Video on the Cosine rule for view factors

Hello everyone, this video will explain how to obtain **the cosine rule** for calculating view factors between two surfaces. This equation is actually quite useful in two different subjects, because view factors are needed for radiation heat transport problems involving infrared or visible light, say in chemical engineering applications, as well as in computer graphics where the simple radiosity method is often used for calculating visible light transport and rendering the image of a scene.

The view factor F_1,2 is the fraction of radiation leaving surface number one which goes on to hit surface number two. The **cosine rule** computes the view factor F_1,2 when the emitting surface, surface number 1, is a diffuse emitter, in fact more precisely when surface number one is a special kind of diffuse emitter called a Lambertian surface. A Lambertian surface has the special property that it appears to have the same brightness when observed from any direction, provided it is observed from the same distance and the illumination of the surface is static. (The observation could be by a human, observing with their eyeball, or by a camera.) Lots of rough or matt surfaces are approximately Lambertian, for example a piece of paper on the desk, under sunlight may look approximately equally bright if you look at it from directly overhead, or from the side at a more oblique angle.

We want to prove equation 1 for the view factor between the two surfaces shown in the diagram.

$$F_{12} = \frac{1}{A_1} \int \int \frac{\cos(\theta_1)\cos(\theta_2)\, dA_1 dA_2}{\pi r^2}$$

First, let's describe a point emitter. An "isotropic point emitter" emits photons radially outwards with spherical symmetry, so one single isotropic point emitter clearly does have the property of being equally bright when viewed from any direction. Let's consider the half of the radiation leaving a point emitter, which heads outwards toward a hemisphere of radius r. If we call the power emitted by point source towards the hemisphere is Q_1 measured in Watts, then the uniform incident radiant flux reaching the hemisphere is $\frac{Q_1}{2\pi r^2}$ Watts per metre squared [equation 2]… and the power reaching an area element on the hemisphere with size δS is called δQ_{1S} and is equal to the uniform incident flux density $\frac{Q_1}{2\pi r^2}$ times by the receiving area δS [equation 3].

Now, a Lambertian surface is an extended surface which also has the special property that it looks equally bright in all directions. This means the incident radiation flux which reaches a distant hemisphere enclosing the Lambertian surface element, is uniformly intense over the hemisphere.

You might be surprised that, while one isotropic point emitter does have the property of producing equal incident radiation flux over an enclosing hemisphere, the Lambertian surface is **<u>not</u>** simply made up of a plane of such isotropic point emitters. In fact, the Lambertian surface does have a (quite easy to calculate) directionality. As we shall now see.

Firstly, if the emitting surface element is Lambertian, and it emits power Q_1 then then the incident radiation flux reaching an enclosing hemisphere must be a spatially uniform amount, $\frac{Q_1}{2\pi r^2}$, exactly as in equation 2. [equation 2b]

Secondly, in the diagram for the Lambertian surface element, we assume that the enclosing hemisphere is distant, meaning the hemisphere has a large radius compared with the diameter of both the emitting surface element (δA_1) and the diameter of the receiving area element on the hemisphere (δS). This large-radius assumption means that the light travelling in the direction of a particular patch δS on the hemisphere travels there as a bundle of essentially parallel rays, because any rays which are not very nearly parallel to this direction would have travelled far enough to diverge and strike a different part of the hemisphere. Let us say the direction from the emitter (da1) to the hemispherical patch (dS) has a polar angle θ_1 from the surface normal, and also an azimuthal angle ϕ_1 (although the azimuthal angle will not really matter). Considering the concept of vector area, a parallel bundle of rays leaving the emitter, which has area δA_1, and travelling in the radial direction (θ_1, ϕ_1), will reach a patch on the hemisphere which has an area $\delta S = \delta A_1 \operatorname{cosine}(\theta_1)$. If this bundle of rays transports an amount of power Q_{1s} , where $\delta Q_{1,s}$ could depend on the emission angle (θ_1) as well as r, then the incident flux density reaching the hemisphere in the direction (θ_1, ϕ_1) is given by $\frac{\delta Q_{1s}(\theta_1)}{\delta A_1 \operatorname{cosine}(\theta_1)}$. [equation 4].

Since we know the flux density in equation 4 is equal to the uniform incident flux required by equation 2, we can see the amount of power emitted from (dA1) to the hemisphere must indeed depend on the emission angle θ_1. In fact, the amount of power emitted towards a the patch dS on the hemisphere must vary in proportion to $\cos(\theta_1)$ if the incident radiation flux is to be uniform. What this means is that because the projected area of (dA1) becomes smaller when it is viewed at a more oblique angle, it must emit proportionally less power in that direction if it is to be viewed as equally bright. [Example with sheet of paper. This is different from the isotropic point emitter, because unlike the point emitter the surface element has a finite projected area which varies with observation angle.]

So, we can write down an equation for the radiation flux of the emission from the Lambertian surface element – call this amount F, at a distance r from the emitting surface in some direction (θ_1, ϕ_1). The emitted radiation flux must be proportional to $\cos(\theta_1)$, so that we can get the Lambertian property of being equally bright in all directions; at a distance r the emitted flux must be proportional to (1/r^2) since the emitted light is spread over a hemisphere with area proportional to r^2; and the emitted flux must be proportional to the amount of emitter surface area, dA1; and let's say there is a constant of proportionality I per unit area… Then we get an equation for emitted radiation flux:

$$F = I \frac{dA_1 \cos(\theta_1)}{r^2}$$

[equation 5]

Now we can get the constant I by equation power emitted by the surface with all the power arriving at the enclosing hemisphere.

The power emitted by the surface element is $J_1 dA_1$ where J_1 is the radiosity or power in Watts emitted per square metre of emitting surface, times by the emitting area $\mathrm{d}A_1$. [equation 6]

$$P_{\text{em}} = J_1 \mathrm{d}A_1$$

The power arriving at the hemisphere is obtained with a bit of integration over the hemispherical surface, as in [equation 7].

$$P_{\text{arr}} = \int F \, \mathrm{d}S$$

$$P_{\text{arr}} = \int I \, \frac{dA_1 \cos(\theta_1)}{r^2} \, . \mathrm{r}^2 \sin(\theta_1) \, \mathrm{d}\theta_1 \, \mathrm{d}\phi_1$$

$$P_{\text{arr}} = \pi \, I \, dA_1$$

So, equating, we get $J_1 dA_1 = \pi I \; dA_1$ So

$$I = \frac{J_1}{\pi}$$

Putting in this value of I to compete equation 7 for the radiation flux emitted in a particular direction, we can now integrate the flux going in the direction of surface 2. Now, the small target surface $\mathrm{d}A_2$ has a projected area of $dA_2\cos(\theta_2)$ on the hemisphere which reaches at its radial position from surface 1, and so the relevant integral for obtaining the power reaching surface 2 is:

$$Q_{12} = \int F\, dS_2 = \frac{J_1}{\pi}\frac{dA_1\cos(\theta_1)}{r^2}\, dA_2\cos(\theta_2)$$

The view factor from surface 1 to surface 2 is the fraction of power which goes to surface 2. Since the total emitted power is $J_1\, A_1$, we can divide through by that to get:

$$F_{12} = \frac{Q_{12}}{Q_1} = \int_{A_1}\int_{A_2} \frac{\cos(\theta_1)\cos(\theta_2)\, dA_1 dA_2}{\pi r^2}$$

Which is equation 1, the cosine rule for evaluating view factors, as required.

Considering the emission direction (θ_1, ϕ_1), we could describe this uniform incident flux reaching the hemisphere as follows.

$$G(r, \theta_1, \phi_1) = \frac{Q_1}{2\pi r^2} \quad \text{which is, of course, independent of } (\theta_1, \phi_1) \qquad (2b)$$

The cosine rule for evaluating radiation view factors (F_{ij})

The view factor F_{12} is the fraction of radiation leaving surface **1** that directly reaches surface **2**

A Lambertian emitter surface appears equally bright when viewed from any direction (at a fixed distance). It is common approximation for "diffuse" surfaces. "Appearing equally bright" means that the rate of arrival of photons at a detector (e.g. a camera, looking directly towards the emitter) is proportional to the apparent size of the emitter seen by the camera.

We want to prove the cosine rule, for a Lambertian emitter surface and a distant receiving surface

$$F_{12} = \frac{1}{A_1} \int\int \frac{\cos(\theta_1)\cos(\theta_2)\,\mathrm{d}A_1\,\mathrm{d}A_2}{\pi r^2} \tag{1}$$

First, consider an isotropic point emitter, which emits a total power $2Q_1$ so that it emits power Q_1 towards a hemisphere. (This definition is useful for comparison with an emitting surface, which can only emit outwards towards a hemisphere.)

An "isotropic point emitter" emits photons radially outwards with spherical symmetry. Clearly, one single isotropic point emitter must have the property of being equally bright when viewed from any direction. The incident radiation flux reaching an enclosing hemisphere of radius r is therefore uniform, and can be called $G(r)$. Equating emitted and arriving power gives the following expression for $G(r)$.

$$G(r) = \frac{Q_1}{2\pi r^2} \tag{2}$$

The power arriving on a surface patch of the hemisphere with area $\mathrm{d}S$ is therefore $\mathrm{d}Q_{1S}$, as follows.

$$\mathrm{d}Q_{1S} = \left(\frac{Q_1}{2\pi r^2}\right)\,\mathrm{d}S \tag{3}$$

This power transport can also be written in terms of the solid angle, $\mathrm{d}\Omega = \mathrm{d}S/r^2$, that is occupied by the patch.

$$\mathrm{d}Q_{1S} = \left(\frac{Q_1}{2\pi}\right)\,\mathrm{d}\Omega \tag{4}$$

Now consider a small area $\mathrm{d}A_1$ of a Lambertian surface emitter that has total area A_1. It is defined as having the property of appearing equally bright when viewed from any direction (at a fixed distance).

The total power emitted by the area element is simply the surface's radiosity (emitted Watts per square metre) multiplied by its area.

$$\mathrm{d}Q_1 = J_1 \mathrm{d}A_1 \tag{5}$$

Now, we need a criterion for what it means for the surface to appear equally bright when viewed from any direction. If we think in terms of photons arriving at a detector which has area A_{det}, then we can write down such a criterion, which we shall call **uniform radiance**. The radiance of the emitting surface element, $\mathrm{d}A_1$, is proportional to the number of photons it sends to the detector, divided by both the detector area and by the apparent size (in solid angle) of the emitting surface when it is viewed from the detector.

In the diagram, imagine the emitting surface $\mathrm{d}A_1$ emits N photons per second towards the detector. The detector may be either vertically above the emitter (which is horizontal and flat), or in general the detector may be at some angle θ_1 from the vertical, but in both cases the detector surface is perpendicular (i.e. square-on) to the arriving rays of light. In the general case of the detector at θ_1, the number of photons emitted in this direction can be defined with an angular dependence $f(\theta_1)$ which describes the rate at which photons are emitted along a trajectory at an angle θ_1 as a fraction of the rate at which they are emitted vertically.

N must be proportional to the emitter area, so it is equal to some constant of proportionality C multiplied by emitter area $\mathrm{d}A_1$ and, by considering equation 4, it must also be proportional to the solid angle $\mathrm{d}\Omega_{\text{det}}$ occupied by the detector.

Considering the vertical detector, the number of photons arriving at the detector is N_{vert} as follows.

$$N_{\text{vert}} = C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\text{det}} \qquad (6)$$

And the radiance of the emitter, considering the detector area A_{det} and the angular size of the emitter, $\mathrm{d}\Omega_1$ is

$$Radiance = \frac{C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\text{det}}}{A_{\text{det}}\ \mathrm{d}\Omega_1} \qquad (7)$$

In the case of the detector at an angle θ_1, the number of detected photons is

$$N_\theta = C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\text{det}}\ f(\theta_1) \qquad (8)$$

Where $f(\theta_1)$ accounts for the surface's angular emission. The angular size of the emitter viewed from the detector is now $\mathrm{d}\Omega_1 \cos(\theta_1)$, so its apparent radiance at the detector is

$$Radiance = \frac{C\ \mathrm{d}A_1\ \mathrm{d}\Omega_{\text{det}}\ f(\theta_1)}{A_{\text{det}}\ \mathrm{d}\Omega_1\ \cos(\theta_1)} \qquad (9)$$

In order for the emitting surface to have the Lambertian property of equal radiance from every observation angle, so that equations (7) and (9) are equal for any θ_1, it must have an angular emission behaviour $f(\theta_1) = \cos(\theta_1)$.

$$f(\theta_1) = \cos(\theta_1) \qquad \text{for a Lambertian emitter} \qquad (10)$$

This is not very intuitive, but it is true. In order for the Lambertian emitter surface to appear equally bright to a detector at any angle, it actually must have an angle-dependent emission – precisely as in equation 10. This is so that the probability of emitting photons in a particular direction decreases along with the decreasing apparent size of the emitter surface at increasingly oblique angles. [The reason why this angular-dependent emission does not apply to the point emitter is that the apparent size of a point emitter is a constant at any angle, because it is determined by diffraction-limited blurring in the camera rather than by its area, because a point has zero area.]

Now that we have obtained an expression for $f(\theta_1)$ in equation 8, we can compute the radiant power $\mathrm{d}Q_{1s}$ emitted from a small surface $\mathrm{d}A_1$ towards a distant surface $\mathrm{d}A_2$. By considering this amount of power as a fraction of the total emission, a view factor is obtained.

Given equation (8) which describes the number of photons arriving at some target surface with angular size $\mathrm{d}\Omega_{\mathrm{det}}$, we can relate this target angular size to the area of a general target surface, $\mathrm{d}A_2$. Using the ideas of vector area, we see a small patch of target surface at distance r from the emitter occupies a solid angle

$$\mathrm{d}\Omega_{\mathrm{det}} = \frac{\mathrm{d}A_2 \cos(\theta_2)}{r^2} \tag{11}$$

Conceptually, the power transported from surface 1 to surface 2 is proportional to the number of photons transported, so the expression in equation (8) can be changed from one in terms of photons per second to one in terms of Watts by changing the constant of proportionality (from C to I_1).

$$\mathrm{d}Q_{1s} = \frac{I_1\, \mathrm{d}A_1 \cos(\theta_1) \mathrm{d}A_2 \cos(\theta_2)}{r^2} \tag{12}$$

The total power emitted by the surface element $\mathrm{d}A_1$ is known from equation (5) to be equal to $J_1 \mathrm{d}A_1$. Since all of this radiant power must be transported to an enclosing hemisphere, for which incident radiation from a small source all arrives perpendicular to the surface ($\theta_2 = 0$ for all received radiation), and $\mathrm{d}A_2 = r^2 \sin(\theta_1)\ \mathrm{d}\theta_1\, \mathrm{d}\phi_1$, we can equate emitted and received power with the following integral over a hemisphere.

$$J_1 \mathrm{d}A_1 = I_1 \mathrm{d}A_1 \int \sin(\theta_1) \cos(\theta_1)\ \mathrm{d}\theta_1\, \mathrm{d}\phi_1 \tag{13}$$

From which we can obtain

$$I_1 = \frac{J_1}{\pi} \tag{14}$$

And hence, since $F_{12} = Q_{1s}/J_1 A_1$

$$F_{12} = \frac{1}{A_1} \int\int \frac{\cos(\theta_1)\cos(\theta_2)\, \mathrm{d}A_1\, \mathrm{d}A_2}{\pi r^2} \tag{15}$$

The very thing that we wanted to demonstrate.

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Further reading

https://ipfs.io/ipfs/QmXoypizjW3WknFiJnKLwHCnL72vedxjQkDDP1mXWo6uco/wiki/Lambert's_cosine_law.html

Now, we assume that the surface elements $\mathrm{d}A_1$ and $\mathrm{d}A_2$ are distant, so that r is large compared with the size of the surface elements. This means that when photons from $\mathrm{d}A_1$ reach a hemisphere at the (large) target distance, the photons travelling towards a small hemispherical patch $\mathrm{d}S$ are travelling along essentially parallel trajectories – any photons with non-parallel trajectories would have had sufficient distance to diverge and strike a different patch on the hemisphere. If we consider the photons emitted from $\mathrm{d}A_1$ with direction (θ_1, ϕ_1), they must strike a patch with the following area.

$$\mathrm{d}S_1(\theta_1, \phi_1) = \mathrm{d}A_1 \cos(\theta_1) \tag{5}$$

If we consider the incident radiation flux at the hemispherical patch at a position (θ_1, ϕ_1), we can describe the incident flux in terms of the power transported to this area element, $\mathrm{d}^2 Q_{1s}(\theta_1, \phi_1)$. (This is written d^2 because $\mathrm{d}A_1$ and its projection $\mathrm{d}S_1$ are both small.)

$$G(r, \theta_1, \phi_1) = \frac{\mathrm{d}^2 Q_{1s}(\theta_1,\phi_1)}{\mathrm{d}S_1(\theta_1,\phi_1)} = \frac{\mathrm{d}^2 Q_{1s}(\theta_1,\phi_1)}{\mathrm{d}A_1 \cos(\theta_1)} \tag{6}$$

The question, now, is how much power is transported to this patch by radiation from a Lambertian surface element.

We need to define the radiant intensity of the emitter surface, $I_{e,\Omega}$, which is the power emitted per unit area of the emitter surface, per unit solid angle of the target. Given the definition of $I_{\mathrm{e},\Omega}$, and knowing the area of the emitter element is $\mathrm{d}A_1$ and the solid angle occupied by the target patch of area S_p is $\mathrm{d}S_\mathrm{p}/r^2$, we can write down an expression for the power transported to the patch.

$$\mathrm{d}Q_{1s} = I_{\mathrm{e},\Omega} \left(\frac{\mathrm{d}S_1}{r^2}\right) \mathrm{d}A_1$$

This radiance may be a directional quantity, $L_{\mathrm{e},\Omega}(\theta_1, \phi_1)$. Given the definition of $L_{\mathrm{e},\Omega}$, and knowing the area of the emitter element is $\mathrm{d}A_1$ and the solid angle occupied by the target patch of area S_p is $\mathrm{d}S_\mathrm{p}/r^2$, we can write down an expression for the power transported to the patch.

$$\mathrm{d}Q_{1s} = L_{\mathrm{e},\Omega} \left(\frac{\mathrm{d}S_1 \cos(\theta_1)}{r^2}\right) \mathrm{d}A_1 \tag{7}$$

From equation (6), the incident radiation flux reaching a patch of area S_p on the hemisphere is therefore

$$G(r, \theta_1, \phi_1) = \frac{L_{\mathrm{e},\Omega} \left(\frac{\mathrm{d}S}{r^2}\right) \mathrm{d}A_1}{\mathrm{d}A_1 \cos(\theta_1)} \qquad ($$

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radiance of the emitter surface, $L_{\mathrm{e},\Omega}$, which is the power emitted per unit area of projected area of the emitter surface, per unit solid angle of the target. This radiance may be a directional quantity, $L_{\mathrm{e},\Omega}(\theta_1, \phi_1)$. Given the definition of $L_{\mathrm{e},\Omega}$, and knowing the area of the emitter element is $\mathrm{d}A_1$ and the solid angle occupied by the target patch of area S_p is $\mathrm{d}S_\mathrm{p}/r^2$, we can write down an expression for the power transported to the patch.