Date: February 17, 2025

AM 5650
Nonlinear Dynamics
Assignment -3

Due : March 20, 2025

1. Consider the system

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1), \quad \dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2).$$

   (a) Show that the system has no fixed points, given that $\omega_1, \omega_2 > 0$ and $K_1, K_2 > 0$.

   (b) Can there be a conserved quantity for this system? If yes, find one. If no, explain why?

   (c) Suppose that $K_1 = K2$. Show that the system can be non-dimensionalized to the form

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}\tau} = 1 + a \sin(\theta_2 - \theta_1), \quad \frac{\mathrm{d}\theta_2}{\mathrm{d}\tau} = \omega + a \sin(\theta_1 - \theta_2).$$

2. Consider the system

$$\dot{\theta}_1 = E - \sin\theta_1 + K \sin(\theta_2 - \theta_1), \quad \dot{\theta}_2 = E + \sin\theta_2 + K \sin(\theta_1 - \theta_2),$$

   where, $E, K \geq 0$.

   (a) Find and classify all the fixed points.

   (b) Show that if E is large enough, the system has periodic solutions on the torus. What type of bifurcations creates the periodic solutions?

   (c) Find the bifurcation curve in (E, K) space at which these periodic solutions are created.

3. Find a period doubling bifurcation for $f(x) = x^3 - ax$.

4. Consider the 1-dimensional map $x_{n+1} = a - x_n^2$.

   (a) Plot the bifurcation diagram with respect to the bifurcation parameter a for $a \in [-1/2, 2]$.

   (b) Identify the value of a at which the bifurcations occur, including the period doubling bifurcations. Verify these analytically.

   (c) Compute the ratios $(a_n - a_{n-1})/(a_{n-1} - a_{n-2})$, where a_k are the values of a at which period doubling bifurcations occur. Show in a table containing three columns, with column 1 indicating period ($= 2^s$). the value of the parameter a at which the s-th period doubling occurs and the last column the ratio computed as above. What do you observe? This was first observed by Feigenbaum in 1978 and has been shown to be indicative of a universality in scaling behaviour of cascades. *A cascade is an infinite sequence of period doubling bifurcations.*