1. Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions:

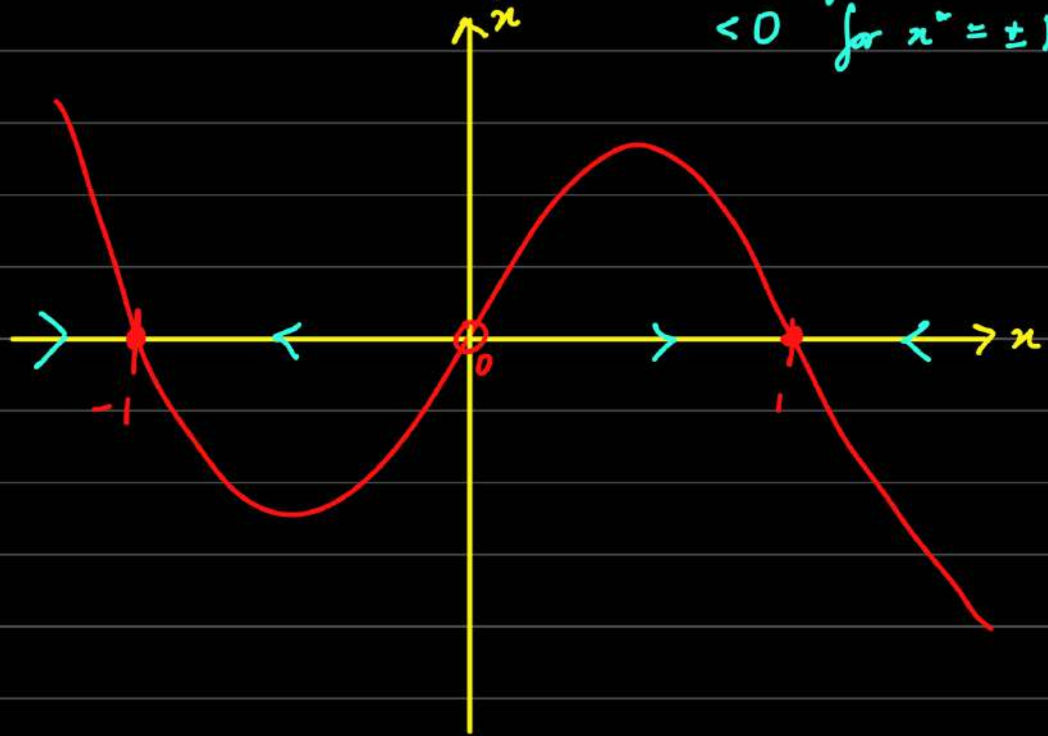
   (a) $\dot{x} = x - x^3$

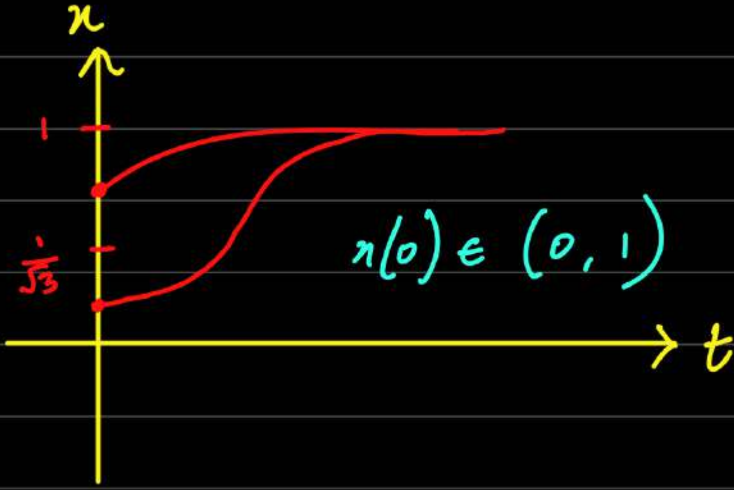
   (b) $\dot{x} = \exp[-x] \sin x$

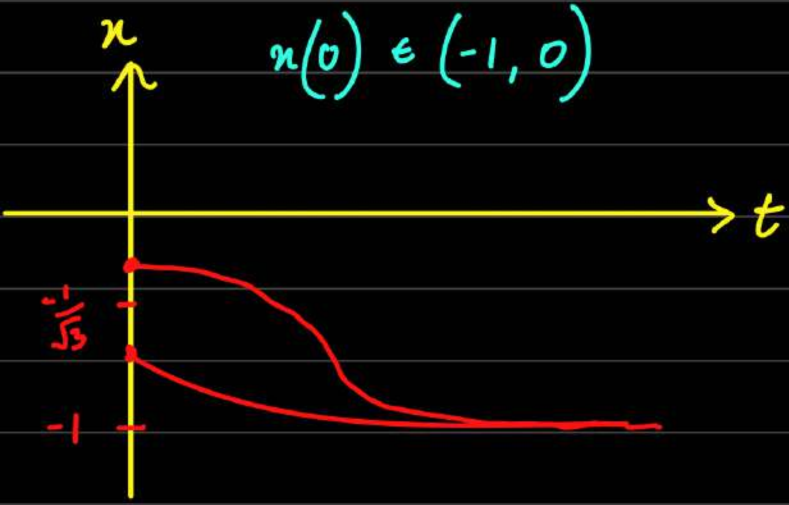
For $\dot{x} = f(x)$, a fixed point is some x^* such that $f(x^*) = 0$.
It is stable if $f'(x^*) < 0$
It is unstable if $f'(x^*) > 0$.

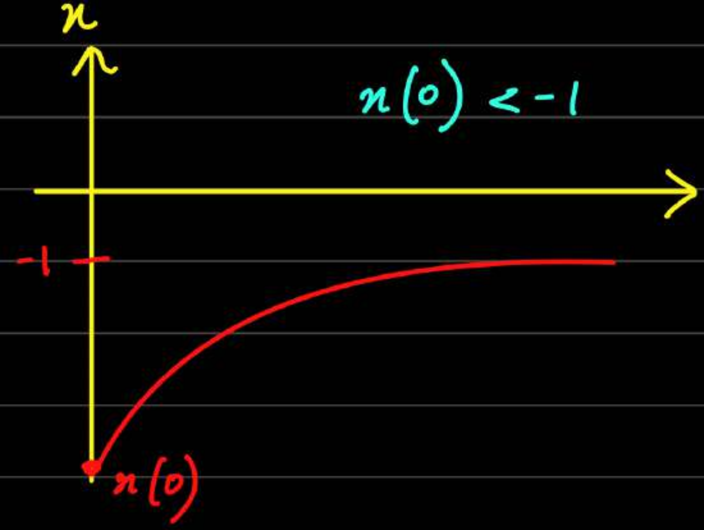
a) $\dot{x} = x - x^3 \Rightarrow f(x) = x - x^3 = 0 \Rightarrow x^* = 0, 1, -1$

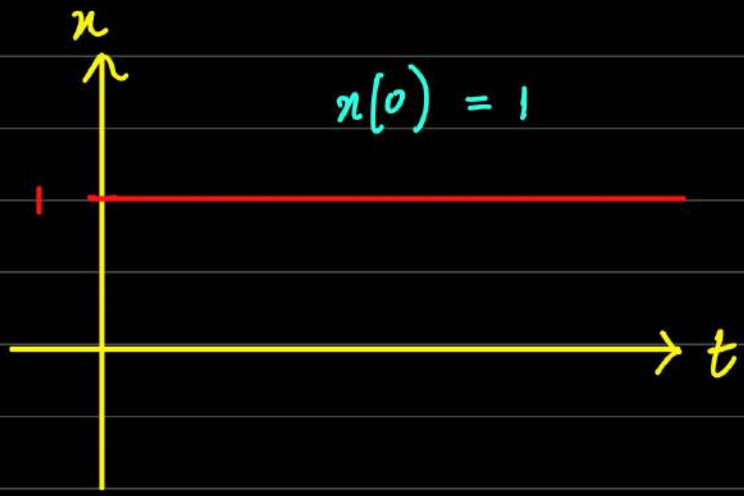
$f'(x) = 1 - 3x^2 > 0$ for $x^* = 0$,
< 0 for $x^* = \pm 1$.

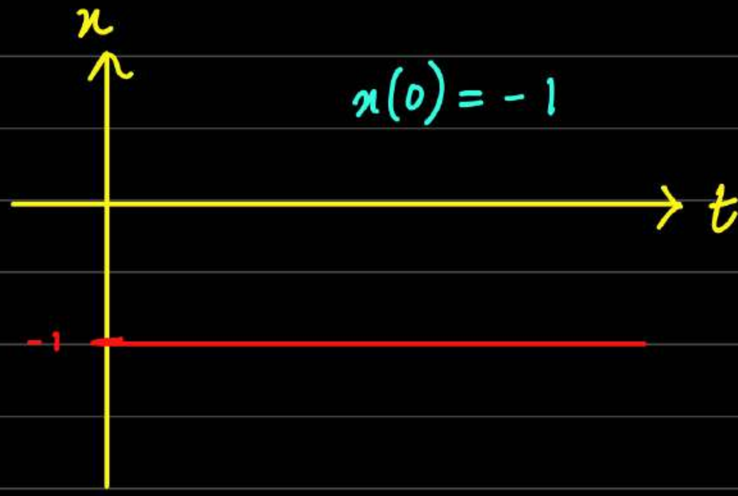


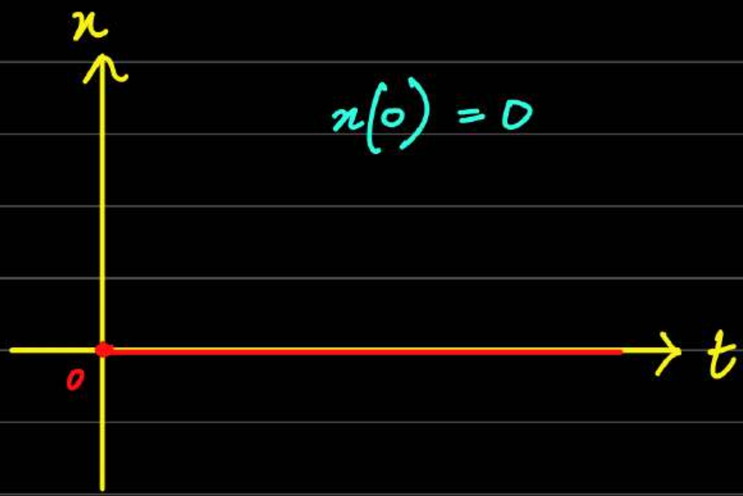












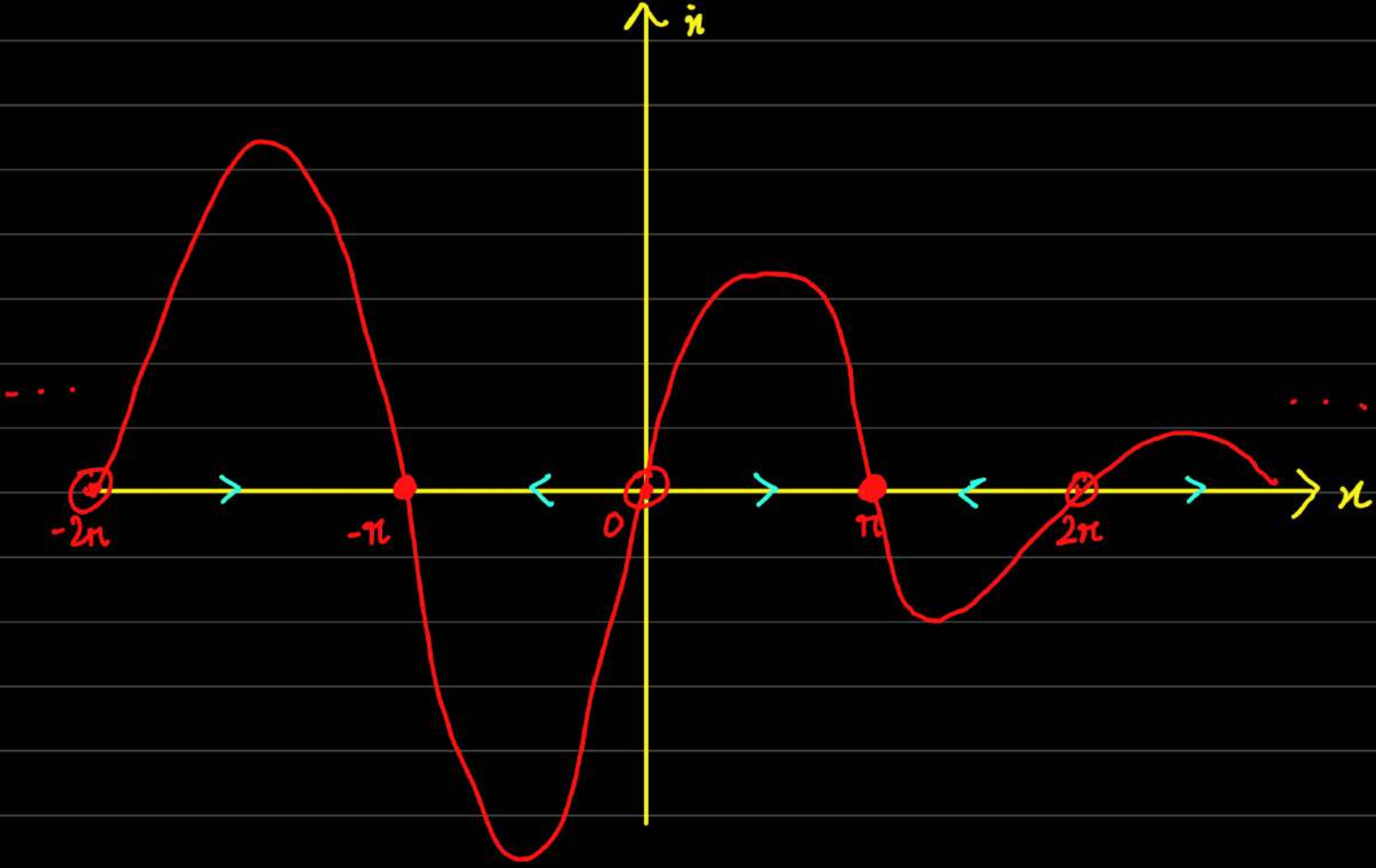
b) $\dot{x} = e^{-x}\sin x \Rightarrow f(x) = e^{-x}\sin x$

$x^* = n\pi$ $\quad$ $n \to$ all integers.

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$

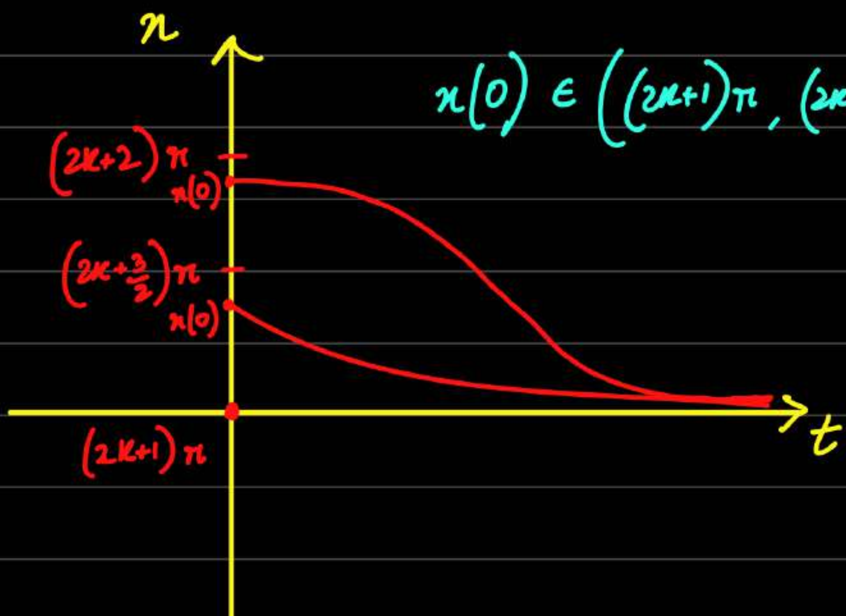
$f'(n\pi) = e^{-n\pi}\cdot(-1)^n$ $\quad < 0$ when n odd

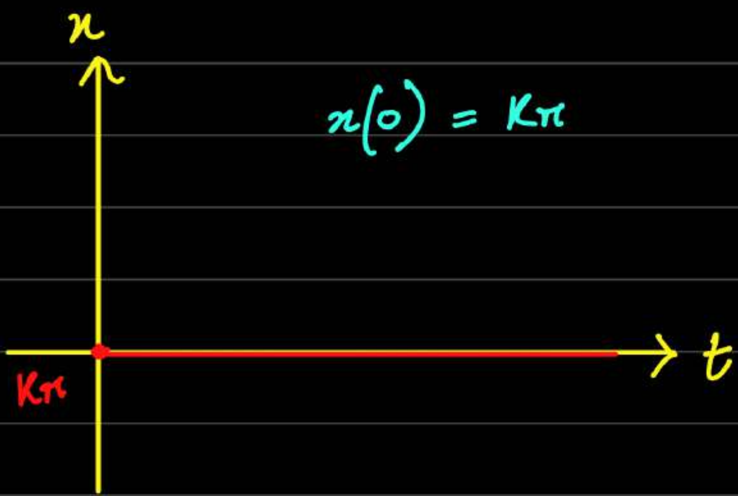
> 0 when n even



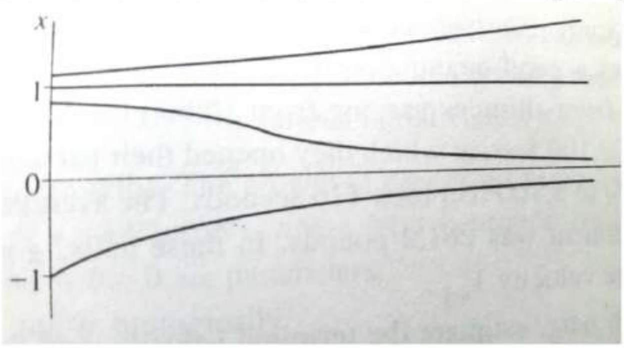
$x(0) \in (2k\pi, (2k+1)\pi)$: 2 cases

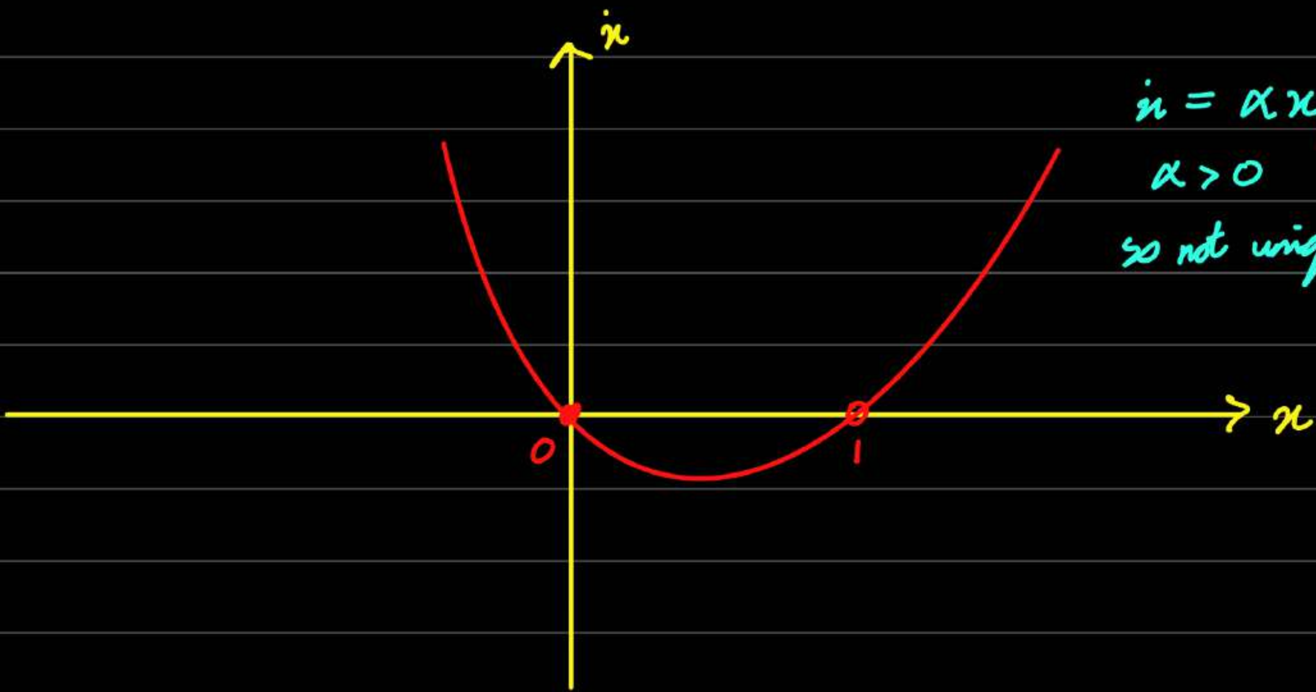
$x(0) \in ((2k+1)\pi, (2k+2)\pi)$: 2 cases





2. Find an equation $\dot{x} = f(x)$ whose solutions $x(t)$ are consistent with those shown in Fig.1. Can there be more than one unique $f(x)$?





$\dot{x} = \alpha x(x-1)$

$\alpha > 0$

so not unique.

3. The velocity $v(t)$ of a skydiver falling to the ground is governed by $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity and $k > 0$ is a constant related to air resistance.

(a) Obtain the analytical solution for $v(t)$, assuming that $v(0) = 0$.

(b) Find the limit of $v(t)$ as $t \to \infty$.

(c) Give a graphical analysis of this problem and thereby rederive a formula for the terminal velocity.

a)

$$\frac{dv}{dt} = g - \frac{k}{m} v^2 \qquad v\sqrt{\frac{k}{m}} = s$$

$$s = v\alpha \qquad \alpha = \sqrt{\frac{k}{m}}$$

$$\frac{dv}{dt} = \frac{1}{\alpha}\frac{ds}{dt}$$

$$\frac{1}{\alpha}\frac{ds}{dt} = g - s^2$$

$$\frac{ds}{g - s^2} = \alpha\, dt$$

$$\frac{1}{2\sqrt{g}} \int_0^s \left(\frac{1}{s+\sqrt{g}} - \frac{1}{s-\sqrt{g}} \right) ds = \alpha \int_0^t dt$$

$$\frac{1}{2\sqrt{g}} \ln\left(\left| \frac{s+\sqrt{g}}{s-\sqrt{g}} \right| \right) \Bigg|_0^s = \alpha t$$

$$\Rightarrow \frac{\sqrt{g}+s}{\sqrt{g}-s} = e^{2\alpha\sqrt{g}t}$$

$$\sqrt{g} + s = \sqrt{g}\, e^{2\alpha\sqrt{g}t} - s\, e^{2\alpha\sqrt{g}t}$$

$$\Rightarrow v\sqrt{\frac{k}{m}} = \sqrt{g}\left(\frac{e^{2\alpha\sqrt{g}t} - 1}{e^{2\alpha\sqrt{g}t} + 1} \right)$$

$$\Rightarrow \boxed{v(t) = \sqrt{\frac{mg}{k}} \left(\frac{\exp\left(2\sqrt{\frac{gk}{m}}\, t\right) - 1}{\exp\left(2\sqrt{\frac{gk}{m}}\, t\right) + 1} \right)}$$

b) $$v(t) = v_T\left(\frac{1-e^{-t/\tau}}{1+e^{-t/\tau}}\right) \quad \text{where } v_T = \sqrt{\frac{mg}{k}}, \quad \tau = \frac{1}{2}\sqrt{\frac{m}{gk}}$$

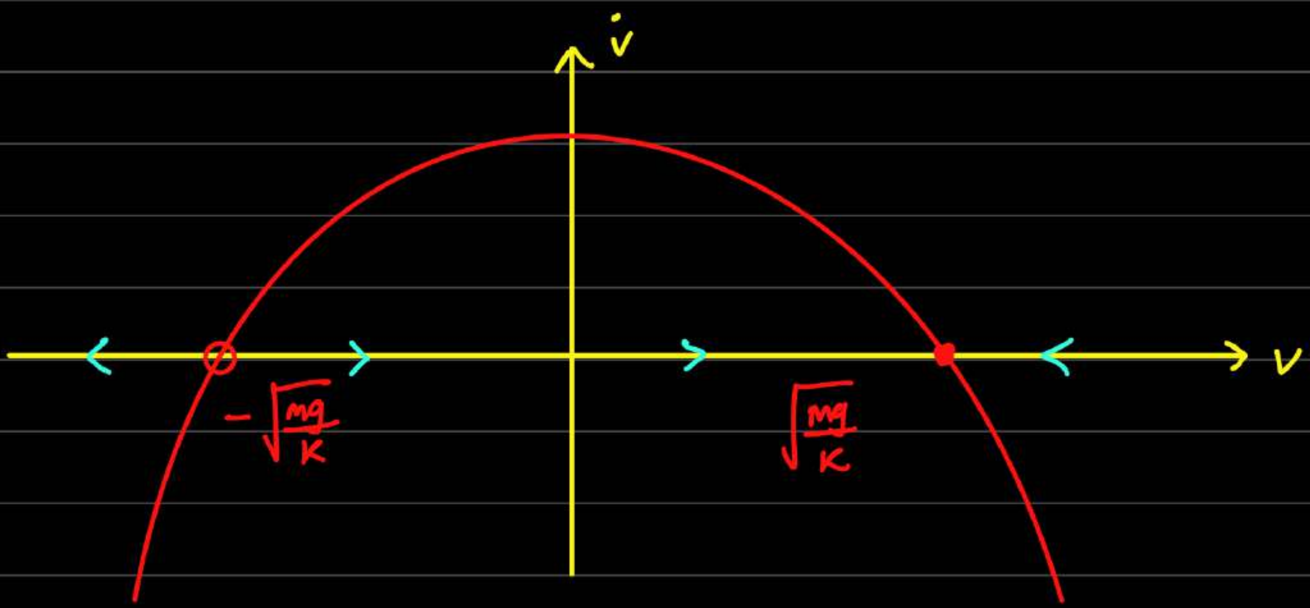
As $t \to \infty$, $$v(\infty) = v_T = \boxed{\sqrt{\frac{mg}{k}}}$$

c) $\dot{v} = g - \frac{k}{m}v^2$ is a 1D flow of the form $\dot{v} = f(v)$ with $f(v) = g - \frac{k}{m}v^2$

So $f(v) = g - \frac{k}{m}v^2$, $\quad v^* = \pm\sqrt{\frac{mg}{k}} \quad (f(v^*) = 0)$

$$f'(v^*) = -\frac{2k}{m}\sqrt{\frac{mg}{k}} < 0 \quad \text{so } v^* = +\sqrt{\frac{mg}{k}} \text{ is stable.}$$

$$= \frac{2k}{m}\sqrt{\frac{mg}{k}} > 0 \quad \text{so } v^* = -\sqrt{\frac{mg}{k}} \text{ is unstable.}$$



4. For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is called the Allee effect and can be explained by the fact that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

(a) Show that the model $\dot{N}/N = r - a(N-b)^2$ provides an example of the Allee effect, if r, a and b satisfy certain constraints.

(b) Find all the fixed points of the system and classify their stability.

(c) Sketch the solutions $N(t)$ for different initial conditions.

(d) Compare the solutions $N(t)$ to those found from the more popular logistic equation. What are the **qualitative** differences?

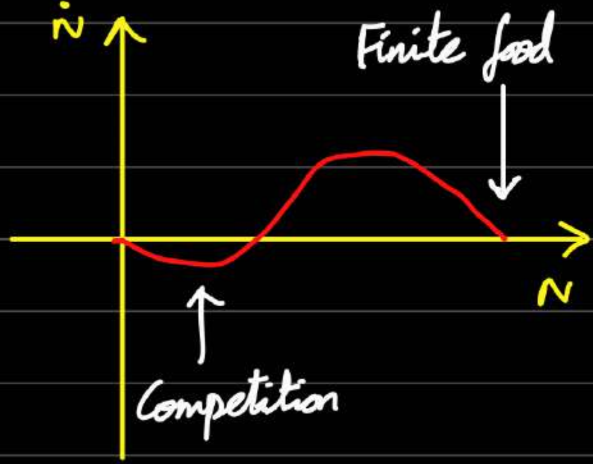
a) $\frac{\dot{N}}{N} = r - a(N-b)^2$, maximum at $N = b$ when $\boxed{r, a, b > 0}$.

$$\dot{N} = N\left[r - a(N-b)^2\right] = f(N)$$

$$f'(N) = \left[r - a(N-b)^2\right] + N\left[-2a(N-b)\right]$$

$$f'(0) = r - ab^2 < 0$$

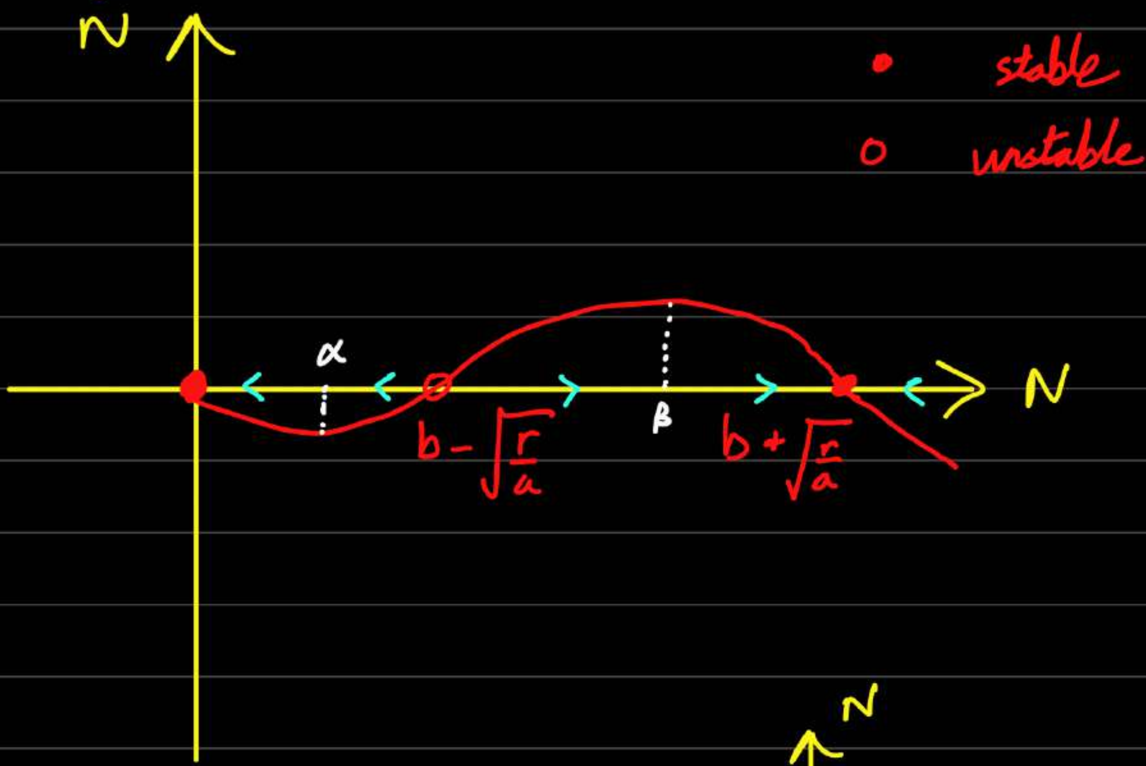
$$\therefore \boxed{r < ab^2}$$



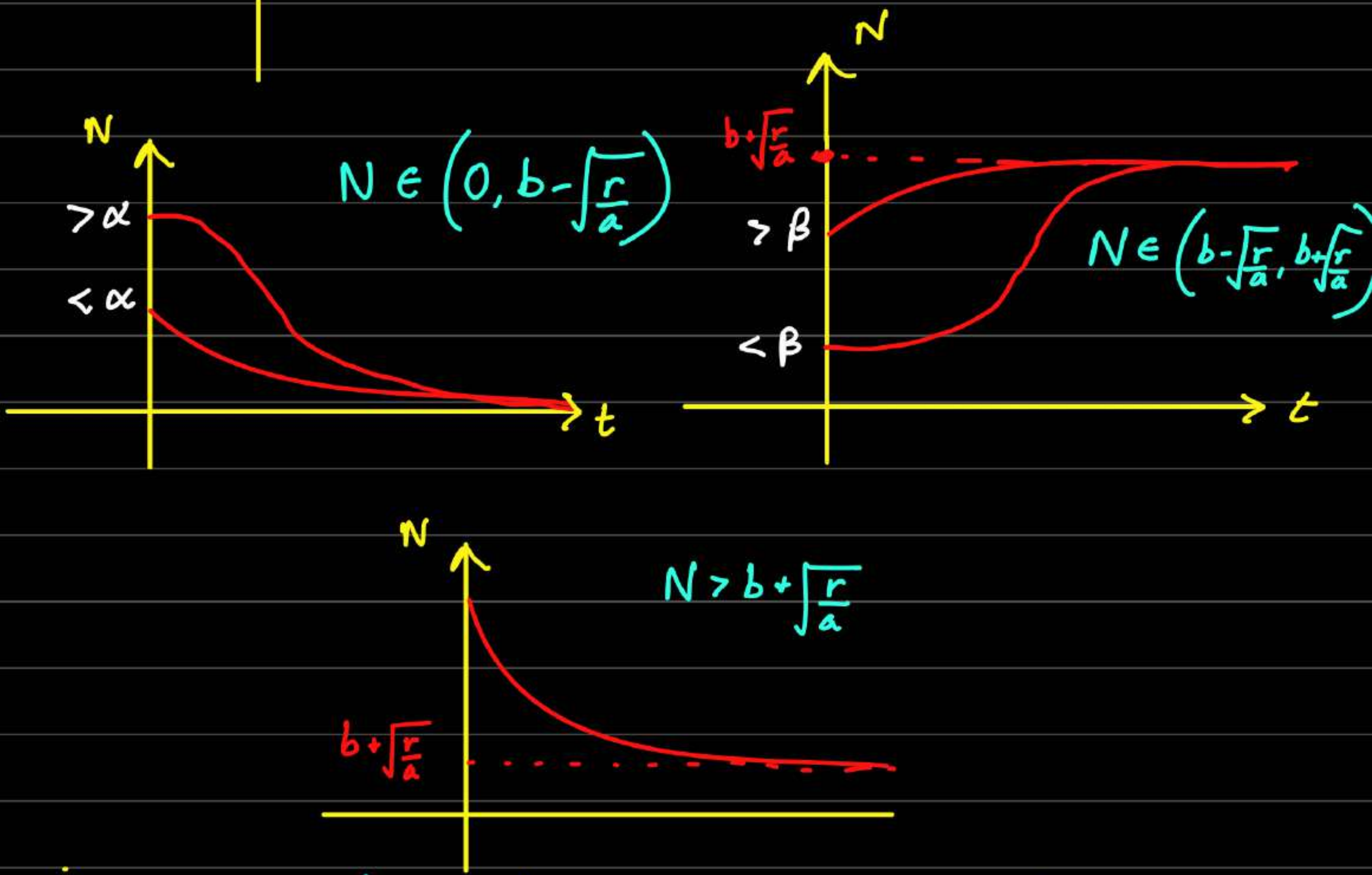
b) $$\dot{N} = N\left(r - aN^2 + 2abN - ab^2\right)$$
$$\dot{N} = -N\left(aN^2 - 2abN + ab^2 - r\right)$$

So, $$N^* = \frac{2ab \pm \sqrt{4a^2b^2 - 4a(ab^2 - r)}}{2a}, \ 0$$

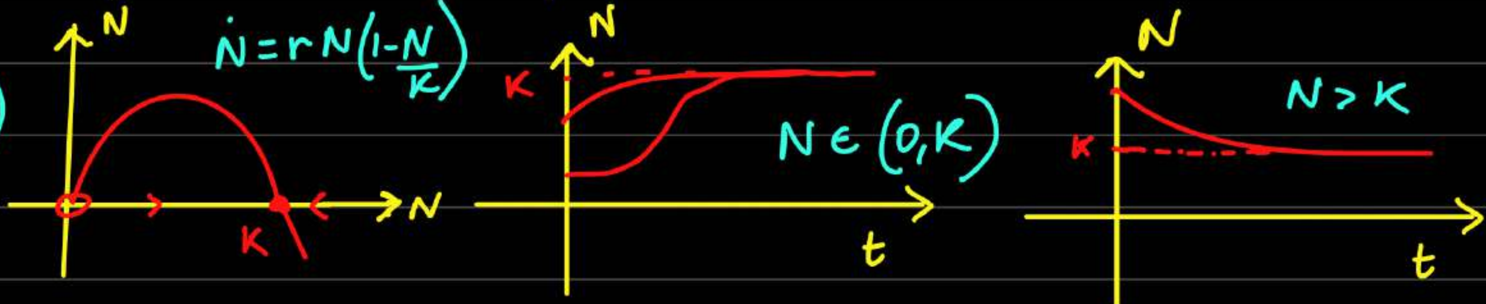
$$\boxed{N^* = b \pm \sqrt{\frac{r}{a}}, \ 0}$$



c)



d)



Only 2 fixed points (instead of 3). Does not model competition or limited food. Small populations increase rapidly and all populations stabilise at K.

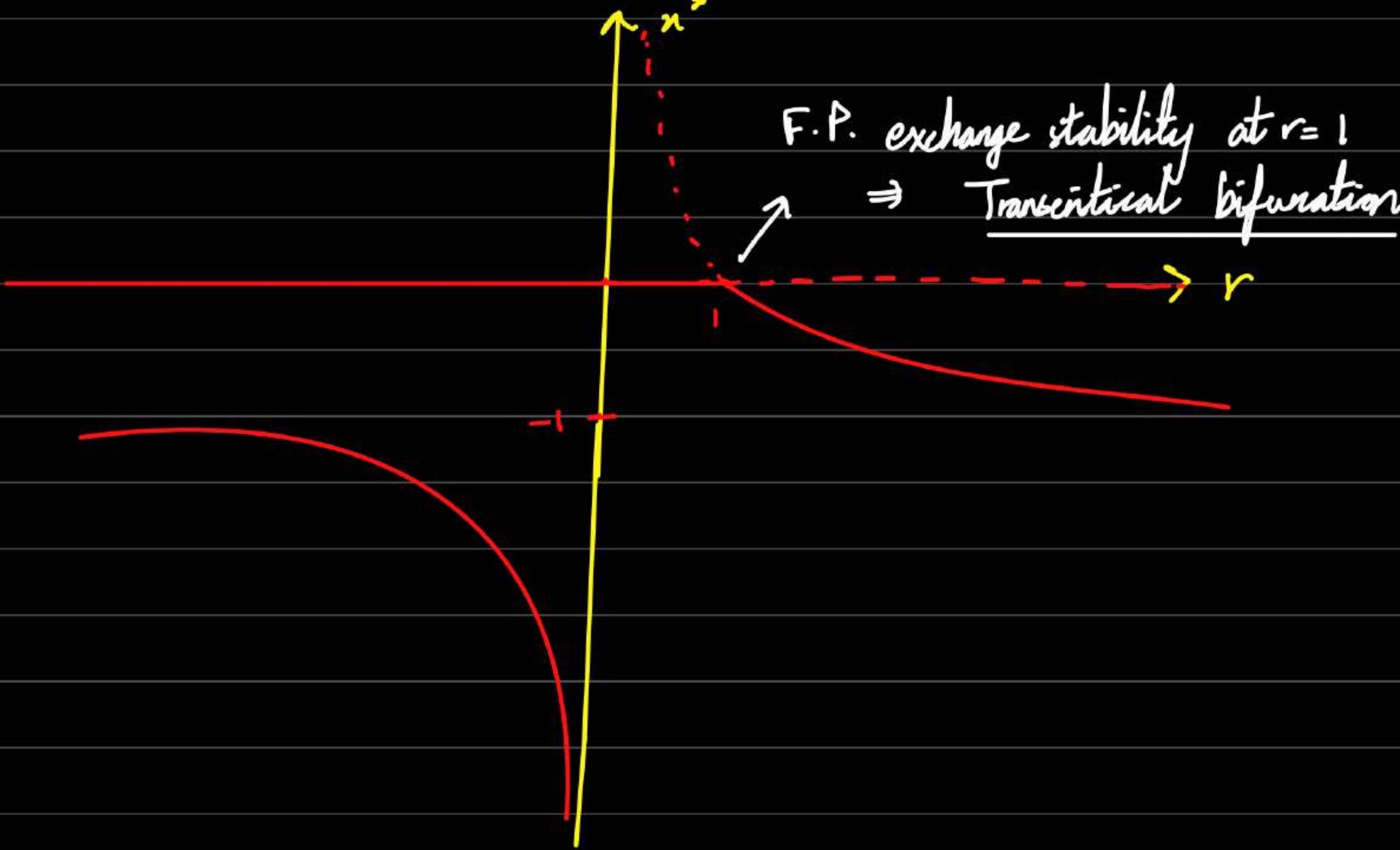
5. For the following problems, find the values of r at which bifurcations occur and classify these bifurcations:

(a) $\dot{x} = rx - x/(1+x)$

a) $\dot{x} = rx - \dfrac{x}{1+x} \Rightarrow x^* = 0, \ \dfrac{1}{r} - 1$

$f'(x) = r - \dfrac{1}{(1+x)^2}$

$f'(0) = r - 1$ $\qquad f'(1/r - 1) = r - r^2 = r(1-r)$

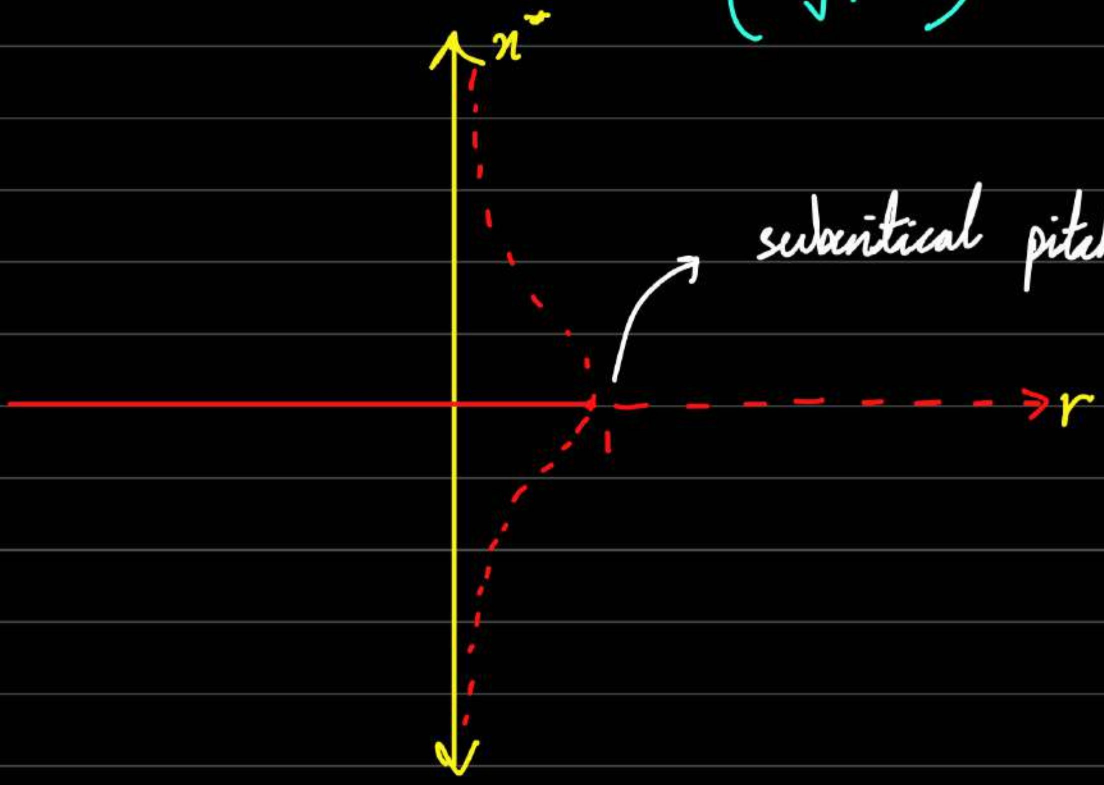


(b) $\dot{x} = rx - x/(1+x^2)$

b) $\dot{x} = x\left(r - \frac{1}{1+x^2}\right)$ so $x^* = 0, \pm\sqrt{\frac{1}{r} - 1}$

$f'(x) = r - \frac{1}{1+x^2} + \frac{2x^2}{(1+x^2)^2}$ $\qquad \frac{1}{1+x^2} = r$

$f'(0) = r - 1$ $\qquad f'\left(\pm\sqrt{\frac{1}{r}-1}\right) = 2r(1-r)$



subcritical pitchfork bifurcation at $r = 1$

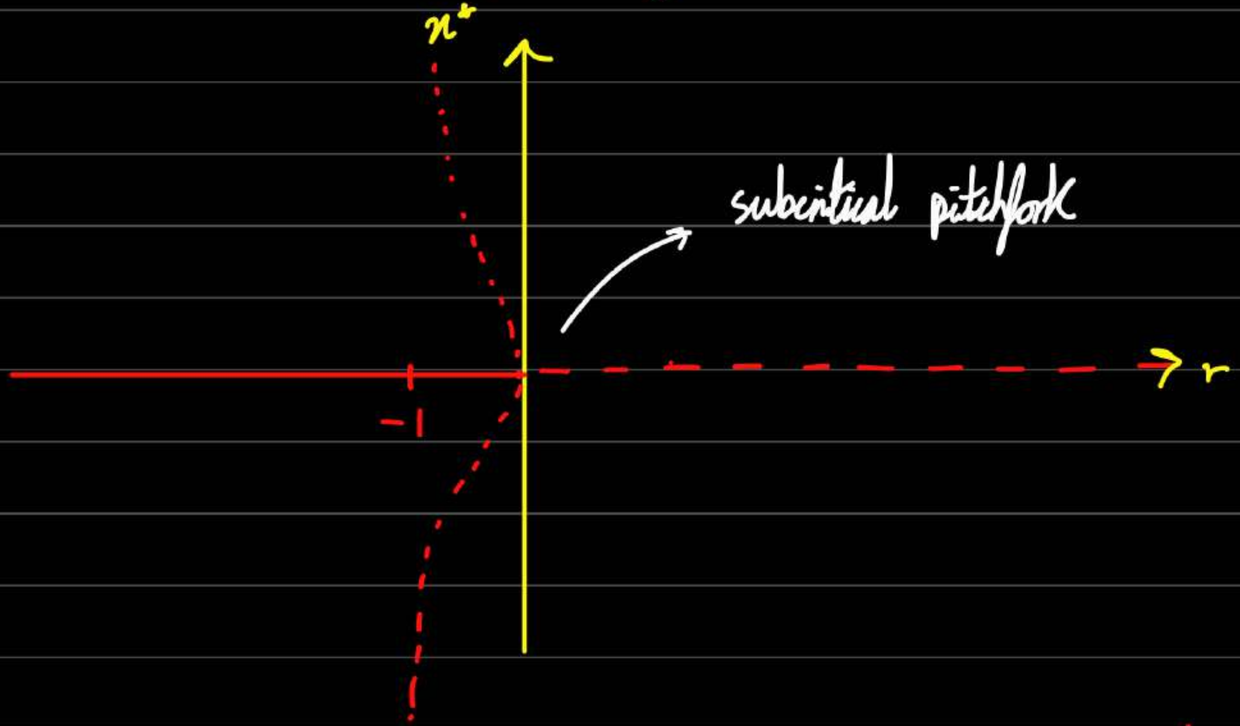
(c) $\dot{x} = rx + x^3/(1+x^2)$

$$\dot{x} = x\left(r + \frac{x^2}{1+x^2}\right) \Rightarrow x^* = 0,\ \pm\sqrt{\frac{-r}{r+1}}$$

$$f'(x) = \left(r + \frac{x^2}{1+x^2}\right) + x\left(\frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2}\right)$$

$$f'(x) = \left(r + \frac{x^2}{1+x^2}\right) + \frac{2x^2}{(1+x^2)^2}$$

$$f'(0) = r \quad , \quad f'\left(\pm\sqrt{\frac{-r}{r+1}}\right) = -2r(r+1)$$

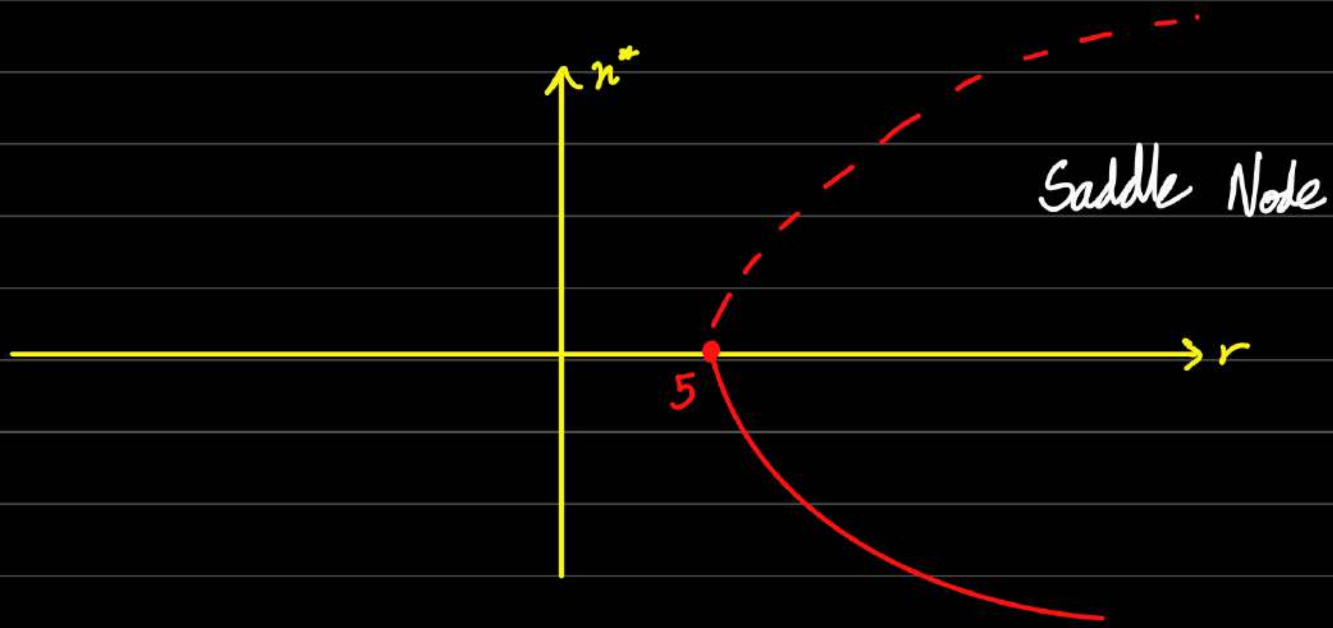


(d) $\dot{x} = 5 - re^{-x^2}$

d) $\dot{x} = 5 - re^{-x^2} \Rightarrow x^* = \pm\sqrt{-\ln\left(\frac{5}{r}\right)}$

$f'(x) = -r(-2x)e^{-x^2} = 2r\,xe^{-x^2}$

$= \pm 10\sqrt{-\ln\left(\frac{5}{r}\right)}$



6. The Maxwell-Bloch equations provide a sophisticated model for a laser and are given by

$$\begin{aligned}\dot{E} &= \kappa(P - E),\\ \dot{P} &= \gamma_1(ED - P),\\ \dot{D} &= \gamma_2(\lambda + 1 - D - \lambda EP),\end{aligned}$$

where, E is the electric field, P denotes the mean polarisation of atoms, D is population inversion, κ is the decay rate in the laser cavity due to beam transmission, γ_1, γ_2 are the decay rates of the atomic polarization and population inversion and λ is the pumping energy parameter.

(a) Assuming $\dot{P} = 0$, $\dot{D} = 0$, express P and D in terms of E, and thereby derive a first order equation for the evolution of E.
This is an acceptable approximation when $\gamma_1, \gamma_2 >> \kappa$, or in other words, the time scale for the evolution of E is significantly faster than the time scales of the other two state parameters.

a) $\dot{P} = 0 \Rightarrow P = ED$, $\dot{D} = 0 \Rightarrow \lambda(1 - EP) + 1 = D$

$$\therefore P = E\left(\lambda(1-EP) + 1\right) = \lambda E(1 - EP) + E$$

$$P = \lambda E - \lambda E^2 P + E$$

$$\Rightarrow \boxed{P = \frac{E(\lambda+1)}{1+\lambda E^2}}$$

$$D = \frac{P}{E} \Rightarrow \boxed{D = \frac{\lambda + 1}{1 + \lambda E^2}}$$

$$\dot{E} = \kappa(P - E) = \kappa E\left(\frac{\lambda+1}{1+\lambda E^2} - 1\right)$$

$$\Rightarrow \dot{E} = \kappa E\left(\frac{\lambda + 1 - 1 - \lambda E^2}{1 + \lambda E^2}\right) = \kappa\lambda E\left(\frac{1 - E^2}{1+\lambda E^2}\right)$$

$$\boxed{\dot{E} = \kappa\lambda E\left(\frac{1 - E^2}{1 + \lambda E^2}\right)}$$

(b) Find all the fixed points of the equation for E.

b) $\dot{E} = 0 \Rightarrow \quad E^* = 0, \ \pm 1$

(c) Draw the bifurcation diagram of E^* with λ as the bifurcation parameter. Identify the stable and unstable branches in the bifurcation diagram.

c) $f'(E) = \dfrac{k\lambda(1-E^2)}{1+\lambda E^2} - \dfrac{2k\lambda E^2}{1+\lambda E^2} - \dfrac{2k\lambda^2 E^2(1-E^2)}{(1+\lambda E^2)^2}$

If $\lambda > 0$

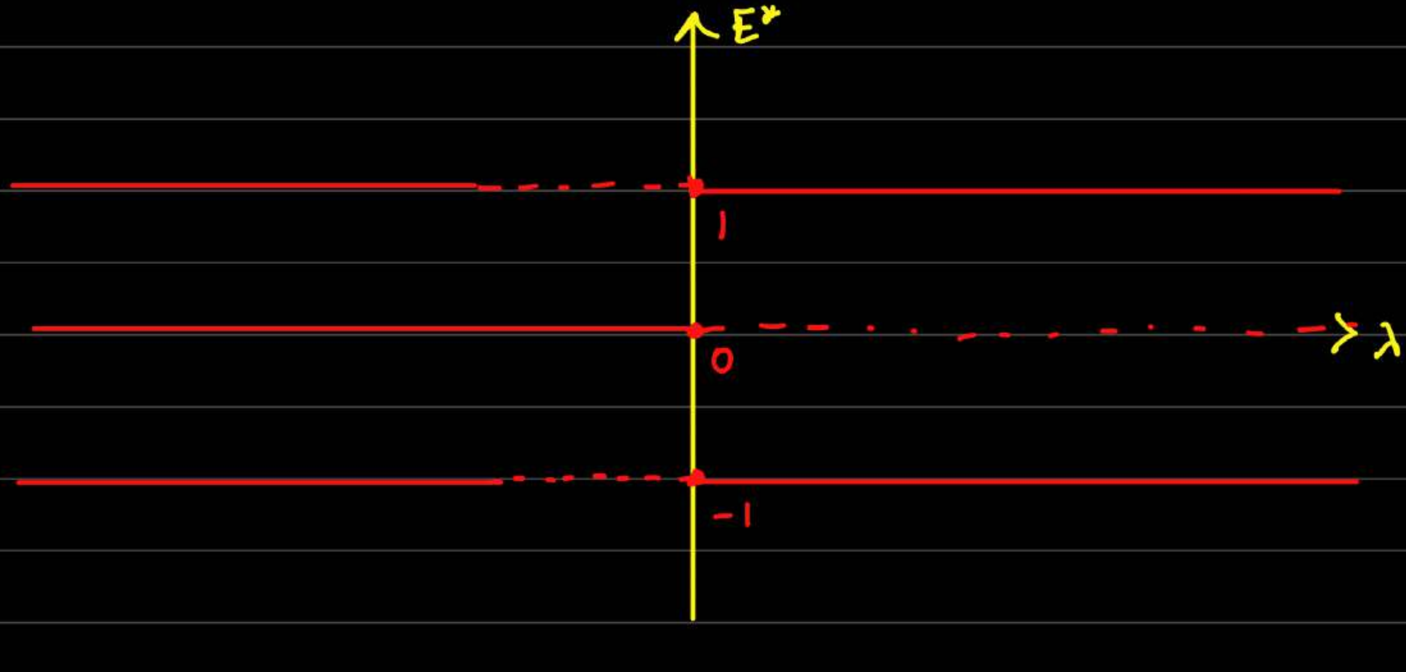
$f'(0) = k\lambda > 0$ so $E^* = 0$ is unstable

$f'(1) = \dfrac{-2k\lambda}{1+\lambda} < 0$ so $E^* = 1$ is stable

$f'(-1) = -\dfrac{2k\lambda}{1+\lambda} < 0$ so $E^* = -1$ is stable.

If $\lambda \in (-1, 0)$, 0 is stable
1 is unstable
−1 is unstable

If $\lambda < -1$, 0 is stable
1 is stable
−1 is stable



7. Show that any matrix of the form $\mathbf{A} = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$, with $b \neq 0$ has only a one-dimensional eigenspace corresponding to the eigenvalue λ. Then solve the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and sketch the phase portrait.

A has one eigenvalue, λ. Let's find the corresponding eigenvector.

$$\begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} \qquad A X = \lambda X$$

$\Rightarrow \quad \lambda x_1 + b x_2 = \lambda x_1 \quad \Rightarrow \quad x_2 = 0$ since $b \neq 0$.

$\therefore$ It has just one eigenvector $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ corresponding to λ.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} \dot{x} = \lambda x + by \\ \dot{y} = \lambda y \end{array}$$

So $y = y_0 e^{\lambda t}$ and $\dfrac{dx}{dt} - \lambda x = b y_0 e^{\lambda t}$

$$e^{-\lambda t} dx - \lambda e^{-\lambda t} x \, dt = b y_0 \, dt$$

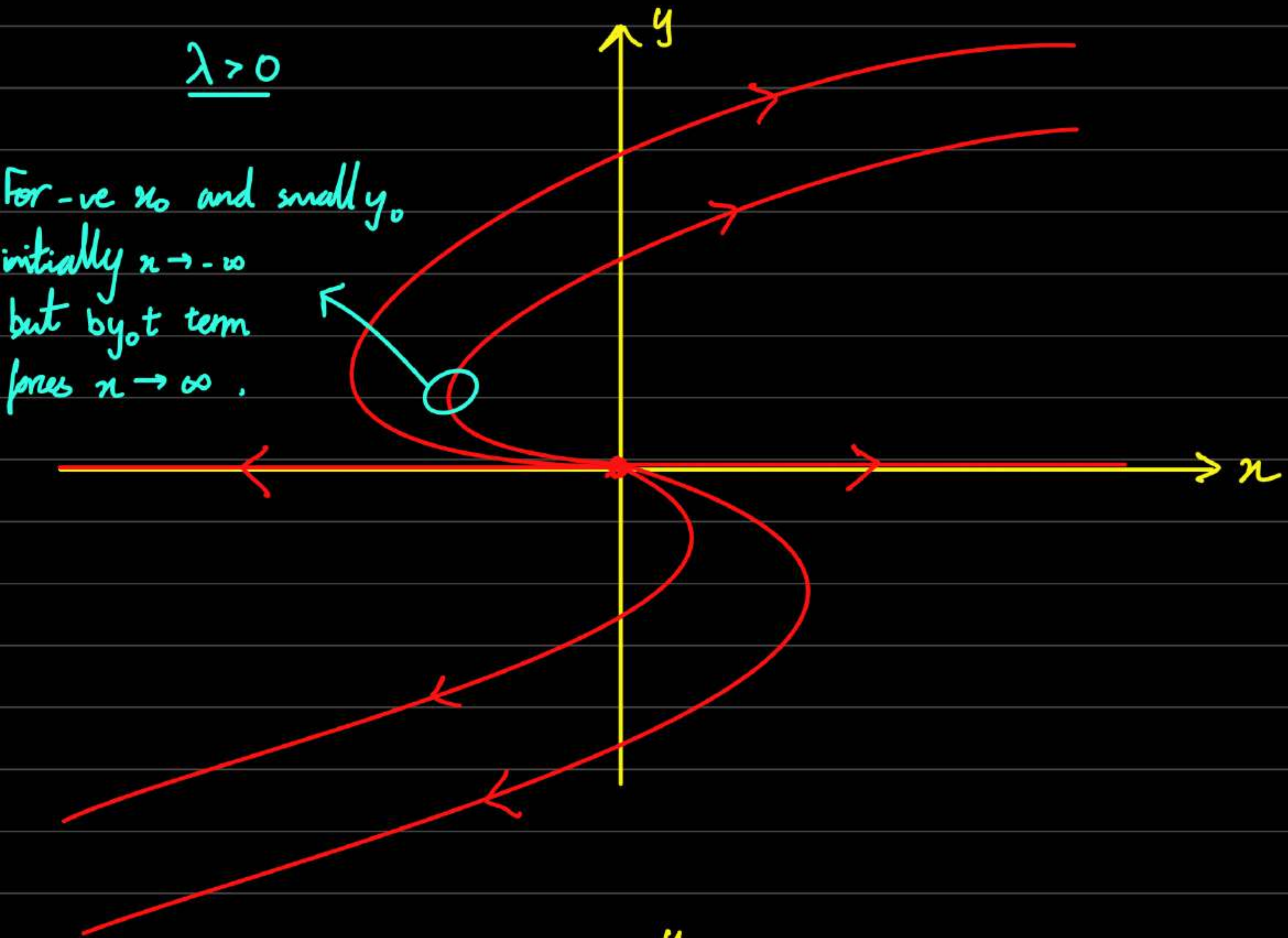
$$\Rightarrow \int d\left(x e^{-\lambda t}\right) = \int b y_0 \, dt$$

$x e^{-\lambda t} - x_0 = b y_0 t$ $\qquad x(0) = x_0$

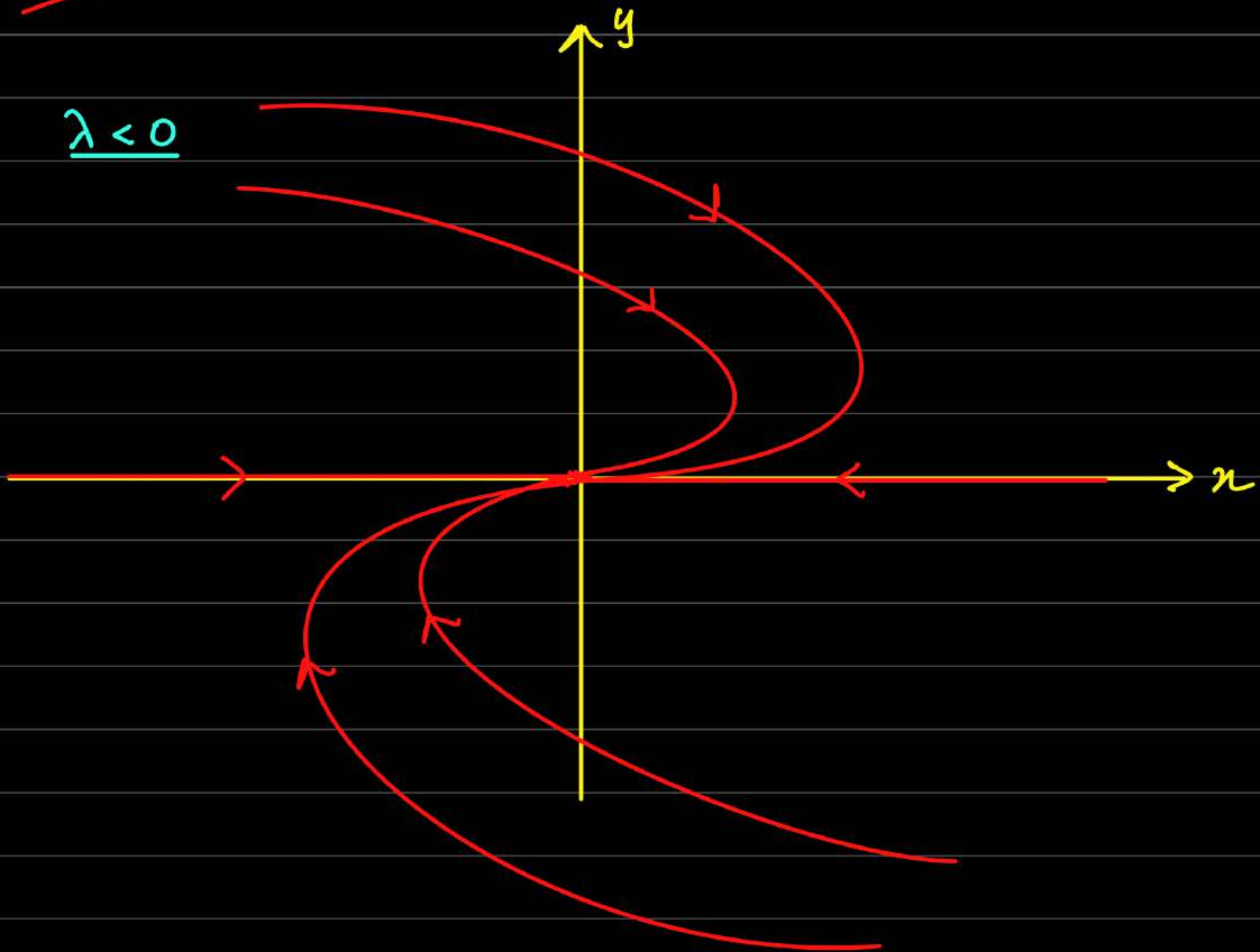
$$x e^{-\lambda t} - x_0 = b y_0 t$$

$$\Rightarrow \boxed{\begin{array}{l} x = (x_0 + b y_0 t) e^{\lambda t} \\ y = y_0 e^{\lambda t} \end{array}}$$

$\underline{\lambda > 0}$



$\underline{\lambda < 0}$



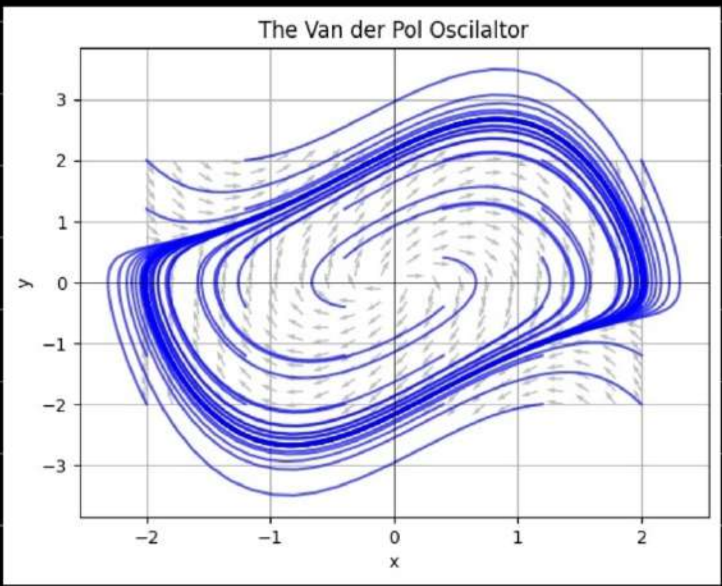
8. Using the computer, plot the phase portraits of the following systems:

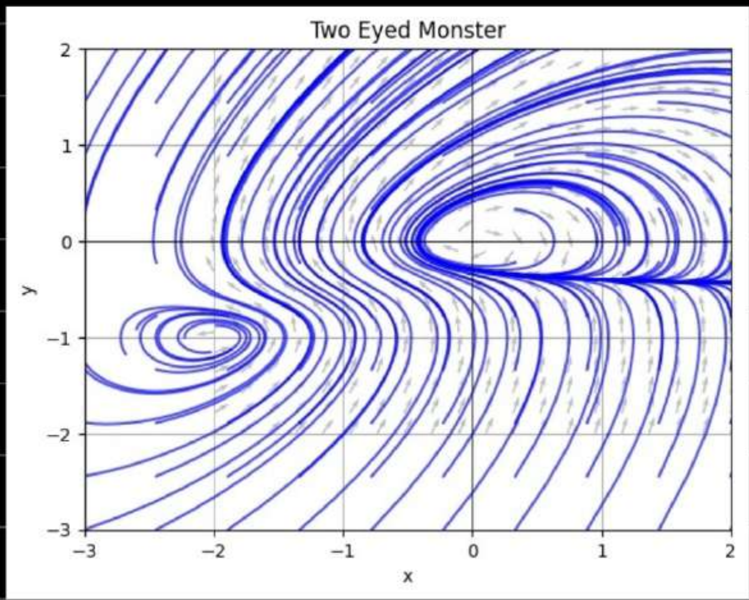
(a) The Van der Pol oscillator: $\dot{x} = y,\ \dot{y} = -x + y(1 - x^2)$

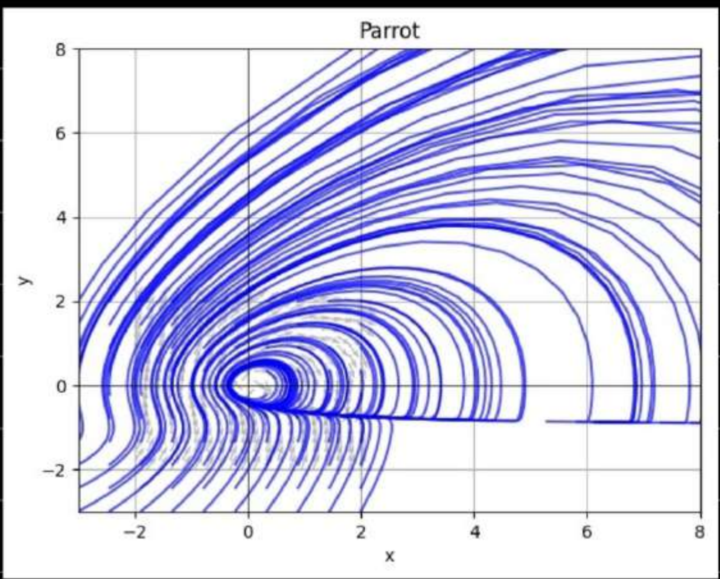
(b) Two eyed monster: $\dot{x} = y + y^2,\ \dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

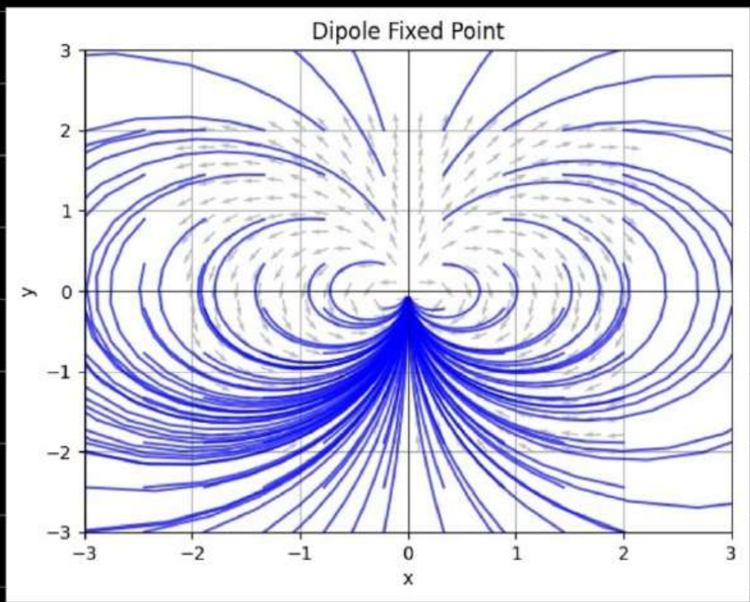
(c) Parrot: $\dot{x} = y + y^2,\ \dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

(d) Dipole fixed point: $\dot{x} = 2xy,\ \dot{y} = y^2 - x^2$









9. Consider the system $\dot{x} = xy,\ \dot{y} = x^2 - y^2$.

(a) Show that the linearisation predicts that the origin is a non-isolated fixed point.

$$\dot{x} = f_1(x,y) = xy$$
$$\dot{y} = f_2(x,y) = x^2 - y^2$$

a) $(0,0)$ is a fixed point.

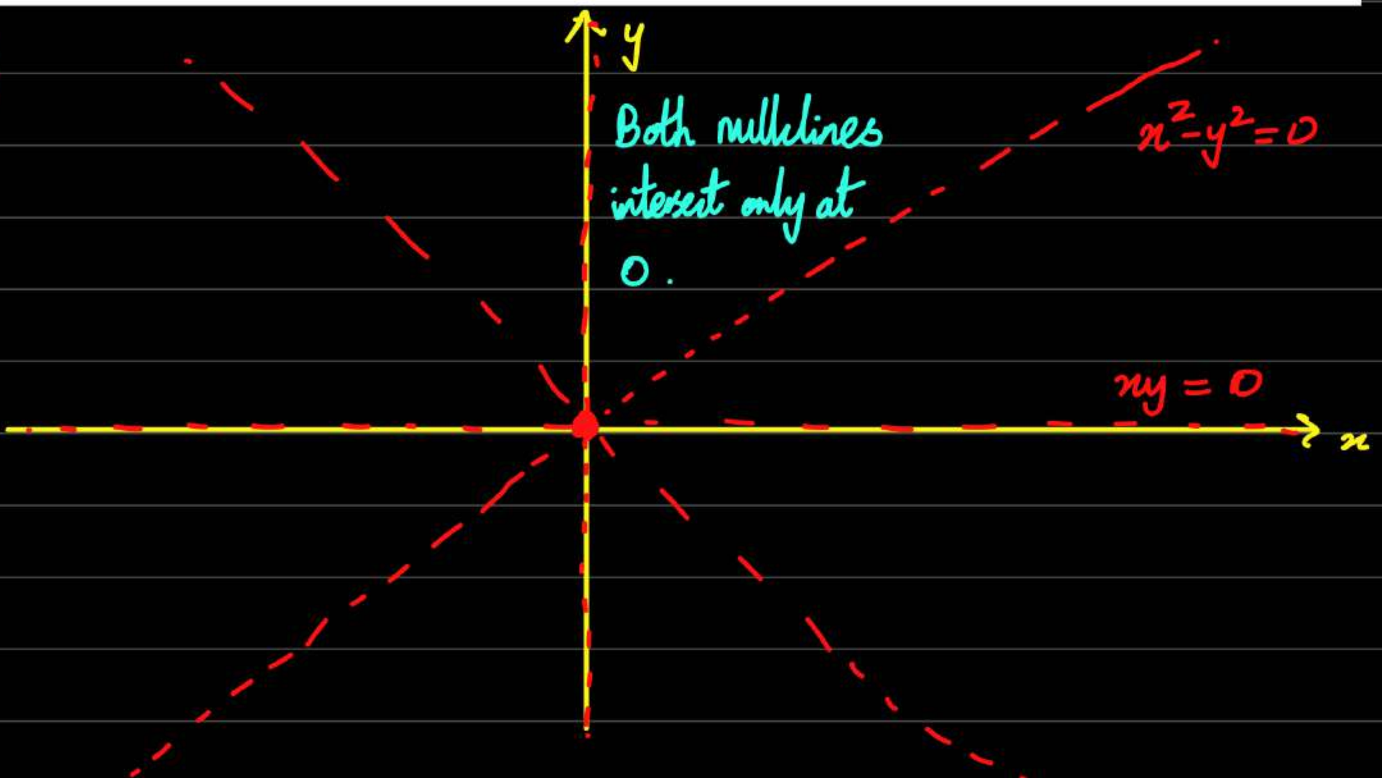
Computing the Jacobian at origin $J = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\right|_{(0,0)}$

$$J = \left.\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}\right|_{0,0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Delta = 0 \Rightarrow$ origin is a non-isolated fixed point (as predicted by linearisation)

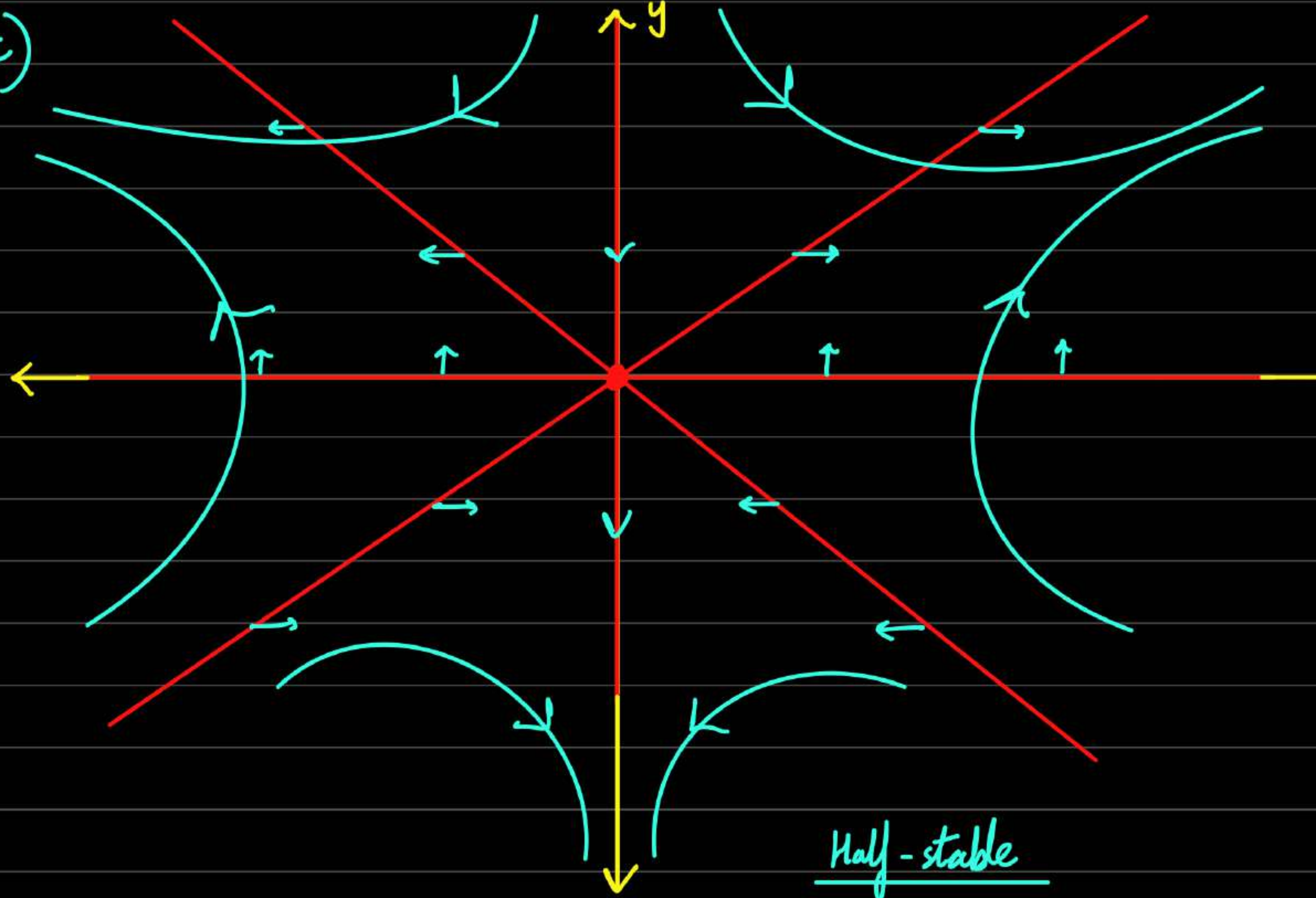
(b) Show that the origin is in fact an isolated fixed point.

b)



(c) Classify the origin in terms of its stability. Sketch the vector field along the null clines and at other points in the phase plane.

c)



Half-stable

(d) Plot a computer generated phase portrait and compare with (c).