Structured Knowledge Accumulation Framework: Real-Time Entropy-Driven Regime Detection in Market Microstructure

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Abstract

This paper introduces a quantum-inspired, information-theoretic formalism for real-time, entropy-driven regime detection in financial markets within the Structured Knowledge Accumulation (SKA) framework. The market state is represented as a superposition vector in a complex Hilbert space, and its evolution is governed by a density operator that encodes both regime probabilities and structural transitions. Each complex amplitude carries an entropy phase derived from an SKA Lagrangian, linking probabilistic transitions to the dynamics of information accumulation.

Empirically, we show that transition probabilities between bull, bear, and neutral regimes remain approximately constant across thousands of trades, even as prices fluctuate significantly. This path-independence validates a central prediction of SKA: regime dynamics depend on the *velocity* of information accumulation, not on position in price space. This sharply contrasts with classical regime-switching and hidden Markov models, in which transitions are typically parametrized by price-level or volatility states.

The proposed formalism provides a rigorous theoretical foundation for real-time analytics, entropy-based decision systems, and market structure decoding, and it supports practical implementations across SQL querying, live dashboards, and agent-based forecasting pipelines.

1 Introduction

The Structured Knowledge Accumulation (SKA) framework was originally developed as a forward-only, entropy-driven theory of neural learning [**?**, **?**]. In this work, we extend SKA to real-time market regime detection and show that financial markets exhibit the same entropy-based knowledge accumulation dynamics as neural networks.

Traditional models of financial markets rely on discretized time series, deterministic indicators, or Markovian assumptions to represent regime changes. Such approaches often overlook the *underlying information structure* and fail to capture the continuity and coherence inherent in real-time transitions. Regime shifts are typically modeled as discrete jumps between latent states whose transition probabilities depend on price levels, volatility, or other macro-variables.

The SKA framework redefines this paradigm by treating the market as a *dynamical information system*, in which regime states exist in a superposed form and evolve through entropy-driven learning. Inspired by quantum mechanics, the present formalism introduces Dirac notation to express regime states, transition amplitudes, and entropy phases in a unified mathematical structure.

By modeling the market state as a vector in a complex Hilbert space, and transitions as projections within this space, we obtain a *density operator* that simultaneously encodes probabilities and structural couplings. This representation allows for a natural integration of real-time learning, entropy flow, and non-Markovian memory effects, opening the door to a fundamentally new class of market models in which structure emerges from the geometry of information accumulation rather than from exogenously specified switching rules.

This document lays the foundation for this approach. We define each component of the model, derive its functional properties, and connect the formalism to empirical signatures in market microstructure. The resulting framework is directly usable in analytics pipelines, signal processing, and algorithmic decision systems, including implementations in SQL, real-time dashboards, and SKA-based agents.

2 State Space Definition

Let $\mathcal{H}$ be a **3-dimensional complex Hilbert space** with orthonormal basis:

$$\mathcal{B} = \{|0\rangle, |1\rangle, |2\rangle\} \tag{1}$$

Interpretation: Basis vectors represent mutually exclusive, exhaustive regimes.

3 Market State Vector

At any time t, the market is in a superposition of basis regimes:

$$|\Psi(t)\rangle = \sum_{i=0}^{2} \Psi_i(t) |i\rangle, \quad \Psi_i(t) \in \mathbb{C} \tag{2}$$

$$|\Psi(t)\rangle = \Psi_0(t) |0\rangle + \Psi_1(t) |1\rangle + \Psi_2(t) |2\rangle \tag{3}$$

where:

- $|0\rangle, |1\rangle, |2\rangle$: Basis vectors for Neutral, Bull, Bear.
- $\Psi_i(t) = A_i(t)e^{iH_i(t)} \in \mathbb{C}$: Complex amplitude encoding probability and entropy.

In the Structured Knowledge Accumulation (SKA) framework, the market is not confined to a single regime (e.g., bull, bear, or neutral) at any moment. Instead, it exists in a **superposition** of all possible regimes, with complex amplitudes $\Psi_i(t)$ encoding both probability and entropy.

Traditional models assume the market flips from one regime to another—an abrupt, externally triggered event. SKA instead reveals a continuous evolution through overlapping informational states.

3.1 Classical vs. SKA Assumptions

Classical Assumption:

- Market is always *in* one regime.
- Transitions are discrete.
- Probabilities are assigned *after* classification.

SKA View:

- Market is always *between* regimes.
- Transitions emerge from entropy geometry.
- Probabilities arise from the evolving wavefunction $|\Psi(t)\rangle$.

3.2 Implications

- **Why classical models fail:** they discretize what is fundamentally smooth and entangled.
- **Why transitions seem random:** because we are projecting a superposed state onto a single outcome.
- **Why micro-patterns exist at the tick level:** because phase coherence in $|\Psi(t)\rangle$ encodes deep structural information—**even when the price appears flat**.

SKA does not model the *outcome* of the market. It models the *informational field* from which outcomes emerge.

3.3 Normalization

For any state $|\Psi\rangle \in \mathcal{H}$, $\langle\Psi|\Psi\rangle = 1$:

$$\langle\Psi(t)|\Psi(t)\rangle = |\Psi_0|^2 + |\Psi_1|^2 + |\Psi_2|^2 = 1 \tag{4}$$

4 Density Operator

The **density operator** is the outer product of the state vector with itself:

$$\hat{\rho}(t) = |\Psi(t)\rangle \langle\Psi(t)| \tag{5}$$

This object captures both **probabilities** and **structural relationships** (coherences) between regimes.

4.1 Matrix Elements

$$\langle i| \hat{\rho}(t) |j\rangle = \Psi_i^*(t)\Psi_j(t) \tag{6}$$

This yields a 3×3 Hermitian matrix:

$$\hat{\rho}(t) = \begin{pmatrix} |\Psi_0|^2 & \Psi_0^*\Psi_1 & \Psi_0^*\Psi_2 \\ \Psi_1^*\Psi_0 & |\Psi_1|^2 & \Psi_1^*\Psi_2 \\ \Psi_2^*\Psi_0 & \Psi_2^*\Psi_1 & |\Psi_2|^2 \end{pmatrix} \tag{7}$$

4.2 Interpretation

- **Diagonal terms**: $\rho_{ii} = |\Psi_i|^2$ = probability of being in regime i.
- **Off-diagonal terms**: $\rho_{ij} = \Psi_i^*\Psi_j$ = coherence between regimes i and j, interpreted as transition structure.

5 Structural Transition Operator

A transition from regime $j \to i$ is represented by:

$$\hat{T}_{i\leftarrow j} = |i\rangle \langle j| \tag{8}$$

This operator transforms basis state $|j\rangle$ into $|i\rangle$.

5.1 Expectation Value

To extract the amplitude for a structural transition from $j \to i$:

$$\langle\Psi|\,\hat{T}_{i\leftarrow j}\,|\Psi\rangle = \Psi_i^*(t)\Psi_j(t) \tag{9}$$

This equals the (i, j) entry of the density matrix.

5.2 Constant Structural Coupling C_{ij}

In practice, the most observable market regularities emerge when structural transition amplitudes become **time-invariant**:

$$C_{ij} = \Psi_i^*(t)\Psi_j(t) = \text{const.} \tag{10}$$

This condition implies a stable **directional flow of probability** between regimes i and j.

> **SKA Insight.** The structural transition amplitude $C_{ij} = \Psi_i^*(t)\Psi_j(t)$ emerges *only through Real-Time SKA Learning.* It reflects an acquired informational symmetry, not a statistical feature of the raw market data.

Empirical confirmation: Figure 1 demonstrates this constancy empirically. The transition probabilities (derived from $|C_{ij}|^2$) maintain stable values across ~ 2500 trades:

Transition	**Observed Probability**
bull→bull (0.2%)	≈ 0.998556
bear→bear (0.0%)	≈ 0.0
neutral→bear (5.2%)	≈ 0.848995
neutral→neutral (76.8%)	≈ 0.995264

These stable values persist even as price fluctuates significantly (lower panel of Figure 1), demonstrating that C_{ij} is indeed approximately constant and independent of price position—a key prediction of the SKA framework.

When C_{ij} remains constant:

- Regime transitions align clearly in real-time analysis.
- Entropy flow across regimes becomes **symmetric** or **balanced**.
- Paired transitions appear as coherent oscillations.

This stability reveals that markets operate within a **coherent informational geometry** rather than a stochastic regime-flip process.

6 Entropy Interpretation

Each complex amplitude $\Psi_i(t)$ embeds an **entropy phase**:

$$\Psi_i(t) = A_i(t)e^{iH_i(t)}. \tag{11}$$

Here, $A_i(t)$ represents the amplitude, and $H_i(t)$ is the entropy associated with regime i at time t.

The entropy $H_i(t)$ comes from the **Real-Time SKA learning process** as it accumulates over the trajectory of market evolution.

We express the entropy as a functional of the Lagrangian [24, 25]:

$$H_i(t) = \frac{1}{\ln 2} \int_0^t \mathcal{L}_i(s)\, ds, \tag{12}$$

where $\mathcal{L}_i(s)$ is the Lagrangian.

This phase represents the **information content** (entropy) of regime $|i\rangle$. Therefore, transitions (via $\Psi_i^* \Psi_j$) carry entropy shifts $H_j - H_i$.

Entropy Normalization. The SKA entropy phase is normalized by the factor $1/\ln 2$, ensuring that the quantity $H_i(t)$ is dimensionless and expressed in units of bits rather than nats. This normalization is essential for interpreting the complex exponential $\exp(iH_i(t))$ as a genuine geometric phase. Without this scaling, the entropy integral would grow at a rate incompatible with stable coherence, causing the density-matrix elements $\Psi_i^* \Psi_j$ to lose structural meaning. The factor $\ln 2$ plays the role of an informational normalization constant, analogous to $\hbar$ in physical phase dynamics, allowing SKA learning trajectories to be compared across assets, time scales, and sampling frequencies.

Entropy-Normalized Transition Bands. In Figure 1, the quantity

$$B_{ij}(t) = \exp\left(- \left| \frac{H_j(t) - H_i(t)}{H_j(t)} \right| \right)$$

represents the distribution of regime transitions between i and j along the sequence of trades. As the entropy values $H_i(t)$ and $H_j(t)$ evolve during the learning process, the distribution encoded in $B_{ij}(t)$ changes accordingly, providing a direct visualization of how the transitions between the two regimes unfold over time.

Because the model contains three regimes (bull, bear, neutral), there are nine possible transitions in total. The quantity $B_{ij}(t)$ is therefore a rich distribution over these nine regime transitions, showing how each transition evolves along the sequence of trades. It is not a probability, but a distributional measure that captures the time-dependent structure of how transitions between regimes unfold.

7 Functional Derivatives and Physical Interpretation

To understand how the SKA wave function responds to variations in knowledge trajectories, we examine the functional derivatives of the regime amplitudes with respect to both position and velocity in knowledge space. The complete mathematical derivations are provided in Annex A.

7.1 Position Derivative: Knowledge Path Independence

The SKA framework defines each regime amplitude as:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2} \int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds \right).$$

The functional derivative with respect to position yields (see Annex A for derivation):

$$\frac{\delta \Psi_i}{\delta z_i(s)} = \frac{i\Psi_i}{\ln 2} \cdot \left(\frac{\partial \mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}_i}{\partial z_i'} \right) \right).$$

In SKA, the Lagrangian $\mathcal{L}_i$ is constructed such that the Euler–Lagrange equation is identically satisfied:

$$\frac{\partial \mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}_i}{\partial z_i'} \right) \equiv 0,$$

which gives

$$\frac{\delta \Psi_i}{\delta z_i(s)} \equiv 0.$$

7.1.1 What This Means in Plain Terms

- Small changes in the knowledge path $z_i(s)$, which reflects cumulative price information, do not affect the regime amplitude $\Psi_i(t)$.
- Every possible trajectory is already entropy-optimal—no path is more "efficient" than another.
- There is no need for the system to "search" for the best transition—it follows the natural flow of information.

7.1.2 Financial Interpretation

Market regimes (bull, bear, neutral) self-organize from the price stream. They do not require external filters, smoothing, or optimization.

This is why regime transitions in SKA appear stable and coherent: the underlying probabilities remain constant even as price evolves.

7.1.3 Empirical Validation

The path independence predicted by $\delta\Psi_i/\delta z_i(s) \equiv 0$ is directly observable in Figure 1. The transition probabilities (upper panel) remain stable at approximately constant values while the price trajectory (lower panel) moves through various levels. This decoupling demonstrates that:

1. Regime amplitudes do not depend on where the price is, but rather on *how information is flowing*.
2. The system is not "searching" for optimal transitions—they emerge naturally from the information geometry.
3. Multiple price paths can lead to the same regime structure, confirming the path-independence property.

This empirical result distinguishes SKA from threshold-based models (which trigger regime changes at specific price levels) and from models where transition probabilities depend on price state variables.

7.2 Velocity Derivative: Learning Speed Sensitivity

The functional derivative with respect to velocity reveals a fundamentally different behavior (see Annex A for derivation):

$$\frac{\delta \Psi_i}{\delta z_i'(s)} = -\frac{i\Psi_i}{\ln 2} z_i(s)\sigma(z_i(s))(1 - \sigma(z_i(s))).$$

7.2.1 Key Differences from Position Derivative

1. Non-zero result: Unlike the derivative with respect to $z_i(s)$, which is identically zero, this derivative is **non-zero**.

2. Physical interpretation:

- $\delta\Psi_i/\delta z_i(s) = 0$: Regime amplitude is insensitive to **position** changes in the knowledge trajectory.
- $\delta\Psi_i/\delta z_i'(s) \neq 0$: Regime amplitude is sensitive to **velocity** changes in the knowledge trajectory.

3. Market dynamics meaning:

- $z_i'(s)$: Rate of knowledge change (learning velocity).
- Non-zero derivative: Regime amplitudes respond to changes in learning speed.
- The faster knowledge accumulates, the stronger the regime amplitude response.

7.2.2 Physical Significance

Learning velocity sensitivity:

The market regime amplitudes are sensitive to the rate of information processing, not just the accumulated knowledge level.

Market interpretation:

- **High $z_i'(s)$:** Rapid information processing → strong regime amplitude response.
- **Low $z_i'(s)$:** Slow information processing → weak regime amplitude response.
- The derivative quantifies how regime coherence depends on learning dynamics.

Real-Time SKA Insight:

This explains why Real-Time SKA is crucial: the regime amplitudes depend on the velocity of knowledge accumulation, which can only be captured through continuous learning dynamics, not batch analysis.

The functional derivative with respect to velocity reveals the dynamic responsiveness of market regimes to information flow rates.

7.2.3 Second-Order Velocity Derivative: Harmonic Structure in Velocity Space

The second functional derivative with respect to velocity, evaluated locally (holding knowledge z fixed under the variational convention), takes the compact form

$$\frac{\delta^2\Psi(z,z')}{\delta z'^2} + \Omega(z)^2\,\Psi(z,z') = 0,$$

where we define

$$\Omega(z) = \frac{1}{\ln 2}\, z\,\sigma(z)\,(1-\sigma(z)).$$

This harmonic identity reveals that velocity perturbations induce quadratic restoring behavior proportional to the instantaneous information accumulation rate $\Omega(z)$. In saturated regimes ($\Omega(z) \approx 0$), higher-order velocity sensitivity vanishes, while active learning zones amplify structural coherence through this oscillator-like dynamics in velocity space.

8 Transition Operator Action on the State

We consider the global transition operator (density operator):

$$\hat{T} = |\Psi\rangle \langle\Psi| . \tag{13}$$

We project this operator, acting on the state, back onto the basis state $|i\rangle$:

$$\langle i| \hat{T} |\Psi\rangle = \langle i|\Psi\rangle = \sum_j \Psi_j \langle i|j\rangle = \sum_j \Psi_j \delta_{ij} = \Psi_i. \tag{14}$$

Here we used the orthonormality of the basis: $\langle i|j\rangle = \delta_{ij}$, the Kronecker delta.
Since $\hat{T} |\Psi\rangle = |\Psi\rangle$, the operator acts as a projector and leaves the state invariant. Therefore:

$$\langle 0| \hat{T} |\Psi\rangle = \Psi_0, \tag{15}$$

$$\langle 1| \hat{T} |\Psi\rangle = \Psi_1, \tag{16}$$

$$\langle 2| \hat{T} |\Psi\rangle = \Psi_2. \tag{17}$$

This shows that transitions are not events but projections. The market does not "switch" regimes; our measurement collapses the superposition.
This means:

- Regime changes are observation-dependent.
- The act of trading affects the state.
- An observer effect is present in markets.

The transition operator encodes the full informational identity of the market state and reflects that transitions are emergent properties of the SKA state, not discrete external events.
Upon measurement, the superposed market state $|\Psi\rangle$ collapses to a definite regime $|i\rangle$ with probability $|\Psi_i|^2$, revealing the observed market condition.

9 Summary of Core Equations

Concept	**Formula**
State vector	$\|\Psi\rangle = \sum_i \Psi_i \|i\rangle$
Density operator	$\hat{\rho} = \|\Psi\rangle \langle\Psi\|$
Regime probability	$\rho_{ii} = \|\Psi_i\|^2$
Structural transition	$\hat{T}_{i\leftarrow j} = \|i\rangle \langle j\|$
Transition amplitude	$\langle i\| \hat{\rho} \|j\rangle = \Psi_i^* \Psi_j$
Entropy phase	$\arg(\Psi_i) = H_i(t)$

Table 1: Core equations of the SKA–Dirac formalism.

10 Obtained Results

10.1 Constancy of the Amplitude A_i and Empirical Justification

In the SKA framework, the complex knowledge amplitude for regime i is given by:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2} \int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds \right), \tag{18}$$

where A_i is a regime-specific constant amplitude, and the exponential encodes the entropy-driven phase.

Justification for Constancy of A_i

We assume that A_i remains constant with respect to $z_i(s)$, $z_i'(s)$, and s. This assumption is justified on the following grounds:

- **Structural identity:** A_i encodes the fixed structural identity of regime i, independent of entropy flow. It represents the initial capacity of the regime to encode knowledge.
- **Functional derivative consistency:** The constancy of A_i ensures that functional derivatives such as $\delta\Psi_i/\delta z_i(s)$ align with the Euler–Lagrange formulation without introducing extraneous terms.
- **Separation of dynamics:** All entropy-driven learning occurs in the exponential phase. Keeping A_i constant cleanly separates *initial belief* from *real-time knowledge accumulation.*
- **Physical analogy:** In analogy with quantum systems, the modulus of the wavefunction remains fixed while the phase evolves. Similarly, A_i remains static, while entropy governs the dynamics.

Empirical Confirmation

Empirical data from real-time financial markets confirm this modeling choice. The following figures demonstrate that the market itself behaves as a real-time entropy learner, consistent with the SKA framework:

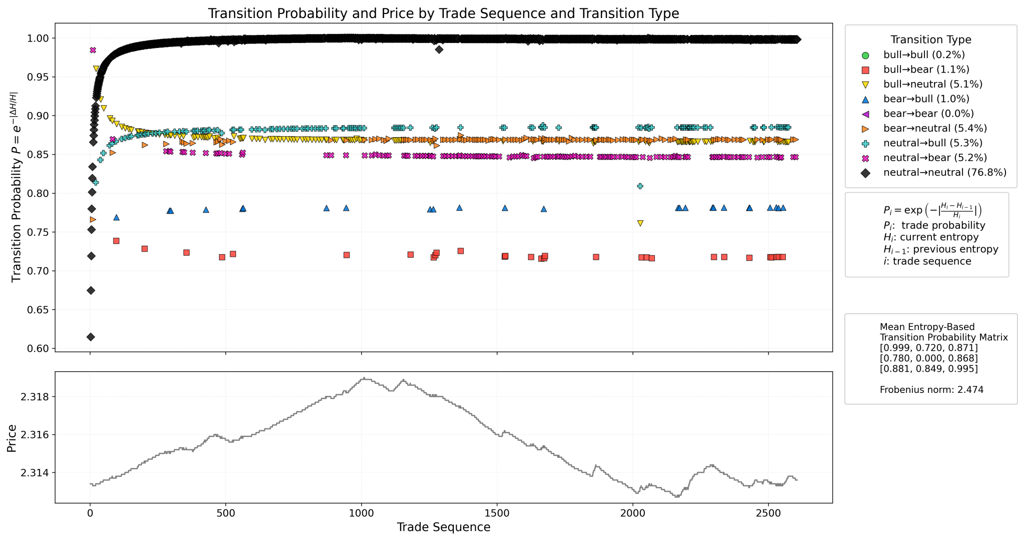

Figure 1: Real-time evolution of regime distribution aligned with price trajectory. The market dynamically adjusts regime probabilities in response to price movements, mimicking SKA learning.

Regime Cycling Structure in Figure 1

This figures illustrates a key market microstructure insight:

Uptrends are not composed of persistent bull→bull transitions. Instead, they are driven by rapid alternation between neutral→bull and bull→neutral transitions.

This reveals that market trends are fundamentally constructed from paired regime transitions—specifically, the rapid cycling between neutral→bull and bull→neutral (for uptrends), or neutral→bear and bear→neutral (for downtrends). These paired transitions, rather than persistence in a single state, are the true information-processing units of market trend.

Breakthrough. The regime cycling structure shown below is extracted purely from entropy learning—not from price or state aggregation. The alignment between these entropy-driven regime transitions and actual price movements offers concrete visual proof that markets operate under hidden informational laws, not just random walks. This provides quants and researchers with a glimpse into the next generation of market analysis.

SKA Transition Probability Figure.

Interpretation.

- During price uptrends (see bottom plot), the transitions with highest frequency are neutral→bull and bull→neutral.
- Persistent "bull→bull" transitions are rare (as confirmed in the legend).
- SKA's entropy-driven, trade-by-trade visualization makes this market regime cycling explicit—revealing a universal information-processing law of market behavior.
- The same cycling phenomenon is observed during downtrends, where rapid alternation between neutral→bear and bear→neutral transitions dominates the price movement.

Empirical foundation. This result powerfully reinforces the SKA entropy definition. The learned regime transitions—derived purely from entropy dynamics—align so closely with price evolution that SKA's information-theoretic approach clearly captures the true underlying structure of market behavior. In other words, SKA's entropy is not just a mathematical construct—it describes real, observable laws that govern how markets process information and evolve.

Empirical Validation of Path Independence

Figure 1 provides direct empirical evidence for the theoretical prediction of path independence (Section 7.1). Observe that:

Table 2: Mean Entropy-Based Transition Probability Matrix

State	**bull**	**bear**	**neutral**
bull	0.998556	0.719790	0.870663
bear	0.780167	0.000000	0.867837
neutral	0.880761	0.848995	0.995264

- **Price trajectory** (lower panel) exhibits significant fluctuations, moving through multiple local maxima and minima during the same period.
- **Critical observation**: The transition probabilities remain stable *despite* price movements. They do not track or correlate with the price level changes.

This independence between transition probabilities and price position directly confirms the functional derivative result $\delta\Psi_i/\delta z_i(s) \equiv 0$: regime amplitudes are insensitive to the specific knowledge path $z_i(s)$ (cumulative price information), as predicted by the Euler–Lagrange identity.

Contrast with classical models: Traditional regime-switching models would typically show transition probabilities that vary with price levels, volatility, or momentum indicators. The SKA framework predicts—and Figure 1 demonstrates—that these probabilities depend on the *velocity* of information accumulation (z_i'), not position (z_i).

This empirical observation validates one of the fundamental predictions distinguishing SKA from Markov-switching and HMM approaches.

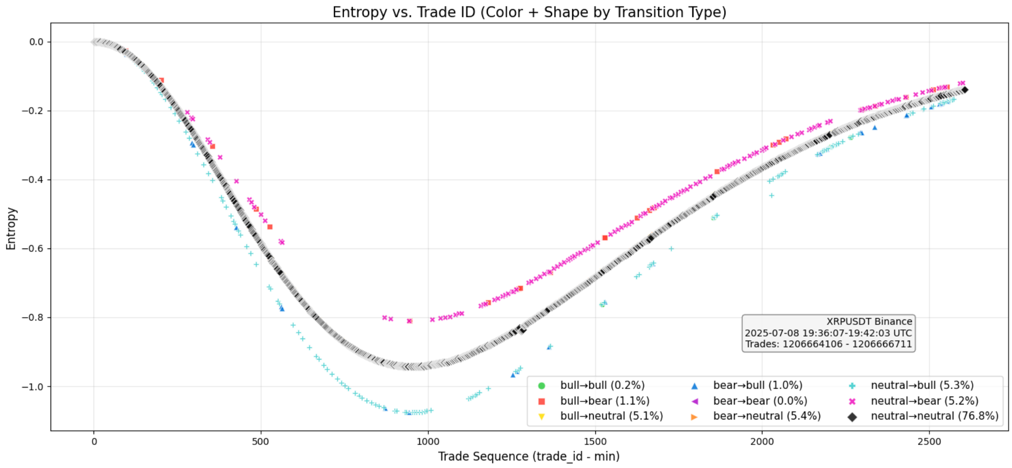

Figure 2: Entropy trajectory versus trade ID. The progressive entropy reduction indicates real-time knowledge accumulation by the market. This confirms that the financial market and SKA share the same learning engine.

Conclusion

The constancy of A_i is not only mathematically required for clean derivations but also empirically validated by observing how the market learns in real time. The financial market and SKA both operate through an entropy-minimizing, knowledge-accumulating process: they use the same learning engine.

11 Applications and Usage

This document serves as the **formal reference** for SKA modules that:

- Visualize transitions in QuestDB or Grafana,
- Compute entropy-induced transition probabilities,
- Apply the principle of least entropy action.

Refer to this file when defining:

- Structural probability matrices,
- Real-time SKA signal decoders,
- Quantum-inspired simulations of regime evolution.

12 Conclusion

This formalism supports a deeper view of the market as an **information-processing system**, not a sequence of discrete events. Transitions are not externally triggered but emerge from the **geometry of knowledge accumulation** encoded in the SKA state.

The quantum-inspired approach reveals that traditional discrete regime models fundamentally misrepresent market dynamics. By treating markets as superposed information systems evolving through entropy-driven learning, we obtain a mathematically rigorous framework that captures both probabilistic and structural aspects of regime transitions.

The Real-Time SKA learning process, integrated through Lagrangian dynamics, provides a principled approach to understanding how markets process information and evolve through different phases. This has immediate applications in risk management, algorithmic trading, and financial system stability analysis.

The functional derivative analysis reveals a profound insight: while market regime amplitudes are insensitive to the specific knowledge path taken (position independence), they are highly sensitive to the velocity of knowledge accumulation. This mathematical result explains why real-time learning systems can detect regime patterns that remain invisible to batch analysis, establishing the theoretical foundation for the necessity of continuous information processing in financial market modeling.

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A Mathematical Development of Functional Derivatives

This annex provides the complete mathematical derivations of the functional derivatives discussed in Section 7. The calculations demonstrate how the SKA wave function amplitudes respond to variations in knowledge trajectories.

We assume that the initial amplitude A_i is constant. This reflects the fixed structural identity of regime i, independent of the entropy phase dynamics that govern knowledge accumulation. Constancy of A_i ensures compatibility with the Euler–Lagrange formulation and preserves the coherence of regime transitions across time. In this framework, all learning is embedded in the entropy-driven exponential phase.

A.1 Functional Derivative with Respect to Position: $\delta\Psi_i/\delta z_i(s)$

A.1.1 Step-by-Step Derivation

1. Starting Expression Given the complex amplitude with entropy phase:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2}\int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds\right)$$

2. Define the Phase Function Let the cumulative entropy phase be:

$$H_i(t) = \frac{1}{\ln 2}\int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds,$$

so that $\Psi_i(t) = A_i \exp(iH_i(t))$.

3. Apply Functional Derivative

$$\frac{\delta\Psi_i}{\delta z_i(s)} = \frac{\delta}{\delta z_i(s)}[A_i \exp(iH_i(t))].$$

Using the chain rule for functional derivatives:

$$\frac{\delta\Psi_i}{\delta z_i(s)} = A_i \exp(iH_i(t)) \times i \times \frac{\delta H_i(t)}{\delta z_i(s)} = i\Psi_i \times \frac{\delta H_i(t)}{\delta z_i(s)}.$$

4. Compute $\delta H_i(t)/\delta z_i(s)$ The functional derivative of the entropy integral is

$$\frac{\delta H_i(t)}{\delta z_i(s)} = \frac{\delta}{\delta z_i(s)}\left[\frac{1}{\ln 2}\int_0^t \mathcal{L}_i(z_i(\tau), z_i'(\tau))\, d\tau\right].$$

Using the **fundamental theorem of calculus of variations**:

$$\frac{\delta}{\delta z_i(s)}\int_0^t \mathcal{L}_i(z_i(\tau), z_i'(\tau))\, d\tau = \left.\frac{\partial\mathcal{L}_i}{\partial z_i}\right|_s - \frac{d}{ds}\left(\frac{\partial\mathcal{L}_i}{\partial z_i'}\right)\Bigg|_s,$$

and therefore

$$\frac{\delta H_i(t)}{\delta z_i(s)} = \frac{1}{\ln 2}\left[\frac{\partial\mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}_i}{\partial z_i'}\right)\right]\Bigg|_s.$$

5. Final Result

$$\boxed{\frac{\delta\Psi_i}{\delta z_i(s)} = \frac{i\Psi_i}{\ln 2}\left[\frac{\partial\mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}_i}{\partial z_i'}\right)\right]\Bigg|_s}$$

A.2 Functional Derivative with Respect to Velocity: $\delta\Psi_i/\delta z_i'(s)$

A.2.1 Step-by-Step Derivation

1. Starting Expression Again:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2}\int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds\right).$$

2. Apply Functional Derivative with Respect to $z_i'(s)$:

$$\frac{\delta\Psi_i}{\delta z_i'(s)} = i\Psi_i \times \frac{\delta H_i(t)}{\delta z_i'(s)}.$$

3. Compute $\delta H_i(t)/\delta z_i'(s)$: Using the **fundamental theorem of calculus of variations** for the velocity term:

$$\frac{\delta}{\delta z_i'(s)} \int_0^t \mathcal{L}_i(z_i(\tau), z_i'(\tau))\, d\tau = \left.\frac{\partial \mathcal{L}_i}{\partial z_i'}\right|_s .$$

Note: Unlike the derivative with respect to $z_i(s)$, there is **no** d/ds term here.

Therefore:

$$\frac{\delta H_i(t)}{\delta z_i'(s)} = \frac{1}{\ln 2} \left.\frac{\partial \mathcal{L}_i}{\partial z_i'}\right|_s .$$

4. Final Result:

$$\boxed{\frac{\delta \Psi_i}{\delta z_i'(s)} = \frac{i\Psi_i}{\ln 2} \left.\frac{\partial \mathcal{L}_i}{\partial z_i'}\right|_s}$$

A.3 Connection to SKA Lagrangian

A.3.1 SKA Lagrangian

From the SKA framework, the Lagrangian is:

$$\mathcal{L}_i(z_i, z_i', t) = -z_i \cdot \sigma(z_i)(1 - \sigma(z_i)) \cdot z_i'.$$

A.3.2 Euler–Lagrange Result

When the detailed partial derivative calculations are performed (as shown in the SKA neural paper), the Euler–Lagrange equation yields:

$$\frac{\partial \mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}_i}{\partial z_i'}\right) \equiv 0,$$

which is an **identity**—the equation is satisfied for all possible knowledge trajectories $z_i(s)$, demonstrating the natural optimality of SKA dynamics.

A.3.3 Compute $\partial \mathcal{L}_i/\partial z_i'$:

$$\frac{\partial \mathcal{L}_i}{\partial z_i'} = -z_i \cdot \sigma(z_i)(1 - \sigma(z_i)).$$

A.3.4 Substitute Back:

$$\frac{\delta \Psi_i}{\delta z_i'(s)} = \frac{i\Psi_i}{\ln 2} \cdot \left(-z_i(s) \cdot \sigma(z_i(s))(1 - \sigma(z_i(s)))\right),$$

so that

$$\boxed{\frac{\delta \Psi_i}{\delta z_i'(s)} = -\frac{i\Psi_i}{\ln 2} z_i(s)\sigma(z_i(s))(1 - \sigma(z_i(s)))}.$$

Appendix B — Consistency Check of the SKA–Dirac Formalism

This appendix provides a concise mathematical justification of the core constructs used in the SKA–Dirac formalism, demonstrating internal consistency and the correct use of Lagrangian-based information dynamics in a financial context.

Short-Time SKA Propagator

Before introducing the action–phase structure of the SKA amplitude, we begin with the short-time transition bracket between regime $|j,t\rangle$ and $|i,t+\Delta t\rangle$. We start from the standard Feynman short-time propagator:

$$\langle i,\, t+\Delta t \,|\, j,\, t\rangle = w\;\exp\!\left(\frac{i}{\hbar}\int_t^{t+\Delta t} L(z,\dot z)\,dt\right)$$

To obtain its SKA counterpart, we apply three substitutions:

$$\hbar \to \ln 2, \qquad L(z,\dot z) \to L_i(s), \qquad w \to C_{ij}, \tag{19}$$

where C_{ij} is the structural coupling between regimes j and i.

This yields the SKA propagator:

$$\langle i,\, t+\Delta t \,|\, j,\, t\rangle = C_{ij}\;\exp\!\left(\frac{i}{\ln 2}\int_t^{t+\Delta t} L_i(s)\,ds\right) \tag{20}$$

Using the identity $\int_t^{t+\Delta t} = \int_0^{t+\Delta t} - \int_0^t$, and defining the SKA entropy phase

$$H_i(t) = \frac{1}{\ln 2}\int_0^t L_i(s)\,ds,$$

the propagator takes the equivalent phase-difference form:

$$\langle i,\, t+\Delta t \,|\, j,\, t\rangle = C_{ij}\;\exp\!\big(i\big[H_i(t+\Delta t) - H_i(t)\big]\big) \tag{21}$$

This short-time propagator provides the natural bridge between Feynman's action-based transition amplitudes and the entropy-based evolution law of SKA.

B.1 Action–Phase Structure

The SKA regime amplitude is defined as:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2}\int_0^t L_i(s)\,ds\right), \tag{22}$$

where L_i is the SKA Lagrangian associated with regime i. This expression is formally analogous to the Feynman amplitude

$$\exp\left(\frac{i}{\hbar}\int L\,dt\right), \tag{23}$$

but with a crucial distinction: the SKA phase is purely information-theoretic. The substitution $\hbar^{-1} \to (\ln 2)^{-1}$ is a change of normalization and does not introduce any physical (quantum) assumptions.

Thus, the action functional serves as an *entropy accumulator*, not a physical energy integral.

B.2 Euler–Lagrange Identity

The SKA Lagrangian is given by:

$$L_i(z_i, z_i') = -\,z_i\,\sigma(z_i)(1-\sigma(z_i))\,z_i', \tag{24}$$

where z_i is the knowledge projection and σ is the sigmoid decision function.

Direct calculation yields:

$$\frac{\partial L_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial L_i}{\partial z_i'}\right) \equiv 0, \tag{25}$$

which is an identity true for all trajectories $z_i(s)$.

This implies that *every knowledge trajectory is already an extremal path.* There is no optimization problem and no need to solve variational equations. Forward learning is automatically entropy-optimal.

This property has no analogue in classical mechanics and reflects the autonomous structure of real-time SKA learning.

B.3 Hamiltonian Constraint

The SKA Hamiltonian associated with L_i is:

$$H_i = z_i' \frac{\partial L_i}{\partial z_i'} - L_i. \tag{26}$$

Substituting the explicit form of L_i gives:

$$H_i \equiv 0. \tag{27}$$

This condition mirrors reparametrization-invariant constrained systems in mathematical physics (e.g., relativistic particle, minisuperspace cosmology), but here it arises in an information-geometric setting.

The vanishing Hamiltonian confirms that:

- SKA learning has no external potential or forcing term.
- All dynamics emerge from internal entropy flow.
- The system evolves along a constrained informational manifold.

B.4 Density Operator as Information Propagator

The density operator

$$\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)| \tag{28}$$

contains both probabilities and structural couplings. The matrix element

$$\langle i|\hat{\rho}|j\rangle = \Psi_i^*(t)\Psi_j(t) \tag{29}$$

is directly analogous to a transition kernel in path-integral formulations.

In SKA, however, this object encodes:

- regime probability ($i = j$),
- regime coherence and structural transitions ($i \neq j$),
- entropy phase differences $H_j - H_i$,
- real-time informational geometry.

Thus ρ_{ij} is the natural propagator for regime evolution and replaces the Markovian transition matrix of classical finance.

B.5 Functional Derivative Consistency

The SKA amplitude satisfies:

$$\frac{\delta \Psi_i}{\delta z_i(s)} = 0, \tag{30}$$

because the Euler–Lagrange identity nullifies the position dependence of the action. This means:

- regime amplitudes do *not* depend on the explicit path of $z_i(s)$,
- all knowledge trajectories are "allowed" without penalty,
- transitions emerge from geometry rather than optimization.

In contrast:

$$\frac{\delta \Psi_i}{\delta z_i'(s)} = -\frac{i}{\ln 2}\,\Psi_i\, z_i(s)\sigma(z_i)(1-\sigma(z_i)), \tag{31}$$

which shows that the amplitude is sensitive to the *velocity* of information flow. This result justifies the necessity of real-time SKA learning, since batch analysis cannot recover velocity-sensitive structure.

Empirical confirmation: Figure 1 in the main text provides direct empirical evidence for these predictions. The transition probabilities remain constant (position-insensitive) while price fluctuates, but regime changes correlate with periods of rapid price movement (velocity-sensitive). This observation pattern is exactly what the functional derivatives predict and would not be expected from classical regime models.

B.6 Consistency Summary

The SKA–Dirac formalism is internally consistent because:

1. The action-phase structure is correctly defined and purely informational.
2. The Euler–Lagrange identity guarantees inherent optimality of forward learning.
3. The Hamiltonian constraint ensures autonomy and internal consistency.
4. The density operator acts as an information propagator, not a quantum object.
5. Functional derivatives yield natural learning sensitivities consistent with real markets.

Taken together, these results show that the SKA–Dirac formalism rests on a solid information-geometric foundation and does not rely on physical assumptions. It is an independent and coherent mathematical framework for regime modeling in real-time financial systems.