Understanding Analysis

Solution of exercise problems.

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Abstract

This is a solution manual for Understanding Analysis, 2nd edition, by Stephen Abbott.

Chapter 5. The Derivative

5.1 **Discussion: Are derivatives continuous?**

The geometric motivation for the derivative is most likely familiar territory. Given a function $g(x)$, the derivative $g'(x)$ is understood to be the slope of the graph of g at each point x in the domain. A graphical picture reveals the impetus behind the mathematical definition

$$g'(x) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

The difference quotient (ratio) $(g(x) - g(c)) / (x - c)$ represents the slope of the line through the two points $(x, g(x))$ and $(c, g(c))$. By taking the limit as x approaches c, we arrive at a well-defined mathematical meaning for the slope of the tangent line at $x = c$.

The myriad applications of the derivative function are the topic of much of the calculus sequence, as well as several other upper-level courses in mathematics. None of these applied questions are pursued here in any length, but it should be pointed out that the rigorous underpinnings of differentiation worked out in this chapter are an essential foundation for any applied study. Eventually, as the derivative is subjected to more and more complex manipulations, it becomes crucial to know precisely how differentiation is defined and how it interacts with other mathematical operations.

Although physical applications are not explicitly discussed, we will encounter several questions of a more abstract quality as we develop the theory. Many of these are concerned with the relationship between differentiation and continuity. Are continuous functions always differentiable? If not, how nondifferentiable can a continuous function be? Are differentiable functions continuous?

Given that a function f has a derivative at every point in its domain, what can we say about the function f'? Is f' continuous? How accurately can we describe the set of all possible derivatives, or are there no restrictions? Put another way, if we are given an arbitrary function g, is it always possible to find a differentiable function f such that $f' = g$, or are there some properties that g must possess for this to occur? In our study of continuity, we saw that restricting our attention to monotone functions had a significant impact on the answers to questions about sets of discontinuity. What effect, if any, does this same restriction have on our questions about potential sets of non-differentiable points? Some of these issues are harder to resolve than the others, and some remain unanswered in any satisfactory way.

A particularly useful class of examples for this discussion are functions of the form

$$g(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

When $n = 0$, we have seen that the oscillations of $\sin(1/x)$ prevent $g_0(x)$ from being continuous at $x = 0$. When $n = 1$, these oscillations are squeezed between $|x|$ and $-|x|$, the result being that g_1 is continuous at $x = 0$. Is $g_1{}'(0)$ defined? Using the preceding definition, we get

$$g_1{}'(0) = \lim_{x \to 0} \frac{g_1(x) - g_1(0)}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$$

which, as we now know, does not exist. Thus, g_1 is not differentiable at $x = 0$. On the other hand, the same calculation shows that g_2 is differentiable at zero. In fact, we have

$$g_2{}'(0) = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

At points different from zero, we can use the familiar rules of differentiation (soon to be justified) to conclude that g_2 is differentiable everywhere in $\mathbf{R}$ with

$$\begin{aligned}
{g_2}'(x) &= \left(x^2 \sin(1/x)\right)' \\
&= 2x \sin(1/x) \ + \ x^2 \cdot \cos(1/x) \cdot \frac{-1}{x^2} \\
&= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)
\end{aligned}$$

when $x \neq 0$ and ${g_2}'(x) = 0$ when $x = 0$.