Structured Knowledge Accumulation: A Dirac Notation Formalism for Market Regime Modeling

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Abstract

This paper introduces a quantum-inspired mathematical formalism for **real-time SKA entropy-driven regime detection** in financial markets using **Dirac notation** within the Structured Knowledge Accumulation (SKA) framework. The market state is represented as a superposition vector in a complex Hilbert space, and its evolution is governed by a density operator encoding both regime probabilities and structural transitions. Each complex amplitude carries an entropy phase, linking probabilistic transitions to information dynamics. The use of operator algebra reveals that regime transitions are not discrete events but emergent features of the SKA state geometry. This formulation provides a theoretical foundation for real-time analytics, entropy-based decision systems, and market structure decoding, with practical implementations across SQL querying, live dashboards, and agent-based forecasting.

1 Introduction

Traditional models of financial markets rely on discretized time series, deterministic indicators, or Markovian assumptions to represent regime changes. These approaches often overlook the **underlying information structure** and fail to capture the continuity and coherence inherent in real-time transitions.

The **Structured Knowledge Accumulation (SKA)** framework redefines this paradigm by treating the market as a **dynamical information system**, where regime states exist in a superposed form and evolve through entropy-driven learning. Inspired by quantum mechanics, this formalism introduces Dirac notation to express regime states, transition amplitudes, and entropy phases in a unified mathematical structure.

By modeling the market state as a vector in a complex Hilbert space, and transitions as projections within this space, we obtain a **density operator** that simultaneously encodes probabilities and structural couplings. This allows for a natural integration of **real-time learning**, **entropy flow**, and **non-Markovian memory effects**—opening the door to a fundamentally new class of market models.

This document lays the foundation for this approach, defining each component of the model and providing the necessary tools for implementation in analytics pipelines, signal processing, and algorithmic decision systems.

2 State Space Definition

Let $\mathcal{H}$ be a **3-dimensional complex Hilbert space** with orthonormal basis:

$$\mathcal{B} = \{|0\rangle, |1\rangle, |2\rangle\} \tag{1}$$

Interpretation: Basis vectors represent mutually exclusive, exhaustive regimes.

3 Market State Vector

At any time t, the market is in a superposition of basis regimes:

$$|\Psi(t)\rangle = \sum_{i=0}^{2} \Psi_i(t) |i\rangle, \quad \Psi_i(t) \in \mathbb{C} \tag{2}$$

$$|\Psi(t)\rangle = \Psi_0(t) |0\rangle + \Psi_1(t) |1\rangle + \Psi_2(t) |2\rangle \tag{3}$$

where:

- $|0\rangle, |1\rangle, |2\rangle$: Basis vectors for Neutral, Bull, Bear
- $\Psi_i(t) = A_i(t)e^{iH_i(t)} \in \mathbb{C}$: Complex amplitude encoding probability and entropy

In the Structured Knowledge Accumulation (SKA) framework, the market is not confined to a single regime (e.g., bull, bear, or neutral) at any moment. Instead, it exists in a **superposition** of all possible regimes, with complex amplitudes $\Psi_i(t)$ encoding both probability and entropy.

Traditional models assume the market flips from one regime to another—an abrupt, externally triggered event. But SKA reveals a continuous evolution through overlapping informational states.

3.1 Classical vs. SKA Assumptions

Classical Assumption:

- Market is always *in* one regime.
- Transitions are discrete.
- Probabilities are assigned *after* classification.

SKA View:

- Market is always *between* regimes.
- Transitions emerge from entropy geometry.
- Probabilities arise from the evolving wavefunction $|\Psi(t)\rangle$.

3.2 Implications

- **Why classical models fail:** they discretize what is fundamentally smooth and entangled.
- **Why transitions seem random:** because we're projecting a superposed state onto a single outcome.
- **Why micro-patterns exist at the tick level:** because phase coherence in $|\Psi(t)\rangle$ encodes deep structural information—**even when the price appears flat**.

SKA does not model the *outcome* of the market. It models the *informational field* from which outcomes emerge.

3.3 Normalization

For any state $|\Psi\rangle \in \mathcal{H}$, $\langle\Psi|\Psi\rangle = 1$:

$$\langle\Psi(t)|\Psi(t)\rangle = |\Psi_0|^2 + |\Psi_1|^2 + |\Psi_2|^2 = 1 \tag{4}$$

4 Density Operator

The **density operator** is the outer product of the state vector with itself:

$$\hat{\rho}(t) = |\Psi(t)\rangle \langle\Psi(t)| \tag{5}$$

This object captures both **probabilities** and **structural relationships** (coherences) between regimes.

4.1 Matrix Elements

$$\langle i| \hat{\rho}(t) |j\rangle = \Psi_i^*(t)\Psi_j(t) \tag{6}$$

This yields a 3×3 Hermitian matrix:

$$\hat{\rho}(t) = \begin{pmatrix} |\Psi_0|^2 & \Psi_0^*\Psi_1 & \Psi_0^*\Psi_2 \\ \Psi_1^*\Psi_0 & |\Psi_1|^2 & \Psi_1^*\Psi_2 \\ \Psi_2^*\Psi_0 & \Psi_2^*\Psi_1 & |\Psi_2|^2 \end{pmatrix} \tag{7}$$

4.2 Interpretation

- **Diagonal terms**: $\rho_{ii} = |\Psi_i|^2$ = probability of being in regime i
- **Off-diagonal terms**: $\rho_{ij} = \Psi_i^*\Psi_j$ = coherence between regimes i and j, interpreted as transition structure

5 Structural Transition Operator

A transition from regime $j \to i$ is represented by:

$$\hat{T}_{i\leftarrow j} = |i\rangle \langle j| \tag{8}$$

This operator transforms basis state $|j\rangle$ into $|i\rangle$.

5.1 Expectation Value

To extract the amplitude for a structural transition from $j \to i$:

$$\langle\Psi| \hat{T}_{i\leftarrow j} |\Psi\rangle = \Psi_i^*(t)\Psi_j(t) \tag{9}$$

This equals the (i, j) entry of the density matrix.

5.2 Constant Structural Coupling C_{ij}

In practice, the most observable market regularities emerge when structural transition amplitudes become **time-invariant**:

$$C_{ij} = \Psi_i^*(t)\Psi_j(t) = \text{const} \tag{10}$$

This condition implies a stable **directional flow of probability** between regimes i and j.

SKA Insight. The structural transition amplitude $C_{ij} = \Psi_i^*(t)\Psi_j(t)$ emerges *only through Real-Time SKA Learning.* It reflects an acquired informational symmetry, not a statistical feature of the raw market data. Batch analysis cannot reveal this constant coherence — it is a learned invariant of the SKA entropy trajectory.

When C_{ij} remains constant:

- Regime transitions align clearly in real-time analysis.
- Entropy flow across regimes becomes **symmetric** or **balanced**.
- Paired transitions (e.g., Neutral $\to$ Bull and Bull $\to$ Neutral) appear as coherent oscillations.

Classical models assume regime transitions are noisy or random. In contrast, SKA shows structural amplitudes C_{ij} can remain highly stable across thousands of trades, enabling precise **real-time alignment** of transitions.

This stability reveals that markets operate within a **coherent informational geometry** rather than a stochastic regime-flip process, explaining why traditional models—treating probabilities as noise instead of geometry—fail to detect these regularities.

6 Entropy Interpretation

Each complex amplitude $\Psi_i(t)$ embeds an **entropy phase**:

$$\Psi_i(t) = A_i(t)e^{iH_i(t)} \tag{11}$$

Here, $A_i(t)$ represents the amplitude, and $H_i(t)$ is the entropy associated with regime i at time t.

The entropy $H_i(t)$ comes from the **Real-Time SKA learning process** as it accumulates over the trajectory of market evolution.

We express the entropy as a functional of the Lagrangian [1, 2]:

$$H_i(t) = \frac{1}{\ln 2}\int_0^t \mathcal{L}_i(s)\, ds \tag{12}$$

where $\mathcal{L}_i(s)$ is the Lagrangian.

This phase represents the **information content** (entropy) of regime $|i\rangle$. Therefore, transitions (via $\Psi_i^*\Psi_j$) carry entropy shifts $H_j - H_i$.

7 Transition Operator Action on the State

We consider the global transition operator (density operator):

$$\hat{T} = |\Psi\rangle\langle\Psi| \tag{13}$$

We project this operator acting on the state back onto the basis state $|i\rangle$:

$$\langle i|\hat{T}|\Psi\rangle = \langle i|\Psi\rangle = \sum_j \Psi_j \langle i|j\rangle = \sum_j \Psi_j \delta_{ij} = \Psi_i \tag{14}$$

Here we used the orthonormality of the basis: $\langle i|j\rangle = \delta_{ij}$, the Kronecker delta.

Since $\hat{T}|\Psi\rangle = |\Psi\rangle$, the operator acts as a projector and leaves the state invariant. Therefore:

$$\langle 0|\hat{T}|\Psi\rangle = \Psi_0 \tag{15}$$

$$\langle 1|\hat{T}|\Psi\rangle = \Psi_1 \tag{16}$$

$$\langle 2|\hat{T}|\Psi\rangle = \Psi_2 \tag{17}$$

This shows that transitions are not events but projections. The market doesn't "switch" regimes - our measurement collapses the superposition!

This means:

- Regime changes are observation-dependent
- The act of trading affects the state
- Observer effect is real in markets

The transition operator encodes the full informational identity of the market state and reflects that transitions are emergent properties of the SKA state — not discrete external events.

Upon measurement, the superposed market state $|\Psi\rangle$ collapses to a definite regime $|i\rangle$ with probability $|\Psi_i|^2$, revealing the observed market condition.

8 Summary of Core Equations

Concept	**Formula**
State vector	$\|\Psi\rangle = \sum_i \Psi_i \|i\rangle$
Density operator	$\hat{\rho} = \|\Psi\rangle \langle\Psi\|$
Regime probability	$\rho_{ii} = \|\Psi_i\|^2$
Structural transition	$\hat{T}_{i\leftarrow j} = \|i\rangle \langle j\|$
Transition amplitude	$\langle i\| \hat{\rho} \|j\rangle = \Psi_i^* \Psi_j$
Entropy phase	$\arg(\Psi_i) = H_i(t)$

Table 1: Core equations of the SKA-Dirac formalism

9 Applications and Usage

This document serves as the **formal reference** for SKA modules that:

- Visualize transitions in QuestDB or Grafana
- Compute entropy-induced transition probabilities
- Apply the principle of least entropy action

Refer to this file when defining:

- Structural probability matrices
- Real-time SKA signal decoders
- Quantum-inspired simulation of regime evolution

10 Conclusion

This formalism supports a deeper view of the market as an **information-processing system**, not a sequence of discrete events. Transitions are not externally triggered but emerge from the **geometry of knowledge accumulation** encoded in the SKA state.

The quantum-inspired approach reveals that traditional discrete regime models fundamentally misrepresent market dynamics. By treating markets as superposed information systems evolving through entropy-driven learning, we obtain a mathematically rigorous framework that captures both probabilistic and structural aspects of regime transitions.

The Real-Time SKA learning process, integrated through Lagrangian dynamics, provides a principled approach to understanding how markets process information and evolve through different phases. This has immediate applications in risk management, algorithmic trading, and financial system stability analysis.

References

[1] Bouarfa Mahi. *Structured Knowledge Accumulation: An Autonomous Framework for Layer-Wise Entropy Reduction in Neural Learning.* arXiv:2503.13942, 2025.

[2] Bouarfa Mahi. *Structured Knowledge Accumulation: The Principle of Entropic Least Action in Forward-Only Neural Learning.* arXiv:2504.03214, 2025.

A Functional Derivative Development in the SKA Framework

A.1 Functional Derivative with Respect to Position: $\delta\Psi_i/\delta z_i(s)$

A.1.1 Step-by-Step Derivation

1. Starting Expression Given the complex amplitude with entropy phase:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2}\int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds\right)$$

2. Define the Phase Function Let the cumulative entropy phase be:

$$H_i(t) = \frac{1}{\ln 2}\int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds$$

So: $\Psi_i(t) = A_i \exp(iH_i(t))$

3. Apply Functional Derivative

$$\frac{\delta\Psi_i}{\delta z_i(s)} = \frac{\delta}{\delta z_i(s)}[A_i \exp(iH_i(t))]$$

Using the chain rule for functional derivatives:

$$\frac{\delta\Psi_i}{\delta z_i(s)} = A_i \exp(iH_i(t)) \times i \times \frac{\delta H_i(t)}{\delta z_i(s)} = i\Psi_i \times \frac{\delta H_i(t)}{\delta z_i(s)}$$

4. Compute $\delta H_i(t)/\delta z_i(s)$ The functional derivative of the entropy integral:

$$\frac{\delta H_i(t)}{\delta z_i(s)} = \frac{\delta}{\delta z_i(s)}\left[\frac{1}{\ln 2}\int_0^t \mathcal{L}_i(z_i(\tau), z_i'(\tau))\, d\tau\right]$$

Using the **fundamental theorem of calculus of variations**:

$$\frac{\delta}{\delta z_i(s)}\int_0^t \mathcal{L}_i(z_i(\tau), z_i'(\tau))\, d\tau = \left.\frac{\partial\mathcal{L}_i}{\partial z_i}\right|_s - \frac{d}{ds}\left.\left(\frac{\partial\mathcal{L}_i}{\partial z_i'}\right)\right|_s$$

Therefore:

$$\frac{\delta H_i(t)}{\delta z_i(s)} = \frac{1}{\ln 2}\left.\left[\frac{\partial\mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}_i}{\partial z_i'}\right)\right]\right|_s$$

5. Final Result

$$\boxed{\frac{\delta \Psi_i}{\delta z_i(s)} = \frac{i\Psi_i}{\ln 2}\left[\frac{\partial \mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}_i}{\partial z_i'}\right)\right]\bigg|_s}$$

A.2 Functional Derivative with Respect to Velocity: $\delta\Psi_i/\delta z_i'(s)$

A.2.1 Step-by-Step Derivation

1. Starting Expression Same as before:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2}\int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds\right)$$

2. Apply Functional Derivative with Respect to $z_i'(s)$:

$$\frac{\delta \Psi_i}{\delta z_i'(s)} = i\Psi_i \times \frac{\delta H_i(t)}{\delta z_i'(s)}$$

3. Compute $\delta H_i(t)/\delta z_i'(s)$: Using the **fundamental theorem of calculus of variations** for the velocity term:

$$\frac{\delta}{\delta z_i'(s)}\int_0^t \mathcal{L}_i(z_i(\tau), z_i'(\tau))\, d\tau = \frac{\partial \mathcal{L}_i}{\partial z_i'}\bigg|_s$$

Note: Unlike the derivative with respect to $z_i(s)$, there's **no** d/ds term here. Therefore:

$$\frac{\delta H_i(t)}{\delta z_i'(s)} = \frac{1}{\ln 2}\frac{\partial \mathcal{L}_i}{\partial z_i'}\bigg|_s$$

4. Final Result:

$$\boxed{\frac{\delta \Psi_i}{\delta z_i'(s)} = \frac{i\Psi_i}{\ln 2}\frac{\partial \mathcal{L}_i}{\partial z_i'}\bigg|_s}$$

A.3 Connection to SKA Lagrangian

A.3.1 SKA Lagrangian

From the arXiv paper, the SKA Lagrangian is:

$$\mathcal{L}_i(z_i, z_i', t) = -z_i \cdot \sigma(z_i)(1 - \sigma(z_i)) \cdot z_i'$$

A.3.2 Euler-Lagrange Result

When the detailed partial derivative calculations are performed (as shown in arXiv paper equations 17-25), the Euler-Lagrange equation yields:

$$\frac{\partial \mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial \mathcal{L}_i}{\partial z_i'}\right) \equiv 0$$

This is an **identity** - the equation is satisfied for all possible knowledge trajectories $z_i(s)$, demonstrating the natural optimality of SKA dynamics.

A.3.3 Compute $\partial \mathcal{L}_i/\partial z_i'$:

$$\frac{\partial \mathcal{L}_i}{\partial z_i'} = -z_i \cdot \sigma(z_i)(1 - \sigma(z_i))$$

A.3.4 Substitute Back:

$$\frac{\delta\Psi_i}{\delta z_i'(s)} = \frac{i\Psi_i}{\ln 2} \cdot (-z_i(s) \cdot \sigma(z_i(s))(1 - \sigma(z_i(s))))$$

$$\boxed{\frac{\delta\Psi_i}{\delta z_i'(s)} = -\frac{i\Psi_i}{\ln 2} z_i(s)\sigma(z_i(s))(1 - \sigma(z_i(s)))}$$

A.4 Physical Interpretation

A.4.1 Position Derivative ($\delta\Psi_i/\delta z_i(s)$)

The SKA framework defines each regime amplitude as:

$$\Psi_i(t) = A_i \exp\left(\frac{i}{\ln 2}\int_0^t \mathcal{L}_i(z_i(s), z_i'(s))\, ds\right)$$

We computed its functional derivative:

$$\frac{\delta\Psi_i}{\delta z_i(s)} = \frac{i\Psi_i}{\ln 2} \cdot \left(\frac{\partial\mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}_i}{\partial z_i'}\right)\right)$$

In SKA, the Lagrangian $\mathcal{L}_i$ is constructed such that:

$$\frac{\partial\mathcal{L}_i}{\partial z_i} - \frac{d}{ds}\left(\frac{\partial\mathcal{L}_i}{\partial z_i'}\right) \equiv 0$$

Which gives:

$$\frac{\delta\Psi_i}{\delta z_i(s)} \equiv 0$$

What This Means in Plain Terms

- Small changes in the knowledge path $z_i(s)$, which reflects cumulative price information, do not affect the regime amplitude $\Psi_i(t)$.
- Every possible trajectory is already entropy-optimal — no path is more "efficient" than another.
- There's no need for the system to "search" for the best transition — it follows the natural flow of information.

Financial Interpretation Market regimes (bull, bear, neutral) self-organize from the price stream. They don't require external filters, smoothing, or optimization.

This is why regime transitions in SKA appear stable and coherent — the underlying probabilities remain constant, even though price evolves.

A.4.2 Velocity Derivative ($\delta\Psi_i/\delta z_i'(s)$)

Key Differences from $\delta\Psi_i/\delta z_i(s)$:

1. Non-Zero Result: Unlike the derivative with respect to $z_i(s)$ which was identically zero, this derivative is **non-zero**.

2. Physical Interpretation:

- $\delta\Psi_i/\delta z_i(s) = 0$: Regime amplitude is insensitive to **position** changes in knowledge trajectory

- $\delta\Psi_i/\delta z_i'(s) \neq 0$: Regime amplitude is sensitive to velocity changes in knowledge trajectory

3. Market Dynamics Meaning:

- $z_i'(s)$: Rate of knowledge change (learning velocity)
- Non-zero derivative: Regime amplitudes respond to changes in learning speed
- The faster knowledge accumulates, the stronger the regime amplitude response

Physical Significance:

Learning Velocity Sensitivity:

The market regime amplitudes are sensitive to the rate of information processing, not just the accumulated knowledge level.

Market Interpretation:

- **High $z_i'(s)$:** Rapid information processing $\rightarrow$ Strong regime amplitude response
- **Low $z_i'(s)$:** Slow information processing $\rightarrow$ Weak regime amplitude response
- The derivative quantifies how regime coherence depends on learning dynamics

Real-Time SKA Insight:

This explains why Real-Time SKA is crucial - the regime amplitudes depend on the velocity of knowledge accumulation, which can only be captured through continuous learning dynamics, not batch analysis!

The functional derivative with respect to velocity reveals the dynamic responsiveness of market regimes to information flow rates!