**Problem 1.** Prove that $\sin x$ is continuous on $\mathbf{R}$.

*Proof.* Fix a point $c \in \mathbf{R}$. Pick an arbitrary $\epsilon > 0$. We are interested to make the distance $|f(x) - f(c)|$ as small as we please. Let's explore the expression $|f(x) - f(c)|$. We have:

$$\begin{aligned}
|f(x) - f(c)| &= |\sin(x) - \sin(c)| \\
&= 2\left|\cos\left(\frac{x+c}{2}\right)\sin\left(\frac{x-c}{2}\right)\right| \\
&\leq 2\left|\sin\left(\frac{x-c}{2}\right)\right| && \{\because |\cos\theta| \leq 1\}
\end{aligned}$$

If we replace the expression $|f(x) - f(c)|$ by an upper bound in the inequality $|f(x) - f(c)| < \epsilon$, we strengthen the condition we are interested to prove. So, let's show that:

$$2\left|\sin\left(\frac{x-c}{2}\right)\right| < \epsilon \tag{1}$$

The Maclaurin's series expansion for $f(x) = \sin(x-c)$ is given by,

$$\begin{aligned}
\sin(x-c) &= (x-c) - \frac{(x-c)^3}{3!} + \frac{(x-c)^5}{5!} - \ldots + \\
&= (x-c) - \left[\frac{(x-c)^3}{3!} - \frac{(x-c)^5}{5!} + \ldots\right]
\end{aligned}$$

Assume that $\delta < 1$.
If $(x-c) \geq 0$, then $\sin(x-c) \leq (x-c) \leq |x-c|$.
If $(x-c) < 0$, then $\sin(x-c) \geq (x-c)$. Therefore, $\sin(x-c) \geq -|x-c|$.
Consequently,

$$-(|x-c|) \leq \sin(x-c) \leq (|x-c|)$$

So, $|\sin(x-c)| \leq |(x-c)|$.
So, if we replace $\left|\sin(\frac{x-c}{2})\right|$ by its upper bound $\left|\frac{x-c}{2}\right|$, we again strengthen the condition we want to prove. Therefore, we must prove that:

$$\begin{aligned}
&2\left|\frac{x-c}{2}\right| < \epsilon \\
\iff &\left|\frac{x-c}{2}\right| < \frac{\epsilon}{2}
\end{aligned}$$

Pick $\delta < \frac{\epsilon}{2}$. Then, $|x-c| < \delta$ implies that $|(x-c)/2| < \delta$. This in turn would mean $|f(x) - f(c)| < \epsilon$. Consequently, $f(x)$ is continuous at $c \in \mathbf{R}$. As the choice of $\delta$ does not depend on the point $c$, $f$ is uniformly continuous on $\mathbf{R}$.

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