Q5) (Lorenz equations for large r) Consider the Lorenz equations in the limit $r \to \infty$. By taking the limit in a certain way, all the dissipative terms in the equations can be removed (Robbins 1979, Sparrow 1982).

(a) Let $\epsilon = r^{-1/2}$, so that $r \to \infty$ corresponds to $\epsilon \to 0$. Find a change of variables involving ϵ such that as $\epsilon \to 0$, the equations become:

$$\dot{X} = Y$$
$$\dot{Y} = -XZ$$
$$\dot{Z} = XY.$$

$r = \tau^{-1}$

$\varepsilon^2 = \tau$ $\quad$ $\varepsilon = \tau^{1/2}$

a)

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x\left(\frac{1}{\varepsilon^2} - z\right) - y$$

$$\dot{z} = xy - bz$$

$$x = \alpha X$$
$$y = \beta Y$$
$$z = \gamma Z$$

$$\dot{X} = \frac{\sigma\beta}{\alpha} Y - \sigma X$$

$$\dot{Y} = \frac{\alpha}{\beta\varepsilon^2} X - \frac{\alpha\gamma}{\beta} XZ - Y$$

$$\dot{Z} = \frac{\alpha\beta}{\gamma} XY - bZ$$

(b) Find two conserved quantities (i.e., constants of the motion) for the new system.

b)
$$x\dot{x} - \dot{z} = 0$$
$$= \frac{d}{dt}\left(\frac{1}{2}x^2 - z\right) = 0$$
$$\therefore \boxed{\alpha = \frac{1}{2}x^2 - z}$$

$$y\dot{y} + z\dot{z} = 0$$
$$\therefore \boxed{\beta = \frac{1}{2}y^2 + \frac{1}{2}z^2}$$

(c) Show that the new system is volume-preserving (i.e., the volume of an arbitrary blob of "phase fluid" is conserved by the time-evolution of the system, even though the shape of the blob may change dramatically).

c) $\nabla \cdot \bar{x} = 0$

(d) Explain physically why the Lorenz equations might be expected to show some conservative features in the limit $r \to \infty$.

d) r represents $T_{upper} - T_{lower}$ in some convection system.
If $r \to \infty$, we have sinks and sources.
So flow becomes steady, unidirectional, less chaotic?

Q6) (Transient chaos) Example 9.5.1 from Strogatz textbook shows that the Lorenz system can exhibit transient chaos for $r = 21$, $\sigma = 10$, and $b = 8/3$. However, not all trajectories behave this way. Using numerical integration, find three different initial conditions for which there is transient chaos, and three others for which there isn't. Give a rule of thumb which predicts whether an initial condition will lead to transient chaos or not.

$(10, 1, 4) \rightarrow$ T.C.

$(1, 1, 2) \rightarrow$ L.C.

$(10, 1, 8) \rightarrow$ TC

$(12, -1, 6) \rightarrow$ TC

$(-12, 1, -6) \rightarrow$ L.C.