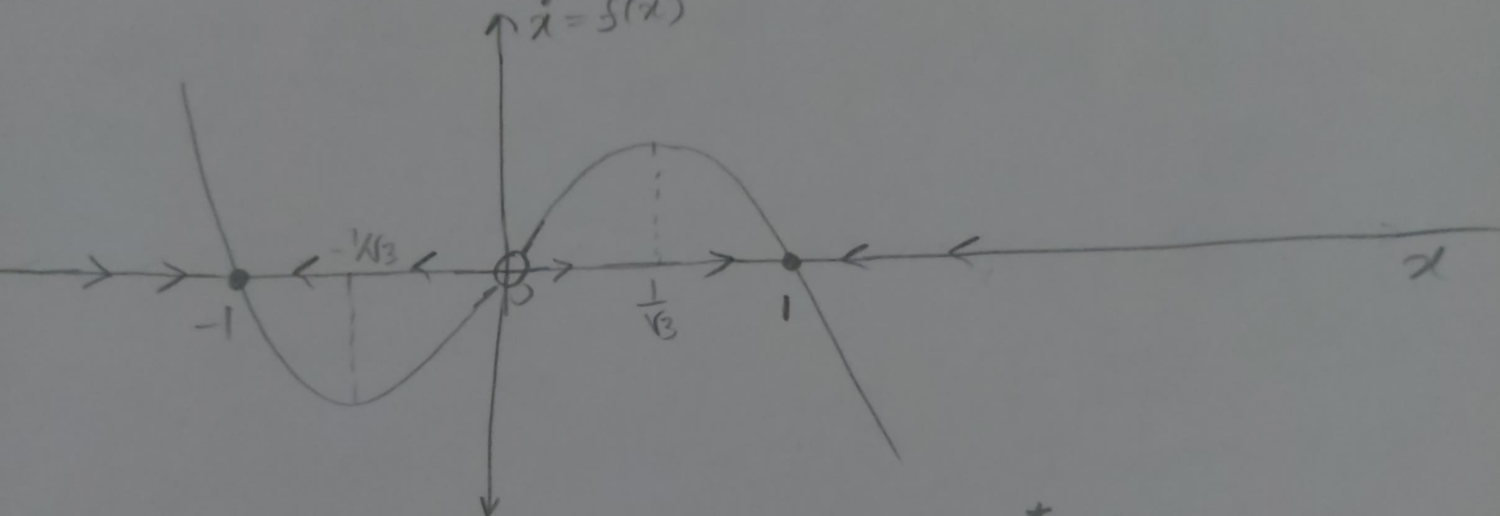
1) $\dot{x} = x - x^3 = x(1-x^2)$

Fixed points :- $x^* = 0, +1, -1$

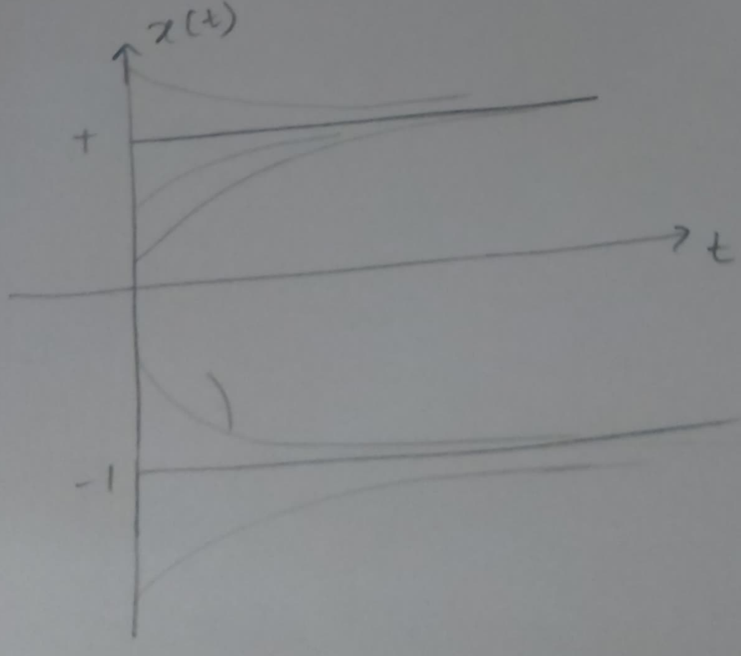
Helpful facts :- derivatives, stationary points

$f'(x) = 1 - 3x^2$, $x = \pm \frac{1}{\sqrt{3}}$



∴ Unstable Fixed point :- $x^* = 0$

stable " " $x^* = \pm 1$



Behavior of trajectories $x(t)$

Analytical expression for $x(t)$

$$\frac{dx}{dt} = x - x^3$$

$$\Rightarrow \int \frac{dx}{x - x^3} = \int dt$$

Need to find $I = \int \frac{dx}{x(1-x^2)} = \int \frac{dx}{x(1+x)(1-x)}$

Use partial fractions

$$I = \int \left[\frac{1}{x} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)} \right] dx$$

$$= \ln(x) - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) = t$$

exponentiating both sides & simplifying,

$$x(t) = \frac{\pm c e^t}{\sqrt{1 + e^{2t} \cdot c^2}}, \quad c = \text{arbitrary const.}$$

(b) $\dot{x} = e^{-x}\sin x$

Analytical sol^{n}s does not exist

Fixed points: where $\sin x = 0$,

i.e. $x^* = n\pi$, $n \in \mathbb{Z}$

Stable FPs $\longrightarrow$ where $f'(x^*)$ is < 0

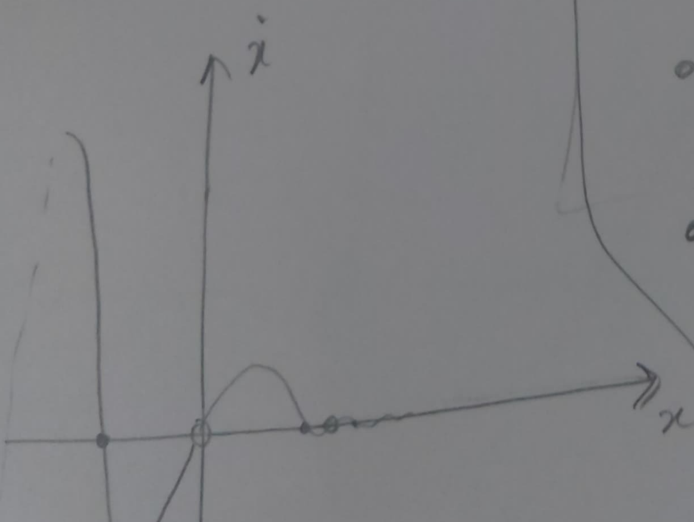
Unstable " $\longrightarrow$ where $f'(x^*)$ is > 0

$\therefore$ stable FPs $\rightarrow x^* = (2n-1)\pi$, $n \in \mathbb{Z}$

Unstable FPs $\rightarrow x^* = 2n\pi$, $n \in \mathbb{Z}$

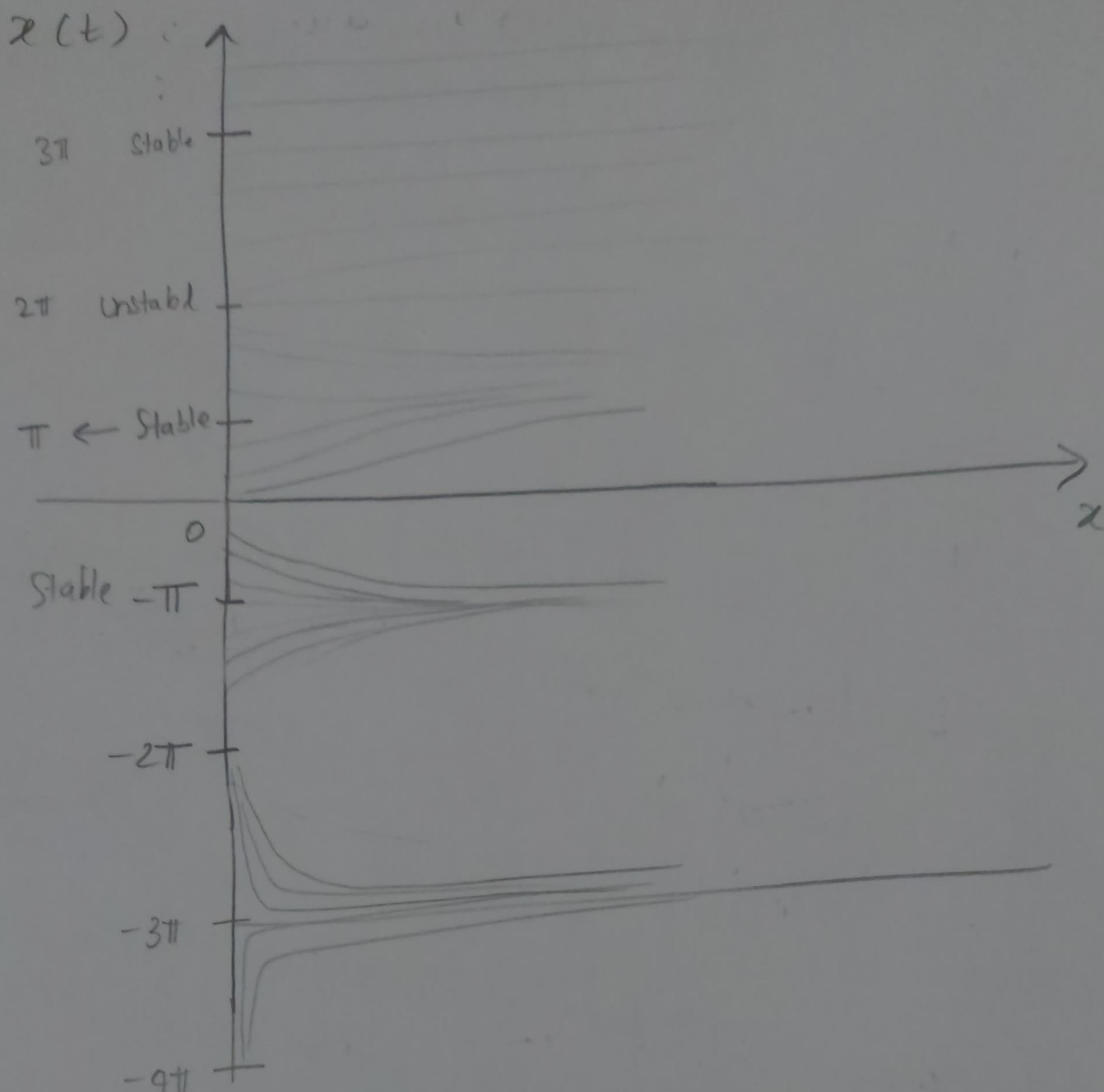
Phase Space

$\dot{x} = e^{-x}\sin x$



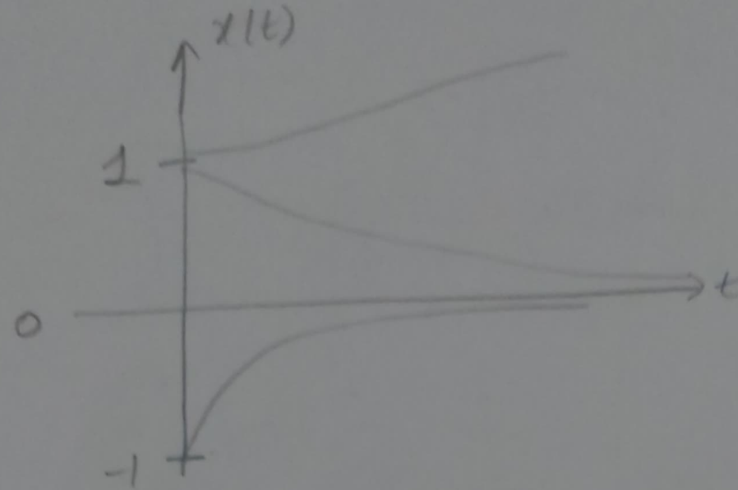
Helpful info for plotting!

- Behavior @ $x \to \infty$, $\dot{x} = 0$
  also, $\dot{x} = 0$ @ $x = n\pi$
- over long times, e^{-x} dominates
- over short times, $\sin(x)$ dominates
- "Explosive" behavior for $-ve$ x
- "Damped" behavior for large $+ve$ x



x(t)
3π
Stable
2π
Unstabl
π ← Stable
0
x
Stable −π
−2π
−3π
−4π

2) $\dot{x} = f(x)$ consistent with



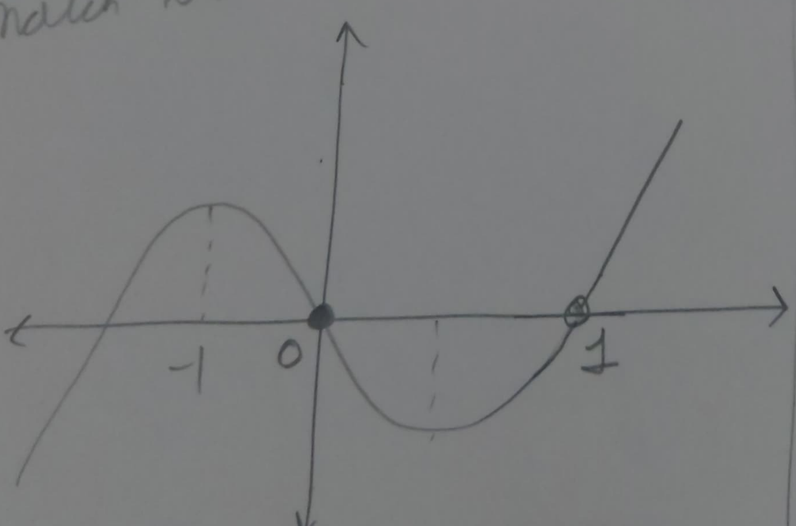
clues :- 1 behaves unstable (But not F.P, as $x(-1)=$ const line $\nexists$)

0 stable F.P

1 unstable F.P

Can fit a cubic with this informa?

ensuring derivatives @ zeroes of polynomial match the above characteristics



One such polynomial :-

$\dot{x} = 7.14x^3 + 6.42x^2$

$- 8.571x$

Yes, there can be more than one $f(x)$

3) $m\dot{v} = mg - kv^2$

$$\dot{v} = g - \frac{k}{m}v^2$$

$$\frac{dv}{dt} = g - \frac{k}{m}v^2 \Rightarrow \int \frac{dv}{g - \frac{k}{m}v^2} = \int dt$$

$$\Rightarrow \frac{1}{2\sqrt{g}} \int \left(\frac{1}{\sqrt{g} + v\sqrt{\frac{k}{m}}} + \frac{1}{\sqrt{g} - v\sqrt{\frac{k}{m}}} \right) dv = \int dt$$

$$\Rightarrow \frac{1}{2\sqrt{g}} \left[\frac{\ln\left(v\sqrt{\frac{k}{m}} + \sqrt{g}\right)}{\sqrt{\frac{k}{m}}} + \frac{\ln\left(\sqrt{g} - v\sqrt{\frac{k}{m}}\right)}{-\sqrt{\frac{k}{m}}} \right] = \int dt$$

$$\Rightarrow \frac{\sqrt{m}}{2\sqrt{gk}} \ln\left(\frac{v\sqrt{\frac{k}{m}} + \sqrt{g}}{\sqrt{g} - v\sqrt{\frac{k}{m}}} \right) = t + c$$

$$\Rightarrow \frac{1}{2}\sqrt{\frac{m}{gk}} \ln\left(\frac{v + \sqrt{\frac{mg}{k}}}{\sqrt{\frac{mg}{k}} - v} \right) = t + c$$

$v(0) = 0 \Rightarrow c = 0$

$$\Rightarrow v(t) = \sqrt{\frac{mg}{k}} \left[\frac{e^{2t\sqrt{\frac{gk}{m}}} - 1}{e^{2t\sqrt{\frac{gk}{m}}} + 1} \right]$$

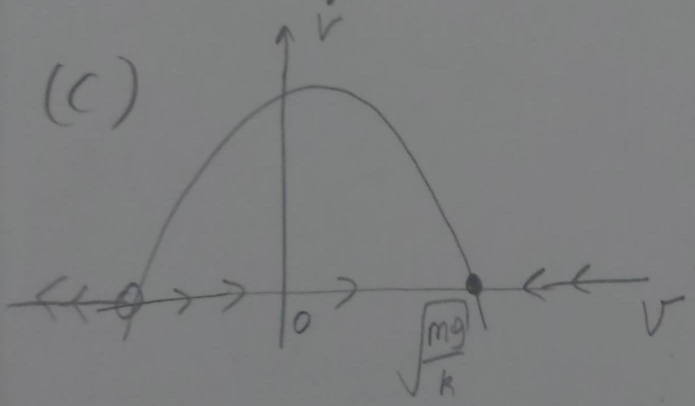
b) $\lim_{t\to\infty} v(t)$ i.e. Terminal Velocity

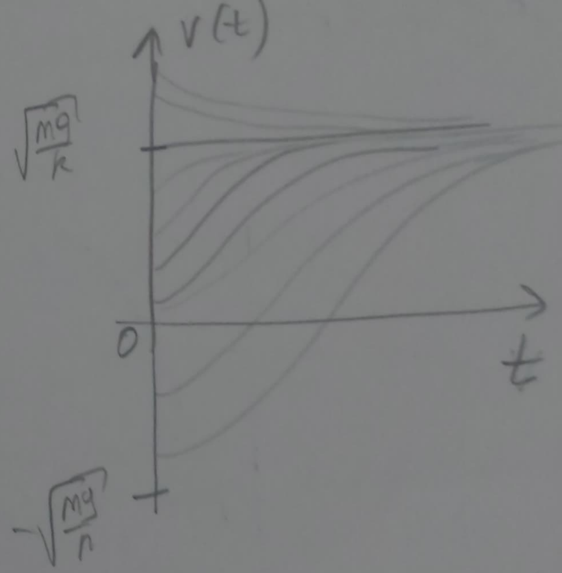
$$\lim_{t\to\infty} \sqrt{\frac{mg}{k}} \left[\frac{e^{2t\sqrt{\frac{gk}{m}}}-1}{e^{2t\sqrt{\frac{gk}{m}}}+1}\right] = \sqrt{\frac{mg}{k}}$$

Sanity check
Should happen when $mg = kv^2$ (forces balance out)

$\Rightarrow v = \sqrt{\frac{mg}{k}}$ $\therefore$ Makes sense

(c)





4) Show that

(a) $\frac{\dot{N}}{N} = r - a(N-b)^2$ can show

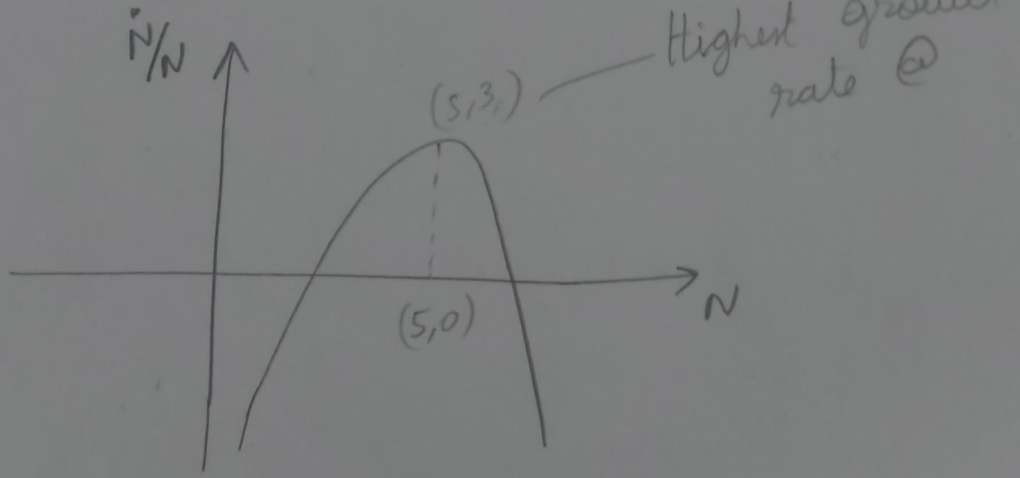
properties of allee effect

(i) maxima occurs at same N^*

(ii) only one maxima

$$\frac{\dot{N}}{N} = f(N) = r - a(N-b)^2 \Big\} \text{ Quadratic eq}^n \text{ in } N$$

Say $(r, a, b) = (3, 1, 5)$



(b) $\dot{N} = N(r - a(N-b)^2)$

FPs $= N^* = 0,\ b + \sqrt{\frac{r}{a}},\ b - \sqrt{\frac{r}{a}}$

5) (a) $\dot{x} = rx - \frac{x}{1+x}$

$\dot{x} = 0 \Rightarrow x = 0$ or $r - \frac{1}{x+1} = 0$

$\Rightarrow \frac{1}{r} - 1 = x$ $\quad\therefore$ Potentially 2 FPs

($x = 0$ & $x = \frac{1}{r} - 1$)

Stability

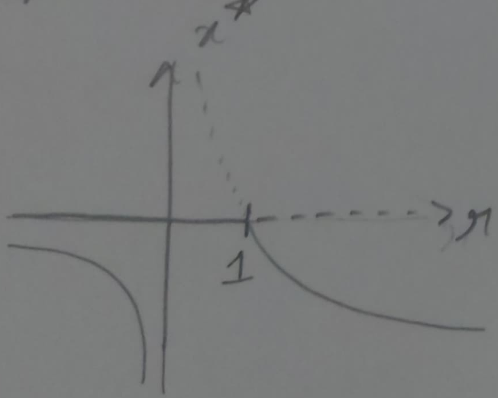
$$r - \frac{(1+x) - x}{(1+x)^2} = r - \frac{1}{(1+x)^2}$$

For $x_* = 0$, $r > 1 \Rightarrow$ unstable & $r < 1 \Rightarrow$ stable

for $x_* = \frac{1}{r} - 1$, when $r - r^2 > 0$, unstable

$r - r^2 < 0$, stable

Bifurca^n diagram



$r = 0$ is a singularity

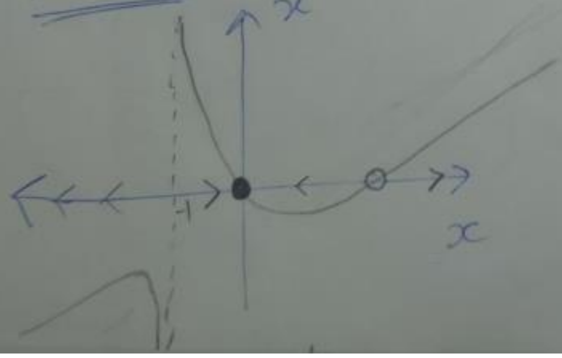
Transcritical Bifurca^n

(b) Similar procedure to part (a)

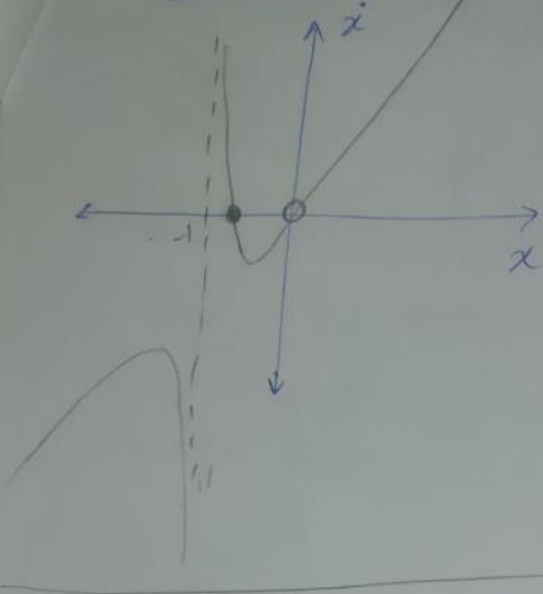
& (c)

Vector field diagram :-

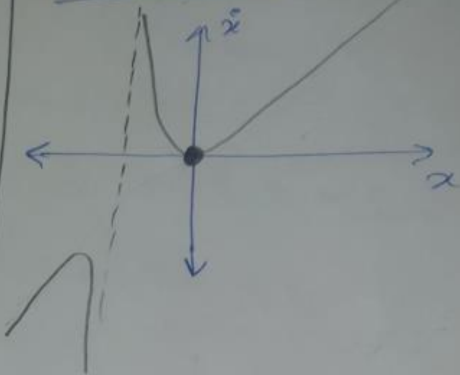
$r < 1$ ($\& \; r > 0$)



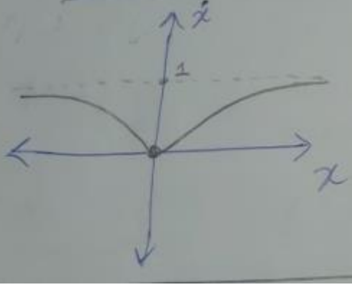
for $r > 1$:-



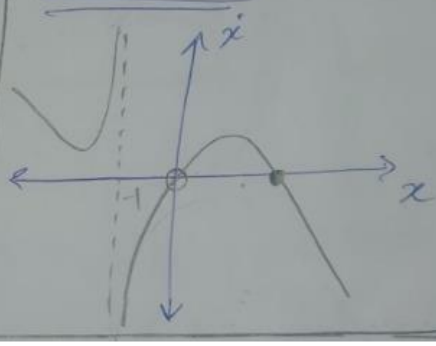
For $r = 1$:-



For $r \neq 0$ (singularity case)



for $r < 0$:-



(d) $\dot{x} = 5 - re^{-x^2}$

FPs :- $5 = re^{-x^2} \Rightarrow \frac{5}{r} = e^{-x^2}$

$\rightarrow e^{x^2} = \frac{r}{5} \Rightarrow x^2 = \ln\left(\frac{r}{5}\right) \Rightarrow x = \pm\sqrt{\ln(r/5)}$

If $r<0$, No FPs

If $r>0$, 2 FPs, each value :- $\pm\sqrt{\ln(r/5)}$

Giving saddle node vibes

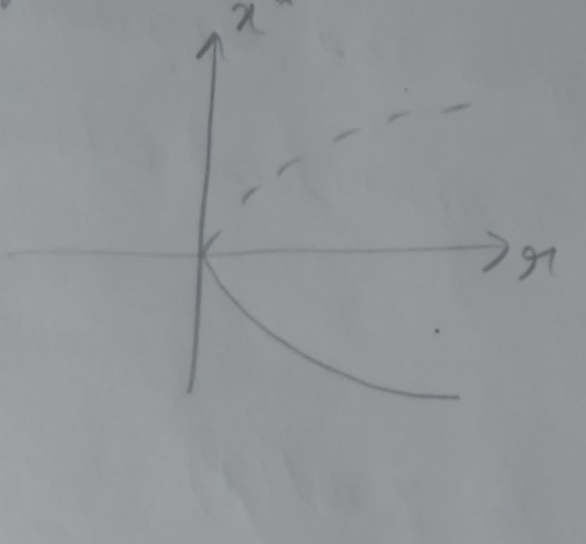
Testing stability :-

$\frac{d}{dx}\left(5 - re^{-x^2}\right) = -re^{-x^2}(-2x)$

$= 2rx\, e^{-x^2}$ ∴ If $xr>0$, unstable,

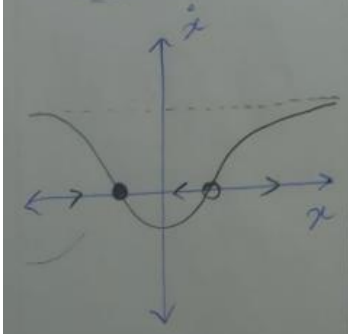
$xr<0$, stable

Bifurca? diagram :-

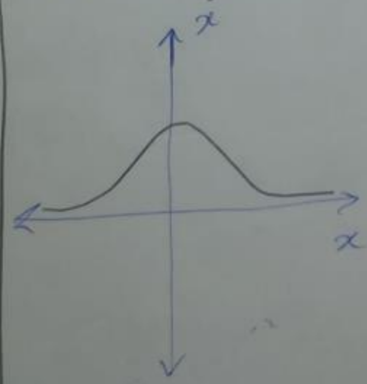


Vector Field diagram :- (For $r < 0$, No ~~[illegible]~~ fixed points)

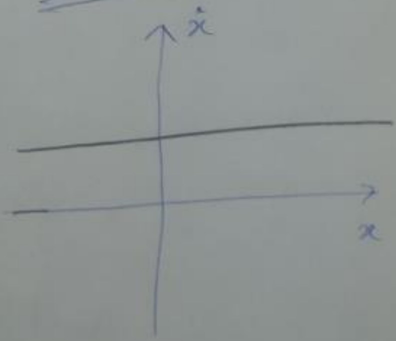
For $r > 0$



For $r < 0$



For $r = 0$



6)

$$\gamma_1(ED-P)=0 \Rightarrow \boxed{P=ED}$$

$$\gamma_2(\lambda+1-D-\lambda EP)=0$$

$$\Rightarrow \lambda+1-D-\lambda EP=0$$

$$\Rightarrow \lambda+1-D-\lambda E^2 D=0$$

$$\Rightarrow D[1+\lambda E^2]=\lambda+1 \Rightarrow \boxed{D=(\lambda+1)/(1+\lambda E^2)}$$

$$\boxed{P=E\left(\frac{1+\lambda}{1+\lambda E^2}\right)}$$

$$\therefore \dot{E}=k\left[E\left(\frac{1+\lambda}{1+\lambda E^2}\right)-E\right]=Ek\left[\frac{1+\lambda}{1+\lambda E^2}-1\right]$$

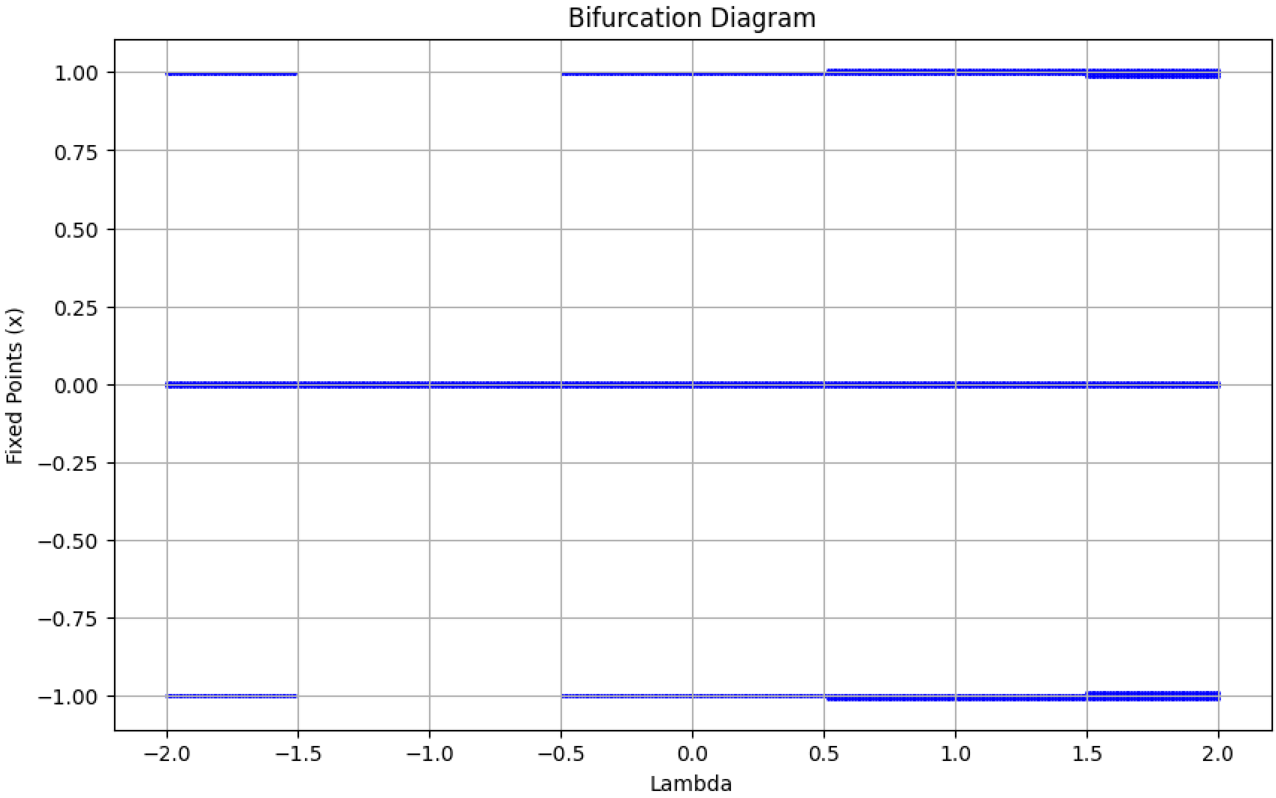
$$\dot{E}=Ek\left[\frac{1+\lambda-1-\lambda E^2}{1+\lambda E^2}\right]=Ek\lambda\left[\frac{1-E^2}{1+\lambda E^2}\right]$$

$$\therefore \dot{E}=Ek\lambda\left(\frac{1-E^2}{1+\lambda E^2}\right)\Big\}\ f(E) \text{ alone}$$

(b) $\dot{E}=0 \Rightarrow \dfrac{E(1-E^2)}{1+\lambda E^2}=0 \Rightarrow E(1-E)(1+E)=0$

$\therefore$ FPs $E^{*}=0, \pm 1$

Bifurcation Diagram:



7) $A = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$

characteristic eqⁿ

$\det(A - tI) = 0$

$\Rightarrow \det \begin{bmatrix} \lambda - t & b \\ 0 & \lambda - t \end{bmatrix} \Rightarrow (\lambda - t)^2 = 0 \Rightarrow t = \lambda$

$\therefore$ eigenvalue $= \lambda$

For eigenvectors, $(A - \lambda I)v = 0$

$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} bx_2 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore x_2 = 0$

$\therefore V = c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad c \in \mathbb{R}$

$\therefore$ one dimensional eigenspace $= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \times c$

$\exists$

Now, $\dot{X} = AX$

$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{matrix} \dot{x}_1 = \lambda x_1 + b x_2 \\ \dot{x}_2 = \lambda x_2 \end{matrix}$

linearly separable

$$\frac{\dot{x}_2}{x_2} = \lambda \Rightarrow x_2(t) = c e^{\lambda t}$$ c = arbit. const.

Substitute $x_2(t)$ into 1st eqn

$$\dot{x}_1 = \lambda x_1 + b c e^{\lambda t}$$

Linear differential eqn

Integration factor is $e^{-\lambda t}$ & solve

$$\boxed{\therefore x_1(t) = c' e^{\lambda t} + b c t e^{\lambda t}}$$

Phase potrait depends on values of c, c', λ & b. Can take diff values, sketch & verify with desmos / python

9)

(a) $J = \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix}$

$J|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

all e-values are 0

$\therefore$ linearisa^n fails to provide info on stability, etc.

(b) $\dot{x} = 0 \Rightarrow x = 0$ or $y = 0$

If $x=0$, $x^2 - y^2 = -y^2 = 0 \Rightarrow y = 0$

If $y=0$, $x^2 - y^2 = x^2 = 0 \Rightarrow x = 0$

$\therefore$ only FP is origin

(c) stability :- Analysis of behavior near it:

1. Along nullclines

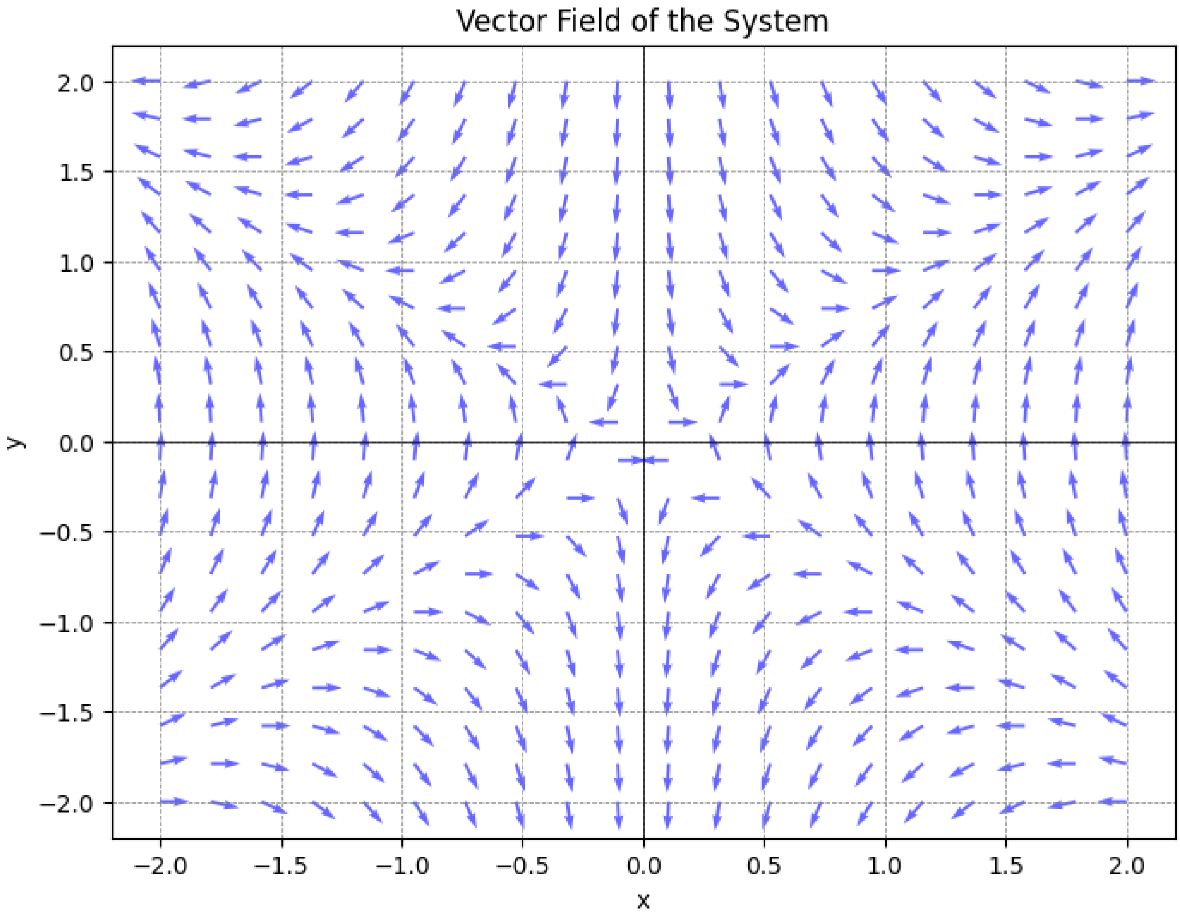
If $x=0$, $\dot{x}=0$, $\dot{y} = -y^2$ (<0 for $y \neq 0$)

$\therefore$ downward along nullcline

If $y=0$, $\dot{y} = x^2$, >0 for $x \neq 0$

$\therefore$ rightward along this nullcline

Question 9


Vector Field of the System
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
y
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x