Functional Limits and Continuity

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Abstract

Solution of the Exercise set 4.4.11

Example 0.1 (Abbot, 4.4.11). Let g be defined on all of $\mathbf{R}$. If B is a subset of $\mathbf{R}$, define the set $g^{-1}(B)$ by

$$g^{-1}(B) = \{x \in \mathbf{R} : g(x) \in B\}$$

Show that g is continuous if and only if $g^{-1}(O)$ is open, whenever $O \subseteq \mathbf{R}$ is an open set.

Proof.

$\Longrightarrow$ direction.

O is an arbitrary open subset of $\mathbf{R}$. Let $y = g(c)$ be an arbitrary point in O

Assume that g is continuous. For all $\epsilon > 0$, there exists $\delta > 0$, such that if $x \in V_\delta(c)$, then $g(x) \in V_\epsilon(g(c))$. Note that, there could be other points in the open interval $(g(c) - \epsilon, g(c) + \epsilon)$, whose pre-image lies in $g^{-1}(O) \setminus V_\delta(c)$. So, $g(V_\delta(c)) \subseteq V_\epsilon(g(c)) \subseteq O$.

g maps everything inside the pre-image $g^{-1}[O]$ to O. Consequently, $V_\delta(c) \subseteq g^{-1}(O)$. As c was arbitrary, this is true for all $c \in g^{-1}[O]$. Consequently, $g^{-1}(O)$ is an open set.

$\Longleftarrow$ direction.

We are told that whenever O is open, then $g^{-1}(O)$ is open.

Pick an arbitrary $\epsilon > 0$. Let c be an arbitrary fixed point in $\mathbf{R}$ and let $y = g(c)$. Consider the open interval $V_\epsilon(y) = (y - \epsilon, y + \epsilon)$.

If O is open, so is the pre-image $g^{-1}(O)$. Considering that $(y-\epsilon, y+\epsilon)$ is open, it's pre-image $X = \{t \in \mathbf{R} : g(t) \in V_\epsilon(y)\}$ is open. Since $c \in X$ and X is open, there exists a δ-neighbourhood around c, $V_\delta(c) = (c - \delta, c + \delta) \subseteq X$ such that $g(V_\delta(c)) \subseteq V_\epsilon(g(c))$.

Pick an arbitrary point $t \in V_\delta(x)$. Then, we are guaranteed that $g(t) \in V_\epsilon(y)$. So, g is continuous at all $c \in \mathbf{R}$.