1. (a) A mass m is attached to the mid-point of an elastic string of length $2a$ and linear stiffness λ (Figure 1). There is no gravity acting, and the tension is zero in the equilibrium position. Obtain the equation of motion for transverse oscillations and sketch the phase paths.

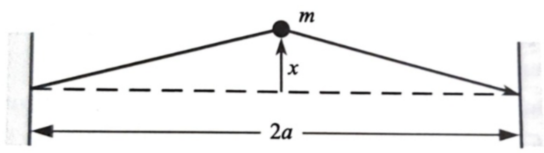
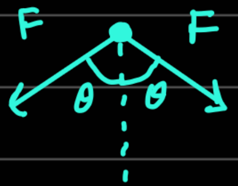
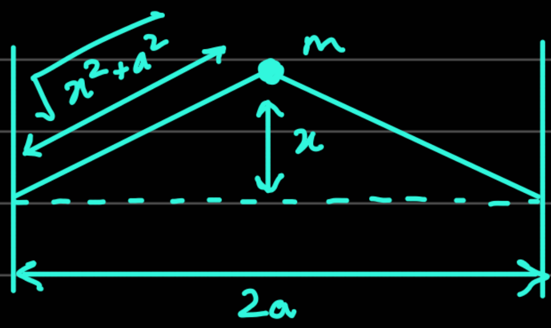


Figure 1: Transverse oscillations

1 a)





Net force = $2F\cos\theta$,

where $F = \lambda\left(\sqrt{x^2+a^2} - a\right)$

$$\cos\theta = \frac{x}{\sqrt{x^2+a^2}}$$

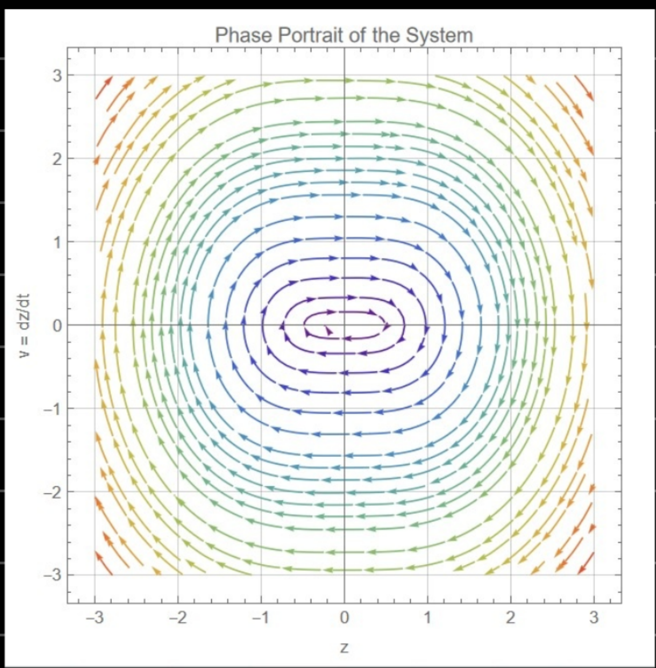
$$\Rightarrow \quad m\ddot{x} = -2\lambda\left(\sqrt{x^2+a^2} - a\right)\frac{x}{\sqrt{x^2+a^2}}$$

$$\ddot{x} = -\frac{2\lambda x}{m}\left(1 - \frac{a}{\sqrt{x^2+a^2}}\right)$$

let $\boxed{z = \frac{x}{a} \text{ and } K = \frac{2\lambda}{m}}$

$$\Rightarrow \quad \boxed{\ddot{z} = -Kz\left(1 - \frac{1}{\sqrt{z^2+1}}\right)}$$

We have,



(b) If stiffness of the string was not linear like in part (a) above, but had a non-linear cubic dependence of the form $f(x) = -\lambda_0 x + \lambda_1 x^3$, obtain the equation of motion for transverse oscillations and sketch the phase paths.

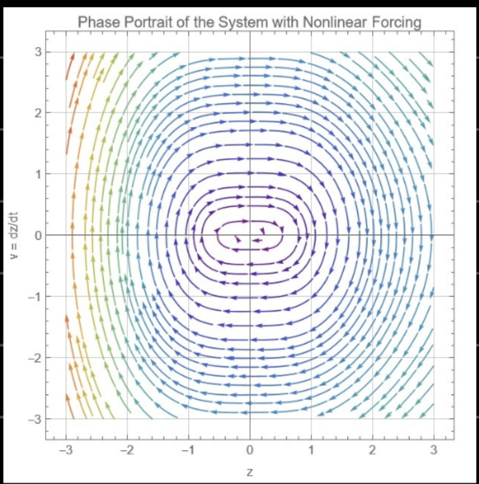
b) $F = \lambda_0\left(\sqrt{x^2+a^2} - a\right) - \lambda_1\left(\sqrt{x^2+a^2} - a\right)^3$

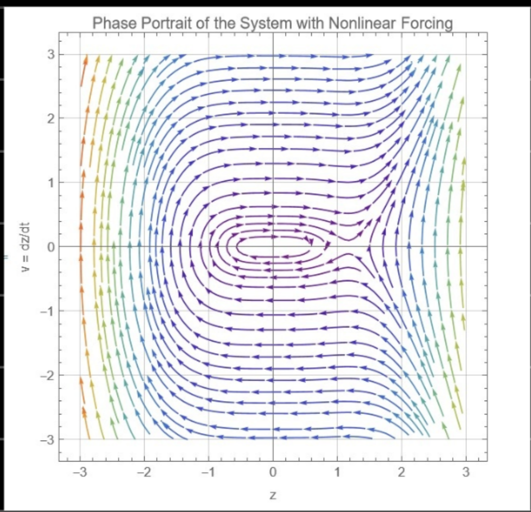
$$m\ddot{x} = \left[-\lambda_0\left(\sqrt{x^2+a^2} - a\right) + \lambda_1\left(\sqrt{x^2+a^2} - a\right)^3\right]\frac{2x}{\sqrt{x^2+a^2}}$$

$$\ddot{x} = -\frac{2\lambda_0 x}{m}\left(1 - \frac{a}{\sqrt{x^2+a^2}}\right) + \frac{2\lambda_1 x}{m}\left(x^2+a^2\right)\left(1 - \frac{a}{\sqrt{x^2+a^2}}\right)^3$$

Again let $\boxed{z = \frac{x}{a}, \quad K_0 = \frac{2\lambda_0}{m}, \quad K_1 = \frac{2\lambda_1 a^2}{m}}$

$$\Rightarrow \boxed{\ddot{z} = -K_0 z\left(1 - \frac{1}{\sqrt{z^2+1}}\right) + K_1\left(1+z^2\right)\left(1 - \frac{1}{\sqrt{z^2+1}}\right)^3}$$





$K_0 > K_1$

Almost resembles linear case.

$K_0 < K_1$

Saddle point emerges.

(c) Is it possible that this cubic system would exhibit chaos for any initial and/or parametric condition?

c) No, you need a minimum of three dimensions (third order system) to see chaos. Both systems are two dimensional (second order).

2. Consider the system

$$\dot{x} = y(2y^2 - 3x^2 + \frac{19}{9}x^4)$$

$$\dot{y} = y^2(3x - \frac{38}{9}x^3) - (4x^3 - \frac{28}{3}x^5 + \frac{40}{9}x^7)$$

Find the locations of its equilibrium points. Verify that the system has four homoclinic paths given by

$$y^2 = x^2 - x^4$$

and

$$y^2 = 2x^2 - \frac{10}{9}x^4$$

Show also that the origin is a higher-order saddle with separatrices in the directions with slopes ± 1, $\pm\sqrt{2}$.

2) As found numerically, the fixed points are $(0,0)$, $\left(\sqrt{\frac{3}{5}}, 0\right)$, $\left(-\sqrt{\frac{3}{5}}, 0\right)$, $\left(\sqrt{\frac{3}{2}}, 0\right)$, $\left(-\sqrt{\frac{3}{2}}, 0\right)$.

As will be shown later, $(0,0)$ is a saddle point, and we must verify the homoclinic orbits.

For it to be a homoclinic orbit, it must pass through $(0,0)$ (trivially seen to be true), and it must be tangential to the flow.

For $y^2 = x^2 - x^4$, $2y\frac{dy}{dx} = 2x - 4x^3 \Rightarrow \frac{dy}{dx} = \frac{x-2x^3}{y}$ — 1

From the ODEs, $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$ and so

$$\frac{\dot{y}}{\dot{x}} = \frac{(x^2-x^4)(3x - \frac{38}{9}x^3) - (4x^3 - \frac{28}{3}x^5 + \frac{40}{9}x^7)}{y(2x^2-2x^4-3x^2+\frac{19}{9}x^4)}$$

$$= \frac{3x^3 - \frac{38}{9}x^5 - 3x^5 + \frac{38}{9}x^7 - 4x^3 + \frac{28}{3}x^5 - \frac{40}{9}x^7}{y(-x^2+\frac{1}{9}x^4)}$$

$$= \frac{-x^3 + \frac{19}{9}x^5 - \frac{2}{9}x^7}{y\left(-x^2 + \frac{1}{9}x^4\right)}$$

and

$$\begin{array}{r|l} & -2x^3 + x \\ \hline \frac{1}{9}x^4 - x^2 & -\frac{2}{9}x^7 + \frac{19}{9}x^5 - x^3 \\ & \overset{+}{-}\frac{2}{9}x^7 \overset{-}{+} 2x^5 \\ \hline & \frac{1}{9}x^5 - x^3 \\ & + \frac{1}{9}x^5 - x^3 \\ \hline & 0 \end{array}$$

$$\Rightarrow \frac{\dot{y}}{\dot{x}} = \frac{-2x^3 + x}{y},$$

which matches with 1

So it is a homoclinic orbit.

Similarly, for $y^2 = 2x^2 - \frac{10}{9}x^4 \Rightarrow \frac{dy}{dx} = \frac{2x - \frac{20}{9}x^3}{y}$ — 2

From the ODEs, $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$

$$\frac{\dot{y}}{\dot{x}} = \frac{\left(2x^2 - \frac{10}{9}x^4\right)\left(3x - \frac{38}{9}x^3\right) - \left(4x^3 - \frac{28}{3}x^5 + \frac{40}{9}x^7\right)}{y\left(4x^2 - \frac{20}{9}x^4 - 3x^2 + \frac{19}{9}x^4\right)}$$

$$= \frac{6x^3 - \frac{76}{9}x^5 - \frac{30}{9}x^5 + \frac{380}{81}x^7 - 4x^3 + \frac{28}{3}x^5 - \frac{40}{9}x^7}{y\left(x^2 - \frac{x^4}{9}\right)}$$

$$= \frac{2x^3 - \frac{22}{9}x^5 + \frac{20}{81}x^7}{y\left(x^2 - \frac{x^4}{9}\right)}$$

$$
\begin{array}{r|l}
 & -\frac{20}{9}x^3 + 2x \\
\hline
-\frac{x^4}{9} + x^2 & \frac{20}{81}x^7 - \frac{22}{9}x^5 + 2x^3 \\
 & -\frac{20}{81}x^7 + \frac{20}{9}x^5 \\
\hline
 & -\frac{2}{9}x^5 + 2x^3 \\
 & +\frac{2}{9}x^5 - 2x^3 \\
\hline
 & 0
\end{array}
$$

$$\Rightarrow \frac{\dot{y}}{\dot{x}} = \frac{2x - \frac{20}{9}x^3}{y}$$

just like in 2

So this one is a homoclinic orbit as well.

Lastly, let's analyse stability at $(0,0)$.

$$\dot{x} = 2y^3 - 3x^2y + \frac{19}{9}x^4y = f(x,y)$$

$$\dot{y} = 3xy^2 - \frac{38}{9}x^3y^2 - 4x^3 + \frac{28}{3}x^5 - \frac{40}{9}x^7 = g(x,y)$$

$$J = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so a linear analysis does not work.

Let's analyse behaviour along any line $y = mx$ near the origin.

$$\dot{x} = mx\left(2m^2x^2 - 3x^2 + \frac{19}{9}x^4\right) \approx m(2m^2 - 3)x^3$$

$$\dot{y} = m^2x^2\left(3x - \frac{38}{9}x^3\right) - \left(4x^3 - \frac{28}{3}x^5 + \frac{40}{9}x^7\right)$$

$$\approx (3m^2 - 4)x^3$$

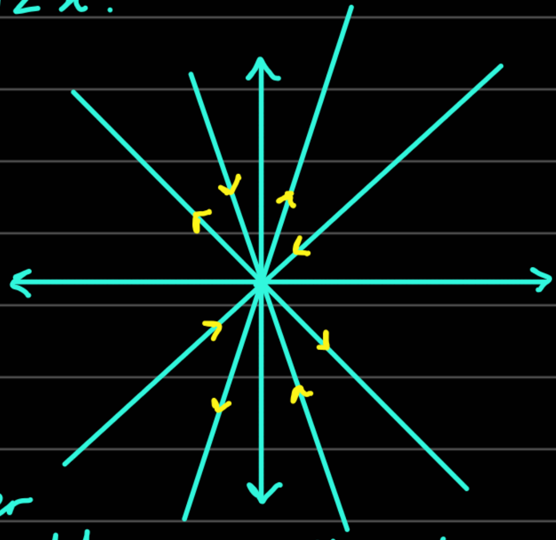
Assume these separatrices exist, then along them, $\frac{dy}{dx} = m$

$$\Rightarrow \quad m = \frac{3m^2 - 4}{m(2m^2 - 3)} \quad \Rightarrow \quad m^4 - 3m^2 + 2 = 0$$

$$\Rightarrow \quad m^2 = 1, 2$$

$$\therefore m = \pm 1, \pm\sqrt{2}$$

This means that near the origin, we have trajectories strictly along the straight lines $y = \pm x, \pm\sqrt{2}x$.



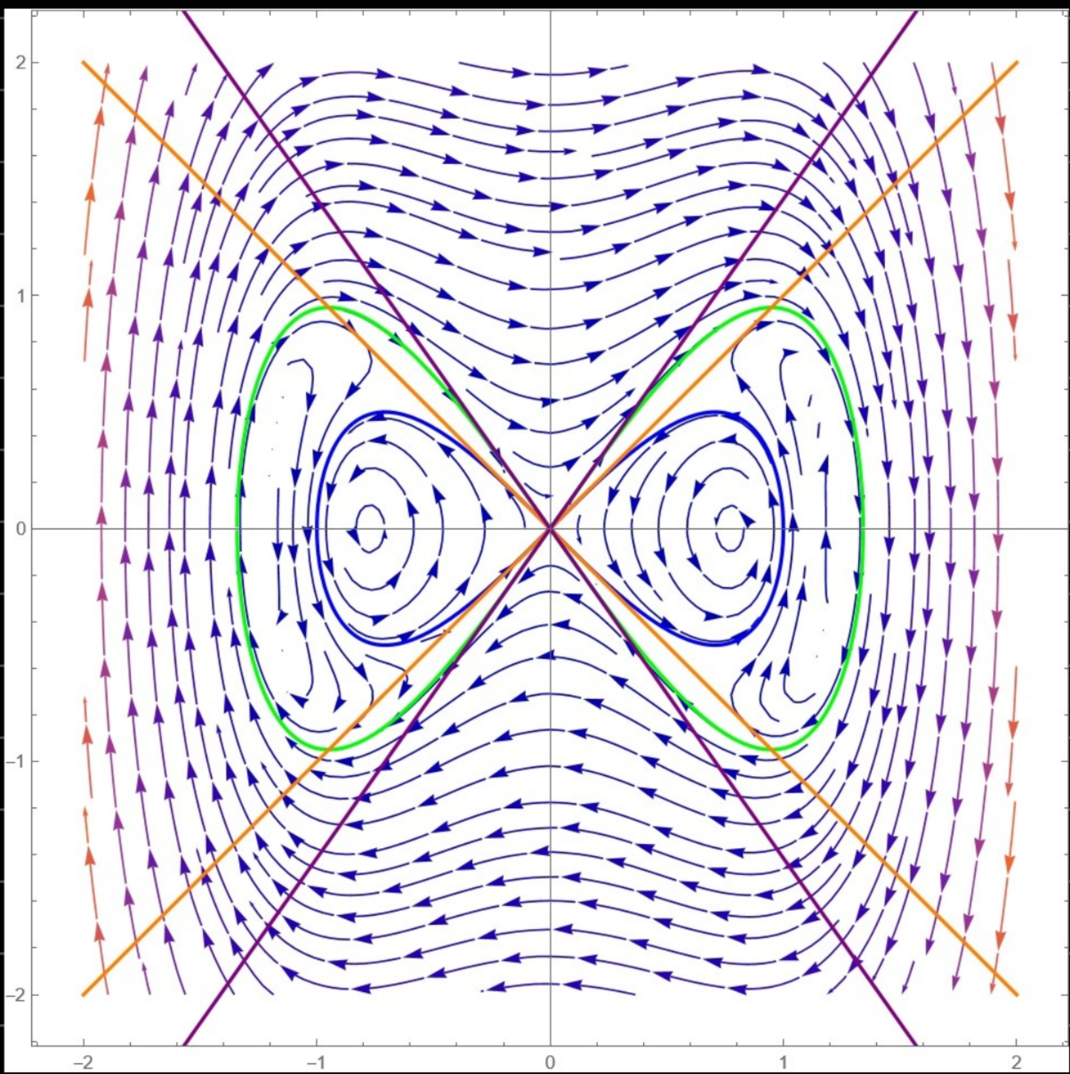
For $m=1$, $\dot{x} \approx -x^3$ $\quad \dot{y} \approx -x^3$

$m=-1$, $\dot{x} \approx x^3$ $\quad \dot{y} \approx -x^3$

$m=\sqrt{2}$, $\dot{x} \approx \sqrt{2}x^3$ $\quad \dot{y} \approx 2x^3$

$m=-\sqrt{2}$, $\dot{x} \approx -\sqrt{2}x^3$ $\quad \dot{y} \approx 2x^3$

So the origin has all charateristics of a higher order saddle. To confirm all answers, we plot our results below.



3. Compute solutions for the Lorenz System

$$\dot{x} = a(y - x)$$

$$\dot{y} = bx - y - xz$$

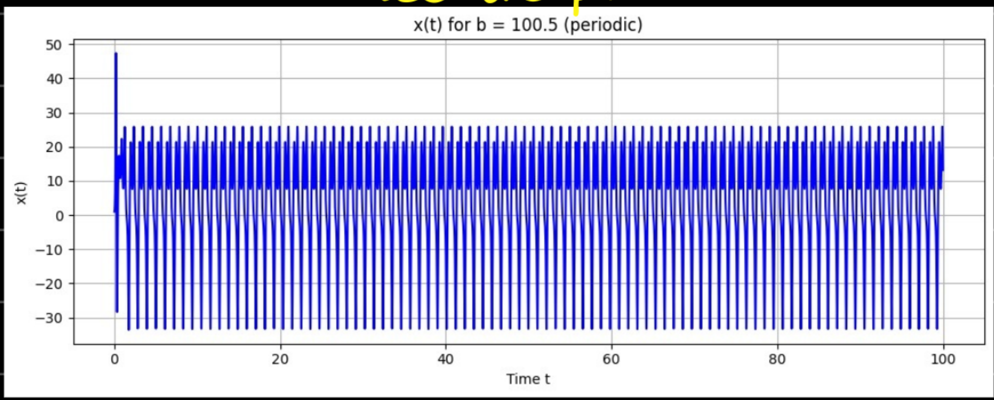
$$\dot{z} = xy - cz$$

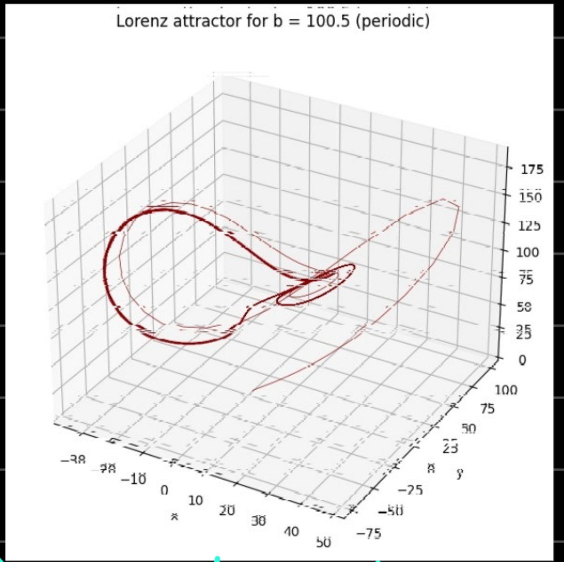
for the parameter section $a = 10, c = \frac{8}{3}$ and various values of b: this is the section frequently chosen to illustrate oscillatory features of the Lorenz attractor. In particular, try $b = 100.5$ and show numerically that there is a periodic attractor. Why will this limit cycle be one of a pair?

Also show that at $b = 166$, the system has a periodic solution, but at $b = 166.1$, the periodic solution is regular for long periods but is then subject to irregular bursts at irregular intervals before resuming its oscillation again. This type of chaos is called **intermittency**.

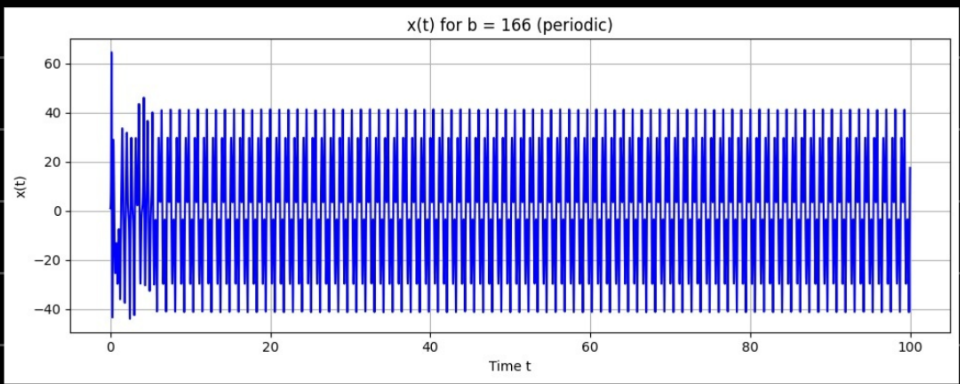
3

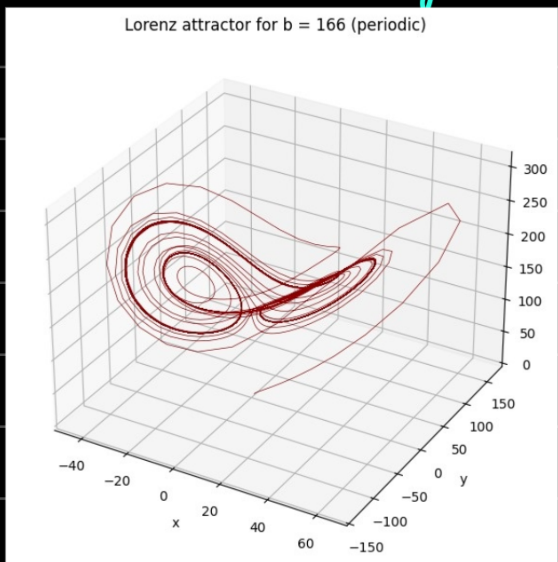
Could also plot Poincaré Section to see one point.

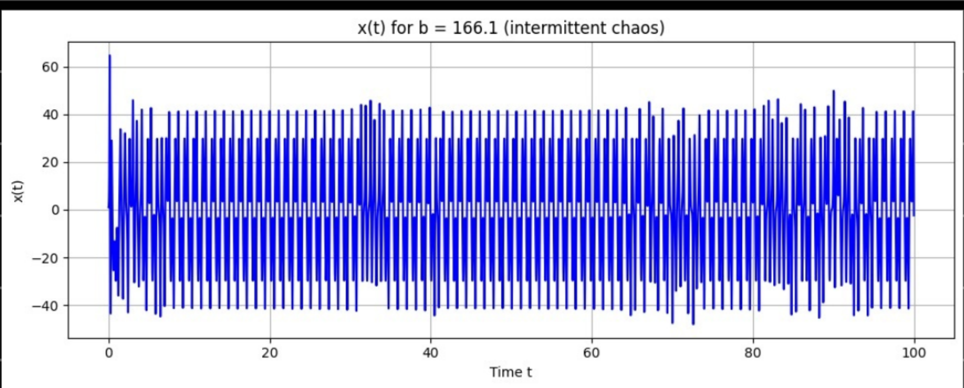


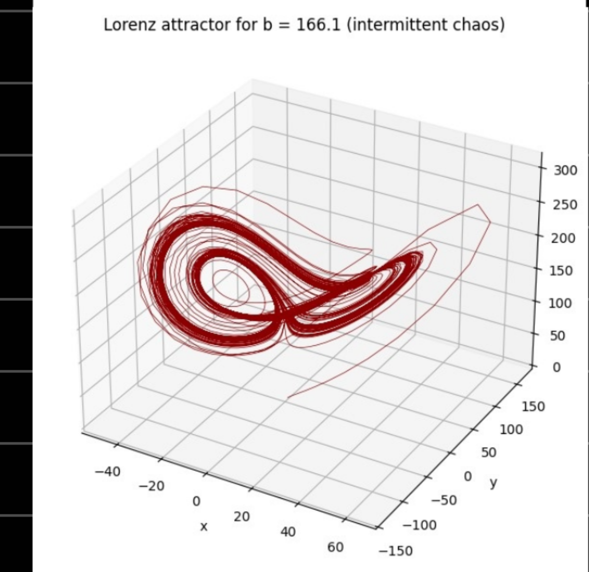


For the Lorenz system, $(x,y,z) \mapsto (-x,-y,z)$ leaves the equations invariant so a limit cycle that is not invariant under such a transformatio must have a twin.







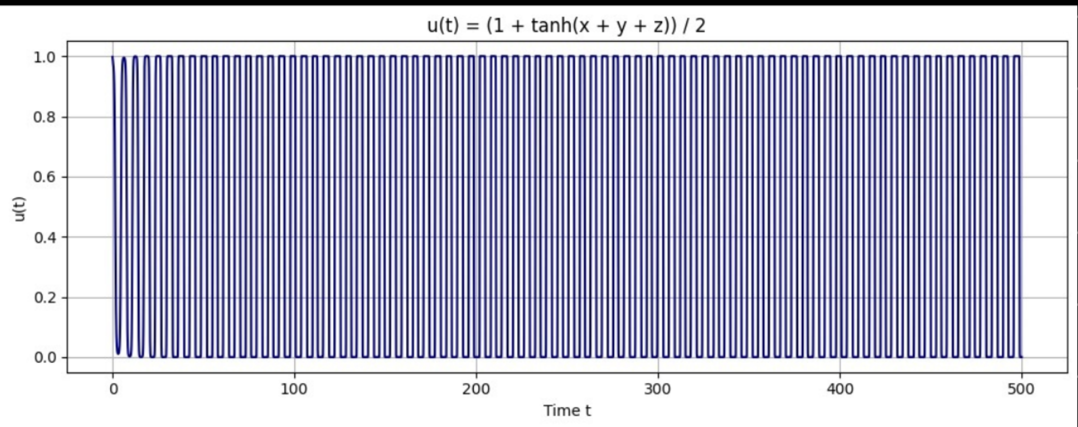


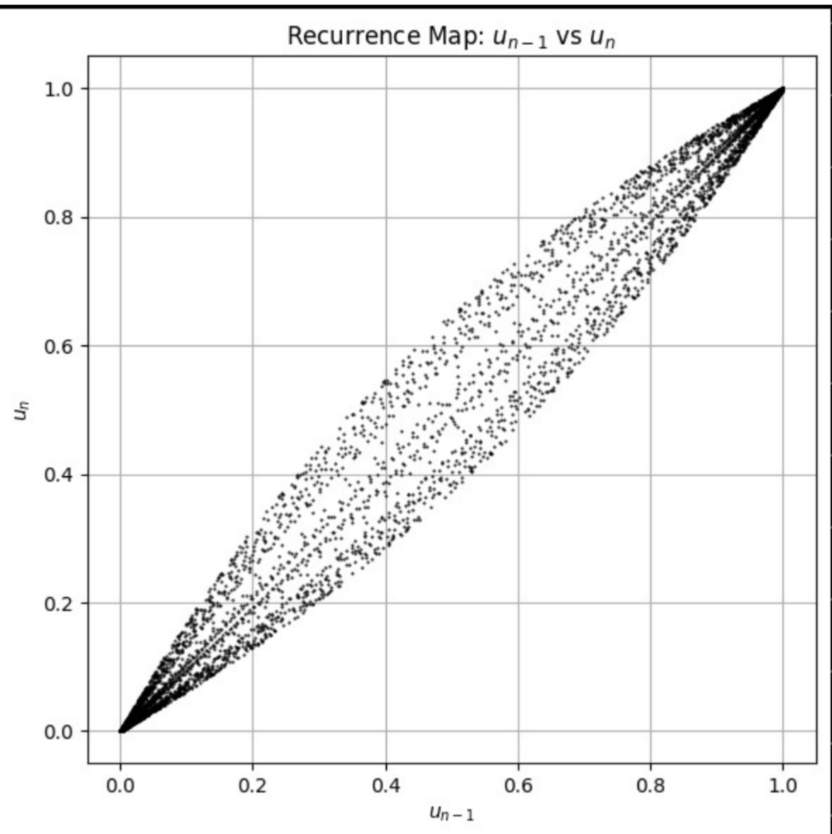
4. Consider the Rössler System

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

Substitute $a = 0.1, b = 0.1$ and $c = 14$. Define a new variable u such that $u = f(x, y, z)$ where f is an analytic function of x, y, z. Find any f such that $u \in [0, 1]$ for all times and for all values of a, b, c. Plot u as a function of time for your chosen function f. Then, plot a recurrence map for u, which is a plot of u_{n-1} (value at the previous time instant) versus u_n (value at the current time instant). Can you comment on the nature of the recurrence plot?

4) Let $u(x,y,z) = \frac{1}{2}\left(1 + \tanh(x+y+z)\right)$ s.t. $u \in [0,1]$





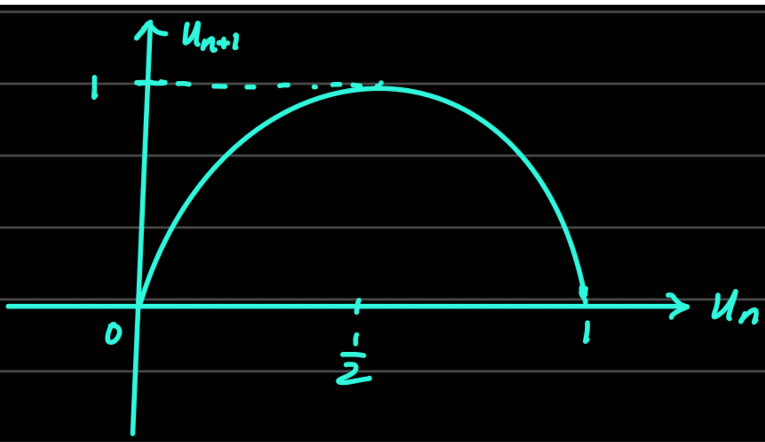
The recurrence map consists of a dense scattering of points, indicative of chaotic behaviour, which is aperiodic on large time scales.

5. (a) Show that the logistic difference equation

$$u_{n+1} = \lambda u_n(1 - u_n)$$

has the general solution $u_n = sin^2(2^n C\pi)$ if $\lambda = 4$, where C is an arbitrary constant (without loss C can be restricted to $0 \leq C \leq 1$). Show that the solution is 2^q-periodic (q any positive integer) if $C = 1/(2^q - 1)$. The presence of *all* these period-doubling solutions indicates chaos.

5 a) For $\lambda = 4$,



Since every $u_n \in [0,1]$, and is also mapped to some $u_{n+1} \in [0,1]$, without loss of generality, we can assume $u_0 = \sin^2\theta_0$ such that

$$u_1 = 4u_0(1-u_0)$$
$$= 4\sin^2\theta_0\cos^2\theta_0 = \sin^2 2\theta_0$$

Similarly, $u_2 = \sin^2 4\theta_0 \Rightarrow u_n = \sin^2 2^n\theta_0$

Again WLOG, choose $\theta_0 = C\pi \Rightarrow \boxed{u_n = \sin^2 2^n C\pi}$ QED

For $C = \dfrac{1}{2^q - 1}$, $u_n = \sin^2 \dfrac{2^n\pi}{2^q-1}$. Now $u_{n+q} = \sin^2 \dfrac{2^{n+q}\pi}{2^q-1}$

$$\Rightarrow u_{n+q} = \sin^2 \frac{2^n(2^q - 1 + 1)\pi}{2^q - 1}$$

$$= \sin^2\left(2^n\pi + \frac{2^n\pi}{2^q-1}\right) = \sin^2 \frac{2^n\pi}{2^q-1} = u_n$$

So u_n is periodic with period q for $C = \dfrac{1}{2^q-1}$

(b) An alternative method of visualizing the structure of difference and differential equations is to plot **return maps** of u_{n-1} versus u_n. For example, a sequence of solutions of the logistic difference equation

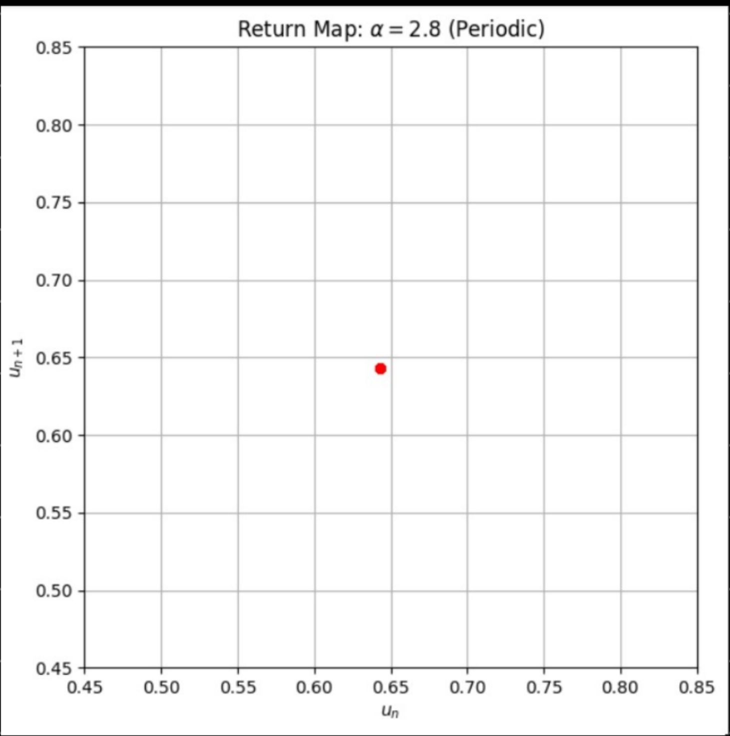
$$u_{n+1} = \alpha u_n(1 - u_n),$$

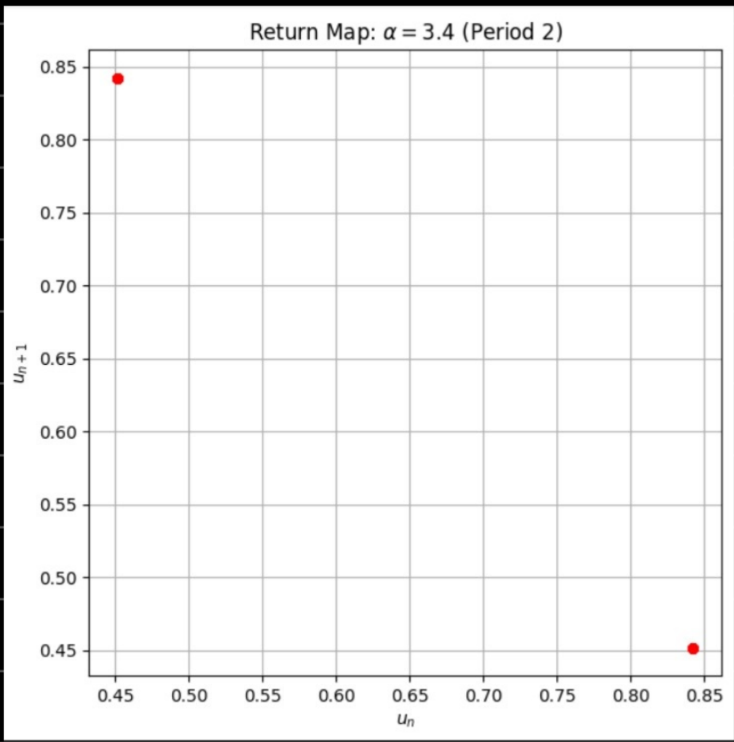
the ordinate would be u_{n-1} and the abscissa u_n. The return map should be plotted after any initial transient returns have died out. If $\alpha = 2.8$, how will the long-term return map appear? Find the return map for $\alpha = 3.4$ also. An exact (chaotic) solution of the logistic equation is

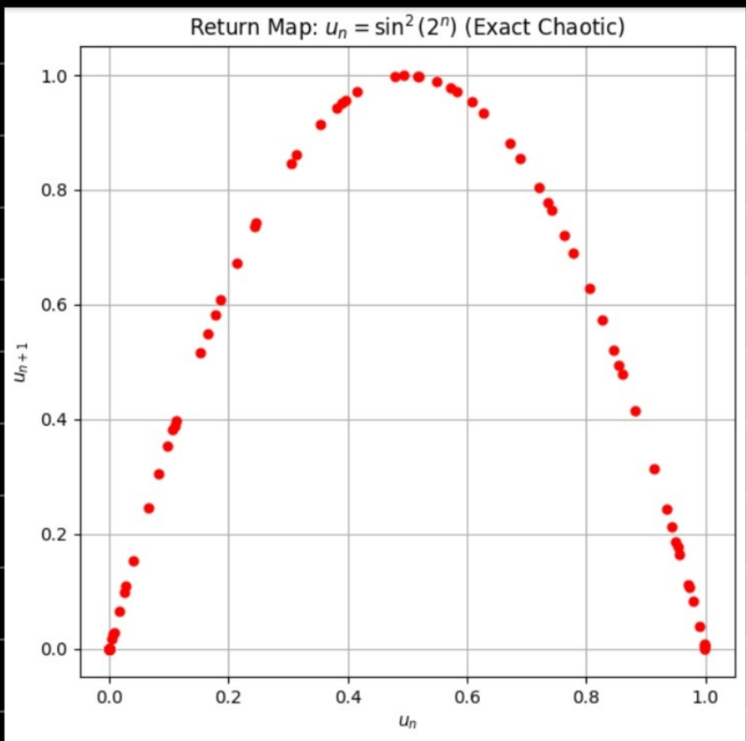
$$u_n = sin^2(2^n).$$

Plot the points (u_n, u_{n-1}) for $n = 1, 2, ..., 200$, say. What structure is revealed?

b) Period 1 for $\alpha \in (2,3)$ and period 2 for $(3, 1+\sqrt{6})$





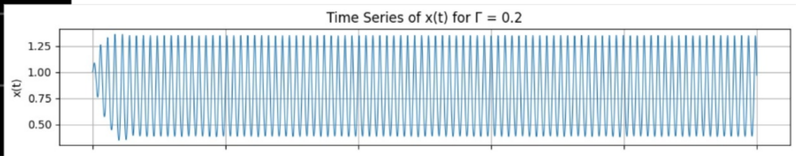


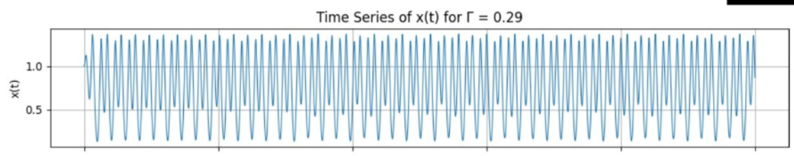
The spread out nature of the third graph is a characteristic of chaos.

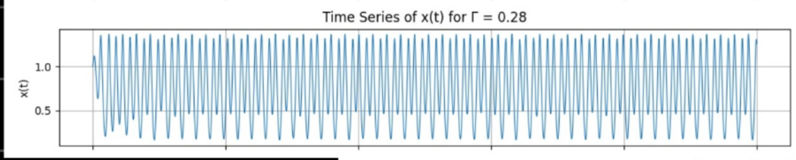
Using a computer program, generate time-series (numerical solution) for the Duffing equation

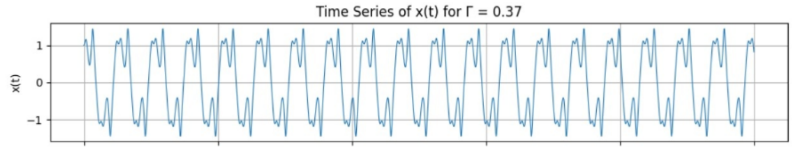
$$\ddot{x} + k\dot{x} - x + x^3 = \Gamma cos(\omega t)$$

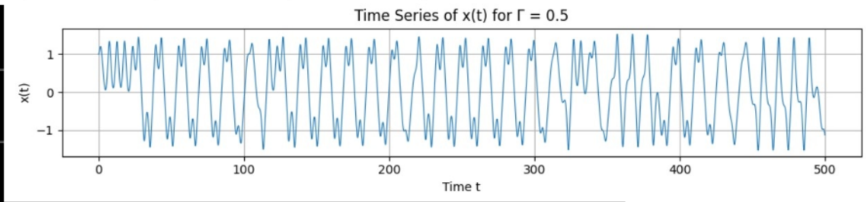
for $k = 0.3, \omega = 1.2$ and selected values of Γ, say $\Gamma = 0.2, 0.28, 0.29, 0.37, 0.5$. Plot transient-free return maps for the interpolated pairs $(x(2\pi n/\omega), x(2\pi(n-1)/\omega))$. For the *chaotic* case $\Gamma = 0.5$, take the time series over an interval $0 \leq t \leq 5000$, say. These return diagrams show that structure is recognizable in chaotic outputs: the returns are not uniformly distributed.

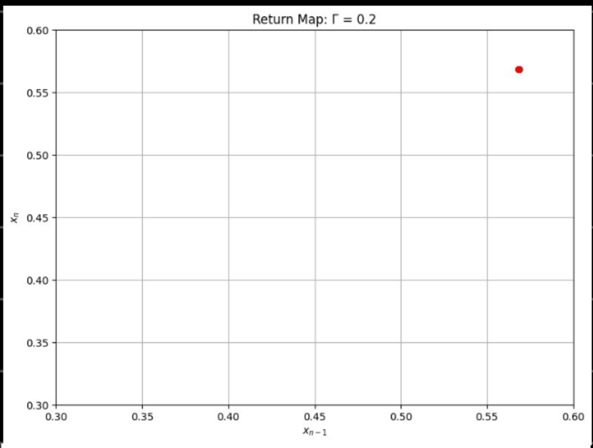




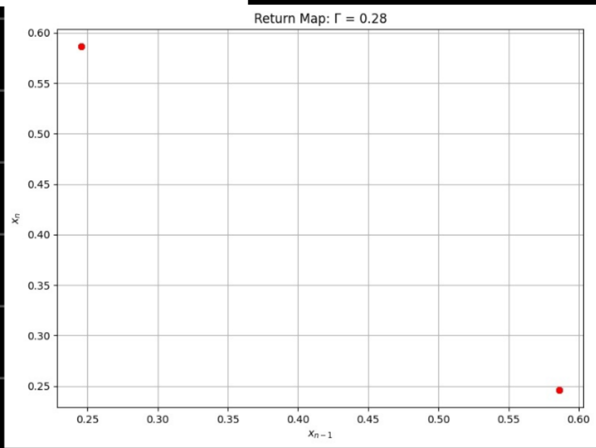


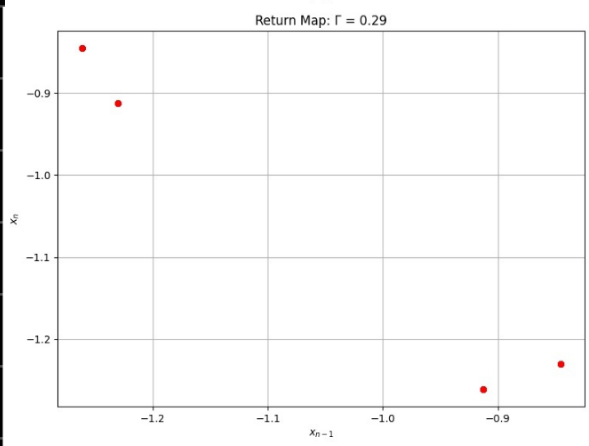


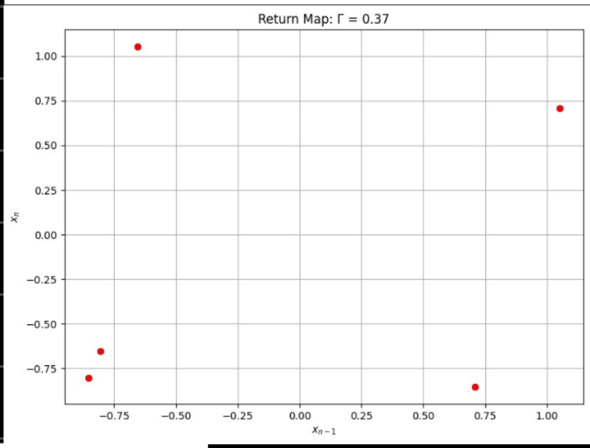


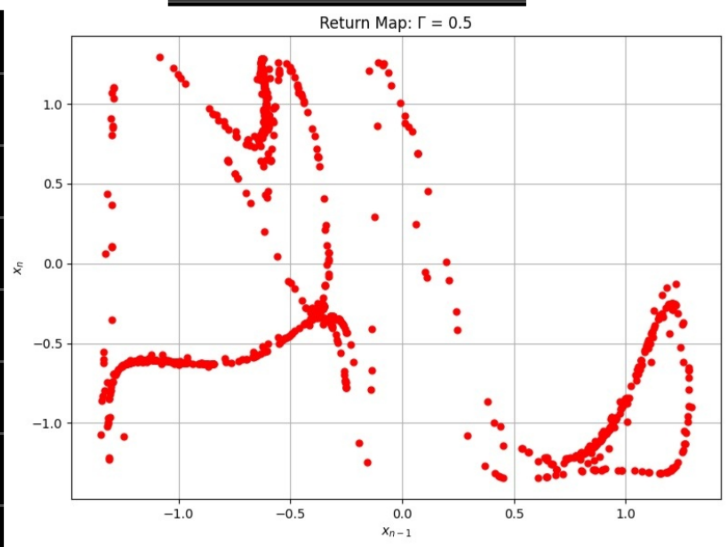


Period doubling cascade towards chaos.









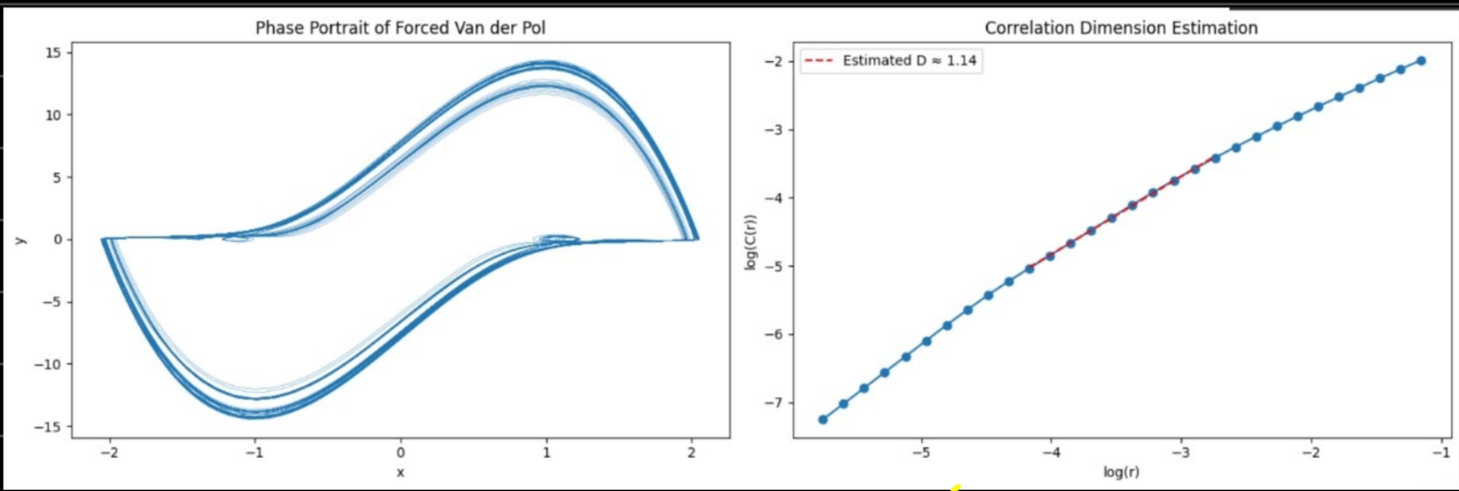
6. Start by finding any system of nonlinear ordinary differential equations that exhibits chaos other than the Rössler, Lorenz and Duffing systems that we discussed in class. Create the phase portrait for this system, for any set of parameters where chaos is exhibited. What is the fractal dimension of the strange attractor for this system?

6) Choosing the Van der Pol oscillator,

$$\ddot{x} - \mu\left(1-x^2\right)\dot{x} + x = A\cos\omega t$$

We choose $\mu = 9.53$, $A = 1.5$, $\omega = \frac{2\pi}{10}$.

This yields a strange attractor of fractal dimension $\approx \boxed{1.14}$



Measures how number of pairs of points within a distance r of each other scales with r.