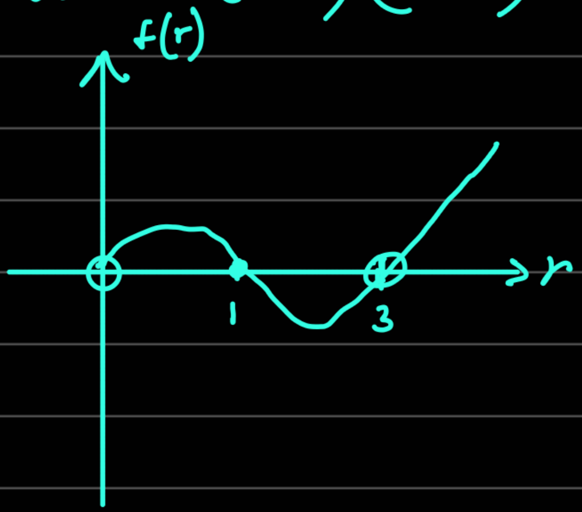
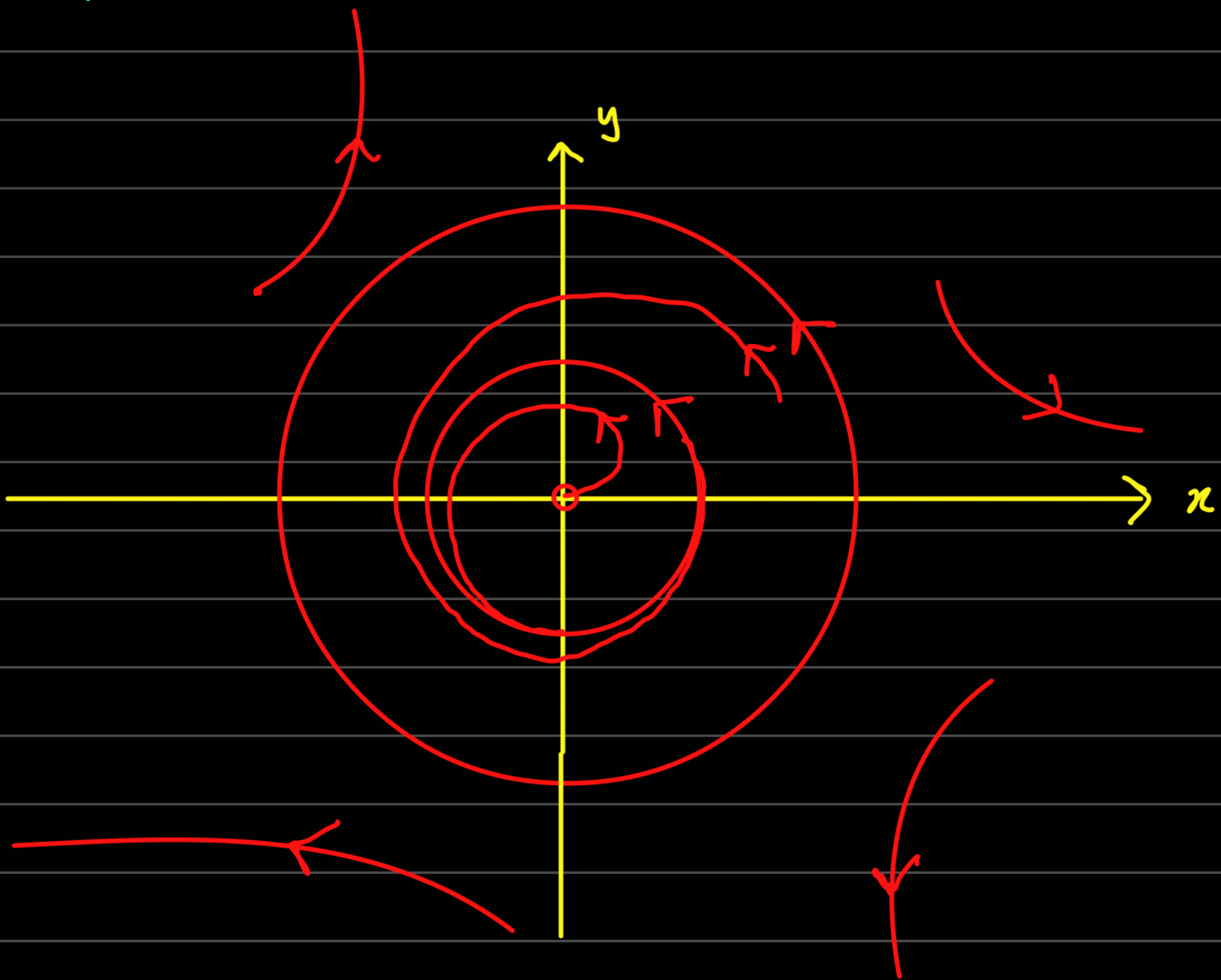
1. Sketch the phase portraits of the following systems:

(a) $\dot{r} = r(1 - r^2)(9 - r^2), \ \dot{\theta} = 1$

a) $f(r) = r(1-r^2)(9-r^2)$, $r^* = 0, 1, 3 \rightarrow$ unstable

$0 \rightarrow$ unstable, $1 \rightarrow$ stable



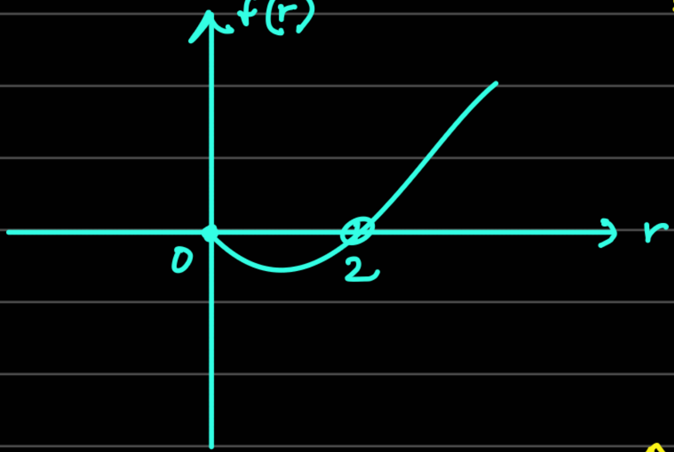


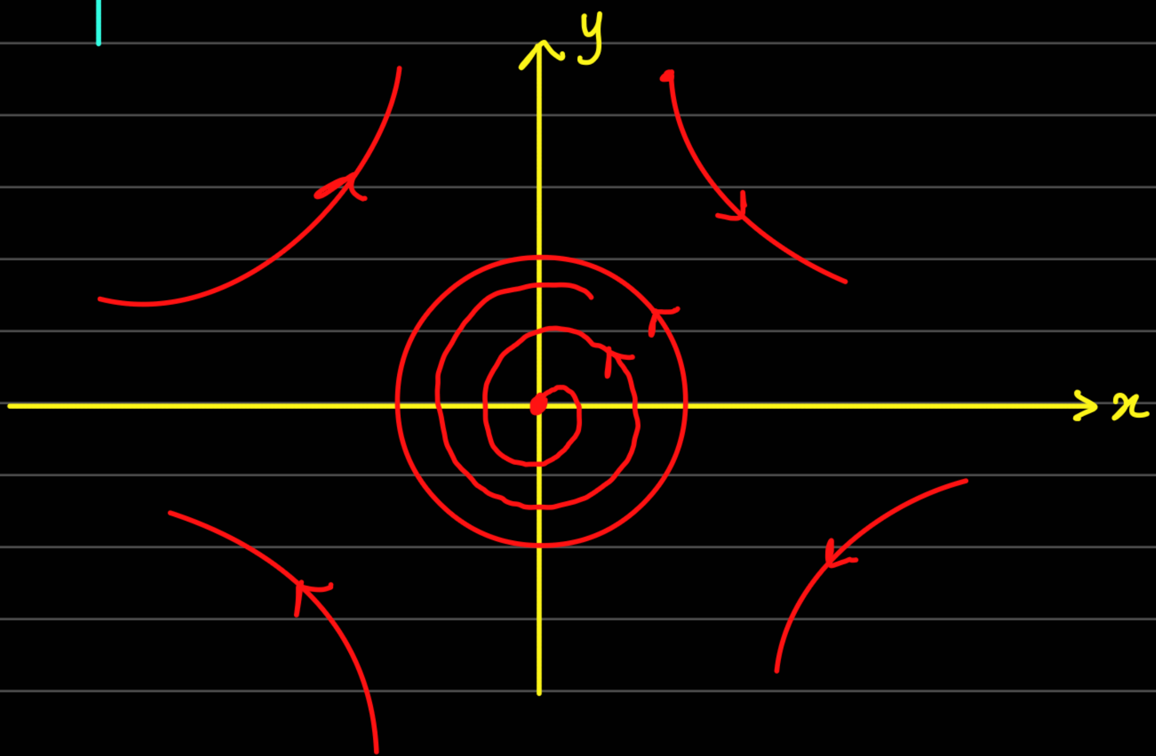
(b) $\dot{r} = r^3 - 4r, \quad \dot{\theta} = 1.$

b) $f(r) = r(r^2 - 4)$

$r^* = 0, 2 \rightarrow$ unstable

$0 \rightarrow$ stable





2. Consider $\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$, where $a > 0$.

(a) Find and classify all the fixed points.

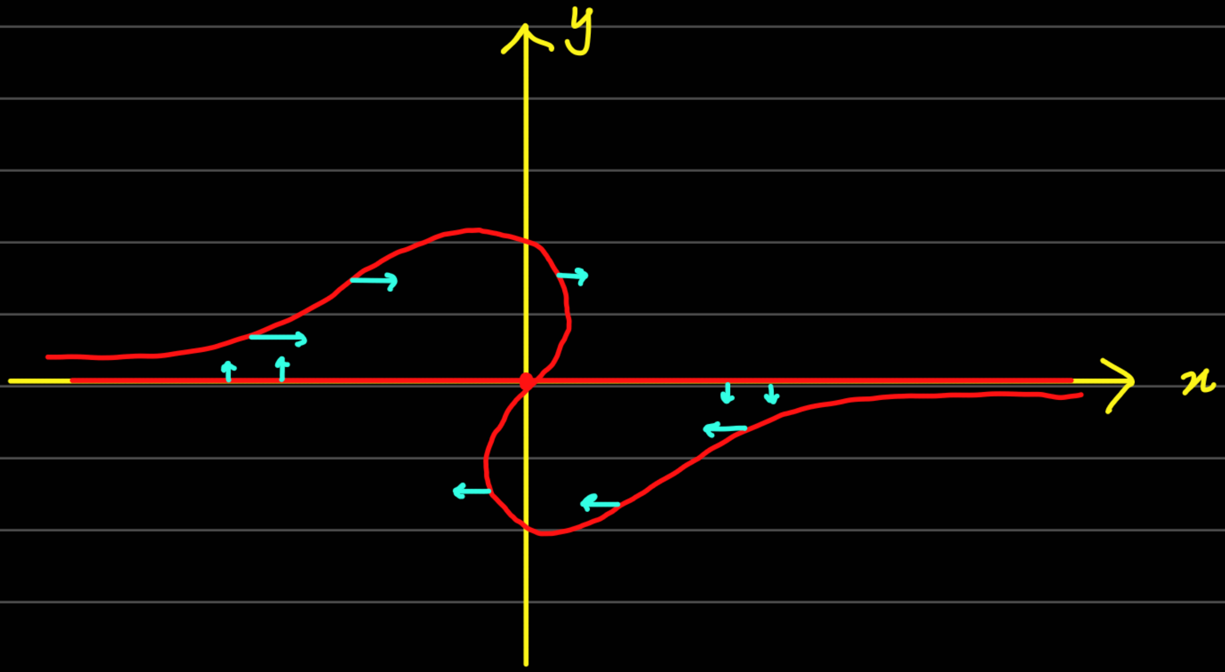
a) $\dot{x} = y \qquad = f(x,y)$

$\dot{y} = -ay(x^2+y^2-1) - x \qquad = g(x,y)$

For f.p., $\boxed{y = 0 \quad , \quad x = 0}$

(b) Show that the system has a circular limit cycle, and find its amplitude and period.

b) Nullclines : $y = 0$ and $ay(x^2+y^2-1) = -x$



So this hints at a circle as a choice for a trapping region, but $(0,0)$ is a f.p. We must show that this is a repeller.

$$J = \begin{pmatrix} 0 & 1 \\ -2axy - 1 & -a(x^2+y^2-1) - 2ay^2 \end{pmatrix}$$

At $(0,0)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$

$\tau = a > 0$

$\Delta = a+1 > 0$

so $(0,0)$ is a repeller.

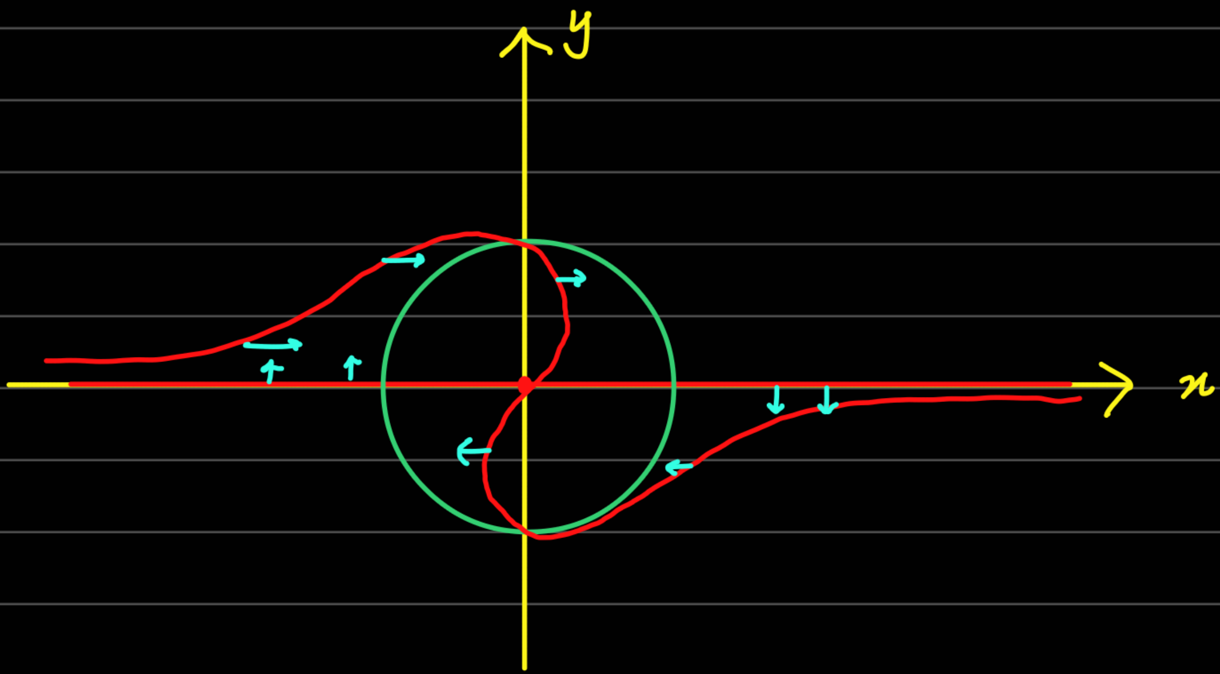
let $x = R\cos\theta$, $y = R\sin\theta$ (fixed R)

Then we get,

$$-R\sin\theta\,\dot{\theta} = R\sin\theta \Rightarrow \dot{\theta} = -1$$

$$R\cos\theta\,\dot{\theta} = -aR\sin\theta\,(R^2-1) - R\cos\theta$$

$$\Rightarrow \quad R = 1 \qquad (\because \dot{\theta} = -1)$$



From judging the direction of arrows, we see that it is stable.

$$\text{Period} = \frac{2\pi}{|\dot{\theta}|} = \boxed{2\pi}, \quad \text{amplitude} = R = \boxed{1}$$

(c) Determine the stability of the limit cycle.

c) From the previous part, the direction of arrows shows that it is stable.

Alternatively,

$$x = (1+\varepsilon)\cos\theta \qquad y = (1+\varepsilon)\sin\theta \qquad |\varepsilon| << 1$$

$$\dot{x} = \dot{\varepsilon}\cos\theta - (1+\varepsilon)\sin\theta\,\dot{\theta} \qquad \dot{y} = \dot{\varepsilon}\sin\theta + (1+\varepsilon)\cos\theta\,\dot{\theta}$$

Converting $\dot{y}$,

$$\Rightarrow \dot{\varepsilon}\sin\theta + (1+\varepsilon)\cos\theta\,\dot{\theta} = -a(1+\varepsilon)\sin\theta\left((1+\varepsilon)^2 - 1\right) - (1+\varepsilon)\cos\theta$$

$$\dot{\varepsilon}\sin\theta + (1+\varepsilon)\cos\theta\,\dot{\theta} = -a(1+\varepsilon)\sin\theta\,(2\varepsilon) - (1+\varepsilon)\cos\theta$$

$$\dot{\varepsilon}\sin\theta + (1+\varepsilon)\cos\theta\,(1+\dot{\theta}) = -2a\varepsilon\sin\theta$$

Converting $\dot{x}$,

$$\dot{\varepsilon}\cos\theta - (1+\varepsilon)\sin\theta\,\dot{\theta} = (1+\varepsilon)\sin\theta$$

$$\dot{\varepsilon}\cos\theta = (1+\varepsilon)\sin\theta\,(1+\dot{\theta})$$

$$\Rightarrow 1+\dot{\theta} = \frac{\dot{\varepsilon}\cos\theta}{(1+\varepsilon)\sin\theta}$$

$$\Rightarrow \dot{\varepsilon}\sin\theta + \frac{\dot{\varepsilon}\cos^2\theta}{\sin\theta} = -2a\varepsilon\sin\theta$$

$\underline{\dot{\varepsilon} = (-2a\sin^2\theta)\,\varepsilon}$ so the perturbation dies out, and $\underline{r=1 \text{ is stable}}$

3. Show that $\dot{x} = -x + 2y^3 - 2y^4$, $\dot{y} = -x - y + xy$ has no periodic solutions. (Hint: Choose a, m and n such that $V = x^m + ay^n$ is a Lyapunov function.

(3)

$$\frac{dV}{dt} = mx^{m-1}\dot{x} + any^{n-1}\dot{y}$$

$$= mx^{m-1}(-x+2y^3-2y^4) + any^{n-1}(-x-y+xy)$$

$$= -mx^m + 2mx^{m-1}y^3 - 2mx^{m-1}y^4 - anxy^{n-1} - any^n + anxy^n$$

$$= -mx^m - any^n + 2mx^{m-1}y^3(1-y) - anxy^{n-1}(1-y)$$

$$\frac{dV}{dt} = -mx^m - any^n + (1-y)(2mx^{m-1}y^3 - anxy^{n-1})$$

So to maintain $\frac{dV}{dt} < 0 \;\forall\, \bar{X} \neq \bar{X}^* = (0,0)$,

$$m=2,\; n=4,\; a=1$$

So $V(x,y) = \boxed{x^2+y^4}$ and $V(x,y) \geq 0 \;\forall\, (x,y) \neq (0,0)$

4. Consider $\dot{x} = x - y - x(x^2+5y^2)$, $\dot{y} = x + y - y(x^2+y^2)$.

(a) Classify the fixed point at the origin.

a) $\frac{\partial f}{\partial x} = 1 - (x^2+5y^2) - 2x^2 = 1 - 3x^2 - 5y^2$ $\qquad \frac{\partial f}{\partial y} = -1 - 10xy$

$\frac{\partial g}{\partial x} = 1 - 2xy$ $\qquad \frac{\partial g}{\partial y} = 1 - (x^2+y^2) - 2y^2 = 1 - x^2 - 3y^2$

$\therefore \quad J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\qquad \tau = 2$

$\Delta = 2$, $\tau^2 - 4\Delta = -4 < 0$

So it is an <u>unstable spiral</u>

(b) Rewrite the system in polar coordinates.

b) $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow \quad \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = r\cos\theta - r\sin\theta - r^3\cos\theta(1+4\sin^2\theta)$ $\quad - 1$

$\dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = r\cos\theta + r\sin\theta - r^3\sin\theta$ $\quad - 2$

1 $\times\cos\theta$ + 2 $\times\sin\theta$: $\boxed{\dot{r} = r - r^3(1+\sin^2 2\theta)}$

1 $\times\sin\theta$ − 2 $\times\cos\theta$: $-r\dot{\theta} = -r - 4r^3\cos\theta\sin^3\theta$

$\Rightarrow \boxed{\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta}$

(c) Determine the circle of maximum radius r_1, centered on the origin such that all trajectories have a radially outward component on it.

(d) Determine the circle of minimum radius r_2, centered on the origin such that all trajectories have a radially inward component on it.

c) $\dot{r} > 0 \Rightarrow r - r^3(1+\sin^2 2\theta) > 0 \Rightarrow \frac{1}{1+\sin^2 2\theta} > r^2$

For largest radius on which $\dot{r} > 0$, and observing that $0 \leq \sin^2 2\theta \leq 1$, we choose $\theta = 0$, such that $r_1 = 1$

d) $\dot{r} < 0 \Rightarrow r - r^3(1+\sin^2 2\theta) < 0 \Rightarrow \dfrac{1}{1+\sin^2 2\theta} < r^2$

With logic similar to above, choose $\theta = \frac{\pi}{4}$ to get a minimum radius on which $\dot{r} < 0$, such that $\boxed{r_2 = \frac{1}{\sqrt{2}}}$

(e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \le r \le r_2$.

e) Let's check for fixed points in this region,

$$\dot{r} = r - r^3\left(1+\sin^2 2\theta\right) = 0 \Rightarrow r^{*2} = \frac{1}{1+\sin^2 2\theta}$$

$$\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta = 0 \Rightarrow r^{*2} = -\frac{1}{4\cos\theta\sin^3\theta}$$

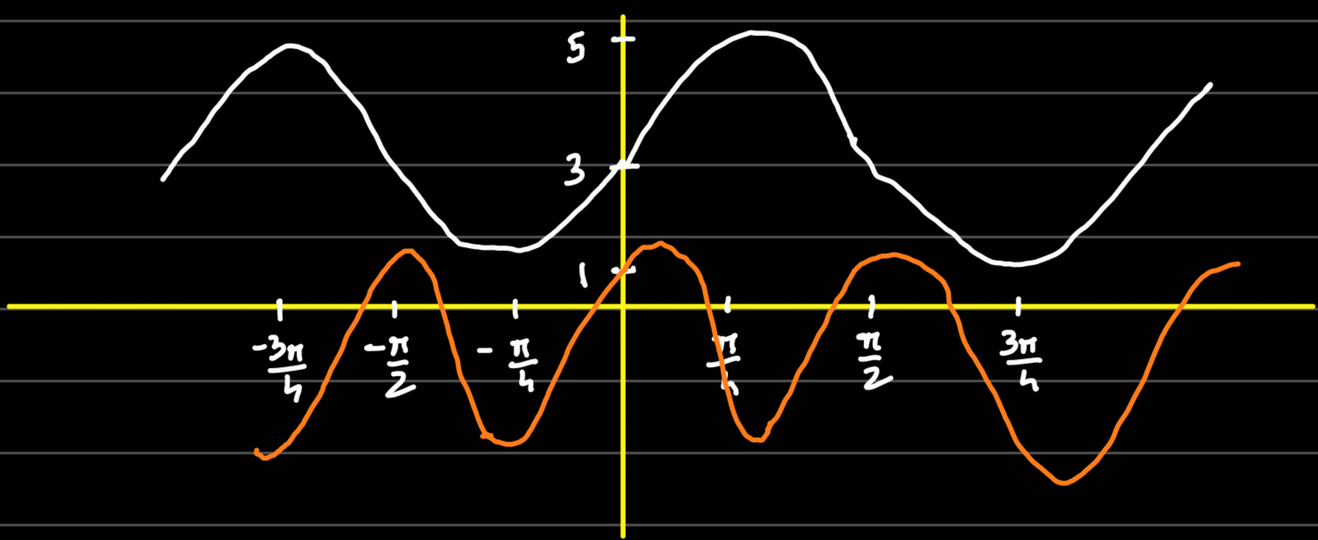
$$\Rightarrow \frac{1}{1+\sin^2 2\theta} = \frac{-1}{4\cos\theta\sin^3\theta}$$

$$\Rightarrow 1+\sin^2 2\theta = -2\sin 2\theta\sin^2\theta$$

$$1+\sin^2 2\theta = \sin 2\theta\left(\cos 2\theta - 1\right)$$

$$3 - \cos 4\theta = \sin 4\theta - 2\sin 2\theta$$

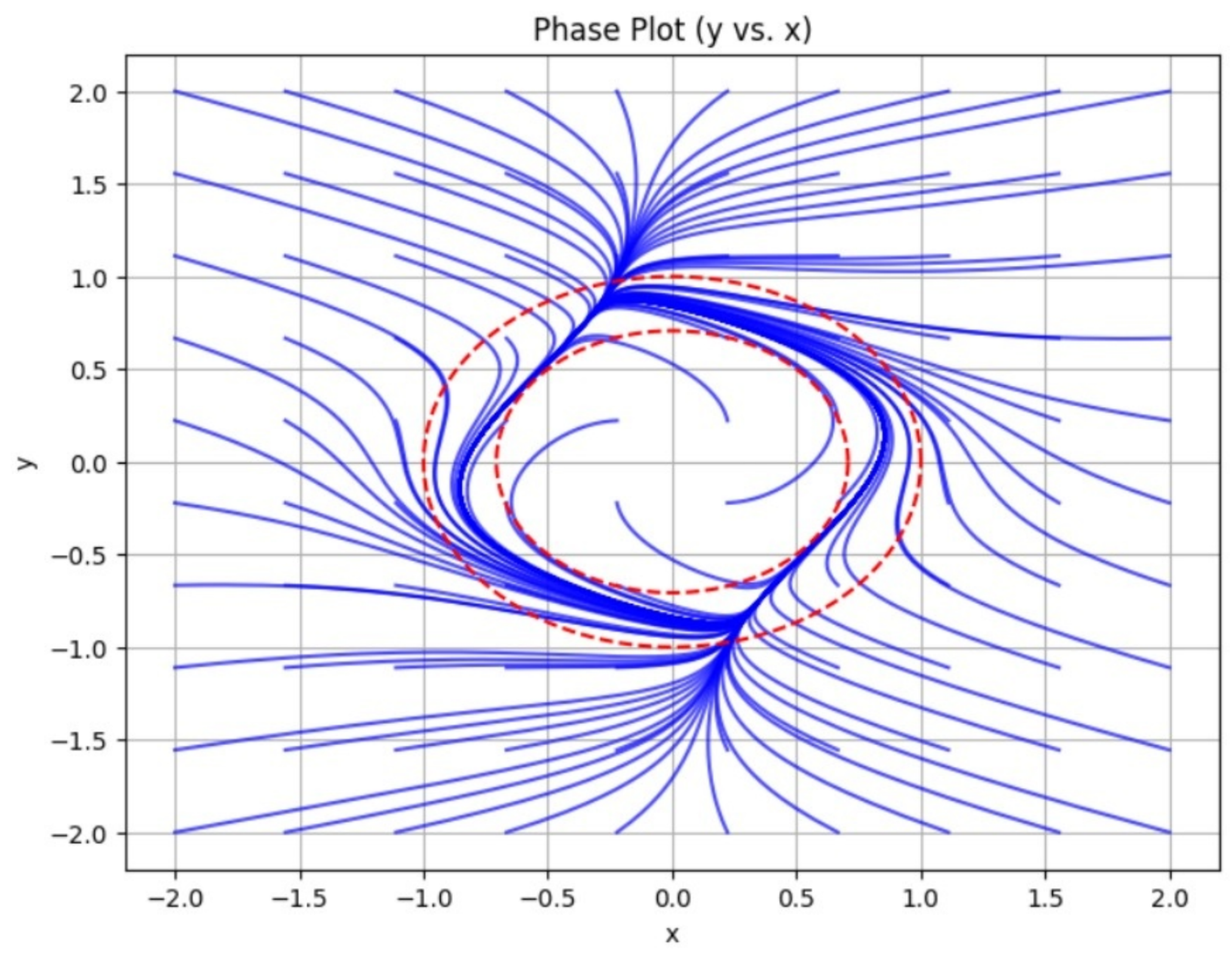
$$3 + 2\sin 2\theta = \sin 4\theta + \cos 4\theta$$



Never cross, so <u>no fixed points</u>.

Since $r \in [r_1, r_2]$ is a trapping region, all vector fields are continuous and there are no fixed points, we have a limit cycle in this region (Poincaré - Bendixson Theorem)

(f) Using numerical integration, compute the limit cycle and verify that it lies in the trapping region constructed in the previous step.



5. Let $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ be a smooth vector field defined on the phase plane.

(a) Show that is this is a gradient system, then $\partial f/\partial y = \partial g/\partial x$.

a) $\dot{x} = f = -\frac{\partial V}{\partial x} \Rightarrow \frac{\partial f}{\partial y} = -\frac{\partial V}{\partial y \partial x}$

$\dot{y} = g = -\frac{\partial V}{\partial y} \Rightarrow \frac{\partial g}{\partial x} = -\frac{\partial V}{\partial x \partial y}$

For smooth vector fields and smooth V, $\boxed{\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}}$

(b) Is the above condition also sufficient?

b) No, just because $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$, it does not mean the system is a gradient system. You still need to satisfy $\frac{\partial f}{\partial x} = -\frac{\partial^2 V}{\partial x^2}$, $\frac{\partial g}{\partial y} = -\frac{\partial^2 V}{\partial y^2}$.

(c) If $f(x, y) = y + 2xy$, $g(x, y) = x + x^2 - y^2$, prove that closed orbits cannot exist.

(d) Find potential function V.

$-\frac{\partial V}{\partial x} = y + 2xy \Rightarrow V = -xy - x^2y + \phi(y)$

$-\frac{\partial V}{\partial y} = x + x^2 - y^2 \Rightarrow V = -xy - x^2y + \frac{y^3}{3} + \psi(x)$

Comparing, we get $\phi(y) = \frac{y^3}{3}$, $\psi(x) = 0$.

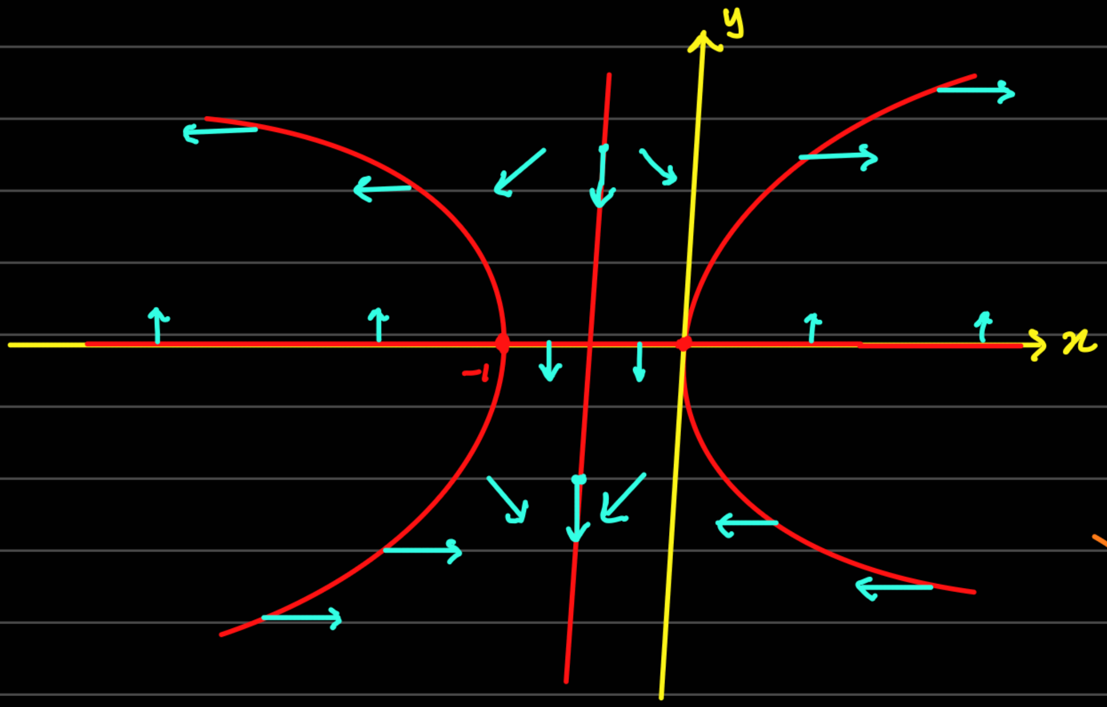
$\therefore$ $\boxed{V(x,y) = -xy - x^2y + \frac{y^3}{3}\text{, the system is a gradient system so closed orbits cannot exist.}}$

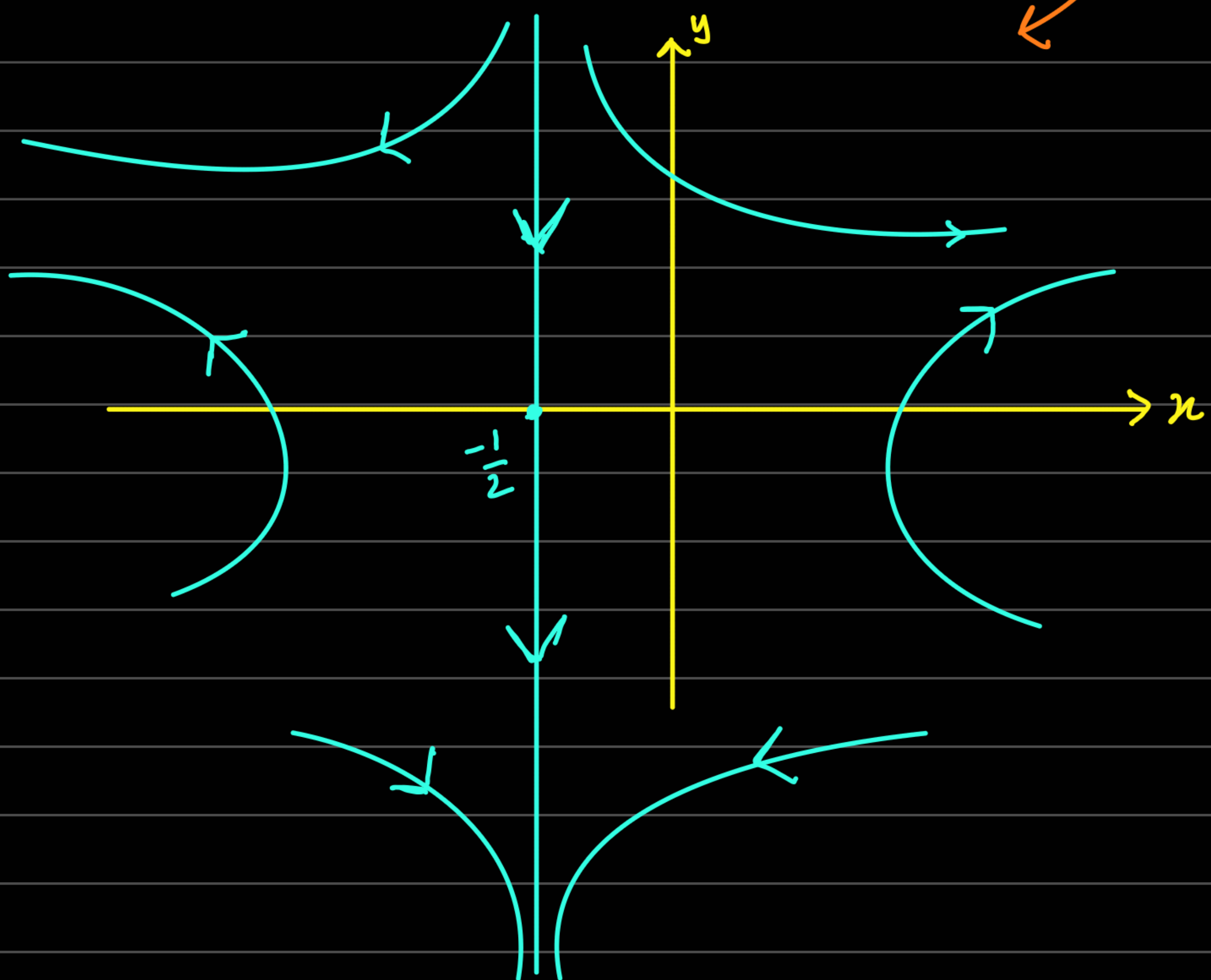
(e) Sketch the phase portrait.

e)

$$\dot{x} = y + 2xy$$
$$\dot{y} = x + x^2 - y^2$$

, nullclines: $y(1+2x) = 0$
$x + x^2 - y^2 = 0$





6. Consider the two-dimensional system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - r^2\mathbf{x}$, where $r = ||\mathbf{x}||$ and $\mathbf{A}$ is a 2×2 constant real matrix with complex eigenvalues $\alpha \pm i\omega$. Prove that there exists at least one limit cycle for $\alpha > 0$ and that there are none for $\alpha < 0$.

6

$$r^2 = ||x||^2 \Rightarrow r\dot{r} = \dot{x} \cdot x$$
$$r\dot{r} = (Ax - r^2 x) \cdot x$$
$$r\dot{r} = Ax \cdot x - r^4$$

Now $||Ax \cdot x|| \leq ||A|| r^2$,

So $\frac{dr}{dt} \leq ||A|| r - r^3 = r(||A|| - r^2)$

For $r > \sqrt{||A||}$, $\frac{dr}{dt} < 0$

Now, for $\alpha > 0$, origin has an unstable spiral, so $r \in (0, \sqrt{||A||})$ is a trapping region.

Then by the Poincare-Bendixson Theorem, there is a limit cycle (for $\alpha > 0$)

Now consider $\alpha < 0$.

$$\nabla \cdot \dot{x} = \nabla \cdot (Ax - r^2 x) = tr(A) - r^4 = 2\alpha - r^4 < 0$$

As per Dulac's criterion, with $g = 1$, $\nabla \cdot (g\dot{x})$ maintains the same sign so there exists no limit cycle for $\alpha < 0$.

7. Consider the system $\dot{r} = r(1-r^2) + \mu r\cos\theta$, $\dot{\theta} = 1$. Using the Poincare-Benedixson theorem, show that a closed orbit exists for all $\mu < 1$, and it lies somewhere in the annulus $0.999\sqrt{1-\mu} < r < 1.001\sqrt{1+\mu}$. (This was shown in the class). Now using the computer, plot the phase portrait for various values of $\mu > 0$. Is there a critical value μ_c at which the closed orbits ceases to exist? If so, estimate it. If not, prove that a closed orbit exists for all $\mu > 0$.

7) Let's search for a circle on which flow is outward.
Then $\dot{r} > 0$, $\quad r(1-r^2) + \mu r\cos\theta > 0$
For an extremal radius, LHS must be +ve even when $\cos\theta$ is -ve enough. So for $\cos\theta = -1$,

$$r(1-r^2) - \mu r = 0$$

$$1 - r^2 - \mu = 0 \quad \Rightarrow \quad r = \sqrt{1-\mu}$$

This must be the min. radius (as flow is outward).

For inward flow, $\dot{r} < 0$

$$r(1-r^2) + \mu r\cos\theta < 0$$

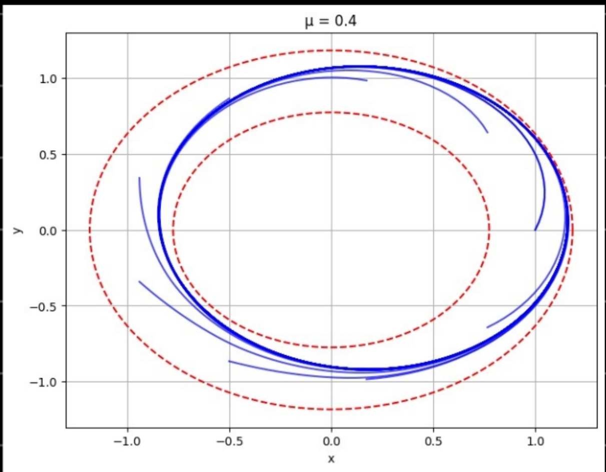
For an extremal radius, LHS must be -ve even when $\cos\theta$ is +ve enough. So for $\cos\theta = +1$,

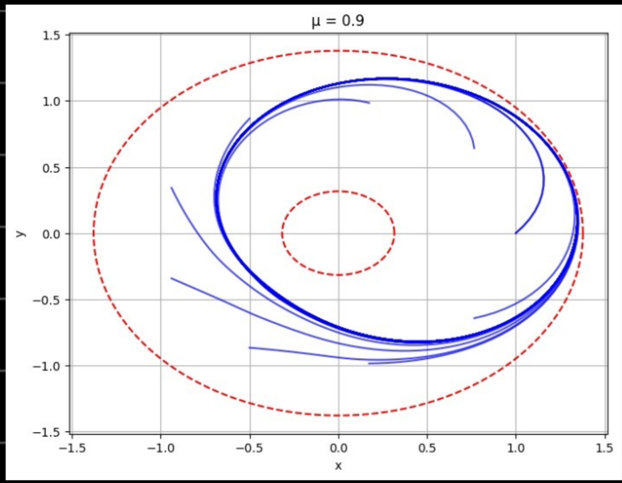
$$r(1-r^2) + \mu r = 0$$

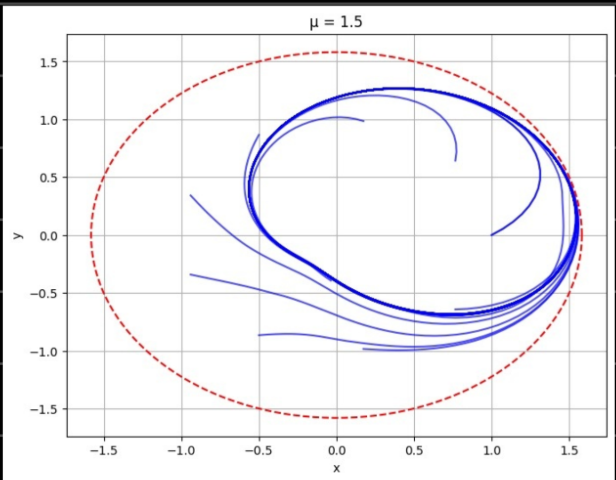
$$\Rightarrow \quad r = \sqrt{1+\mu}$$

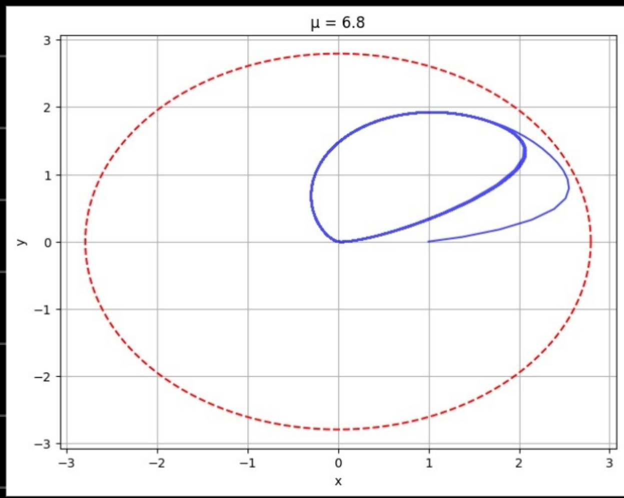
This must be the max. radius (as flow is inward).

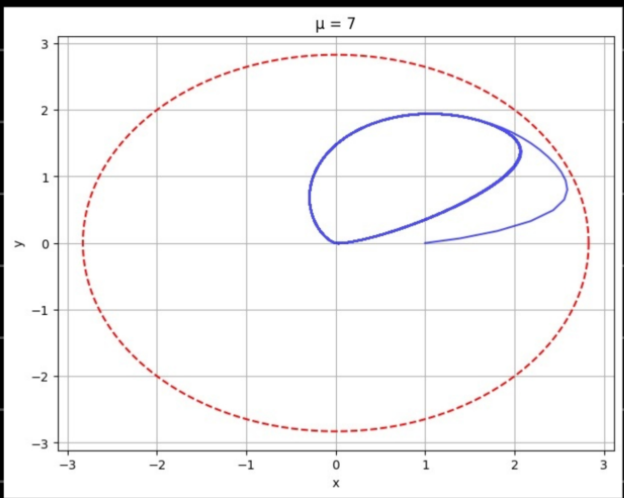
So we have constructed a trapping region $\sqrt{1-\mu} < r < \sqrt{1+\mu}$, there are no fixed points in this region and all fields are smooth enough.
$\therefore$ As per the Poincare-Bendixson Theorem, there exists a limit cycle in this region for $\mu < 1$.



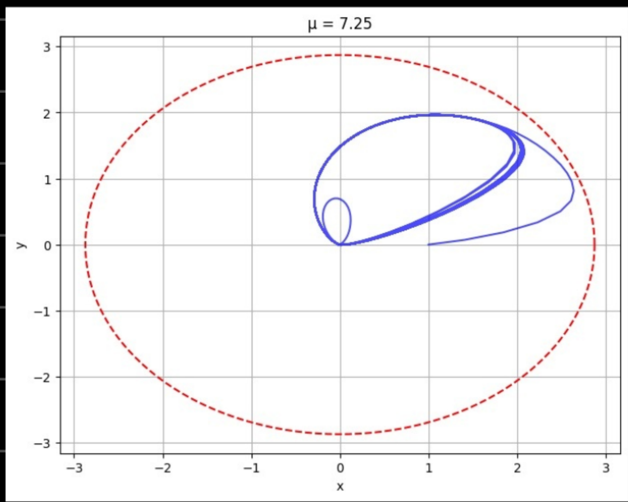


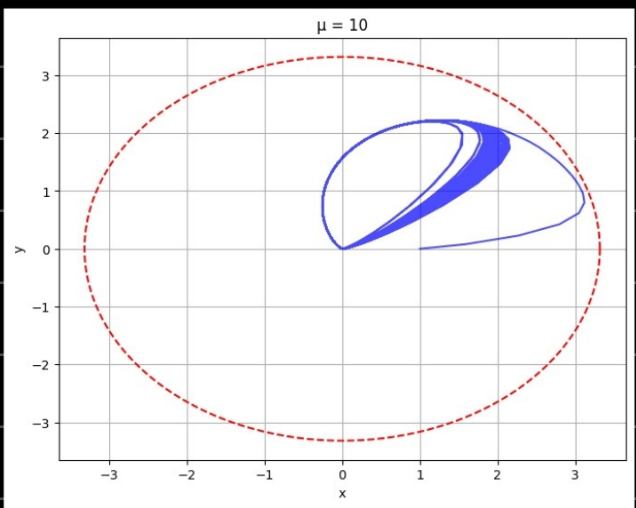


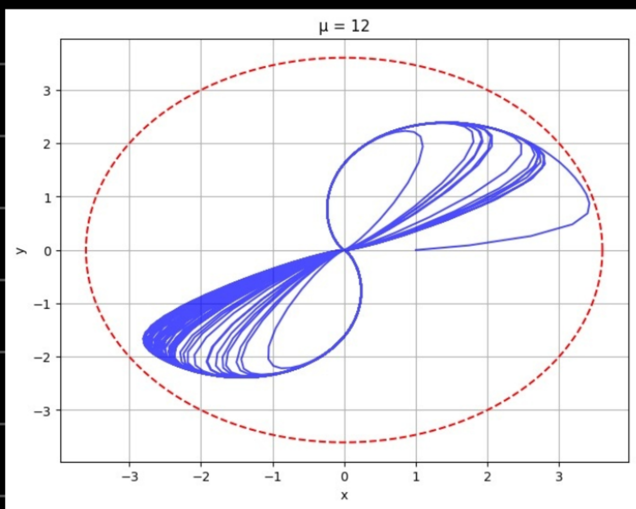




So $\boxed{\mu_c \approx 7}$, after which precessions are observed.





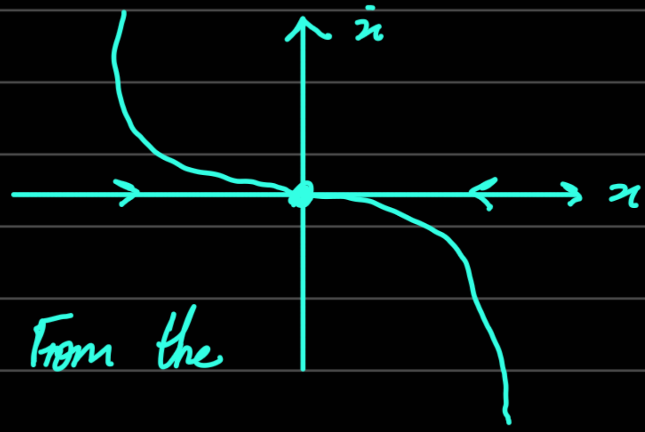


8. There is a theorem which states that: *Let $\bar{v}$ be an equilibirum of $\dot{\mathbf{v}} = \mathbf{f}(\mathbf{v})$. If the real part of each eigenvalue of $\mathbf{Df}(\bar{\mathbf{v}})$ is strictly negative, then $\bar{\mathbf{v}}$ is asymptotically stable. If the real part of at least one eigenvalue is strictly positive, then $\bar{\mathbf{v}}$ is unstable.*

Now the one dimensional system $\dot{x} = -x^3$ has an equilibrium at $x = 0$. Decide whether $x = 0$ is an asymptotically stable. Does this equation have unique solutions? Find all solutions that satisfy $x(0) = 1$.

(8) $f(x) = -x^3$, with $x^* = 0$

$$\left.\frac{df}{dx}\right|_{x=0} = -3x^2\Big|_{x=0} = 0$$



So the theorem does not really apply here. From the graph, we see it is asymptotically stable.

$$\frac{dx}{dt} = -x^3 \Rightarrow \frac{dx}{x^3} = -dt$$

$$\left.\frac{x^{-2}}{-2}\right|_{x_0}^{x} = -t$$

$$\Rightarrow \frac{1}{x^2} - \frac{1}{x_0^2} = 2t$$

$$\Rightarrow \frac{1}{x^2} = \frac{1}{x_0^2} + 2t$$

$$\frac{1}{x^2} = 1+2t \Rightarrow \boxed{x = \frac{1}{\sqrt{1+2t}}}$$

So solutions are unique.

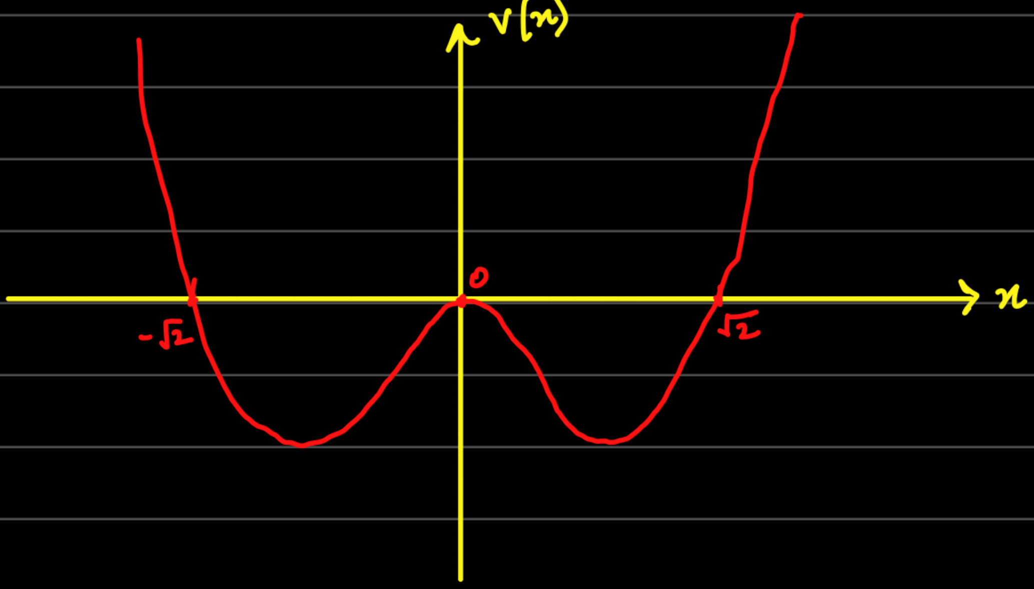
9. Consider the forced damped Duffing oscillator given by $\ddot{x} + 0.1\dot{x} - x + x^3 = 2\sin t$. Show that the corresponding undamped unforced system has a double well potential. Plot the potential function.

Write a computer program to plot numerical solutions of the forced damped double-well in the $(x - \dot{x})$-plane. In particular, locate and plot the attracting periodic orbit of period 2π and the two attracting periodic orbits of period 6π that lie in the region $-5 \leq x, \dot{x} \leq 5$.

9

$$\ddot{x} - x + x^3 = 0$$

Comparing with $\ddot{x} + \frac{dV}{dx} = 0$, $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$



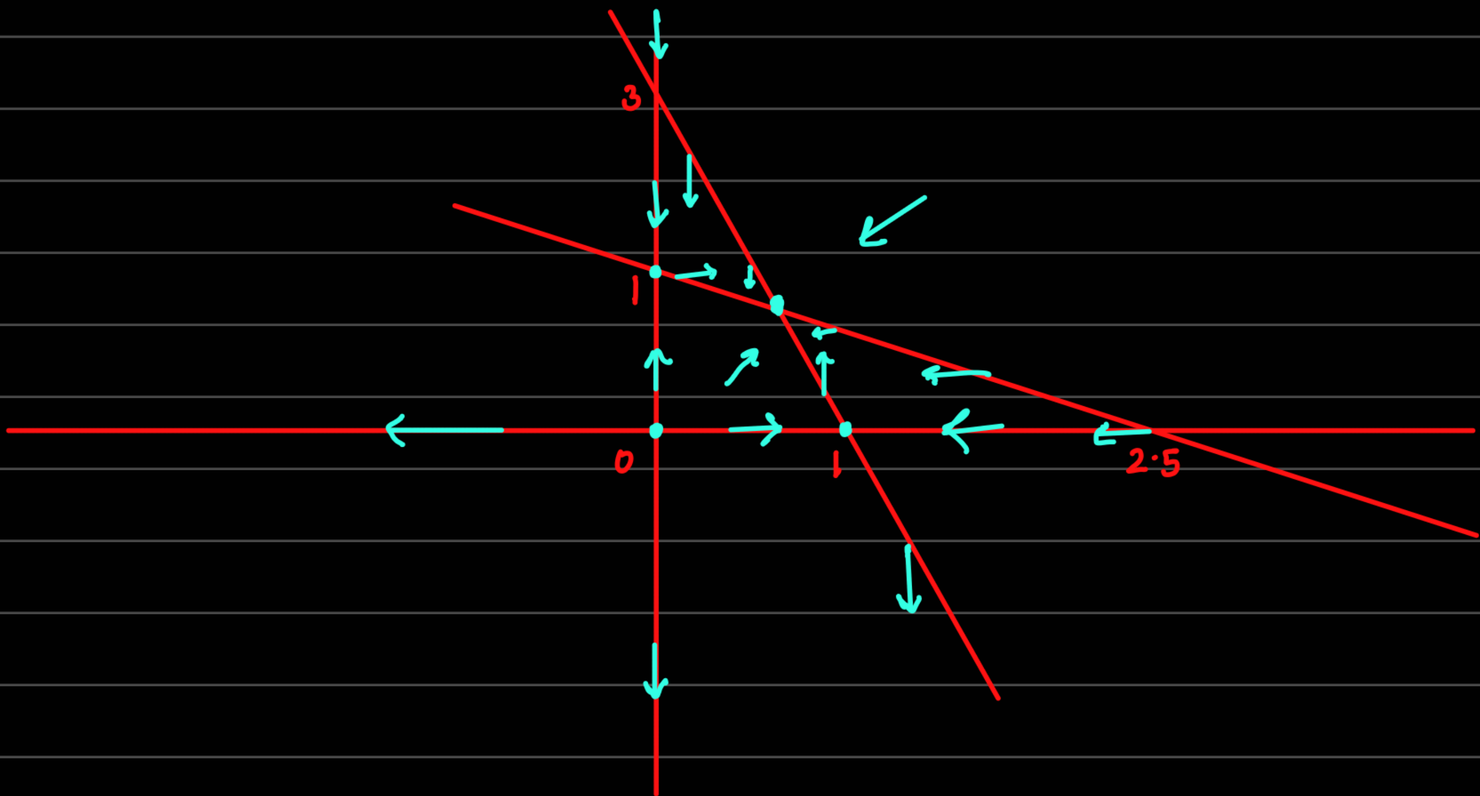
10. Use the method of nullclines to determine the global behaviour of solutions for the following system

$$\begin{aligned}\dot{x} &= 3x(1-x) - xy \\ \dot{y} &= 5y(1-y) - 2xy.\end{aligned}$$

Describe sets of initial conditions that evolve to distinct final states.

10 Nullclines: $x(3-3x-y) = 0 \Rightarrow x=0,\ 3x+y=3$

$y(5-5y-2x) = 0 \Rightarrow y=0,\ 2x+5y=5$

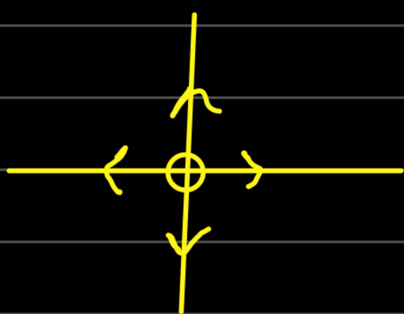


$$J = \begin{pmatrix} 3-6x-y & -x \\ -2x & 5-10y-2x \end{pmatrix}$$

At $(0,0)$, $J = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ $\quad \tau = 8 \quad \Delta = 15$

$\tau^2 - 4\Delta > 0 \Rightarrow$ unstable node

Has eigenvectors along x-y axes



At $(1,0)$: $J = \begin{pmatrix} -3 & -1 \\ -2 & 3 \end{pmatrix}$ $\quad \tau = 0, \ \Delta = -11$

$\lambda = \pm\sqrt{11}$

$$\begin{pmatrix} -3 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sqrt{11}\,a \\ \sqrt{11}\,b \end{pmatrix}$$

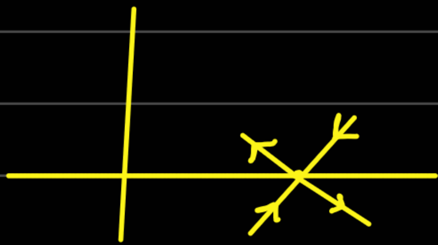
$\Rightarrow -3a - b = \sqrt{11}\,a$

$-2a + 3b = \sqrt{11}\,b$

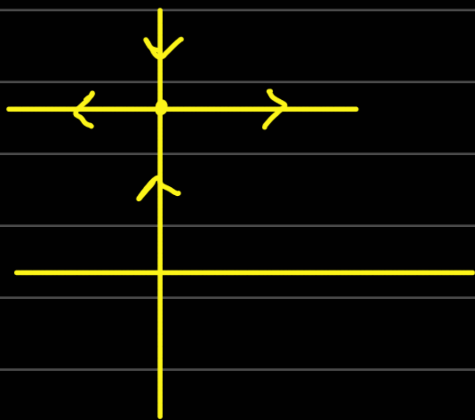
$\Rightarrow \frac{a}{b} = \frac{3-\sqrt{11}}{2}$

so $\lambda = \sqrt{11}$, $\begin{pmatrix} \frac{3-\sqrt{11}}{2} \\ 1 \end{pmatrix}$

$\lambda = -\sqrt{11}$, $\begin{pmatrix} \frac{3+\sqrt{11}}{2} \\ 1 \end{pmatrix}$



At $(0,1)$: $J = \begin{pmatrix} 2 & 0 \\ 0 & -5 \end{pmatrix}$

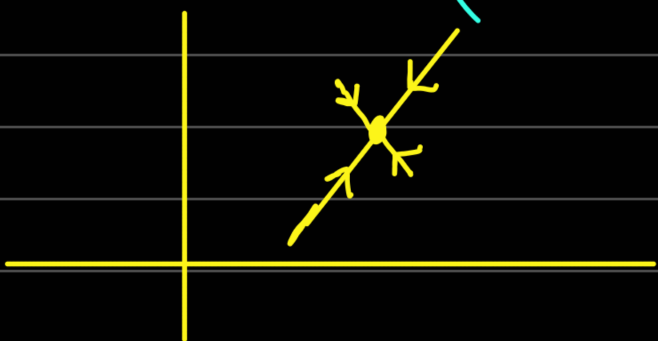


At $\left(\frac{10}{13}, \frac{9}{13}\right)$: $J = \begin{pmatrix} 3 - \frac{60}{13} - \frac{9}{13} & -\frac{10}{13} \\ -\frac{20}{13} & 5 - \frac{90}{13} - \frac{20}{13} \end{pmatrix}$

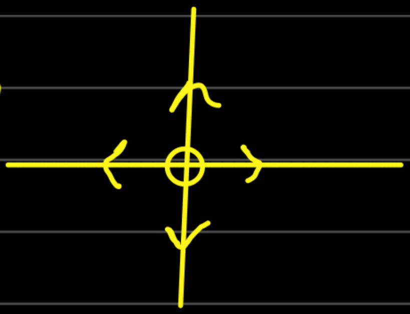
$= \begin{pmatrix} -30/13 & -10/13 \\ -20/13 & -45/13 \end{pmatrix}$

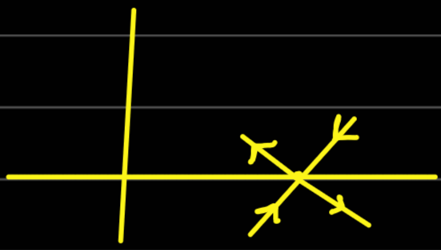
$\tau = -75/13$

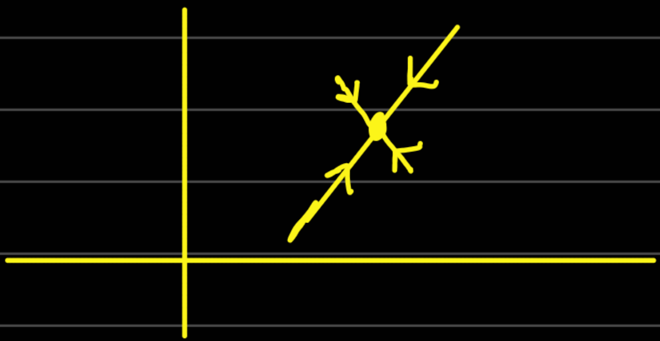
$\Delta = \frac{30\cdot 45 - 200}{169}$ 70



So,







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