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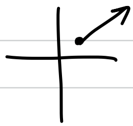
Today
- CS Analyst Groups
- Questions about the HW
- Linear Algebra

- Scalar : 20
- Vector : $\bar{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$



- Matrix : $\underset{\text{rows}}{n} \times \underset{\text{columns}}{m}$

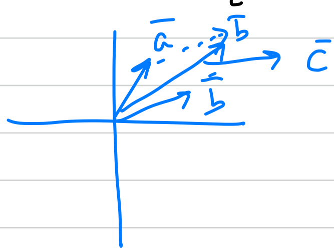
$$\begin{bmatrix} 1 & \\ 2 & \dots \\ 3 & \end{bmatrix}$$

Vectors:

Addition, subtraction, multiplication by a scalar

$\bar{v} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\bar{v} + \bar{x} = \begin{bmatrix} 1+3 \\ 2+5 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$



Matrix Operations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 21+16 & \dots \\ 4+1 & \end{bmatrix} = \begin{bmatrix} 37 & 5 \\ 39 & 5 \end{bmatrix}$$

$$\begin{bmatrix}3 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}3+4\\ 2+1\end{bmatrix} = \begin{bmatrix}7\\ 3\end{bmatrix}$$

Matrix are linear transformations

vector $\longrightarrow$ vector

$U \to V$

U = domain, vector space

V = codomain

Transpose

$$\begin{bmatrix}2 & 3\\ 1 & 1\end{bmatrix}^T = \begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix}$$

Determinants:

Matrix $\to$ scalar

$$\begin{bmatrix}0 & -2/3\\ 3/2 & 0\end{bmatrix} = 0 \cdot 0 - 3/2 \cdot (-2/3)$$
$$= 0 - (-1) = 1$$

Stretch vertically by $3/2$
Squeezing horizontally by $2/3$
Counterclockwise $90°$ rotation

area/volume will be the sam

Inverses & Identity

$$M \cdot M^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Dot product / inner product:

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + b \cdot d$$

$$= \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} c \\ a \end{bmatrix}$$

Vector Norm:

$$\|v\| = \sqrt{\sum_{i \text{ in } v} v_i^2}$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 \cdots}$$

Linear Algebra Review

Eigenvectors / Eigenvalues

$A\vec{x} = \vec{b}$ $\qquad$ A $m \times n$, x $n \times 1$, b $m \times 1$

$A\vec{v} = \lambda \vec{v}$ $\qquad$ λ eigenvalue, $\vec{v}$ eigenvector

$A\vec{v} - \lambda\vec{v} = 0$

$(A - \lambda I)\vec{v} = 0$ $\qquad$ $\det(A - \lambda I) = 0$

$(A - \lambda I)\vec{v} = 0$ $\qquad$ $\begin{bmatrix} A - \lambda I \\ \\ \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$(A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

$$\begin{aligned} \det(A - \lambda I) &= (-\lambda)(-3-\lambda) + 2 \\ &= \lambda^2 + 3\lambda + 2 \\ &(\lambda+2)(\lambda+1) \quad \lambda = -2, -1 \end{aligned}$$

$\lambda = -1$ $\qquad$ $A - \lambda I \rightarrow A + I$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1} \begin{array}{ccc} x_1 & x_2 & \\ \end{array} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_2$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Eigen Decomposition

$$A\vec{x} = \vec{b} \qquad \vec{v}_1, \dots, \vec{v}_n, \lambda_1, \dots, \lambda_n$$

$$A\vec{v}_1 = \lambda\vec{v}_1$$

$$A\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}\begin{bmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

$$Q = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix} \qquad \boldsymbol{\lambda} = \begin{bmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

$$AQ = Q\boldsymbol{\lambda}$$

$$A\,\underbrace{\cancel{QQ^{-1}}}_{I} = Q\lambda Q^{-1}$$

$$A = Q\lambda Q^{-1}$$

Singular Value Decomposition

Data Reduction -> PCA, Linear Regression, etc.

$$\underline{A = U \Sigma V^T} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ & & & \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & \cdots & \\ 0 & \sigma_2 & & \\ 0 & & \ddots & \\ 0 & & & \\ \vdots & & & \sigma_n \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ & & & \end{bmatrix}$$

left singular vectors — singular values — right singular vectors

U, V = orthogonal

$UU^T = I$, $U^T = U^{-1}$

$$\boxed{A^T A} = \underline{(U \Sigma V^T)^T (U \Sigma V^T)}$$
$$= V \Sigma^T U^T U \Sigma V^T = \underline{V \Sigma^T \Sigma V^T} = V(\Sigma^T \Sigma)V^{-1}$$

$$\underline{AV = U\Sigma}$$

$V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \\ & & \end{bmatrix}$ where v_i is an eigenvector of $A^T A$

$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$ where $\sigma_i = \sqrt{\lambda_i}$ where λ_i is an eigenvalue of $A^T A$

if A is invertible: Σ is $n \times n$

$$A^{-1} = (U \Sigma V^T)^{-1} = V \Sigma^{-1} U^T$$

$$A^{-1} = V \Sigma^{-1} U^T$$

else: Σ is $m \times n$, $m \neq n$

pseudoinverse

$A^+ = V \Sigma^+ U^T$ where $\underline{\Sigma^+ \Sigma} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{bmatrix}$ (m 1's)

Linear Regression Example

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \quad y = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 4 \end{bmatrix} \rightarrow \boxed{\begin{aligned} b_0 + b_1 &= 3 \\ b_0 + 3b_1 &= 2 \\ b_0 + 4b_1 &= 5 \\ b_0 + 5b_1 &= 4 \end{aligned}}$$

$\underline{5} \quad \underline{x} \qquad \underline{y}$

$$\underline{X}\,\underline{\beta} = y$$

$$\varepsilon = y - x\beta$$

X	4×2
y	4×1
β	2×1

$$MSE = \frac{1}{n}\varepsilon^T\varepsilon$$

$$= \frac{1}{n}\left(y - x\beta\right)^T\left(y - x\beta\right)$$

$$\underbrace{(1\times4)(4\times2)}_{(1\times2)}(2\times1)$$

$$(1\times2)(2\times1)$$

$$(1\times1)$$

$$= \frac{1}{n}\left(\left(y^T - \beta^T x^T\right)\left(y - x\beta\right)\right)$$

$$= \frac{1}{n}\left(y^Ty - \underline{y^Tx\beta} - \underline{\beta^Tx^Ty} + \beta^Tx^Tx\beta\right)$$

note: $\underline{y^Tx\beta = \left(\beta^Tx^Ty\right)^T}$ and both are 1×1

so $y^Tx\beta = \beta^Tx^Ty$

$$\rightarrow \frac{1}{n}\left(y^Ty - 2\beta^Tx^Ty + \beta^Tx^Tx\beta\right)$$

$$\nabla MSE = \frac{1}{n}\left(\nabla y^T y - 2\nabla \beta^T X^T y + \nabla \beta^T X^T X \beta\right)$$
w.r.t β

$$= \frac{1}{n}\left(0 - 2X^T y + 2X^T X \beta\right)$$

$$= \frac{2}{n}\left(X^T X \beta - X^T y\right)$$

$$0 = \cancel{\frac{2}{n}}\left(X^T X \beta - X^T y\right)$$

$$X^T y = X^T X \beta$$

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

Using SVD:

$$X\beta = y$$

$$\hat{\beta} = X^{+} y$$