Functional Limits and Continuity

Quasar Chunawala

Abstract

Solution of the Exercise set 4.4.13

Example 0.1 (Abbot, 4.4.13). (**Continuous Extension Theorem**). (a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbf{R}$ is uniformly continuous and $(x_n) \subseteq A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

(b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b), if and only if it is possible to define the values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$. (In the forward direction, first produce candidates for $g(a)$ and $g(b)$ and then show the extended g is continuous.)

Proof.

(a) Pick an arbitrary $\epsilon > 0$.

If f is uniformly continuous, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $x, y \in A$ satisfying $|x - y| \leq \delta$, we have $|f(x) - f(y)| < \epsilon$.

If $(x_n) \subseteq A$ is Cauchy, for all $\delta > 0$, there exists $N \in \mathbf{N}$, such that for all $n > m \geq N$, the distance $|x_n - x_m| < \delta$. So, $|x_n - x_m|$ can be made as small as we please.

Consequently, there exists $N \in \mathbf{N}$, such that for all $n > m > N$, $|f(x_n) - f(x_m)| < \epsilon$. Thus, $(f(x_n))$ is Cauchy.

(b) $\implies$ direction.

We are told that, g is uniformly continuous on (a, b). Pick an arbitrary $\epsilon > 0$. There exists a $\delta > 0$, such that for all $x, y \in (a, b)$ satisfying $|x - y| < \delta$, it follows that $|g(x) - g(y)| < \epsilon$.

Consider an arbitrary sequence $(x_n) \subseteq (a, b)$ such that $(x_n) \to a$. Since (x_n) is convergent, (x_n) is Cauchy. As g is uniformly continuous over (a, b), g preserves Cauchy sequences in (a, b). Consequently, $g(x_n)$ is Cauchy and convergent. Define $g(a) := L = \lim_{x_n \to a} g(x_n)$.

To show that the extended g is indeed continuous on $[a, b]$, we proceed by contradiction. Assume that g is not continuous at a. Then, there exist two sequences (x_n) and (y_n), such that $\lim x_n = \lim y_n = a$, but $\lim g(x_n) \neq \lim g(y_n)$.

$\lim x_n = \lim y_n = a$. For the prescribed δ, there exists $N_1 \in \mathbf{N}$, such that $|x_n - a| < \delta/2$. There exists $N_2 \in \mathbf{N}$ such that $|y_n - a| < \delta/2$. Select $N = \max\{N_1, N_2\}$. Then,

$$\begin{aligned} |x_n - y_n| &= |(x_n - a) - (y_n - a)| \\ &\leq |(x_n - a)| + |(y_n - a)| \\ &= \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

But, from the definition of uniform continuity, for all $n \geq N$, it implies $|g(x_n) - g(y_n)| < \epsilon$. So, $\lim(g(x_n) - g(y_n)) = 0$. Both, $g(x_n)$ and $g(y_n)$ are Cauchy sequences and hence convergent. So, by the Algebraic limit theorem, $\lim(g(x_n) - g(y_n)) = \lim g(x_n) - \lim g(y_n) = 0$. Consequently, $\lim g(x_n) = \lim g(y_n)$.

Hence, our initial assumption is false. g is continuous at a. In a similar fashion, we can show that g is continuous at b.

Define

$$h(x) = \begin{cases} \lim_{x_n \to a} g(x_n), \quad \forall (x_n) \subseteq (a,b) \text{ such that } (x_n) \to a, & \text{if } x = a \\ g(x), \ \text{if } x \in (a,b) \\ \lim_{y_n \to b} g(y_n), \quad \forall (y_n) \subseteq (a,b) \text{ such that } (y_n) \to b, & \text{if } y = b \end{cases}$$

$\Longleftarrow$ direction. If it is possible to define the values $g(a)$ and $g(b)$ at the endpoints, so that the extended function g is continuous at a and b, then g is continuous over a compact set K. A function that is continuous on a compact set is uniformly continuous on K. If g is uniformly continuous on K, it is uniformly continuous over any subset of K including (a,b).