A Discussion On

Chaos and Routes to Chaos in Coupled Duffing Oscillators with Multiple Degrees of Freedom [1]

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1. Introduction

Literature on studying routes to chaos in low dimensional systems is quite abundant. However the authors felt more work could be done for increasingly complex multidimensional systems, in the hopes of seeing how chaotic performance is affected by the same. Hence the choice of system they have used is a string of coupled Duffing oscillators with varying parameters, investigated by performing different numerical simulations.

The authors choose to study 2,3,4,5, and 6 coupled oscillators, by utilising numerical methods in time, frequency, and phase space through observing the Lyapunov Spectrum, Power Spectral Density, and the Poincaré section, respectively. Wolf's method [2][3] was used for determining the Lyapunov Spectrum, and for the Poincaré sections, a Runge Kutta integrator assisted in strobing the system at every period, with the first 1600 transients discarded for a more informative output.

Increasing system complexity through increasing number of degrees of freedom, different kinds of forcing excitations, higher order coupling, etc., has been studied. The results show diversity in the routes to chaos, such as period doubling cascades, crises, and torus breakdowns. Crisis replaces period doubling as the dominant route to chaos in higher-dimensional systems. Other such key results have been discussed later.

In this report, we summarise and explain the model used in the paper, look at key takeaways, explore a new setting related to the paper, and conclude with some comments and questions to ponder over. Relevant code shall be submitted as a Jupyter notebook.

2. Description of Model

We recall the prototypical equation for any Duffing oscillator:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos \omega t$$

Developing on this, the paper describes an example of 3 oscillators, two of which are coupled linearly and the remaining one is provided with an external forcing. We must note the dashpot-like damping for each mass to make it more realistic. The equations of motion are as given below:

$$m\ddot{x}_1 + Vx_1^3 + c\dot{x}_1 - V(x_2 - x_1)^3 - c(\dot{x}_2 - \dot{x}_1) = P\cos \omega t$$

$$m\ddot{x}_3 + Vx_3^3 + c\dot{x}_3 + K_l(x_3 - x_2) = 0$$

$$m\ddot{x}_2 + V(x_2 - x_1)^3 + c(\dot{x}_2 - \dot{x}_1) - K_l(x_3 - x_2) = 0$$

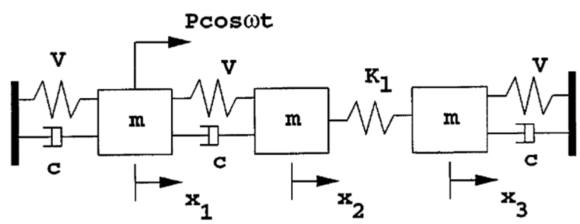


Fig. 1. Three linearly coupled Duffing oscillators.

with V being the cubic stiffness, K_l is the linear coupling, and c is the damping coefficient. Each equation is the application of Newton's Second Law on each mass. Note the nonlinearity of the Duffing springs, the relative dashpot damping and how the first two masses would have been independent from the third if not for the linear coupling between mass 2 and mass 3.

The equations were subsequently nondimensionalised and split into 2N + 1 ordinary differential equations, where N is the number of oscillators and the one extra equation is the trivial $\dot{y} = \omega$ equation for time, a trick to make the system seem autonomous. Again for 3 masses, the equations (which may be logically extrapolated to more oscillators) now look like

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = \left[B\cos y_3 - ky_2 - y_1^3 + (y_4 - y_1)^3 + k(y_5 - y_2)\right]$$
$$\dot{y}_3 = \omega$$
$$\dot{y}_4 = y_5$$
$$\dot{y}_5 = \left[k_c(y_6 - y_4) - k(y_5 - y_2) - (y_4 - y_1)^3\right]$$
$$\dot{y}_6 = y_7$$
$$\dot{y}_7 = \left[-k_c(y_6 - y_4) - y_6^3 - ky_7\right]$$

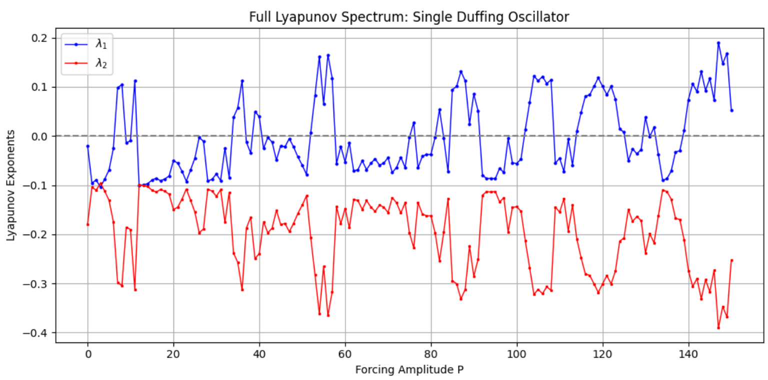
We are now primarily interested in varying the forcing amplitude B, and seeing what routes to chaos the system takes. For all graphs, $k = 0.2, k_c = 10$ *and* $\omega = 1$.

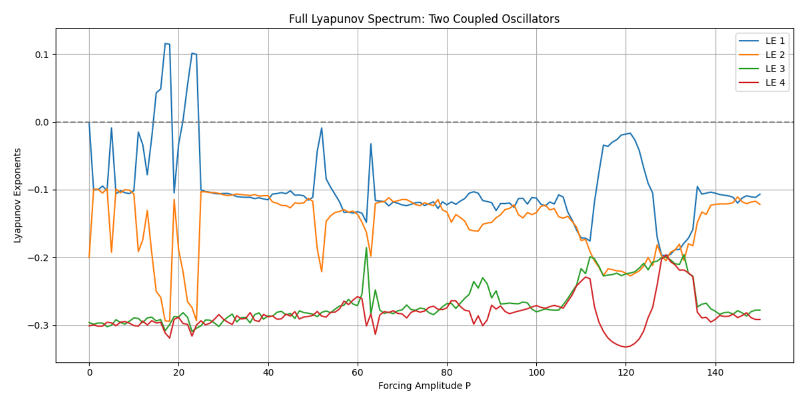
3. Key Takeaways: Routes to Chaos, Quasiperiodicity and More

The paper presents a great many empirical results, and we shall summarise the most important discussions here. Probably the two key takeaways are

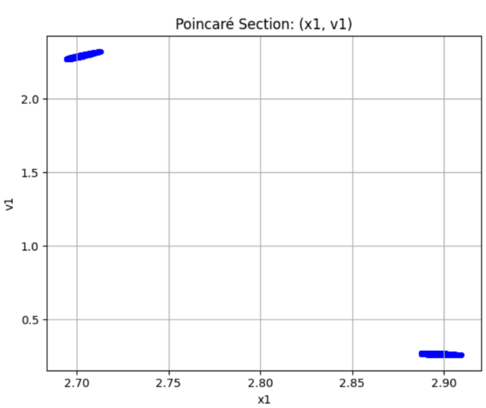
"Chaos decreases with more degrees of freedom"
and
"Crises and Torus Breakdowns become quite prominent"

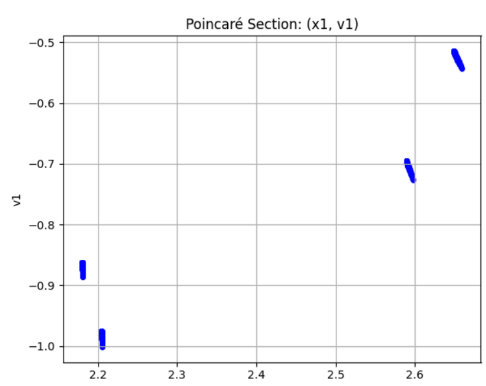
Let's examine each result, and more, individually. We plot the Lyapunov exponent for each degree of freedom as a function of B, the forcing amplitude, and notice that the regions of chaos significantly decrease.

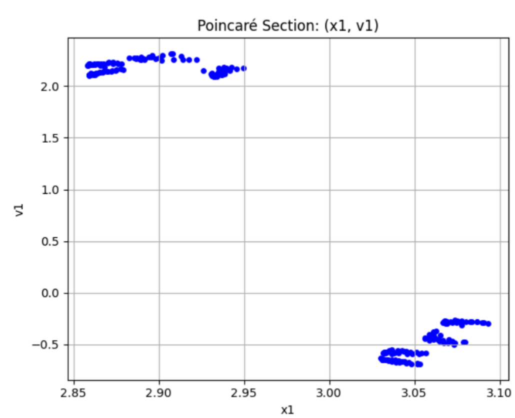


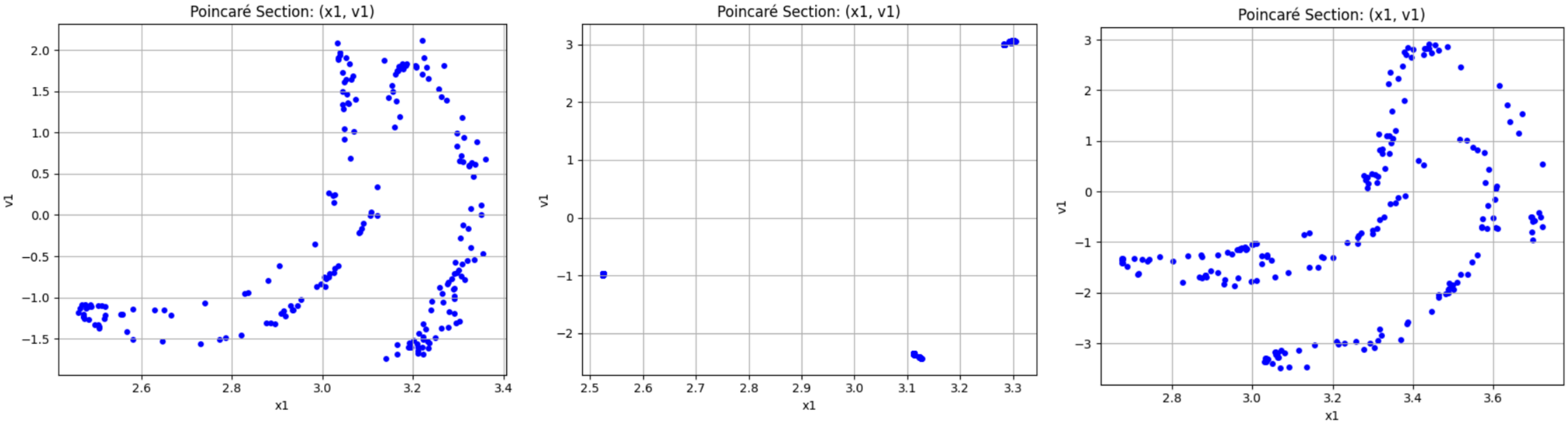


To clarify, chaos occurs when the largest Lyapunov exponent for a particular value of B is positive. For the single Duffing oscillator (with same parameter values), chaos is widespread (as seen by the behaviour of the blue line and how it crosses the zero mark several times) while on the other hand for just two oscillators, chaos get severely restricted to just the B =18-26 range. To get a feel for what's really happening around the critical regions in this Lyapunov exponent map, let's see the Poincaré sections for (any) one of the oscillators around B=18-26.



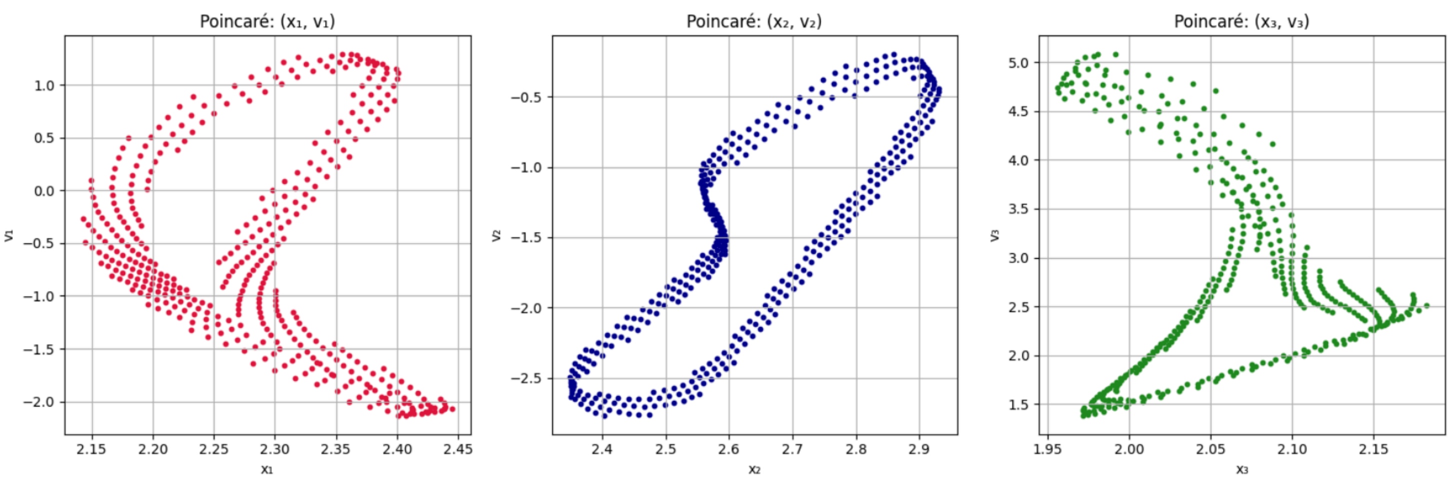






The above set of figures show the variation as $B: 12 \to 14 \to 14.5 \to 18 \to 20 \to 22.5$. Notice how the system follows a period doubling cascade initially, and a gradual descent into chaos as marked by B=18. But the route to chaos for higher B is different: the system has period 3 for a while till ~21, and then suddenly we see the appearance of a strange attractor without warning. This is the mark of a *crisis event*, a phenomenon that becomes more widespread in higher dimensional systems and one that can be problematic if it appears in your control system.

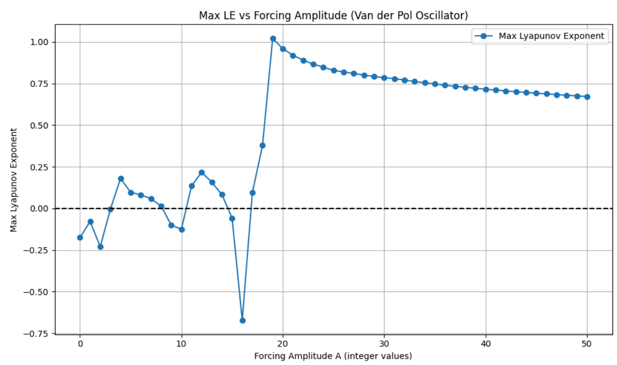
The next most interesting feature of these systems is the appearance of quasiperiodicity, either just as a cameo (in 3 coupled oscillators) or as a route to chaos (for symmetrically coupled oscillators), which is termed as *torus breakdown.* Quasiperiodicity is expressed as a closed curve on a Poincaré section, as shown below for B=24.8 and 3 coupled oscillators.

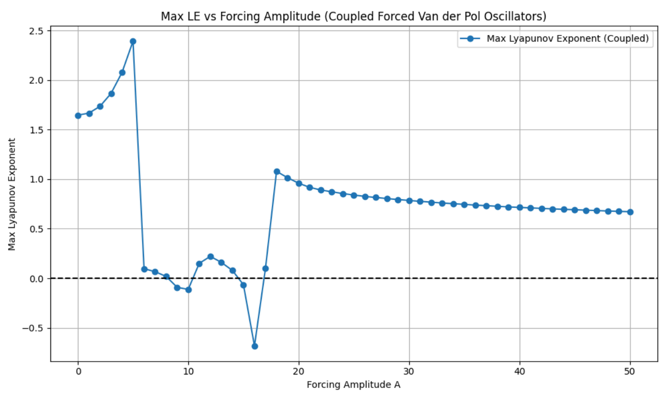


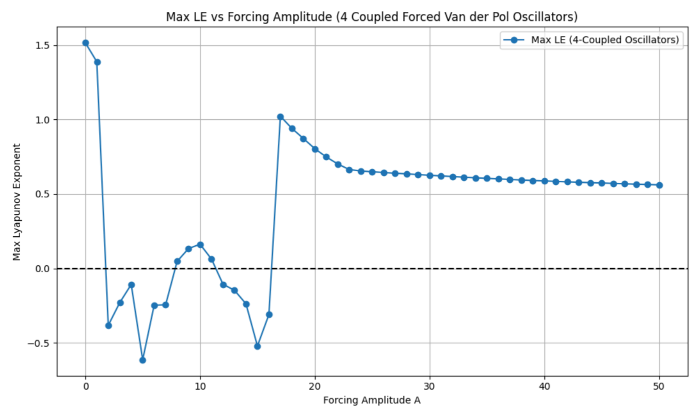
Aside from these, the authors also mention other variations. It turns out that for a given B, increasing the linear coupling coefficient or making the coupling cubic significantly increases the number of chaotic windows. Another variation was to distribute excitations by forcing another mass as well. For frequency ratios of 0.2 and 0.7, no chaotic motion at all was observed for 2 coupled oscillators, and is ascribed partly to mode locking, wherein oscillations lock onto a particular frequency and do not go astray. More number of oscillators showed only one or two new features of interest, and the main features have been explained above.

4. Self Exploration and Results

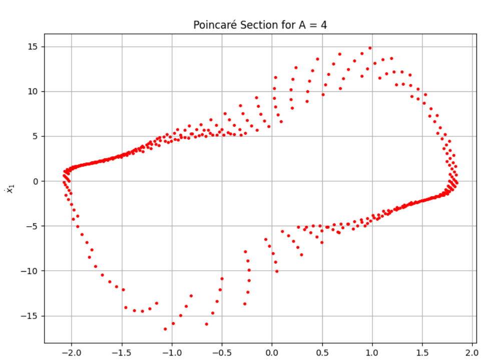
Probably the most non-intuitive result is that chaos decreases when you add more complexity to the system. Naturally, I wanted to see if this was generalisable to other oscillators. So I chose the Van der Pol oscillator and performed similar analysis. Shown below are results for the variation of the maximum Lyapunov exponent for 1, 2 and 4 symmetrically coupled oscillators as forcing amplitude varies from 1 to 50.

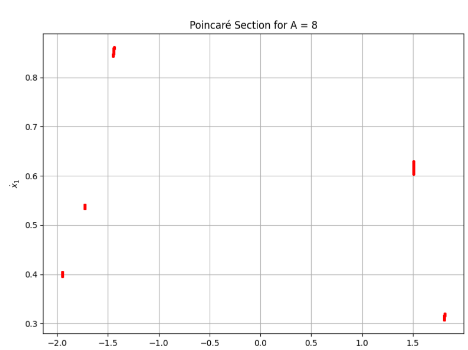


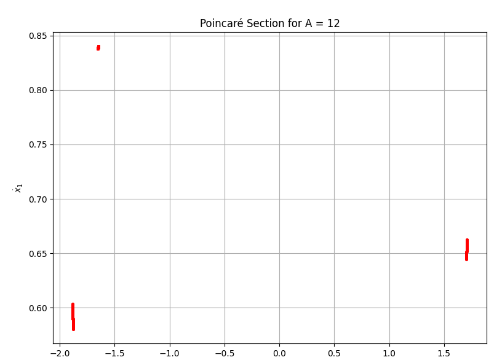




Interestingly all three cases show a flatlining MLE after A≈18. Clearly the MLEs are more positive for two oscillators than four or one, and it appears as if in general we cannot say that coupling oscillators together always leads to a reduced number of chaotic windows. This then begs the question: what really causes a system's chaotic regimes to increase or decrease? Is the choice of parameters special, and can all this be backed mathematically? I could not cover the latter here, and this could be left for a future work. Lastly, given below are the Poincaré sections for two coupled oscillators, also seeming to show crisis.







5. Conclusion and Comments

Through this paper I learnt several new tools for time series analysis. Earlier my immediate instinct was to judge system behaviour just by looking at the phase space behaviour or possible bifurcation diagrams, but now I see the importance of plotting Poincaré sections and seeing variation in the Lyapunov exponents. Crisis as a route to chaos was also new to me, and I enjoyed playing around with the parameters to see it happen before my eyes.

My main contention with the paper was that the authors did not include linear stiffness in the ODEs, and the cubic springs initially confused me into thinking that maybe that was the coupling they wanted to achieve. It is also a paper that his little to no mathematical analysis, but makes up for this by going quite deep into presenting empirical data that matches pretty well with existing literature.

Coupled Duffing oscillators may have been studied since antiquity, but nonetheless they have practical importance in many fields such as cardiography [4], electrical circuits [5], etc. This paper also suggests that for the applications of such nonlinear systems, to avoid unpredictable chaos we could decrease coupling, introduce mode locking or increase dimensionality.

The Duffing oscillator is one that always has things to hide, with probably many more secrets lurking in the dark, waiting for us to discover them. This paper only touched the tip of a humongous iceberg.

6. References

[1]D.E. Musielak, Z.E. Musielak, J.W. Benner, Chaos and routes to chaos in coupled Duffing oscillators with multiple degrees of freedom, Chaos, Solitons & Fractals, Volume 24, Issue 4, 2005, Pages 907-922, ISSN 0960-0779, https://doi.org/10.1016/j.chaos.2004.09.1
[2] Wolf A, Swift JB, Swinney HL, Vasano JA. Physica D 1985;7:285.
[3] Wolf A. Quantifying chaos with Lyapunov exponents. In: Holden AV, editor. Chaos. Princeton: Princeton University Press; 1986. [22] Welch PD. IEEE Trans Audion Electroacoust 1967;AU-15:70.
[4] Fonkou, Rodrigue F., and Marcelo A. Savi. 2023. "Heart Rhythm Analysis Using Nonlinear Oscillators with Duffing-Type Connections" Fractal and Fractional 7, no. 8: 592.
[5] https://doi.org/10.3390/fractalfract7080592 3. https://doi.org/10.14483/issn.2248-4728