

# Endsem: AM5650 Nonlinear Dynamics

Due at 6pm on May 15, 2025. Please upload electronically on moodle.

May 14, 2025

1. (a) A mass  $m$  is attached to the mid-point of an elastic string of length  $2a$  and linear stiffness  $\lambda$  (Figure 1). There is no gravity acting, and the tension is zero in the equilibrium position. Obtain the equation of motion for transverse oscillations and sketch the phase paths.

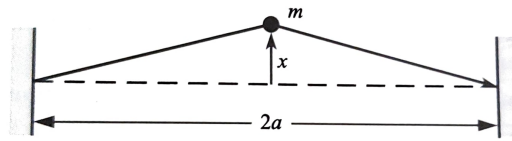


Figure 1: Transverse oscillations

- (b) If stiffness of the string was not linear like in part (a) above, but had a non-linear cubic dependence of the form  $f(x) = -\lambda_0 x + \lambda_1 x^3$ , obtain the equation of motion for transverse oscillations and sketch the phase paths.
  - (c) Is it possible that this cubic system would exhibit chaos for any initial and/or parametric condition?
2. Consider the system

$$\begin{aligned}\dot{x} &= y(2y^2 - 3x^2 + \frac{19}{9}x^4) \\ \dot{y} &= y^2(3x - \frac{38}{9}x^3) - (4x^3 - \frac{28}{3}x^5 + \frac{40}{9}x^7)\end{aligned}$$

Find the locations of its equilibrium points. Verify that the system has four homoclinic paths given by

$$y^2 = x^2 - x^4$$

and

$$y^2 = 2x^2 - \frac{10}{9}x^4$$

Show also that the origin is a higher-order saddle with separatrices in the directions with slopes  $\pm 1, \pm\sqrt{2}$ .

3. Compute solutions for the Lorenz System

$$\begin{aligned}\dot{x} &= a(y - x) \\ \dot{y} &= bx - y - xz \\ \dot{z} &= xy - cz\end{aligned}$$

for the parameter section  $a = 10, c = \frac{8}{3}$  and various values of  $b$ : this is the section frequently chosen to illustrate oscillatory features of the Lorenz attractor. In particular, try  $b = 100.5$  and show numerically that there is a periodic attractor. Why will this limit cycle be one of a pair?

Also show that at  $b = 166$ , the system has a periodic solution, but at  $b = 166.1$ , the periodic solution is regular for long periods but is then subject to irregular bursts at irregular intervals before resuming its oscillation again. This type of chaos is called **intermittency**.

4. Consider the Rössler System

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

Substitute  $a = 0.1, b = 0.1$  and  $c = 14$ . Define a new variable  $u$  such that  $u = f(x, y, z)$  where  $f$  is an analytic function of  $x, y, z$ . Find any  $f$  such that  $u \in [0, 1]$  for all times and for all values of  $a, b, c$ . Plot  $u$  as a function of time for your chosen function  $f$ . Then, plot a recurrence map for  $u$ , which is a plot of  $u_{n-1}$  (value at the previous time instant) versus  $u_n$  (value at the current time instant). Can you comment on the nature of the recurrence plot?

5. (a) Show that the logistic difference equation

$$u_{n+1} = \lambda u_n(1 - u_n)$$

has the general solution  $u_n = \sin^2(2^n C\pi)$  if  $\lambda = 4$ , where  $C$  is an arbitrary constant (without loss  $C$  can be restricted to  $0 \leq C \leq 1$ ). Show that the solution is  $2^q$ -periodic ( $q$  any positive integer) if  $C = 1/(2^q - 1)$ . The presence of *all* these period-doubling solutions indicates chaos.

(b) An alternative method of visualizing the structure of difference and differential equations is to plot **return maps** of  $u_{n-1}$  versus  $u_n$ . For example, a sequence of solutions of the logistic difference equation

$$u_{n+1} = \alpha u_n(1 - u_n),$$

the ordinate would be  $u_{n-1}$  and the abscissa  $u_n$ . The return map should be plotted after any initial transient returns have died out. If  $\alpha = 2.8$ , how will the long-term return map appear? Find the return map for  $\alpha = 3.4$  also. An exact (chaotic) solution of the logistic equation is

$$u_n = \sin^2(2^n).$$

Plot the points  $(u_n, u_{n-1})$  for  $n = 1, 2, \dots, 200$ , say. What structure is revealed?

Using a computer program, generate time-series (numerical solution) for the Duffing equation

$$\ddot{x} + k\dot{x} - x + x^3 = \Gamma \cos(\omega t)$$

for  $k = 0.3, \omega = 1.2$  and selected values of  $\Gamma$ , say  $\Gamma = 0.2, 0.28, 0.29, 0.37, 0.5$ . Plot transient-free return maps for the interpolated pairs  $(x(2\pi n/\omega), x(2\pi(n-1)/\omega))$ . For the *chaotic* case  $\Gamma = 0.5$ , take the time series over an interval  $0 \leq t \leq 5000$ , say. These return diagrams show that structure is recognizable in chaotic outputs: the returns are not uniformly distributed.

6. Start by finding any system of nonlinear ordinary differential equations that exhibits chaos other than the Rössler, Lorenz and Duffing systems that we discussed in class. Create the phase portrait for this system, for any set of parameters where chaos is exhibited. What is the fractal dimension of the strange attractor for this system?