The cosine rule for evaluating radiation view factors (F_{ij})

The view factor F_{12} is the fraction of radiation leaving surface **1** that directly reaches surface **2**

A Lambertian emitter surface appears equally bright when viewed from any direction (at a fixed distance). It is common approximation for “diffuse” surfaces. “Appearing equally bright” means that the rate of arrival of photons at a detector (e.g. a camera, looking directly towards the emitter) is proportional to the apparent size of the emitter seen by the camera.

We want to prove the cosine rule, for a Lambertian emitter surface and a distant receiving surface

$$F_{12} = \frac{1}{A_1} \int\int \frac{\cos(\theta_1)\cos(\theta_2)\,\mathrm{d}A_1\,\mathrm{d}A_2}{\pi r^2} \tag{1}$$

First, consider an isotropic point emitter, which emits a total power $2Q_1$ so that it emits power Q_1 towards a hemisphere. (This definition is useful for comparison with an emitting surface, which can only emit outwards towards a hemisphere.)

An “isotropic point emitter” emits photons radially outwards with spherical symmetry. Clearly, one single isotropic point emitter must have the property of being equally bright when viewed from any direction. The incident radiation flux reaching an enclosing hemisphere of radius r is therefore uniform, and can be called $G(r)$. Equating emitted and arriving power gives the following expression for $G(r)$.

$$G(r) = \frac{Q_1}{2\pi r^2} \tag{2}$$

The power arriving on a surface patch of the hemisphere with area $\mathrm{d}S$ is therefore $\mathrm{d}Q_{1S}$, as follows.

$$\mathrm{d}Q_{1S} = \left(\frac{Q_1}{2\pi r^2}\right)\,\mathrm{d}S \tag{3}$$

This power transport can also be written in terms of the solid angle, $\mathrm{d}\Omega = \mathrm{d}S/r^2$, that is occupied by the patch.

$$\mathrm{d}Q_{1S} = \left(\frac{Q_1}{2\pi}\right)\,\mathrm{d}\Omega \tag{4}$$

Now consider a small area $\mathrm{d}A_1$ of a Lambertian surface emitter that has total area A_1. It is defined as having the property of appearing equally bright when viewed from any direction (at a fixed distance).

The total power emitted by the area element is simply the surface’s radiosity (emitted Watts per square metre) multiplied by its area.

$$\mathrm{d}Q_1 = J_1\mathrm{d}A_1 \tag{5}$$

Now, we need a criterion for what it means for the surface to appear equally bright when viewed from any direction. If we think in terms of photons arriving at a detector which has area A_{det}, then we can write down such a criterion, which we shall call **uniform radiance**. The radiance of the emitting surface element, $\mathrm{d}A_1$, is proportional to the number of photons it sends to the detector, divided by both the detector area and by the apparent size (in solid angle) of the emitting surface when it is viewed from the detector.

In the diagram, imagine the emitting surface $\mathrm{d}A_1$ emits N photons per second towards the detector. The detector may be either vertically above the emitter (which is horizontal and flat), or in general the detector may be at some angle θ_1 from the vertical, but in both cases the detector surface is perpendicular (i.e. square-on) to the arriving rays of light. In the general case of the detector at θ_1, the number of photons emitted in this direction can be defined with an angular dependence $f(\theta_1)$ which describes the rate at which photons are emitted along a trajectory at an angle θ_1 as a fraction of the rate at which they are emitted vertically.

N must be proportional to the emitter area, so it is equal to some constant of proportionality C multiplied by emitter area $\mathrm{d}A_1$ and, by considering equation 4, it must also be proportional to the solid angle $\mathrm{d}\Omega_{\mathrm{det}}$ occupied by the detector.

Considering the vertical detector, the number of photons arriving at the detector is N_{vert} as follows.

$$N_{\mathrm{vert}} = C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\mathrm{det}} \qquad (6)$$

And the radiance of the emitter, considering the detector area A_{det} and the angular size of the emitter, $\mathrm{d}\Omega_1$ is

$$Radiance = \frac{C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\mathrm{det}}}{A_{\mathrm{det}}\ \mathrm{d}\Omega_1} \qquad (7)$$

In the case of the detector at an angle θ_1, the number of detected photons is

$$N_\theta = C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\mathrm{det}}\, f(\theta_1) \qquad (8)$$

Where $f(\theta_1)$ accounts for the surface's angular emission. The angular size of the emitter viewed from the detector is now $\mathrm{d}\Omega_1 \cos(\theta_1)$, so its apparent radiance at the detector is

$$Radiance = \frac{C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\mathrm{det}}\, f(\theta_1)}{A_{\mathrm{det}}\ \mathrm{d}\Omega_1 \cos(\theta_1)} \qquad (9)$$

In order for the emitting surface to have the Lambertian property of equal radiance from every observation angle, so that equations (7) and (9) are equal for any θ_1, it must have an angular emission behaviour $f(\theta_1) = \cos(\theta_1)$.

$$f(\theta_1) = \cos(\theta_1) \qquad \text{for a Lambertian emitter} \qquad (10)$$

This is not very intuitive, but it is true. In order for the Lambertian emitter surface to appear equally bright to a detector at any angle, it actually must have an angle-dependent emission – precisely as in equation 10. This is so that the probability of emitting photons in a particular direction decreases along with the decreasing apparent size of the emitter surface at increasingly oblique angles. [The reason why this angular-dependent emission does not apply to the point emitter is that the apparent size of a point emitter is a constant at any angle, because it is determined by diffraction-limited blurring in the camera rather than by its area, because a point has zero area.]

Now that we have obtained an expression for $f(\theta_1)$ in equation 8, we can compute the radiant power $\mathrm{d}Q_{1s}$ emitted from a small surface $\mathrm{d}A_1$ towards a distant surface $\mathrm{d}A_2$. By considering this amount of power as a fraction of the total emission, a view factor is obtained.

Given equation (8) which describes the number of photons arriving at some target surface with angular size $\mathrm{d}\Omega_{\mathrm{det}}$, we can relate this target angular size to the area of a general target surface, $\mathrm{d}A_2$. Using the ideas of vector area, we see a small patch of target surface at distance r from the emitter occupies a solid angle

$$\mathrm{d}\Omega_{\mathrm{det}} = \frac{\mathrm{d}A_2 \cos(\theta_2)}{r^2} \tag{11}$$

Conceptually, the power transported from surface 1 to surface 2 is proportional to the number of photons transported, so the expression in equation (8) can be changed from one in terms of photons per second to one in terms of Watts by changing the constant of proportionality (from C to I_1).

$$\mathrm{d}Q_{1s} = \frac{I_1\, \mathrm{d}A_1 \cos(\theta_1) \mathrm{d}A_2 \cos(\theta_2)}{r^2} \tag{12}$$

The total power emitted by the surface element $\mathrm{d}A_1$ is known from equation (5) to be equal to $J_1 \mathrm{d}A_1$. Since all of this radiant power must be transported to an enclosing hemisphere, for which incident radiation from a small source all arrives perpendicular to the surface ($\theta_2 = 0$ for all received radiation), and $\mathrm{d}A_2 = r^2 \sin(\theta_1)\ \mathrm{d}\theta_1\, \mathrm{d}\phi_1$, we can equate emitted and received power with the following integral over a hemisphere.

$$J_1 \mathrm{d}A_1 = I_1 \mathrm{d}A_1 \int \sin(\theta_1) \cos(\theta_1)\ \mathrm{d}\theta_1\, \mathrm{d}\phi_1 \tag{13}$$

From which we can obtain

$$I_1 = \frac{J_1}{\pi} \tag{14}$$

And hence, since $F_{12} = Q_{1s}/J_1 A_1$

$$F_{12} = \frac{1}{A_1} \int\int \frac{\cos(\theta_1)\cos(\theta_2)\, \mathrm{d}A_1\, \mathrm{d}A_2}{\pi r^2} \tag{15}$$

The very thing that we wanted to demonstrate.

EJR

Further reading

https://ipfs.io/ipfs/QmXoypizjW3WknFiJnKLwHCnL72vedxjQkDDP1mXWo6uco/wiki/Lambert's_cosine_law.html