1. Consider the system

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1), \quad \dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2).$$

(a) Show that the system has no fixed points, given that $\omega_1, \omega_2 > 0$ and $K_1, K_2 > 0$.

a) $\dot{\theta}_1 = \dot{\theta}_2 = 0$, so $\theta_1 = \alpha$
$\theta_2 = \beta$

$$0 = \omega_1 + K_1 \sin(\beta - \alpha)$$
$$0 = \omega_2 - K_2 \sin(\beta - \alpha) \Rightarrow \frac{\omega_2}{K_2} = \frac{-\omega_1}{K_1}$$

This is not possible since ω_i and $K_i > 0$.
So one of them might stop but the other would still be in motion.

(b) Can there be a conserved quantity for this system? If yes, find one. If no, explain why?

b)
$$\frac{\dot{\theta}_1 - \omega_1}{K_1} = \frac{\omega_2 - \dot{\theta}_2}{K_2}$$

$$\Rightarrow \frac{\dot{\theta}_1}{K_1} + \frac{\dot{\theta}_2}{K_2} = \frac{\omega_1}{K_1} + \frac{\omega_2}{K_2}$$

or
$$\boxed{S = \dot{\theta}_1 + \frac{K_1}{K_2}\dot{\theta}_2} = \omega_1 + \frac{K_1}{K_2}\omega_2 \text{ is a constant.}$$

In the specific case of $K_1 = K_2$,

$$S = \dot{\theta}_1 + \dot{\theta}_2 = \omega_1 + \omega_2 \text{ is a constant.}$$

(c) Suppose that $K_1 = K2$. Show that the system can be non-dimensionalized to the form

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}\tau} = 1 + a\sin(\theta_2 - \theta_1), \quad \frac{\mathrm{d}\theta_2}{\mathrm{d}\tau} = \omega + a\sin(\theta_1 - \theta_2).$$

c)

$$\dot{\theta}_1 = \omega_1 + K\sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K\sin(\theta_1 - \theta_2)$$

Define $\tau = \omega_1 t$, $a = \frac{K}{\omega_1}$, $\omega = \frac{\omega_2}{\omega_1}$

Then
$$\boxed{\begin{aligned} \frac{d\theta_1}{d\tau} &= 1 + a\sin(\theta_2 - \theta_1) \\ \frac{d\theta_2}{d\tau} &= \omega + a\sin(\theta_1 - \theta_2) \end{aligned}}$$

2. Consider the system

$$\dot{\theta}_1 = E - \sin\theta_1 + K\sin(\theta_2 - \theta_1), \quad \dot{\theta}_2 = E + \sin\theta_2 + K\sin(\theta_1 - \theta_2),$$

where, $E, K \geq 0$.

(a) Find and classify all the fixed points.

a)

$$E = \sin\theta_1 - K\sin(\theta_2 - \theta_1)$$
$$E = -\sin\theta_2 - K\sin(\theta_1 - \theta_2)$$

$$\sin\theta_1 + \sin\theta_2 = K\sin(\theta_2-\theta_1) - K\sin(\theta_1-\theta_2)$$

$$2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) = -4K\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

So if $\frac{\theta_1-\theta_2}{2} = (2n+1)\frac{\pi}{2} \Rightarrow \theta_1 - \theta_2 = (2n+1)\pi$

$$\theta_1 = \theta_2 + (2n+1)\pi$$

$\Rightarrow \dot{\theta}_1 = E + \sin\theta_2 = 0 \Rightarrow$ $\boxed{\theta_2^* = -\sin^{-1}E}$ and $\boxed{\theta_1^* = \pi - \sin^{-1}E}$ (only $n=0$, flow on circle)

If $\frac{\theta_1-\theta_2}{2} \neq (2n+1)\frac{\pi}{2}$,

$$\sin\left(\frac{\theta_1+\theta_2}{2}\right) = -2K\sin\left(\frac{\theta_1-\theta_2}{2}\right)$$

$$\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2} = -2K\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + 2K\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}$$

$$(2K+1)\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} = (2K-1)\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}$$

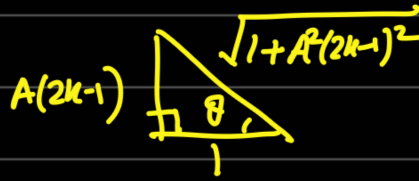
Multiply by $\sin\frac{\theta_2}{2}\cos\frac{\theta_1}{2}$,

$$(2K+1)\sin\theta_1\cos\theta_2 = (2K-1)\sin\theta_2\cos\theta_1$$

$$\Rightarrow \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{2k-1}{2k+1} = \begin{array}{l}\theta_1 = \tan^{-1}\big(A(2k-1)\big)\\ \theta_2 = \tan^{-1}\big(A(2k+1)\big)\end{array}$$

$$E = \sin\theta_1 - K\sin(\theta_2-\theta_1)$$

$$E = -\sin\theta_2 - K\sin(\theta_1-\theta_2)$$



$$E = \sin\theta_1 - k\sin(\theta_2-\theta_1)$$

$$2E = \sin\theta_1 - \sin\theta_2$$

$$E = \frac{A(2k-1)}{\sqrt{1+A^2(2k-1)^2}} - k\sin\left(\tan^{-1}\left(\frac{2A}{1+A^2(4k^2-1)}\right)\right)$$

$$E = \frac{A(2k-1)}{''} - k\,\frac{\left(\frac{2A}{1+A^2(4k^2-1)}\right)}{\sqrt{1+\frac{4A^2}{\left(1+A^2(4k-1)^2\right)^2}}}$$

$$E = \frac{A(2k-1)}{\sqrt{1+A^2(2k-1)^2}} - k\,\frac{2A}{\sqrt{\left(1+A^2(4k-1)^2\right)^2+4A^2}}$$

(b) Show that if E is large enough, the system has periodic solutions on the torus. What type of bifurcations creates the periodic solutions?

b)

3. Find a period doubling bifurcation for $f(x) = x^3 - ax$.

3

$f(x) = x \Rightarrow x^3 - (a+1)x = 0$

$x^2 - (a+1) = 0$

$\Rightarrow x^* = \pm\sqrt{1+a}\ ,\ 0$

$\lambda = f'(x) = 3x^2 - a = 3(a+1) - a$

$= 2a + 3$ for $x^* = \pm\sqrt{1+a}$

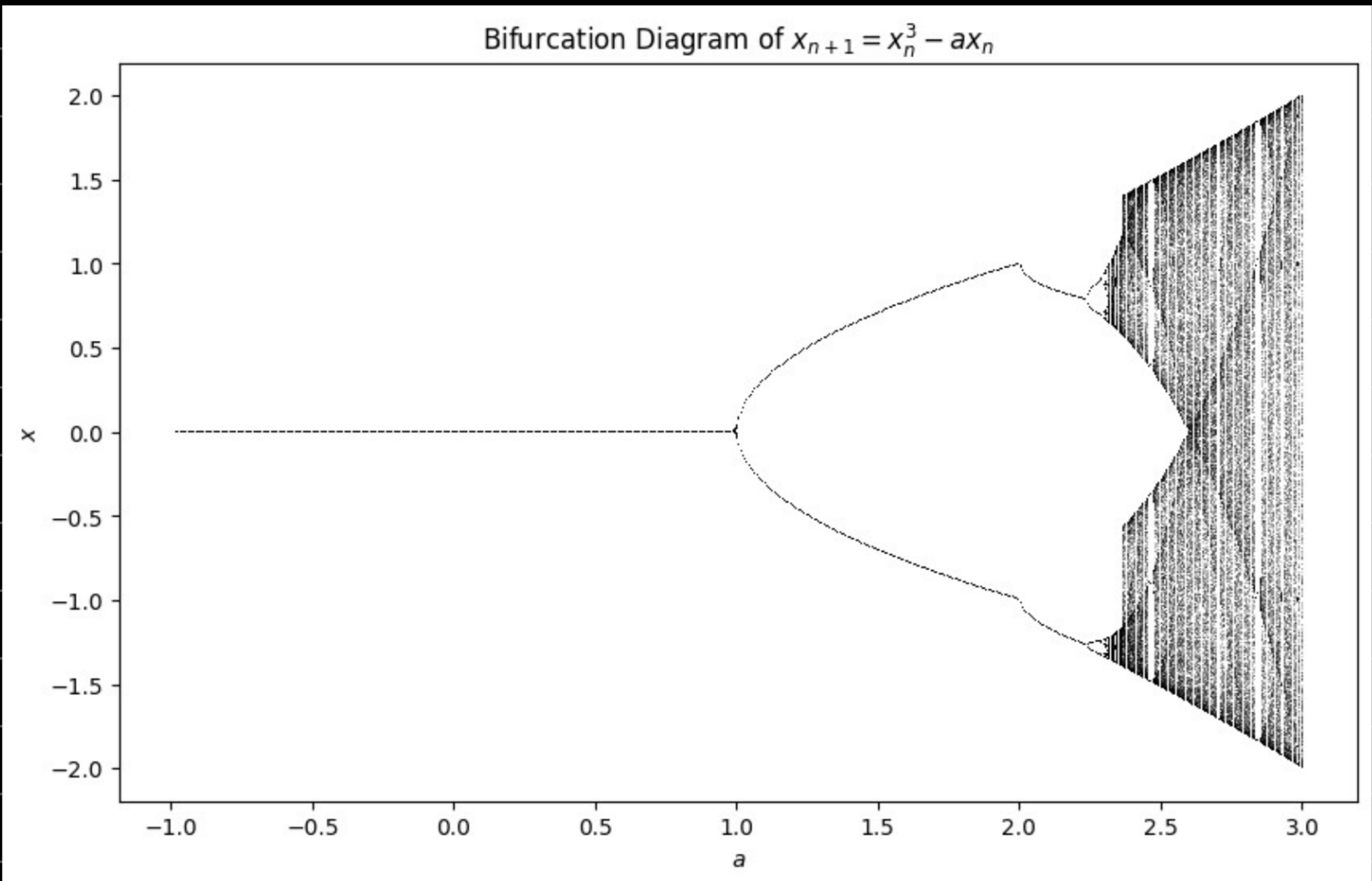
$a \geq -1 \leftarrow$

$\Rightarrow \lambda \geq 1$, always unstable

$= -a$ for $x^* = 0$

Stable for $a \in [-1, 1]$.

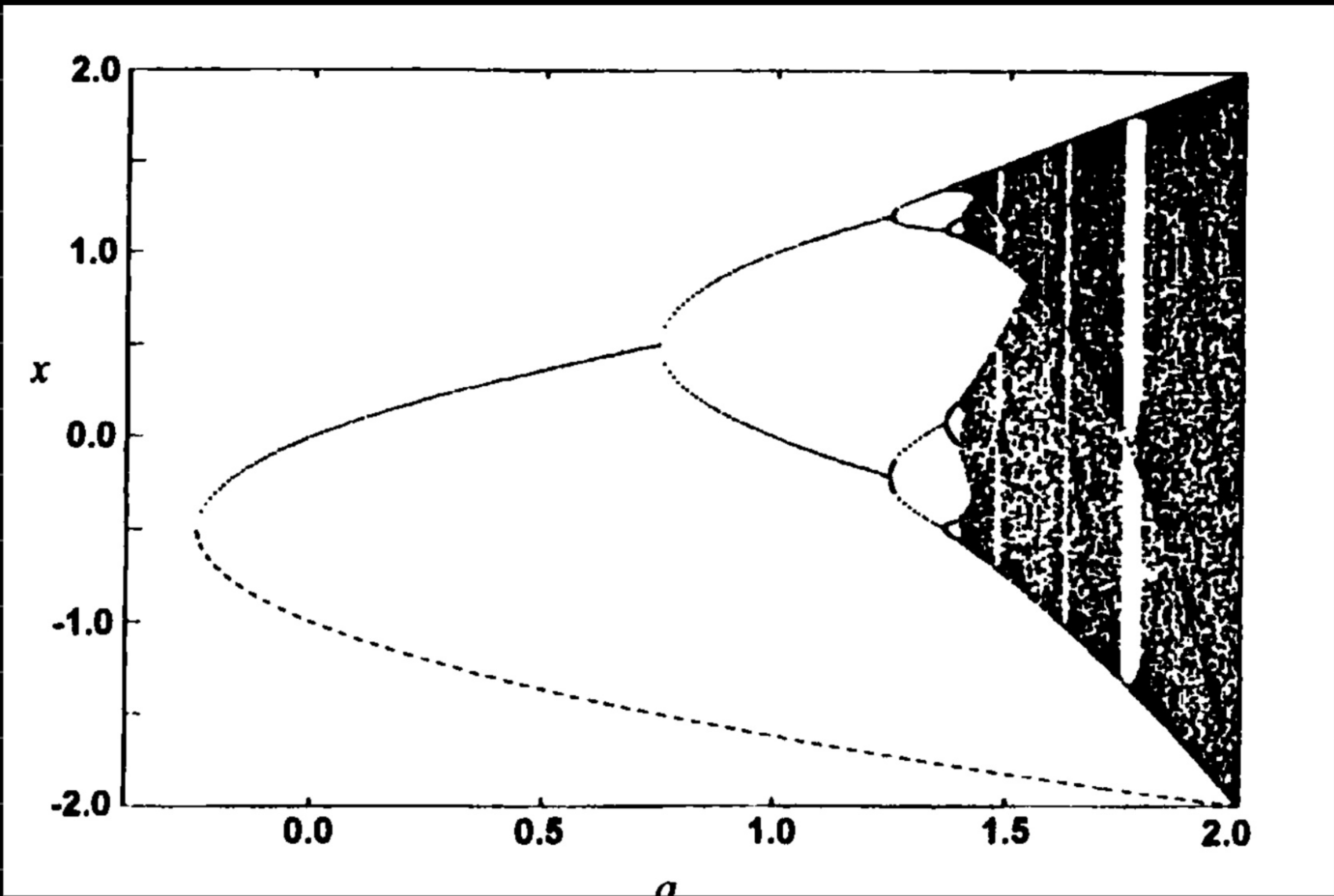
Then for $\boxed{a > 1}$, $\lambda < -1$ so period doubles.



4. Consider the 1-dimensional map $x_{n+1} = a - x_n^2$. e.g: 5.10

(a) Plot the bifurcation diagram with respect to the bifurcation parameter a for $a \in [-1/2, 2]$.

4 a)



(b) Identify the value of a at which the bifurcations occur, including the period doubling bifurcations. Verify these analytically.

b) $f(x) = x$

$a - x^2 = x \Rightarrow x_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + a}$

$x_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} + a}$

$\lambda = -2x^* = -1 \Rightarrow x^* = 1/2$, occurs for x_1 when $a = 3/4$

$\therefore$ First period doubling at $\boxed{a_1 = 3/4}$

(Note $\lambda > 1$ always for x_2 so unstable)

One may do the above repeatedly to get each period doubling a_n, but this is very lengthy.

Alternatively, let's transform this map to the logistic map, since we already know the a_n for that one.

$$x_{n+1} = a - x_n^2$$

Let $x_n = \alpha y_n + \beta$

$$\Rightarrow \alpha y_{n+1} + \beta = a - (\alpha y_n + \beta)^2$$
$$\alpha y_{n+1} + \beta = a - \alpha^2 y_n^2 - 2\alpha\beta y_n - \beta^2$$
$$\Rightarrow y_{n+1} = \frac{a}{\alpha} - \alpha y_n^2 - 2\beta y_n - \frac{\beta^2}{\alpha} - \frac{\beta}{\alpha}$$
$$y_{n+1} = \frac{a - \beta - \beta^2}{\alpha} - 2\beta y_n\left(1 + \frac{\alpha}{2\beta} y_n\right)$$

Compare with $z_{n+1} = r z_n (1 - z_n)$

Then $a = \beta^2 + \beta$, $2\beta = -r$, $\frac{\alpha}{2\beta} = -1$

$$\Rightarrow \alpha = r$$

and $a = \frac{1}{4}(\alpha^2 - 2\alpha)$

$$\Rightarrow \boxed{a = \frac{1}{4}(r^2 - 2r)}$$

So if we know values of r at which period doubling happens, we know it for corresponding a.

We shall get numbers in the next problem.

(c) Compute the ratios $(a_n - a_{n-1})/(a_{n-1} - a_{n-2})$, where a_k are the values of a at which period doubling bifurcations occur. Show in a table containing three columns, with column 1 indicating period ($= 2^s$). the value of the parameter a at which the s-th period doubling occurs and the last column the ratio computed as above. What do you observe? This was first observed by Feigenbaum in 1978 and has been shown to be indicative of a universality in scaling behaviour of cascades. *A cascade is an infinite sequence of period doubling bifurcations.*

c)

s	r_s	a_s
1	3	0.75
2	3.449	1.2494
3	3.544	1.368
4	3.564	1.3935
5	3.569	1.3999
6	3.570	1.4012

Verification on next page

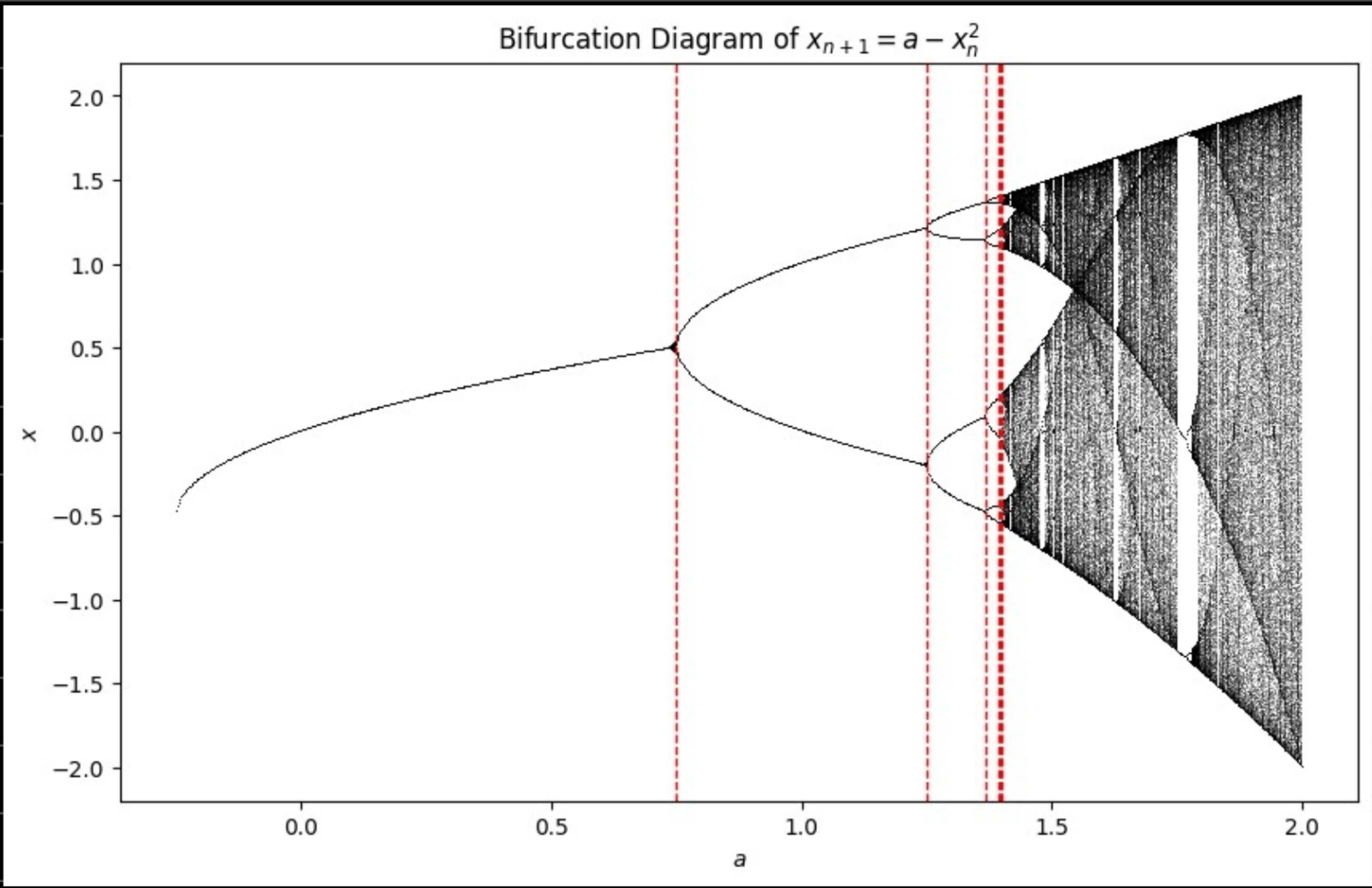
$$F_3 = \frac{a_3 - a_2}{a_2 - a_1} = \frac{1.368 - 1.2494}{1.2494 - 0.75} = 0.2378$$

$$F_4 = \frac{a_4 - a_3}{a_3 - a_2} = 0.215$$

$$F_5 = 0.25$$

$$F_6 = \frac{1.4012 - 1.3999}{1.399 - 1.3935} = 0.203125$$

With better numbers, we would expect $F_{n\to\infty} \to \frac{1}{\delta} \approx 0.2141$



Bifurcation Diagram of $x_{n+1} = a - x_n^2$
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
x
0.0
0.5
1.0
1.5
2.0
a