Supplementary Note

Shannon Entropy as a Path Integral of SKA Entropy Along the Sigmoid Trajectory

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Abstract

This note provides a compact derivation showing that Shannon's binary entropy arises as a path integral of the SKA local entropy differential when the decision field is linked to the knowledge variable by the sigmoid map. Specifically, the SKA entropy increment is a differential 1-form proportional to $-z\,dD$, and under the sigmoid trajectory $D = \sigma(z)$ its integral admits a closed-form primitive equal (up to a constant) to Shannon's binary entropy.This establishes a precise bridge between an SKA learning-trajectory formulation and classical information measures.

1 Setup and Notation

Let $z \in \mathbb{R}$ denote a (scalar) knowledge variable and let the decision probability be produced by a sigmoid:

$$D = \sigma(z) \equiv \frac{1}{1+e^{-z}} \in (0,1). \tag{1}$$

The inverse map is the logit,

$$z = \log \frac{D}{1-D}. \tag{2}$$

In SKA, the local entropy density is introduced in differential form as the alignment between knowledge and decision change. In the simplest scalar setting, the infinitesimal entropy increment (per learning step / along a trajectory) is

$$dh \;\equiv\; -\frac{1}{\ln 2}\, z\, dD. \tag{3}$$

In vector form, for $z \in \mathbb{R}^n$ and $D \in (0,1)^n$,

$$dh \;\equiv\; -\frac{1}{\ln 2}\, z \cdot dD \;=\; -\frac{1}{\ln 2} \sum_{i=1}^{n} z_i\, dD_i. \tag{4}$$

This note focuses first on the scalar case, then states the extension.

2 Statement of the Result

Claim. Along the sigmoid trajectory $D = \sigma(z)$, the path integral of the SKA entropy differential (3) yields Shannon's *binary entropy* (up to an additive constant):

$$H_{\mathrm{bin}}(D) \;=\; -\frac{1}{\ln 2}\Big(D \ln D + (1-D)\ln(1-D)\Big), \tag{5}$$

so that

$$\boxed{\int_{\gamma:\, D=\sigma(z)} dh \;=\; H_{\mathrm{bin}}(D) \;+\; \text{const.}} \tag{6}$$

3 Derivation

Starting from the SKA entropy differential (3),

$$dh = -\frac{1}{\ln 2}\, z\, dD, \tag{7}$$

and using the sigmoid inverse (2),

$$dh = -\frac{1}{\ln 2}\, \log \frac{D}{1-D}\, dD. \tag{8}$$

We now integrate (8) with respect to D (i.e., along the trajectory induced by the sigmoid). Define the primitive

$$\mathcal{H}(D) \;\equiv\; -\frac{1}{\ln 2} \int \log \frac{D}{1-D}\, dD. \tag{9}$$

Split the logarithm:

$$\log \frac{D}{1-D} = \ln D - \ln(1-D), \tag{10}$$

so

$$\int \log \frac{D}{1-D}\, dD = \int \ln D\, dD \;-\; \int \ln(1-D)\, dD. \tag{11}$$

Compute each term:

$$\int \ln D\, dD = D \ln D - D + C_1, \tag{12}$$

$$\int \ln(1-D)\, dD = -(1-D)\ln(1-D) + (1-D) + C_2, \tag{13}$$

hence

$$\int \log \frac{D}{1-D}\, dD = \Big(D \ln D + (1-D)\ln(1-D)\Big) + C, \tag{14}$$

where constants and linear terms combine into a single constant C because the $-D$ and $+(1-D)$ cancel up to a constant.

Substituting into (9) gives

$$\mathcal{H}(D) = -\frac{1}{\ln 2}\Big(D \ln D + (1-D)\ln(1-D)\Big) + \text{const}. \tag{15}$$

Choosing the constant so that $\mathcal{H}(0) = \mathcal{H}(1) = 0$ yields precisely the Shannon binary entropy (in bits) in (5), establishing (6).

4 Interpretation: "Entropy as a Path Integral"

Equation (3) shows that SKA entropy is naturally expressed as a *differential* quantity:

$$dh \propto -z\, dD. \tag{16}$$

The sigmoid (1) selects a distinguished trajectory in decision space by enforcing a specific relationship between internal knowledge z and decision probability D. Under this constraint, the SKA differential 1-form becomes exact and admits the closed-form state function (5). In this sense, Shannon entropy emerges as the integrated trace of the SKA entropy differential along the sigmoid-defined learning curve.

5 Vector Extension (Component-wise Binary Entropy)

For $z \in \mathbb{R}^n$ and $D \in (0,1)^n$ with independent sigmoid components

$$D_i = \sigma(z_i) \quad \Leftrightarrow \quad z_i = \log \frac{D_i}{1 - D_i}, \tag{17}$$

the SKA entropy differential (4) becomes

$$dh = -\frac{1}{\ln 2} \sum_{i=1}^{n} \log \frac{D_i}{1 - D_i} \, dD_i. \tag{18}$$

Integrating component-wise yields

$$H(D) \; = \; \sum_{i=1}^{n} H_{\mathrm{bin}}(D_i) \; = \; -\frac{1}{\ln 2} \sum_{i=1}^{n} \Big(D_i \ln D_i + (1 - D_i) \ln(1 - D_i) \Big), \tag{19}$$

again up to an additive constant fixed by boundary conditions.

6 Remarks and Limits

1. **Units.** The factor $1/\ln 2$ converts natural logarithms (nats) into bits.
2. **Boundary constants.** The integral produces Shannon entropy up to an additive constant; choosing $H(0) = H(1) = 0$ fixes it.
3. **Why sigmoid matters.** The derivation uses only the logit identity (2). Other decision maps would generate different primitives, i.e., different entropy-like state functions.
4. **SKA viewpoint.** In this formulation, Shannon entropy is not postulated by axioms; it appears as the primitive of a trajectory-level SKA entropy differential constrained by a bounded decision nonlinearity.

7 Conclusion

Under the sigmoid decision map, the SKA entropy differential $dh = -(1/\ln 2)\, z \, dD$ integrates to Shannon's binary entropy. This provides a direct mathematical bridge between SKA's trajectory-based entropy formulation and classical information measures, supporting the interpretation that Shannon entropy can be viewed as a path integral along the knowledge trajectory induced by the sigmoid.

Keywords: SKA, sigmoid, logit, Shannon entropy, path integral, learning trajectory.