Date: February 17, 2025

AM 5650
Nonlinear Dynamics
Assignment -2

Due : Feb 24, 2025

1. Sketch the phase portraits of the following systems:
   (a) $\dot{r} = r(1-r^2)(9-r^2),\ \dot{\theta} = 1$
   (b) $\dot{r} = r^3 - 4r,\ \ \dot{\theta} = 1.$

2. Consider $\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$, where $a > 0$.
   (a) Find and classify all the fixed points.
   (b) Show that the system has a circular limit cycle, and find its amplitude and period.
   (c) Determine the stability of the limit cycle.

3. Show that $\dot{x} = -x + 2y^3 - 2y^4,\ \ \dot{y} = -x - y + xy$ has no periodic solutions. (Hint: Choose a, m and n such that $V = x^m + ay^n$ is a Lyapunov function.

4. Consider $\dot{x} = x - y - x(x^2 + 5y^2),\ \ \dot{y} = x + y - y(x^2 + y^2)$.
   (a) Classify the fixed point at the origin.
   (b) Rewrite the system in polar coordinates.
   (c) Determine the circle of maximum radius r_1, centered on the origin such that all trajectories have a radially outward component on it.
   (d) Determine the circle of minimum radius r_2, centered on the origin such that all trajectories have a radially inward component on it.
   (e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \leq r \leq r_2$.
   (f) Using numerical integration, compute the limit cycle and verify that it lies in the trapping region constructed in the previous step.

5. Let $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ be a smooth vector field defined on the phase plane.
   (a) Show that is this is a gradient system, then $\partial f/\partial y = \partial g/\partial x$.
   (b) Is the above condition also sufficient?
   (c) If $f(x, y) = y + 2xy,\ g(x, y) = x + x^2 - y^2$, prove that closed orbits cannot exist.
   (d) Find potential function V.
   (e) Sketch the phase portrait.

6. Consider the two-dimensional system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - r^2\mathbf{x}$, where $r = ||\mathbf{x}||$ and $\mathbf{A}$ is a 2×2 constant real matrix with complex eigenvalues $\alpha \pm i\omega$. Prove that there exists at least one limit cycle for $\alpha > 0$ and that there are none for $\alpha < 0$.

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7. Consider the system $\dot{r} = r(1-r^2) + \mu r \cos\theta$, $\dot{\theta} = 1$. Using the Poincare-Benedixson theorem, show that a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$. (This was shown in the class). Now using the computer, plot the phase portrait for various values of $\mu > 0$. Is there a critical value μ_c at which the closed orbits ceases to exist? If so, estimate it. If not, prove that a closed orbit exists for all $\mu > 0$.

8. There is a theorem which states that: *Let $\bar{v}$ be an equilibirum of $\dot{\mathbf{v}} = \mathbf{f}(\mathbf{v})$. If the real part of each eigenvalue of $\mathbf{Df}(\bar{\mathbf{v}})$ is strictly negative, then $\bar{\mathbf{v}}$ is asymptotically stable. If the real part of at least one eigenvalue is strictly positive, then $\bar{\mathbf{v}}$ is unstable.*

   Now the one dimensional system $\dot{x} = -x^3$ has an equilibrium at $x = 0$. Decide whether $x = 0$ is an asymptotically stable. Does this equation have unique solutions? Find all solutions that satisfy $x(0) = 1$.

9. Consider the forced damped Duffing oscillator given by $\ddot{x} + 0.1\dot{x} - x + x^3 = 2\sin t$. Show that the corresponding undamped unforced system has a double well potential. Plot the potential function.

   Write a computer program to plot numerical solutions of the forced damped double-well in the $(x-\dot{x})$-plane. In particular, locate and plot the attracting periodic orbit of period 2π and the two attracting periodic orbits of period 6π that lie in the region $-5 \leq x, \dot{x} \leq 5$.

10. Use the method of nullclines to determine the global behaviour of solutions for the following system

$$\begin{aligned} \dot{x} &= 3x(1-x) - xy \\ \dot{y} &= 5y(1-y) - 2xy. \end{aligned}$$

   Describe sets of initial conditions that evolve to distinct final states.