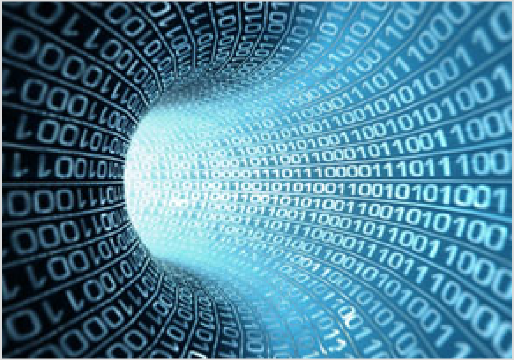
Hierarchical Modeling: Mixed Effects vs Bayesian

SageSure Data Science Symposium 2022

W. D. Brinda

Quantitations LLC

11 Nov 2022



The general formulation of the multiple linear model with iid Normal errors:

$$Y_i = \beta_0 + \beta_1 x_i^{(1)} + \ldots + \beta_d x_i^{(d)} + \epsilon_i$$

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Linear model vs linear mixed model

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In contrast, the general formulation of the linear *mixed* model with iid Normal errors:

$$Y_i = \beta_0 + \beta_1 x_i^{(1)} + \ldots + \beta_d x_i^{(d)} + \underline{b_1 z_i^{(1)} + \ldots + b_m z_i^{(m)}} + \epsilon_i$$

with $\epsilon_1, \ldots, \epsilon_n$ iid Normal and the “*random effects*" $b_1, \ldots, b_m$ are assumed to be drawn from a multivariate Normal distribution.

Example: Blood pressure measurements

Suppose that n_f females and n_m males each had their [systolic] blood pressures measured. Assume that the distributions of female and male blood pressures are both Normal with the same standard deviation σ_w. Let their expectations be denoted μ_f and μ_m respectively. With $x_1, \ldots, x_n$ representing sex (the only explanatory variable) and $Y_1, \ldots, Y_N$ representing blood pressures, we can represent this scenario with the linear model

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Define $z_i^{(j)}$ to be to be the indicator variable for whether observation i came from subject j. Then, we might want to modify our model to

$$Y_i = \mu_f + (\mu_m - \mu_f)\mathbb{I}(x_i = \text{"}M\text{"}) + b_1 z_i^{(1)} + \ldots + b_m z_i^{(m)} + \epsilon_i.$$

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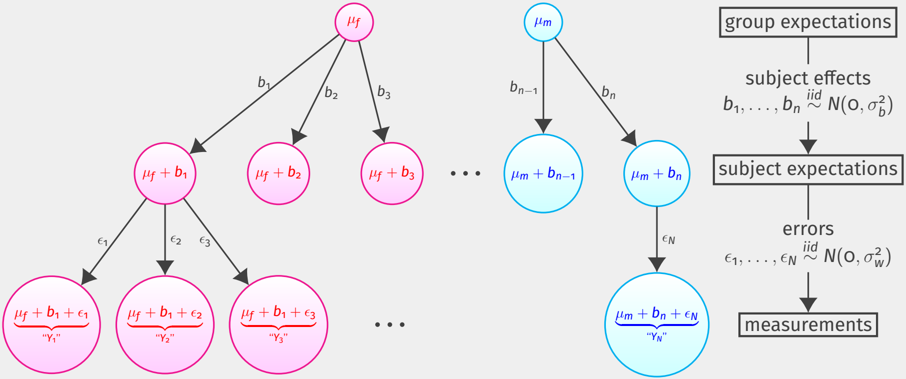
with iid Normal *random effects* $b_1, \ldots, b_m$.
To better understand what this model means, suppose observation 12 came from subject 5 who is a female: the corresponding equation simplifies to

$$Y_{12} = \mu_f + b_5 + \epsilon_{12}$$

Model Diagram



R Script

Now, I'll turn to an R script to simulate blood pressure data according to the mechanism just described and draw some plots to visualize it. (We'll continue through this script at later points in the talk.)

If you'd like, you can download the file and execute the code along with me:
quantitations.com/static/symposium.r

Health Insurance Decision Example

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Data: HDC selected 11 members at random this year (5 female, 6 male) to have their blood pressures measured, between 1 and 4 times each.

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Manager: ... What? ...

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Manager: And do you think the parameter estimates *are* correct?

Frequentist: No.

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Frequentist: No.
[Uncomfortable silence...]

Classic Bayes Theorem example

Classic counter-intuitive example demonstrating Bayes' Theorem:

Suppose the prevalence of a disease in the general population is 1 in 5000. A test for this disease has a 1% false positive rate and a 2% false negative rate. If this test gives you a positive result, find the probability that you have the disease.

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P-values are useful when your objective is to control false-positive rates. (Even then, it can fail! e.g. selective submission of experimental results.)

In most contexts, this is an objective we shouldn't care much about.

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Critics' objections against Bayesian statistics: A **prior belief** must be brought into the analysis:

- can be *subjective*
- can be *difficult to quantify*

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Manager: [unable to speak... weeping tears of joy]

End

Thank you, SageSure!

Any questions?