AM5650 Lorentz System Problems

Q1) Recreate Lorenz's predicament by writing a computer program to solve the equations in 8-bit arithmetic.

Write a computer program to numerically solve the Lorenz system:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(r - z) - y$$
$$\frac{dz}{dt} = xy - bz$$

Implement it using 8-bit floating-point or fixed-point arithmetic instead of higher precision.

Observe how errors accumulate over time and how they affect the system's dynamics.

Q2) For the lorentz system, generate the bifurcation diagram and show regions of stability in the full three-parameter space.

Q3) For each of the values of r given below, use a computer to explore the dynamics of the Lorenz system, assuming $\sigma = 10$ and $b = 8/3$ as usual. In each case, plot $x(t)$, $y(t)$, and x vs. z.

(i) $r = 166.3$ (intermittent chaos)

(ii) $r = 212$ (noisy periodicity)

(iii) The interval $145 \leq r \leq 166$ (period-doubling)

Q4) (Hysteresis between a fixed point and a strange attractor) Consider the Lorenz equations with $\sigma = 10$ and $b = \frac{8}{3}$. Suppose that we slowly "turn the r knob" up and down. Specifically, let $r = 24.4 + \sin(\omega t)$, where ω is small compared to typical orbital frequencies on the attractor. Numerically integrate the equations, and plot the solutions in whatever way seems most revealing. You should see a striking hysteresis effect between an equilibrium and a chaotic state.

Q5) (Lorenz equations for large r) Consider the Lorenz equations in the limit $r \to \infty$. By taking the limit in a certain way, all the dissipative terms in the equations can be removed (Robbins 1979, Sparrow 1982).

(a) Let $\epsilon = r^{-1/2}$, so that $r \to \infty$ corresponds to $\epsilon \to 0$. Find a change of variables involving ϵ such that as $\epsilon \to 0$, the equations become:

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= -XZ \\ \dot{Z} &= XY. \end{aligned}$$

(b) Find two conserved quantities (i.e., constants of the motion) for the new system.

(c) Show that the new system is volume-preserving (i.e., the volume of an arbitrary blob of "phase fluid" is conserved by the time-evolution of the system, even though the shape of the blob may change dramatically).

(d) Explain physically why the Lorenz equations might be expected to show some conservative features in the limit $r \to \infty$.

(e) Solve the system in part (a) numerically. What is the long-term behavior? Does it agree with the behavior seen in the Lorenz equations for large r?

Q6) (Transient chaos) Example 9.5.1 from Strogatz textbook shows that the Lorenz system can exhibit transient chaos for $r = 21$, $\sigma = 10$, and $b = 8/3$. However, not all trajectories behave this way. Using numerical integration, find three different initial conditions for which there is transient chaos, and three others for which there isn't. Give a rule of thumb which predicts whether an initial condition will lead to transient chaos or not.