contradiction, that K is not a bounded set. The idea is to produce a sequence in K that marches off to infinity in such a way that it cannot have a convergent subsequence as the definition of compactness requires. To do this, notice that because K is not bounded there must exist an element $x_1 \in K$ satisfying $|x_1| > 1$. Likewise, there must exist $x_2 \in K$ with $|x_2| > 2$, and in general, given any $n \in \mathbf{N}$, we
can produce $x_n \in K$, such that $|x_n| > n$.

Now because, K is assumed to be compact, every sequence in K should have a subsequence that converges to a limit that is also in K. But the elements of the subsequence satisfy $|x_{n_k}| > n_k$, and consequently, (x_{n_k}) is unbounded. Unbounded sequences are divergent (contrapositive of the fact that convergent sequences are bounded). Thus, we have a contradiction. So, K must atleast be a bounded set.

Next, we will show that K is also closed. To see that K contains its limit points, let $x = \lim x_n$, where (x_n) is any sequence contained in K, and argue that x must be in K as well. By definition, the sequence (x_n) has a convergent subsequence (x_{n_k}) and using the result every subsequence of a convergent sequence converges to the same limit as the original sequence. Consequently, $(x_{n_k}) \rightarrow x$ and $x \in K$. Thus, K is closed.

The proof of the converse statement is request in exercise 3.3.3.

There may be a temptation to consider closed intervals as being kind of standard archetype for compact sets, but this is misleading. The structure of compact sets can be much more intricate and interesting. For instance, the Cantor set is closed and bounded, hence it is compact. It is more useful to think of compact sets as generalizations of closed intervals. Whenever a fact involving closed intervals is true, it is often the case that the same result holds when we replace *closed interval* with *compact set*.

Theorem. Nested Compact set Property.
If

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \ldots$$

is a nested sequence of non-empty compact sets, then the intersection $\displaystyle\bigcap_{n=1}^{\infty} K_n$ is not empty.

Proof.

In order to take advantage of the compactness of each K_n, we are going to produce a sequence that is eventually in each of these sets. Thus, for each $n \in \mathbf{N}$, pick a point $x_n \in K_n$. Because the compact sets are nested, it follows that the sequence (x_n) is contained in K_1. By the definition of compact sets, (x_n) has a convergent subsequence (x_{n_k}) whose limit $x = \lim x_{n_k}$ is an element of K_1.

Now, the tail $(x_2, x_3, x_4, \ldots)$ is contained in the set K_2, the tail $(x_3, x_4, \ldots)$ is contained in the set K_3, and in fact, given a particular $n_0 \in \mathbf{N}$, the tail of the sequence (x_n) are contained in K_{n0}, as long as $n \geq n_0$. Ignoring the finite number of terms for which $n_k < n_0$, the same subsequence (x_{n_k}) is then also contained in K_{n0}. The conclusion is that $x = \lim x_{n_k}$ is a element of K_{n0}. Because, n_0 was arbitrary, $x \in \bigcap_{n=1}^{\infty} K_n$.

Open Covers.

Defining compactness for sets in $\mathbf{R}$, is remniscent of the situation we encountered with completeness in that there are a number of equivalent ways to describe this phenomenon. We demonstrated the equivalence of two such characterisations in the theorem above. What this theorem implies is that we could have decided to define compact sets to be sets that are closed and bounded, and then proved that sequences contained in compact sets have convergent subsequences with limits in the set. There are some larger issues involved in deciding what the definition should be, but what is important at this moment is that we be versatile enough to use whatever description of compactness is most appropriate for a given situation.

Although, the theorem above is sufficient for most of our purposes, there is a third important characterization of compactness, equivalent to the two others, which is described in terms of open covers and finite subcovers.

Definition. Let $A \subseteq \mathbf{R}$. An *open cover* for A, is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ whose union contains the set A; that is $A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$. Given an open cover for A, a *finite subcover* is a finite subcollection of open sets from the original open cover whose union still manages to completely contain A.

Example. Consider the open interval $(0,1)$. For each point $x \in (0,1)$, let O_x be the open interval $(x/2, 1)$. Taken together, the infinite collection $\{O_x : x \in (0,1)\}$ forms an open cover for the open interval $(0,1)$. Notice, however, that it is impossible to find a finite subcover. Given any proposed finite subcollection

$$\{O_{x_1}, O_{x_2}, O_{x_3}, \ldots, O_{x_N}\}$$

set $x' = \min\{x_1, x_2, \ldots, x_N\}$ and observe that any real number y satisfying $0 < y \leq x'/2$ is not contained in the union $\bigcup_{i=1}^{n} O_{x_i}$.

Now consider a similar cover for the closed interval $[0, 1]$. For $x \in (0, 1)$, the sets $O_x = (x/2, 1)$ do a fine job covering $(0, 1)$, but in order to have an open cover of the closed interval $(0, 1)$, we must cover the end-points. To remedy this, we could fix $\epsilon > 0$ and let $O_0 = (-\epsilon, \epsilon)$ and $O_1 = (1-\epsilon, 1+\epsilon)$. Then the collection

$$\{O_0, O_1, O_x : x \in (0, 1)\}$$

is an open cover for $[0, 1]$. But, this time, notice there is a finite subcover. Because of the addition of the set O_0, we can choose an x' so that $x'/2 < \epsilon$. It follows, that $\{O_0, O_1, O_{x'}\}$ is a finite subcover for $[0, 1]$.

Theorem. (Heine-Borel Theorem). Let K be a subset of $\mathbf{R}$. All of the following statements are equivalent in the sense that any one of them implies the two others.

(i) K is compact.
(ii) K is closed and bounded.
(iii) Every open cover for K has a finite subcover.

Proof.
The equivalence of (i) and (ii) is the content of theorem on characterization of compact sets. What remains to show is that (iii) is equivalent to (i) and (ii). Let's first assume (iii), and prove that it implies (ii) (and thus (i) as well).

To show that K is bounded, we construct an open cover for K by defining O_x to be an open interval of radius 1 around each point $x \in K$. In the language of neighbourhoods, $O_x = V_1(x)$. The open cover $\{O_x : x \in K\}$ then must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. Because, K is contained in a finite union of bounded sets, K itself must be bounded.

The proof that K is closed is more delicate and we argue by contradiction. Let (y_n) be a Cauchy sequence contained in K, with $\lim y_n = y$. To show that K is closed, we must demonstrate that $y \in K$ and so assume for contradiction that this is not the case. If $y \notin K$, then every $x \in K$ is some positive distance away from y. We now construct an open cover by taking O_x to be an interval of radius $|x - y|/2$ around each point x in K. Because, we are assuming (iii), the resulting open cover $\{O_x : x \in K\}$ must have a finite subcover $\{O_{x_1}, O_{x_2}, \ldots, O_{x_n}\}$. The contradiction arises when we realize that, in the spirit of the example 3.3.7, that this finite subcover does not contain all the elements of the sequence (y_n). To make this explicit, set

$$\epsilon_0 = \min\left\{\frac{|x-y|}{2} : 1 \le i \le n\right\}$$

Because, $(y_n) \to y$, we can certainly find a term y_N of the sequence satisfying $|y_N - y| < \epsilon_0$. But, such a y_N must necessarily be excluded from each O_{x_i}, meaning that

$$y_N \notin \bigcup_{i=1}^{n} O_{x_i}$$

Thus, our supposed subcover does not actually cover all of K. This contradiction implies that $y \in K$, and hence K is closed and bounded.

The proof that (ii) implies (iii) is outlined in exercise 3.3.9. To be historically accurate, it is this particular implication that is appropriately referred to as the Heine-Borel Theorem.

Exercise Problems.

1. [Abbott, 3.3.1] Show that if K is compact and non-empty, then $\sup K$ and $\inf K$ both exist and are elements of K.

Proof.
By the Heine-Borel theorem, as K is compact and non-empty, K is a closed and bounded subset of $\mathbf{R}$. So, there exists $M > 0$, such that $|x| < M$, for all $x \in K$. Consequently, $-M < x < M$ for all $x \in K$.

By the Axiom of Completeness (AoC), every subset of $\mathbf{R}$ bounded above has a least upper bound, and subset of $\mathbf{R}$, bounded below has a greatest lower bound. Thus, both $\sup K$ and $\inf K$ exist.

Let $s = \sup K$. Consider a neighbourhood of the point s having radius 1. By the definition of supremum, there exists an $x_1 \in K$, such that $s - 1 < x_1 < s < s + 1$. There exists an $x_2 \in K$, such that $s - \frac{1}{2} < x_2 < s < s + \frac{1}{2}$. In general, there exists $x_n \in K$, such that

$$s - \frac{1}{n} < x_n < s < s + \frac{1}{n}$$

Given an arbitrary $\epsilon > 0$, if we pick $N > \frac{1}{\epsilon}$, then for all $n \ge N$, we have:

$$x_n \in V_\epsilon(s)$$

So, the sequence $(x_n) \rightarrow s$. As K is closed, K contains all its limit points. Consequently, $s \in K$.

We can similarly argue for $l = \inf K$.

2. [Abbott, 3.3.2] Decide which of the following sets are compact. For those, that are not compact, show how the definition of compactness breaks down. In other words, give an example of a sequence contained in the given that does not possess a subsequence converging to a limit in the set.

(a) $\mathbf{N}$.

$\mathbf{N}$ is unbounded. So, $\mathbf{N}$ is not compact.

Consider the sequence $(a_n) = (1, 2, 3, \ldots)$. There is no subsequence of (a_n) that converges to a limit in $\mathbf{N}$.

(b) $\mathbf{Q} \cap [0, 1]$.

Consider the sequence of rationals in $[0, 1]$:

$$\left(\frac{1}{1}, \frac{1}{1+\frac{1}{1!}}, \frac{1}{1+\frac{1}{1!}+\frac{1}{2!}}, \frac{1}{1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}}, \ldots, \frac{1}{\sum_{r=0}^{n} \frac{1}{r!}}, \ldots\right)$$

Every subsequence of this sequence converges to e^{-1}. Hence, $Q \cap [0, 1]$ is not compact.

(c) The Cantor Set.

The Cantor Set is formed by removing the open middle one thirds. The inifinite union of open sets is open, so their complement is a closed set. Thus, C is uncountable, closed and bounded subset of $\mathbf{R}$. Thus, C is compact.

(d) $\left\{1 + 1/2^2 + 1/3^2 + \ldots + 1/n^2 : n \in \mathbf{N}\right\}$

Let (s_n) be sequence of partial sums of the infinite series $\sum \frac{1}{n^2}$. We know, that $(s_n) \rightarrow \frac{\pi^2}{6}$.

Therefore, the given set is not closed and it is not a compact set.

(e) $\{1, 1/2, 2/3, 3/4, 4/5, \ldots\}$

The given set is

$$\left\{1 - \frac{1}{n} : n \in N\right\}$$

The limit of the sequence $(a_n) = 1 - (1/n)$ is 1. Since the limit is in the set, it is closed and bounded. Hence, it is compact.

3. [Abbot, 3.3.3] Prove the converse of the theorem 3.3.4 by showing that if a set $K \subseteq \mathbf{R}$ is closed and bounded, then it is compact.

Proof.

Since $K \subseteq \mathbf{R}$, is bounded, by the Bolzanno Weirstrass theorem, every sequence (x_n) contained in K, has atleast one convergent subsequence (x_{n_k}).

From the definition of closed sets, a set is closed, if and only if every Cauchy sequence contained in K has a limit that is also an element of K. Since, K is closed, the (x_{nk}) converges to a limit that is also in K.

By definition, we conclude that K is compact.

4. [Abbott, 3.3.4] Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both or neither.

(a) $K \cap F$.

Since K compact, K is closed and bounded. The intersection of closed sets is closed. So, $K \cap F$ is definitely closed. Moreover, $K \cap F \subseteq K$, so is bounded. Hence, $K \cap F$ is also compact.

(b) $\overline{F^C \cup K^C}$

The closure of any set is closed. So, $\overline{F^C \cup K^C}$ is definitely a closed set. Since, K^C is an unbounded subset of $\mathbf{R}$, the resulting set is also unbounded. Hence, it is not a compact set.

(c) $K \setminus F = \{x \in K : x \notin F\}$

(d) $\overline{K \cap F^C}$

This is definitely a closed set. Moreover, $K \cap F^C$ is a bounded subset of $\mathbf{R}$. Hence, $\overline{K \cap F^C}$ is also bounded. So, it is compact.

5. [Abbott, 3.3.5] Decide whether the following propositions are true or false. If the claim is valid, supply a short proof, and if the claim is false, provide a counterexample.

(a) The arbitrary intersection of compact sets is compact.

This proposition is true.

Justification. Compact sets are closed and bounded. The arbitrary intersection of closed and bounded sets is also closed and bounded. Hence, the abitrary intersection of compact sets is compact.

(b) The arbitrary union of compact sets is compact.

This proposition is false.

Counterexample. Consider singleton sets of rational numbers between 0 and 1. Let $I_{m,n} := \{m/n\}$, where $m, n \in \mathbf{N}$ and $0 \leq m \leq n$, $n \neq 0$ and m, n have no common factors. The arbitrary union of these sets

$$\bigcup I_{m,n} = \bigcup \left\{\frac{m}{n}\right\} = \mathbf{Q} \cap [0, 1]$$

$I_{m,n}$ is a closed set. But, $\mathbf{Q} \cap [0, 1]$ is not closed.

(c) Let A be arbitrary and let K be compact. Then, the intersection $A \cap K$ is compact.

This proposition is false.

Counterexample. Consider $A = (0, 1)$, $K = [0, 1]$. Then, $A \cap K = (0, 1)$, which is not a compact set.

(d) If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ \dots$ is a nested sequence of nonempty closed sets, then the intersection

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

This proposition is false.

Counterexample.

Consider $A_n = [n, \infty)$ where $n \in \mathbf{N}$. A_n is a closed set. Clearly,

$$\bigcap_{n}^{\infty} A_n = \emptyset$$

6. [Abbott, 3.3.6] This exercise is meant to illustrate the point made in the opening paragraph to the section 3.3. Verify that the following three statements are true if every blank is filled with the word "finite". Which are true, if every blank is filled in with the word "compact"? Which are true, if every blank is filled in with the word closed?

(a) Every _________ set has a maximum.

Finite.
This proposition is true.

Compact.
This proposition is true.

Let K be a compact set. By the theorem on characterization of compact sets, if K is compact, then K is closed and bounded. Since K is bounded subset of $\mathbf{R}$, by the least upper bound property, K has a least upper bound.

Let $s = \sup K$. By definition, for all $\epsilon > 0$, there exists an $a \in K$, such that $s - \epsilon < a < s$. Pick $\epsilon = \dfrac{1}{n}$, then there exists a_n, such that $s - \dfrac{1}{n} < a_n < s < s + \dfrac{1}{n}$. Consequently, we have produced a sequence $(a_n) \subseteq K$, such that $\lim a_n = s$. So, s is a limit point of K. Since, K is closed, $s \in K$. Thus, K has a maximal element.

Closed.

This proposition is false. $\mathbf{R}$ is closed, but has no maximal element.

(b) If A and B are ______, then $A + B = \{a + b : a \in A, b \in B\}$ is also ______.

Finite.
The cartesian product of finite sets A and B, $A \times B$ is finite. Since, $+ : A \times B \to A + B$ is a well-defined function on a finite set, each element in the domain, maps to one and only one element in the co-domain. It is onto, the number of elements $A \times B \geq$ the number of elements of $A + B$. Since the Cartesian product $A \times B$ is finite, the image set $A + B$ is also finite.

Compact.
This proposition is true.

First, since A and B are bounded, $|a| \leq M$ for all $a \in A$ and $|b| \leq N$ for all $b \in B$. Thus,

$$|a + b| \leq |a| + |b| \leq M + N$$

for all $a \in A,\ b \in B$.

Thus, $A + B$ is bounded.

Let's prove that $A + B$ is closed.

Since, $A + B$ is bounded, by the Bolzanno Weierstrass Theorem, every sequence has atleast one convergent subsequence contained in $A + B$. Let (c_n) be an convergent subsequence in $A + B$. Suppose $(c_n) \to L$. Every subsequence (c_{n_k}) of (c_n) also converges to L. We are interested to prove that $L \in A + B$.

From the definition of $A + B$, we can write:

$$c_n = a_n + b_n$$

As A is a compact set, (a_n) has a subsequence (a_{n_k}), which converges to a limit $\lim a_{n_k} = a'$ which is also in A. We can pick $b_{n_k} = (c_{n_k} - a_{n_k})$. By definition, such a subsequence lies in B. Since (c_{n_k}) and (a_{n_k}) are convergent, their sum is also convergent. Since, B is closed, $\lim(b_{n_k}) = b'$ lies in B.

Now, $L = \lim(c_{nk})$. As (a_{n_k}) and (b_{n_k}) are convergent, by the algebraic limit theorem, $L = \lim(c_{n_k}) = \lim(a_{n_k}) + \lim(b_{n_k}) = a' + b'$ which belongs to $A + B$. So, $A + B$ is compact.

Closed.

This proposition is false.

★ TODO.

(c) If $\{A_n : n \in \mathbf{N}\}$ is a collection of ______ sets with the property that every finite subcollection has a non-empty intersection, then $\bigcap_{n=1}^{\infty} A_n$ is non-empty as well.

Finite.
Since finite sets are compact, we conclude that their intersection is non-empty.

Compact.
Let

$$\begin{aligned}
A_1' &= A_1 \\
A_2' &= A_1 \cap A_2 \\
A_3' &= A_1 \cap A_2 \cap A_3 \\
&\vdots \\
A_n' &= \bigcap_{k=1}^{n} A_k
\end{aligned}$$

Since, the arbitrary intersection of compact sets is compact, $A_1', A_2', \ldots$ are non-empty compact sets. We have sequence of non-empty nested compact sets -

$$A_1' \supseteq A_2' \supseteq A_3' \supseteq \ldots$$

By the nested compact set property, their intersection is non-empty.

Closed.

This proposition is false, because a nested sequence of non-empty closed sets do not always have a non-empty intersection.

7. [Abbott, 3.3.7] As some more evidence of the surprising nature of the Cantor set, follow these steps to show that the sum $C + C = \{x + y : x, y \in C\}$ is equal to the closed interval $[0, 2]$. Keep in mind that C has zero length and contains no intervals.

Because $C \subseteq [0,1]$, $C + C \subseteq [0,2]$, so we only need to prove the reverse inclusion $[0,2] \subseteq \{x+y : x, y \in C\}$. Thus, given $s \in [0,2]$, we must find two elements $x, y \in C$, satsfying $x + y = s$.

(a) Show that there exists $x_1, y_1 \in C_1$ for which $x_1 + y_1 = s$. Show in general that, for arbitrary $n \in \mathbf{N}$, we can always find $x_n, y_n \in C_n$, for which $x_n + y_n = s$.

Proof.
We proceed by mathematical induction.

(a) **Base case** $n = 1$.

Fix $s \in [0,2]$. We want to find $x_1, y_1 \in C_1$, such that $x_1 + y_1 = s$. We know that $C_1 = [0,1/3] \cup [2/3,1]$. Then, we have that:

$$\begin{aligned} [0,1/3] + [0,1/3] &= [0,2/3] \\ [0,1/3] + [2/3,1] &= [2/3,4/3] \\ [2/3,1] + [2/3,1] &= [4/3,2] \end{aligned}$$

Hence, $C_1 + C_1 = [0,2/3] \cup [2/3,4/3] \cup [4/3,1] = [0,2]$, so for any $s \in [0,2]$, we can find $x_1, y_1 \in C_1$ such that $x_1 + y_1 = s$.

A convenient way to visualize this result in the (x,y)-plane is to shade in the four squares corresponding to the components of $C_1 \times C_1$ and observe that, for each $s \in [0,2]$. For each n we can draw a similar picture (with increasing number of smaller squares), and our job is to argue that $x + y = s$ continues to intersect atleast one of the smaller squares.

To argue by induction, suppose that we can find $x_n,\ y_n \in C_n$, such that $x_n + y_n = s$. To show that this must hold for $n+1$, let's focus our attention on a square from the nth stage where $x_n + y_n = s$ holds (that is $x + y = s$ intersects an nth stage square). Moving to the $n+1$th stage, means removing the open middle one third of this shaded region. But, this is self-similar and results in a situation precisely like the one in the figure below, implying that the straight line $x + y = s$ must intersect a $(n+1)$st stage square. This shows that there exists $x_{n+1} + y_{n+1} \in C_{n+1}$ where $x_{n+1} + y_{n+1} = s$.

(b) We have (x_n) and (y_n) with $x_n, y_n \in C_n$ and $x_n + y_n = s$ for all n. The sequence (x_n) doesn't converge, but (x_n) is bounded so by the Bolzanno-Weierstrass Theorem, there exists a convergent subsequence (x_{nk}). Now, look at the corresponding subsequence $y_{nk} = s - x_{nk}$. Using the Algebraic Limit Theorem, we see that this subseqeuence converges to

$y = \lim(s - x_{n_k}) = s - x$. This shows that $x + y = s$. We need to argue that $x, y \in C$.

One temptation is to say that because C is closed, $x = \lim(x_{n_k})$ must be in C. However, we don't know (and it probably isn't true) that (x_{nk}) is in C. We can say that (x_{nk}) is in C_1, because

8. [Abbott 3.3.8] Let K and L be non-empty compact sets, and define

$$d = \inf\{|x - y| : x \in K \text{ and } y \in L\}$$

This turns out to be a reasonable definition of the distance between K and L.

(a) If K and L are disjoint, show $d > 0$ and that $d = |x_0 - y_0|$ for some $x_0 \in K$ and $y_0 \in L$.

Proof.

Proof.

(a) Consider the set

$$A = \{|x - y| : x \in K \text{ and } y \in L\}$$

I'm interested to show that A is a compact set and therefore $\inf A \in A$.

First, let's dispose off the boundedness. Since K and L are bounded, we have

$$\begin{aligned} |x - y| &\leq |x| + |y| \\ &\leq M + N \end{aligned}$$

So, A is a bounded set.

Every sequence contained in A is bounded and therefore has atleast one convergent subsequence

(a_n), by the Bolzanno Weierstrass Theorem. Let $\lim a_n = a$. We are interested to prove that $a \in A$.

Any subsequence of (a_n), also converges to a. Let a_{n_k} be a subsequence of (a_n).

Since K is a compact set, every sequence (x_n) in K has a subsequence (x_{n_k}) such that $x = \lim(x_{n_k})$ belongs to K.

We pick (y_{n_k}) such that,

$$a_{n_k} = |x_{n_k} - y_{n_k}|$$

By definition, (y_{n_k}) is contained in L. Moreover, (a_{n_k}) is convergent, so the absolute value sequence $(|x_{n_k} - y_{n_k}|)$ is also convergent. We also know that, (x_{n_k}) is convergent. We don't know, if (y_{n_k}) is convergent. But, as L is compact, (y_{n_k}) has a convergent subsequence $(y_{n_{k_l}})$ that has a limit in L. Consequently,

$$a_{nk_l} = |x_{nk_l} - y_{nk_l}|$$

Since, all they are all convergent sequences, by the Algebraic Limit Theorem,

$$\begin{aligned} \lim\ a_{nk_l} &= |\lim\ x_{nk_l} - \lim\ y_{nk_l}| \\ a &= x - y \end{aligned}$$

As $x \in K$ and $y \in L$, we have $a \in A$. So, A is compact. Thus, $\inf A$ exists and belongs to A.

Consequently, there exists $x_0 \in K$, $y_0 \in L$ such that

$$d = |x_0 - y_0|$$

Claim. $d > 0$.
We can always construct a sequence $a_n = |x_n - y_n|$ that converges to $\inf A$, where $x_n \in K,\ y_n \in L$. Suppose $d = \inf A = 0$ and assume that $(x_n) \to x$. We argue that $(y_n) \to x$.

Since, (a_n) is a convergent sequence, for all $\epsilon > 0$, there exists $N \in \mathbf{N}$, such that $|a_n - 0| < \epsilon$

for all $n \geq N$.

$$\begin{aligned}
|a_n - 0| &< \epsilon \\
|x_n - y_n| &< \epsilon \\
|(x_n - x) - (y_n - x)| &< \epsilon \\
\therefore\ |(x_n - x)| + |(y_n - x)| &< \epsilon \\
\frac{\epsilon}{2} + |(y_n - x)| &< \epsilon \\
|y_n - x| &< \frac{\epsilon}{2}
\end{aligned}$$

for all $n \geq N$. So, $(y_n) \rightarrow x$. Since, K and L are compact, $x \in K$ and $x \in L$. Therefore, $x \in K \cap L$. This contradicts the fact that $K \cap L = \emptyset$. So, $d > 0$.

(b) Show that it's possible to have $d = 0$ if we assume only that the disjoint sets K and L are closed.

★ TODO.

9. Follow these steps to prove the final implication in Theorem 3.3.8.

Assume K satisfies (i) and (ii), and let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover for K. For contradiction, let's assume that no finite subcover exists. Let I_0 be a closed interval containing K.

(a) Show that there exists a nested sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \ldots$ with the property that, for each n, $I_n \cap K$ cannot be finitely covered and $\lim |I_n| = 0$.

Proof.

3.4 **Perfect Sets and Connected Sets.**

One of the underlying goals of topology is tro strip away all of the extraneous information that comes with our intuitive picture of the real numbers and isolate just those properties that are responsible for the phenomenon we are studying. For example, we were quick to observe that any closed interval is a compact set. The content of the Heine Borel Theorem, however, is that the compactness of a closed interval has nothing to do with the fact that the set is an interval but is a consequence of the set being bounded and closed. In Chapter 1, we argued that the set of real numbers between 0 and 1 is an uncountable set. This turns out to the case for any nonempty closed set that does not contain isolated points.

Definition 3.4.1 (Perfect Set). A set $P \subseteq \mathbf{R}$, is perfect if it is closed and contains no isolated points.

Closed intervals (other than the singleton sets $[a, a]$) serve as the most obvious class of perfect sets, but there are more interesting examples.

Example 3.4.2 (Cantor Set). It is not too hard to see that the Cantor set is perfect. In Section 3.1, we defined the Cantor set as the intersection

$$C = \bigcap_{n=0}^{\infty} C_n$$

where each C_n is a finite union of closed intervals. Using the result that the arbitrary intersection of clsoed intervals is closed, each C_n is closed and therefore C is closed as well. It remains to show that no point in C is isolated.

Let $x \in C$ be arbitrary. To convince ourselves that x is not isolated, we must construct a sequence (x_n) of points in C, different from x, that converges to x. From our earlier discussion, we know that C contains the endpoints of the intervals that make up each C_n. In the exercise we sketch the argument that these are all that is needed to construct the sequence (x_n).

One argument for the uncountability of the Cantor set was presented in Section 3.1. Another perhaps more satisfying argument for the same conclusion can be obtained from the next theorem.

Theorem 3.4.3. A non-empty perfect set is uncountable.

Proof.

If P is perfect and nonempty, then it must be uncountable because otherwise it would consist only of isolated points. Let's assume, for contradiction, that P is countable. Thus, we can write

$$P = \{x_1, x_2, x_3, \dots\}$$

where every element of P appears on the list. The idea is to construct a sequence of nested compact sets K_n, all conained in P, with the property that $x_1 \notin K_2$, $x_2 \notin K_3$, $x_3 \notin K_4$, $\dots$ Some care must be taken to ensure that each K_n is non-empty, for then we can use the nested compact set property to produce an

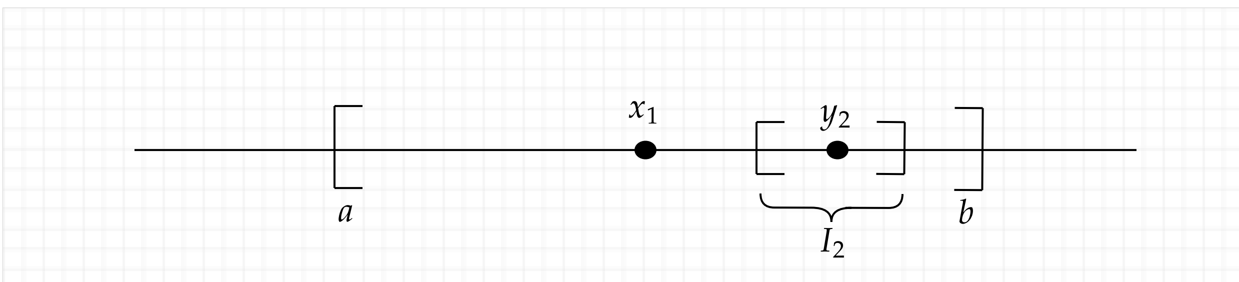
$$x \in \bigcap_{n=1}^{\infty} K_n \subseteq P$$

that cannot be on the list $\{x_1, x_2, x_3, \ldots\}$.

Let I_1 be a closed interval that contains x_1 in its interior (that is x_1 is not an endpoint of I_1). Now, x_1 is not isolated, so there exists some other point $y_2 \in P$ that is also in the interior of I_1. Construct a closed interval I_2, centered on y_2, so that $I_2 \subseteq I_1$, but $x_1 \notin I_2$. More explicitly, if $I_1 = [a, b]$, let

$$\epsilon = \min\{y_2 - a, b - y_2, |x_1 - y_2|\}$$

Then, the interval $I_2 = [y_2 - \epsilon/2, y_2 + \epsilon/2]$ has the desired properties.



This process can be continued. Because $y_2 \in P$ is not isolated, there must exist another point $y_3 \in P$ in the interior of I_2, and we may insist that $y_3 \neq x_2$. Now construct I_3 centered on y_3 and small enough so that $x_2 \notin I_3$ and $I_3 \subseteq I_2$. Observe that $I_3 \cap P \neq \emptyset$, because this intersection contains atleast y_3.

If we carry out this construction inductively, the result is a sequence of closed intervals I_n satisfying -

(1) $I_{n+1} \subseteq I_n$.

(2) $x_n \notin I_{n+1}$.

(3) $I_n \cap P \neq \emptyset$.

To finish the proof, we let $K_n = I_n \cap P$. For each $n \in \mathbf{N}$, we have that K_n is closed because it is the intersection of closed sets, and bounded because it is contained in the bounded set I_n. Hence K_n is compact. By construction, K_n is non-empty and $K_{n+1} \subseteq K_n$. Thus, we can employ the nested compact set property to conclude that the intersection

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset$$

But, each K_n is a subset of P, and the fact that $x_n \notin I_{n+1}$ leads us to the conclusion that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, which is the sought-after contradiction. □

Connected Sets.

Although the two open intervals $(1, 2)$ and $(2, 5)$ have the limit point $x = 2$ in common, there is still some space between them in the sense that no limit point of one of these intervals is actually contained in the other. Said another way, the closure of $(1, 2)$ is disjoint from $(2, 5)$ and the closure of $(2, 5)$ does not intersect $(1, 2)$. Notice that this same observation can be made about $(1, 2]$ and $(2, 5)$, even though these latter sets are disjoint.

Definition 3.4.4. (Separated sets). Two non-empty sets $A, B \subseteq \mathbf{R}$ are separated if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty. A set $E \subseteq \mathbf{R}$ is disconnected if it can be written as $E = A \cup B$, where A and B are non-empty separated sets. A set that is not disconnected is called a connected set.

Example 3.4.5. (i) If we let $A = (1, 2)$ and $B = (2, 5)$, then it is not difficult to verify that $E = (1, 2) \cap (2, 5)$ is disconnected. Notice that the sets $C = (1, 2]$ and $D = (2, 5)$ are not separated because $C \cap \overline{D} = \{2\}$ is non-empty. This should be comforting. The union $C \cup D$ is equal to the interval $(1, 5)$ which better not qualify as a disconnected set. We will prove in a moment that every interval is a connected subset of $\mathbf{R}$ and vice versa.

(ii) Let's show that the set of rational numbers is disconnected. If we let

$$A = \mathbf{Q} \cap \left(-\infty, \sqrt{2}\right) \quad \text{and} \quad B = \mathbf{Q} \cap \left(\sqrt{2}, \infty\right)$$

then we certainly have $\mathbf{Q} = A \cup B$. The fact that $A \subseteq \left(\infty, \sqrt{2}\right)$ implies (by the Order Limit Theorem) that any limit point of A will necessarily fall in $\left(-\infty, \sqrt{2}\right]$. Because this is disjoint from B, we get $\overline{A} \cap B = \emptyset$. We can similarly show that $A \cap \overline{B} = \emptyset$, which implies that A and B are separated.

The definition of connected is stated as the negation of disconnected, but a little care with the logical negation of the quantifiers in definition 3.4.4 results in a positive characterization of connectedness. Essentially, a set is connected, if no matter how it is partitioned into two non-empty disjoint sets, it is always possible to show that the limit points of at least one of the sets contains a limit point of the other.

Theorem 3.4.6. A set $E \subseteq \mathbf{R}$ is connected if and only if, for all nonempty disjoint sets A and B satisfying $E = A \cup B$, there always exists a convergent sequence $(x_n) \to x$ with (x_n) contained in one of A or B, and x an element of the other.

Proof.

See exercise 3.4.6.

The concept of connectedness is more relevant when working with subsets of the plane and other higher-dimensional spaces. This is because in $\mathbf{R}$, the connected sets coincide precisely with the collection of intervals (with the understanding that unbounded intervals such as $(-\infty, 3)$ and $[0, \infty)$ are included.

Theorem 3.4.7. A set $E \subseteq \mathbf{R}$ is connected if and only if whenever $a < c < b$ with $a, b \in E$, it follows that $c \in E$ as well.

Proof.

$(\implies)$ Assume that E is connected, and let $a, b \in E$ and $a < c < b$. Set

$$A = (-\infty, c) \cap E \quad \text{and} \quad B = (c, \infty) \cap E$$

Because $a \in A$ and $b \in B$, neither set is empty and, just as in example 3.4.5 (ii), neither set contains a limit point of the other. If $E = A \cup B$, then we would have that E is disconnected, which it is not. It must be that $A \cup B$ is missing some element of E and c is the only possibility. Thus, $c \in E$.

$(\impliedby)$ Conversely, assume that E is an interval in the sense that whenever $a < c < b$ for some c, then $c \in E$. Our intent is to use the characterisation of connected sets in Theorem 3.4.6, so let $E = A \cup B$ where A and B are non-empty and disjoint. We need to show that one of these sets contains a limit point of the other.

Pick $a_0 \in A$ and $b_0 \in B$, and for the sake of argument assume $a_0 < b_0$. Because E is itself an interval, the interval $I_0 = [a_0, b_0]$ is contained in E. Now, bisect I_0 into two equal halves. The midpoint of I_0 must either be in A or B, so choose $I_1 = [a_1, b_1]$ to be the half that allows us to have $a_1 \in A$ and $b_1 \in B$, and so choose $I_1 = [a_1, b_1]$ to be the half that allows us to have $a_1 \in A$ and $b_1 \in B$. Continuing this process yields a sequence of nested intervals $I_n = [a_n, b_n]$, where $a_n \in A$, $b_n \in B$ and the length $(b_n - a_n) \to 0$. The remainder of this argument should feel familiar. By the Nested Interval Proeprty, there exists an

$$x \in \bigcap_{n=0}^{\infty} I_n$$

and it is straightfoward to show that the sequences of endpoints each satisfy $\lim a_n = x$ and $\lim b_n = x$. But now $x \in E$ must belong to either A or B, making it a limit point of the other. This completes the argument. $\square$

Exercises.

1. [Abbott, 3.4.1] If P is a perfect set and K is compact, is the intersection $P \cap K$ always compact? Always perfect?

Since P is a perfect set, it is closed and it does not contain isolated points. Since, K is compact it is closed and bounded. The arbitrary intersection of closed sets is closed, so $P \cap K$ is closed. $P \cap K \subseteq K$ so it is bounded. Therefore, $P \cap K$ is always compact.

Let $P = [0, 1]$ and $K = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\} \cup \{0\}$. Here, $P \cap K = K$ and K has isolated points. So, $P \cap K$ is not always perfect.

2. [Abbott, 3.4.2] Does there exist a perfect set consisting of only rational numbers?

Proof.

A set $P \subseteq \mathbf{R}$ is perfect, if it closed and has no isolated points.

A finite non-empty subset of $\mathbf{Q}$ consists of isolated points. So, it is not perfect. Consider a countable subset of $\mathbf{Q}$ for example $\mathbf{Q} \cap [0, 1]$. Every element of this set is a limit point, for if x is an element of the set, for all $\epsilon > 0$, the open interval $(x - \epsilon, x + \epsilon)$ intersects $\mathbf{Q}$ in some point other than x. But, such a set is not closed under $\mathbf{R}$.

Thus, in either case, a non-empty subset of $\mathbf{Q}$ is not perfect.

In general, a nonempty perfect set is uncountable. So, any subset of $\mathbf{Q}$ which is a countable set is not perfect.

3. [Abbott, 3.4.3] Review the portion of the proof given in Example 3.4.2 to show that the Cantor set is perfect and follow these steps to complete the argument.

(a) Because $x \in C_1$, argue that there exists an $x_1 \in C \cap C_1$ with $x_1 \neq x$ satisfying $|x - x_1| \leq 1/3$.

(b) Finish the proof by showing that for each $n \in \mathbf{N}$, there exists $n \in \mathbf{N}$, there exists $x_n \in C \cap C_n$, different from x, satisfying $|x - x_n| \leq 1/3^n$.

Proof.

Let $x \in C$ be an arbitrary element. By the construction of the Cantor set $C = \bigcap_{n=0}^{\infty} C_n$, so $x \in C_n$ for each n.

C_n is a union of 2^n disjoint closed intervals of length 3^{-n}. We can pick x_n to be the end-point of the interval to which x belongs. Even if x is one of the end-points, the other end-point is at a distance at most 3^{-n}.

Now, consider the sequence $(x_n) = (x_1, x_2, \ldots)$. Given any arbitrary $\epsilon > 0$, pick n_0 such that

$$\frac{1}{3^{n_0}} < \epsilon \quad \text{or} \quad n_0 > \left(\log \frac{1}{\epsilon}\right) / (\log 3)$$

then for $n \geq n_0$, we have $|x_n - x| < \epsilon$. So, $(x_n) \to x$.

Thus, C has no isolated points. Consequently, C is a perfect set.

4. [Abbott, 3.4.4] Repeat the Cantor construction from section 3.1 starting with the interval $[0, 1]$. This time, however, remove the open middle fourth from each component.

(a) Is the resulting set compact? Perfect?

Proof.

We have, $I_0 = [0, 1]$. Removing the open middle fourth, we have:

$$I_1 = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$$

Removing the open middle fourth again, we have,

$$I_2 = \left[0, \frac{1}{16}\right] \cup \left[\frac{3}{16}, \frac{4}{16}\right] \cup \left[\frac{12}{16}, \frac{13}{16}\right] \cup \left[\frac{15}{16}, 1\right]$$

Let $I = \bigcap_{n=0}^{\infty} I_n$. Then, since the arbitrary intersection of closed intervals is closed, I is closed, bounded. So, I is compact. I does not contain any isolated points, so I is perfect.

(b) Using the algorithms from Section 3.1, compute the length and dimension of this Cantor-like set.

The length of the open intervals removed is