Understanding Transverse Momentum Distributions (TMDs)

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1 Introduction

This document aims to provide a detailed understanding of Transverse Momentum Distributions (TMDs), their theoretical underpinnings in Quantum Chromodynamics (QCD), their relevance in processes like Semi-Inclusive Deep Inelastic Scattering (SIDIS), and the associated formalism.

2 Foundational Understanding and SIDIS

2.1 Feynman Diagrams for TMDs in SIDIS

2.1.1 Identifying Key Feynman Diagrams for SIDIS

Semi-Inclusive Deep Inelastic Scattering (SIDIS), where a lepton scatters off a nucleon and a specific hadron is detected in the final state (e.g., $e + N \to e' + h + X$), is a crucial process for studying the internal structure of hadrons, including their Transverse Momentum Distributions (TMDs). The fundamental interaction at the parton level is the scattering of the virtual photon (emitted by the lepton) off a quark (or antiquark) within the nucleon. This struck quark then fragments into the observed hadron h.

The most basic Feynman diagram representing this process at the lowest order in perturbative QCD is often referred to as the "handbag diagram." This diagram depicts the lepton emitting a virtual photon, which is then absorbed by a quark inside the target nucleon. The scattered quark subsequently fragments into the detected hadron.

However, to understand TMDs, we need to go beyond this simple picture and consider the transverse momentum of the partons. TMDs account for the intrinsic transverse momentum of quarks and gluons confined within the nucleon, as well as the transverse momentum generated during the fragmentation process. The Feynman diagrams relevant for a TMD description of SIDIS incorporate these aspects and involve more complex QCD interactions.

Specifically, the diagrams that contribute to the definition and evolution of TMDs include those with gluon emissions and absorptions by the active parton (the quark being struck or the quark fragmenting). These gluon exchanges are crucial for understanding the gauge-invariant definition of TMDs, which requires the inclusion of Wilson lines (gauge links). These Wilson lines represent the resummation of soft and collinear gluon interactions between the active parton and the remnants of the nucleon or the other products of fragmentation.

In the context of TMD factorization for SIDIS at low transverse momentum of the produced hadron (P_{hT}), the process can be visualized as:

1. A virtual photon interacts with a quark inside the nucleon. This quark possesses not only a longitudinal momentum fraction x but also an intrinsic transverse momentum k_T relative to the nucleon's direction. This is described by a TMD Parton Distribution Function (TMD PDF).
2. The struck quark, after interacting with the photon, fragments into the observed hadron. During this fragmentation process, a transverse momentum p_T (relative to the fragmenting quark's direction) can be generated. This is described by a TMD Fragmentation Function (TMD FF).

Diagrammatically, one considers the hard scattering part (photon-quark interaction) and connects it to non-perturbative TMD correlators. These correlators are represented by quark (or gluon) fields separated by a light-like distance and connected by gauge links. The gauge links are path-ordered exponentials of the gluon field and are essential to ensure gauge invariance. Their specific path depends on the process and the color flow. For SIDIS, the gauge links for

TMD PDFs typically involve future-pointing and/or past-pointing staples, reflecting the initial and final state interactions of the struck quark with the target remnants.

2.1.2 Role of Feynman Diagrams in Understanding TMDs in SIDIS

The Feynman diagrams in the context of TMDs and SIDIS serve multiple crucial roles:

1. **Visualizing the Interaction:** At the most basic level, they provide a visual representation of the scattering process. The handbag diagram shows the lepton-quark interaction mediated by a virtual photon and the subsequent fragmentation. More complex diagrams illustrate gluon radiation and absorption, which are fundamental to understanding the transverse momentum of partons.

2. **Defining TMD Correlators:** The operator definition of TMDs involves quark and gluon fields separated by a certain distance and connected by gauge links. Feynman diagrams help in understanding how these non-local operators arise from summing up classes of diagrams involving soft and collinear gluon exchanges. The structure of the gauge link (e.g., its path in spacetime) is dictated by the process and can be derived by analyzing the relevant Feynman diagrams that contribute to the leading power behavior of the cross-section in the TMD regime.

3. **Calculating Perturbative Components:** While TMDs themselves are non-perturbative objects that describe the long-distance structure of hadrons, their evolution with energy scales (like Q^2 or the TMD scale ζ) and their matching onto collinear parton distributions at high transverse momentum can be calculated perturbatively. Feynman diagrams are the primary tool for performing these perturbative QCD calculations. For instance, calculating the TMD evolution kernels (like the Collins-Soper kernel) or the perturbative coefficients in TMD factorization theorems involves evaluating loop diagrams and real emission diagrams.

4. **Understanding Factorization:** TMD factorization theorems state that, under certain kinematic conditions (typically when the observed transverse momentum is much smaller than the hard scale Q), the SIDIS cross-section can be written as a convolution of hard scattering parts (calculable in perturbation theory), TMD PDFs, and TMD FFs. The proof and understanding of these factorization theorems rely heavily on analyzing the singularity structure of Feynman diagrams in various momentum regions (hard, collinear, soft). The diagrams help identify which contributions factorize into universal TMDs and which belong to the hard process-dependent part.

5. **Illustrating Gauge Invariance:** The need for gauge links in the definition of TMDs becomes clear when considering Feynman diagrams in a gauge theory like QCD. Without appropriate gauge links, the matrix elements defining TMDs would not be gauge invariant. The diagrams show how gluon attachments to the quark lines, when resummed, lead to these path-ordered exponentials of the gluon field.

In essence, Feynman diagrams provide the bridge between the abstract operator definitions of TMDs and their phenomenological application in describing experimental observables like SIDIS. They are indispensable for deriving the theoretical framework of TMDs, including their definitions, evolution equations, and factorization properties.

2.2 Definition and Origin of TMDs

2.2.1 Detailed Definition of Transverse Momentum Distributions (TMDs)

Transverse Momentum Distributions (TMDs), also known as unintegrated parton distribution functions (uPDFs) when referring to the initial state, are fundamental quantities in Quantum Chromodynamics (QCD) that describe the three-dimensional momentum structure of partons (quarks and gluons) inside a hadron. Unlike the more familiar collinear Parton Distribution Functions (PDFs), which describe the probability of finding a parton carrying a certain longitudinal momentum fraction x of the parent hadron (integrated over all transverse momenta), TMDs provide additional information about the parton's transverse momentum k_T relative to the direction of the parent hadron.

A generic TMD PDF, denoted as $f(x, k_T^2, \mu, \zeta)$, depends on:

- x: The longitudinal momentum fraction of the parton relative to the hadron.
- k_T^2: The square of the parton's transverse momentum.
- μ: The renormalization scale, or factorization scale, typically related to the hard scale of the process (e.g., Q^2 in DIS).
- ζ (or μ_T or b_T in Fourier conjugate space): The TMD evolution scale, also known as the Collins-Soper scale or rapidity scale. This scale is related to the energy dependence of the TMD that is not captured by the standard DGLAP evolution associated with μ.

The formal definition of a quark TMD PDF, for example, involves a non-local matrix element of quark fields within the hadron state. For an unpolarized quark in an unpolarized hadron, the relevant correlator $\Phi_q(x, k_T)$ is defined as:

$$\Phi_q(x, k_T; n, S) = \int \frac{d\xi^- d^2\xi_T}{2(2\pi)^3} e^{ixP^+\xi^- - ik_T\cdot\xi_T} \langle P, S|\bar{\psi}_q(0)W(0,\xi)\gamma^+\psi_q(\xi)|P, S\rangle\Big|_{\xi^+=0} \quad (1)$$

Where:

- $|P, S\rangle$ represents the hadron state with momentum P and spin S.
- $\psi_q(\xi)$ and $\bar{\psi}_q(0)$ are quark field operators at spacetime positions ξ and 0, respectively.
- γ^+ is a Dirac gamma matrix projecting onto the "good" light-cone components.
- P^+ is the large light-cone momentum component of the hadron.
- $\xi = (\xi^+, \xi^-, \xi_T)$ is the spacetime separation vector, with $\xi^+ = 0$ meaning the fields are separated at equal light-cone time.
- $W(0, \xi)$ is a Wilson line (or gauge link) connecting the points 0 and ξ. This is a path-ordered exponential of the gluon field, $W = \mathcal{P}\exp(-ig \int A \cdot dl)$, ensuring the gauge invariance of the definition. The precise path of the Wilson line depends on the process being considered (e.g., Drell-Yan vs. SIDIS) due to initial/final state interactions. For SIDIS TMD PDFs, the gauge link typically includes a staple extending to light-cone infinity in the direction opposite to the hadron's motion, representing final-state interactions of the struck quark.

This correlator can be decomposed into various TMD functions depending on the polarization of the quark and the hadron. For instance, $f_1(x, k_T^2)$ is the unpolarized TMD PDF, while functions like $g_{1L}(x, k_T^2)$ (helicity TMD) or $h_1^\perp(x, k_T^2)$ (Sivers function, correlating hadron spin with parton transverse momentum) describe spin-dependent effects.

Similarly, TMD Fragmentation Functions (TMD FFs), denoted $D(z, p_T^2, \mu, \zeta)$, describe the probability for a parton to fragment into a specific hadron with longitudinal momentum fraction z (relative to the parent parton) and transverse momentum p_T (relative to the fragmenting parton's direction). Their definition also involves non-local matrix elements of quark/gluon fields, but this time describing the hadronization process, and includes appropriate gauge links.

2.2.2 Origin of TMDs in the Theoretical Framework of QCD

TMDs arise naturally within QCD when one considers processes where the transverse momentum of partons plays a significant role and is not integrated over. Their origin can be understood from several perspectives:

1. **Intrinsic Parton Motion:** Partons inside a hadron are not static point particles moving collinearly with the hadron. Due to their confinement within a finite volume (approximately 1 femtometer), the Heisenberg uncertainty principle implies that they must possess some intrinsic transverse momentum. This is a non-perturbative effect inherent to the bound-state nature of hadrons.

2. **QCD Dynamics and Radiative Effects:** Even if one were to imagine a parton with zero intrinsic transverse momentum at some low scale, QCD interactions, particularly gluon radiation, would generate transverse momentum. When a quark emits a gluon, both the quark and the gluon acquire transverse momentum. The resummation of soft and collinear gluon emissions is a key aspect of TMD physics and contributes to their scale dependence (evolution).

3. **Factorization of Cross Sections:** In high-energy scattering processes like SIDIS or Drell-Yan production at low transverse momentum of the final state system (e.g., the produced hadron pair or lepton pair), QCD factorization theorems are developed to separate the short-distance (hard) part of the interaction from the long-distance (soft/collinear) parts. When the transverse momentum q_T of the observed system is much smaller than the hard scale Q (i.e., $q_T \ll Q$), the factorization involves TMDs. The derivation of these factorization theorems from first principles in QCD, by analyzing the singular behavior of Feynman diagrams, shows that the cross-sections can be expressed in terms of these TMD objects. The gauge links in the TMD definition are a direct consequence of ensuring that these factorized long-distance parts are process-independent (universal, up to calculable process-dependent Wilson lines) and gauge invariant.

4. **Operator Product Expansion (OPE) in a Non-Collinear Regime:** While collinear PDFs are related to light-cone operators in the OPE, TMDs can be seen as a generalization that retains information about transverse momentum. They are defined via matrix elements of bilocal quark or gluon operators separated by a light-like distance but also allowing for transverse separation, Fourier transformed with respect to this transverse separation.

In summary, TMDs originate from the fundamental quantum and relativistic nature of partons confined within hadrons, combined with the dynamics of QCD interactions (gluon radiation and absorption). They are formally defined through gauge-invariant non-local matrix elements within the framework of QCD and emerge as essential components in the factorization of hadronic cross sections in specific kinematic regimes where transverse momentum effects are prominent.

2.2.3 Clarifying the Two Scales in TMDs: The Hard Scale (μ) and the TMD Scale (ζ)

Transverse Momentum Distributions (TMDs) are characterized by their dependence on two distinct types of scales, which is a crucial feature distinguishing them from collinear Parton Distribution Functions (PDFs) that depend on a single factorization/renormalization scale μ.

1. **The Renormalization/Factorization Scale (μ):** This scale is analogous to the scale found in collinear PDFs and is typically related to the hard scale of the process in which the TMD is being probed. For example, in Deep Inelastic Scattering (DIS) or SIDIS, μ is often chosen to be of the order of Q, the virtuality of the photon. This scale arises from the renormalization of ultraviolet (UV) divergences that appear in the perturbative calculations of quantities involving TMDs. The dependence of TMDs on μ is governed by evolution equations that are similar in spirit to the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations for collinear PDFs, but they also include terms specific to the TMD nature. This evolution resums logarithms of μ^2/k_T^2 or μ^2/Λ_{QCD}^2.

2. **The TMD Evolution Scale (ζ or related variables like μ_T, b_T, or rapidity parameter η):** This second scale is unique to TMDs and is essential for regulating and resumming a different class of divergences known as rapidity divergences or light-cone divergences. These divergences appear in the definition of TMDs due to the presence of light-like Wilson lines extending to infinity. They are not UV or purely collinear divergences but are associated with the energy sharing between collinear and soft gluon emissions that contribute to the transverse momentum.

   The TMD scale ζ (often expressed as $\zeta = M_h^2 Q^2/(xP^+)^2$ or related to a cutoff in rapidity space) effectively separates contributions from different rapidity regions. The evolution of TMDs with respect to ζ is governed by the Collins-Soper (CS) equation. This equation resums large logarithms of Q^2/k_T^2 (or more precisely, logarithms involving the ratio of the hard scale to the transverse momentum scale, which can also be seen as logarithms of large rapidity differences). The CS equation describes how the shape of the TMD in k_T changes as the energy of the process (related to Q) changes, even if k_T itself is small.

Why are two scales necessary?

- **Regulating Different Divergences:** Collinear PDFs are defined by integrating over all transverse momenta, which effectively smears out the issues related to soft gluon radiation that become problematic when k_T is explicitly considered. The light-like Wilson lines in their definition, necessary for gauge invariance, introduce rapidity divergences when one calculates loop corrections. These are distinct from the UV divergences handled by the μ scale. The ζ scale (or its equivalent) is introduced to regulate these rapidity divergences.

- **Resumming Large Logarithms:** In processes where $Q^2 \gg k_T^2$, large logarithms of Q^2/k_T^2 appear in perturbative calculations. These logarithms can spoil the convergence of the perturbative series. The TMD evolution formalism, particularly the CS equation for ζ-evolution and the DGLAP-like evolution for μ-evolution, allows for the resummation of these large logarithms to all orders in perturbation theory, leading to more reliable predictions.

- **Factorization Requirements:** The TMD factorization theorems, which separate the SIDIS (or Drell-Yan) cross-section into a hard part and TMDs, require a consistent framework to handle all types of divergences. The two-scale formalism provides this consistency. The hard part depends on Q and μ, while the TMDs depend on x, k_T, μ, and ζ. The dependence on μ and ζ in the TMDs cancels the corresponding dependence in the hard part and the soft factor (if not absorbed into the TMD definition), ensuring that the physical cross-section is independent of these unphysical scales.

- **Connecting to Collinear Factorization:** At large transverse momentum ($k_T \sim Q$), TMD factorization should smoothly match onto collinear factorization. The μ evolution plays a role here, but the ζ evolution and the structure of TMDs at large k_T (where they can be expressed as a convolution of a perturbative coefficient and a collinear PDF) are also crucial for this matching. The Operator Product Expansion (OPE) can be used to show that for $k_T \gg \Lambda_{QCD}$, a TMD can be related to collinear PDFs via a perturbatively calculable coefficient function. This region is governed by DGLAP-like evolution.

In practice, TMDs are often defined in transverse position space (b_T-space), which is the Fourier conjugate to k_T-space. In b_T-space, the Collins-Soper equation takes a simpler (multiplicative) form. The b_T variable acts as a probe of the transverse separation, and the evolution in ζ (or Q) describes how the b_T distribution changes.

In summary, the two scales μ and ζ in TMDs reflect the need to handle different types of divergences (UV/collinear vs. rapidity) and to resum different classes of large logarithms that appear when describing processes sensitive to parton transverse momentum. This two-scale structure is a hallmark of TMD physics and is essential for the consistency and predictive power of TMD factorization.

2.3 Relating TMD Definition to the SIDIS Process

2.3.1 Connecting TMD Definitions to the Semi-Inclusive Deep Inelastic Scattering (SIDIS) Process

The formal definitions of Transverse Momentum Distributions (TMDs) as gauge-invariant, non-local matrix elements of quark and gluon fields are directly connected to the experimental process of Semi-Inclusive Deep Inelastic Scattering (SIDIS) through the framework of TMD factorization. SIDIS, where a lepton scatters off a nucleon and a specific hadron is detected in the final state ($e + N \to e' + h + X$), provides a key experimental avenue to probe and extract TMDs.

TMD Factorization in SIDIS

In the kinematic regime where the transverse momentum of the produced hadron P_{hT} (or, more precisely, the transverse momentum q_T of the virtual photon-hadron system, where $q_T = P_{hT}/z$ at leading order, with z being the hadron's momentum fraction) is much smaller than the hard scale Q (the virtuality of the photon, i.e., $q_T^2 \ll Q^2$), the SIDIS cross-section can be factorized. This TMD factorization theorem states that the cross-section can be expressed as a convolution of a hard scattering part (calculable in perturbative QCD) and non-perturbative TMDs (both TMD Parton Distribution Functions for the initial nucleon and TMD Fragmentation Functions for the produced hadron).

The generic structure of the SIDIS cross-section in the TMD regime looks like:

$$\frac{d\sigma}{dx_B dQ^2 dz_h d^2 P_{hT}} \approx H(Q^2, \mu) \sum_q e_q^2 \int d^2 k_T d^2 p_T \delta^2(z_h k_T + p_T - P_{hT}) \; f_{q/N}(x_B, k_T^2; \mu, \zeta_F) D_{h/q}(z_h, p_T^2; \mu, \zeta_D) + \tag{2}$$

Where:

- $H(Q^2, \mu)$ is the hard scattering factor, representing the lepton-quark scattering at the lowest order (virtual photon exchange). It is calculable perturbatively and depends on the hard scale Q and the renormalization scale μ.
- e_q is the electric charge of the quark of flavor q.

- $f_{q/N}(x_B, k_T^2; \mu, \zeta_F)$ is the TMD PDF for finding a quark q inside the nucleon N with longitudinal momentum fraction x_B (Bjorken-x) and transverse momentum k_T. It depends on the scales μ and ζ_F (the TMD evolution scale for the PDF).
- $D_{h/q}(z_h, p_T^2; \mu, \zeta_D)$ is the TMD FF for a quark q to fragment into a hadron h with longitudinal momentum fraction z_h (relative to the quark) and transverse momentum p_T (relative to the fragmenting quark's direction). It depends on μ and ζ_D (the TMD evolution scale for the FF).
- The delta function $\delta^2(z_h k_T + p_T - P_{hT})$ enforces the transverse momentum conservation: the observed hadron's transverse momentum P_{hT} results from the quark's intrinsic transverse momentum k_T (scaled by z_h) and the transverse momentum p_T generated during fragmentation.
- The Y-term represents corrections that become important at larger q_T and ensures a smooth matching to collinear factorization regimes. It is often suppressed at very low q_T.
- The sum is over all active quark and antiquark flavors.

The Role of Gauge Links in SIDIS TMDs

The specific structure of the Wilson lines (gauge links) in the operator definition of $f_{q/N}$ and $D_{h/q}$ is crucial and process-dependent. For SIDIS TMD PDFs, the gauge link typically involves a staple extending to light-cone infinity along a path that accounts for the final-state interactions between the struck quark and the target remnants. These final-state interactions are responsible for time-reversal odd TMDs, such as the Sivers function (which describes a correlation between the nucleon's transverse spin and the quark's transverse momentum) and the Boer-Mulders function (correlating quark transverse spin and transverse momentum in an unpolarized nucleon).

For TMD FFs in SIDIS, the gauge link structure accounts for the initial-state interactions of the fragmenting quark (which was produced in the hard scattering) before it hadronizes. This can lead to T-odd TMD FFs like the Collins function (correlating the fragmenting quark's transverse spin with the produced hadron's transverse momentum relative to the quark).

Extracting TMDs from SIDIS Observables

By measuring various azimuthal asymmetries in the SIDIS cross-section, one can access different TMDs. The cross-section can be written more generally as a sum of structure functions, each multiplied by a specific azimuthal angular dependence (e.g., $\cos(\phi_h - \phi_S)$, $\sin(\phi_h - \phi_S)$, $\cos(2\phi_h)$, etc., where ϕ_h is the azimuthal angle of the produced hadron and ϕ_S is the azimuthal angle of the nucleon's spin vector). Each of these structure functions is a convolution of different TMD PDFs and TMD FFs.

For example:

- The unpolarized cross-section is primarily sensitive to the unpolarized TMD PDF f_1 and the unpolarized TMD FF D_1.
- A $\sin(\phi_h - \phi_S)$ asymmetry in the scattering of a transversely polarized nucleon is proportional to the convolution of the Sivers TMD PDF $h_1^\perp$ and the unpolarized TMD FF D_1.
- A $\sin(\phi_h + \phi_S)$ asymmetry (Collins asymmetry) when the nucleon is transversely polarized is proportional to the convolution of the transversity TMD PDF h_1 and the Collins TMD FF $H_1^\perp$.

By performing global fits to SIDIS data (along with data from other processes like Drell-Yan and e^+e^- annihilation), phenomenologists can extract these non-perturbative TMD functions. The evolution equations (Collins-Soper for ζ and DGLAP-like for μ) are essential for relating TMDs measured at different scales and in different experiments.

Thus, the formal operator definitions of TMDs, including their specific gauge link structures, provide the theoretical basis for interpreting SIDIS measurements. The factorization theorem allows us to connect these fundamental QCD quantities to observable cross-sections and asymmetries, enabling the experimental study of the 3D momentum structure of hadrons.

2.4 TMD Evolution Equations

2.4.1 Necessity and Origin of TMD Evolution Equations

Transverse Momentum Dependent (TMD) parton distribution functions (PDFs) and fragmentation functions (FFs) are not static quantities but evolve with the energy scales involved in the scattering process. This evolution is a fundamental consequence of Quantum Chromodynamics (QCD) and is essential for making precise predictions and for relating measurements made at different energy scales. The necessity for TMD evolution equations arises from several key aspects of QCD and the definition of TMDs:

1. **Scale Dependence of Quantum Field Theories:** In any quantum field theory, including QCD, physical observables should be independent of unphysical regularization and renormalization scales introduced to handle divergences in calculations. However, the calculated quantities that are not themselves direct physical observables, like PDFs or TMDs, do depend on these scales. Evolution equations describe how these quantities change as the scales are varied, ensuring that the scale dependence cancels out in the calculation of physical cross-sections.

2. **Resummation of Large Logarithms:** Perturbative QCD calculations for processes involving TMDs often generate large logarithms of ratios of scales. For example, in SIDIS, if the hard scale Q is much larger than the typical transverse momentum k_T (i.e., $Q^2 \gg k_T^2$), logarithms like $\ln(Q^2/k_T^2)$ appear. These logarithms can be large and spoil the convergence of fixed-order perturbative expansions. TMD evolution equations, such as the Collins-Soper (CS) equation and DGLAP-like equations, resum these large logarithms to all orders in the strong coupling constant α_s, leading to more reliable theoretical predictions.

3. **Rapidity Divergences and Wilson Lines:** As discussed earlier, TMDs are defined with light-like Wilson lines to ensure gauge invariance. These Wilson lines, extending to infinity, introduce special types of divergences called rapidity divergences (or light-cone divergences) when loop corrections are calculated. These are not standard ultraviolet (UV) or collinear divergences. The TMD evolution scale ζ (or related variables like the Collins-Soper parameter) is introduced to regulate these rapidity divergences. The evolution with respect to ζ (governed by the CS equation) specifically addresses these rapidity logarithms.

4. **Universality and Factorization:** TMD factorization theorems, which allow us to write cross-sections as convolutions of hard parts and universal TMDs, rely on a consistent framework for handling all scale dependencies. The evolution equations ensure that the TMDs extracted from one process at a certain set of scales can be evolved and used to predict another process at different scales, thus preserving the universality of TMDs.

In essence, the origin of TMD evolution equations lies in the need to systematically account for radiative corrections (gluon emissions and absorptions) that modify the parton distributions as a function of the probing scales. These corrections lead to the running of the strong coupling α_s and the scale dependence of the TMDs themselves.

2.4.2 Key Features of TMD Evolution Equations (e.g., Collins-Soper Equation)

The evolution of TMDs is governed by a set of coupled equations, primarily:

1. **Collins-Soper (CS) Equation:** This equation describes the evolution of TMDs with respect to the TMD scale ζ (or the Collins-Soper evolution parameter, which is related to the rapidity difference between the hadron and the hard interaction). The CS equation is typically written for TMDs in transverse position space (b_T-space, the Fourier conjugate of k_T-space), where b_T represents the transverse separation between the quark fields in the TMD definition. The CS equation has the form:

$$\frac{d \ln f(x, b_T; \mu, \zeta)}{d \ln \sqrt{\zeta}} = K(b_T; \mu) \tag{3}$$

   Where $f(x, b_T; \mu, \zeta)$ is the TMD in b_T-space, and $K(b_T; \mu)$ is the Collins-Soper evolution kernel. This kernel is perturbatively calculable. At small b_T (corresponding to large k_T), $K(b_T; \mu)$ can be computed in powers of $\alpha_s(\mu)$. At large b_T (small k_T), the kernel becomes non-perturbative and needs to be modeled or fitted from data. The CS equation resums logarithms of ζ (or $Q^2 b_T^2$).

2. **Renormalization Group Equations for μ-dependence:** TMDs also depend on the renormalization/factorization scale μ, similar to collinear PDFs. This μ-evolution is governed by DGLAP-like equations, but with modifications due to the TMD nature. The evolution equation for the μ-dependence can be written as:

$$\frac{d \ln f(x, b_T; \mu, \zeta)}{d \ln \mu} = \gamma_F(\mu, \zeta/\mu^2) \tag{4}$$

   Where γ_F is an anomalous dimension. This anomalous dimension itself depends on $\alpha_s(\mu)$ and potentially on the ratio ζ/μ^2. This part of the evolution resums logarithms of μ^2/k_T^2 (or $\mu^2 b_T^2$).

The combined evolution in μ and ζ allows TMDs to be related across different hard scales Q and different intrinsic transverse momentum scales. The solution to these evolution equations involves an exponential factor (the Sudakov factor) that resums the large logarithms. This Sudakov factor suppresses TMDs at very large b_T (very small k_T) if Q is large, reflecting the fact that it is difficult for a parton to remain at very low transverse momentum when subjected to a very hard probe due to increased radiation.

The TMD evolution framework is crucial for phenomenological applications, allowing for consistent analysis of data from different experiments and kinematic regions. It forms a cornerstone of precision QCD studies involving transverse momentum.

2.5 Factorization in Processes Involving TMDs

2.5.1 The Concept of TMD Factorization

TMD factorization is a fundamental concept in Quantum Chromodynamics (QCD) that allows for a systematic separation of short-distance (hard, perturbative) physics from long-distance (soft/collinear, non-perturbative) physics in high-energy scattering processes where the transverse momentum of partons is explicitly resolved and is relatively small compared to the hard interaction scale. It asserts that the cross-section for such processes can be written as a convolution of a process-dependent hard scattering part, calculable in perturbative QCD, and process-independent (or universal, up to well-defined process-dependent Wilson lines) Transverse Momentum Distributions (TMDs). These TMDs encapsulate the non-perturbative information about the three-dimensional momentum structure of partons within hadrons (TMD PDFs) or about the hadronization of partons into observed hadrons (TMD FFs).

The essence of factorization is to isolate the parts of the interaction that occur at different spacetime scales. The hard part involves large momentum transfers (characterized by a hard scale Q) and occurs over short distances. The TMDs describe the physics at longer distance scales, related to the confinement of partons and their intrinsic motion, characterized by transverse momenta k_T and the hadron mass scale Λ_{QCD}.

2.5.2 Conditions Under Which TMD Factorization is Expected to Apply

TMD factorization is not universally applicable to all processes or all kinematic regimes. Its validity relies on specific conditions:

1. **Presence of at Least Two Well-Separated Scales:** The most crucial condition is the existence of a hard scale Q (e.g., the virtuality of the photon in DIS/SIDIS, or the mass of the lepton pair in Drell-Yan) and a smaller transverse momentum scale q_T (e.g., the transverse momentum of the produced hadron system in SIDIS, or the lepton pair in Drell-Yan), such that $q_T^2 \ll Q^2$. This hierarchy of scales is what allows for the separation of dynamics.
2. **Leading Power Approximation:** TMD factorization theorems are typically proven at the leading power in q_T/Q. Corrections to the factorization formula are suppressed by powers of $(q_T/Q)^n$.
3. **Structure of Interactions:** The process must be such that the long-distance interactions (soft and collinear gluon exchanges) can be systematically disentangled from the hard interaction and absorbed into the definition of universal TMDs. This involves demonstrating that interactions that could spoil universality (e.g., those connecting partons from different hadrons in a way that is not simply resummed into gauge links) are power-suppressed.
4. **Gauge Invariance and Wilson Lines:** The definition of TMDs must include appropriate Wilson lines (gauge links) to ensure their gauge invariance. The specific path of these Wilson lines is process-dependent and reflects the color flow and the history of soft gluon interactions (initial-state or final-state interactions). For example, the Wilson lines for Drell-Yan TMD PDFs differ from those for SIDIS TMD PDFs due to the different final/initial state color environments.
5. **Absence of Uncancelled Infrared Singularities in the Hard Part:** After factorization, the hard scattering part must be free of infrared (soft and collinear) singularities that are not absorbed into the TMDs. All such singularities associated with the external hadrons should be consistently factorized into the TMDs.

2.5.3 How to Know if Factorization Applies to a Certain Process

Determining whether TMD factorization applies to a specific process involves rigorous theoretical analysis within QCD:

1. **Diagrammatic Analysis (Feynman Diagrams):** This is a cornerstone of factorization proofs. Physicists analyze the momentum regions of loop integrations in Feynman diagrams contributing to the process. They identify "leading regions" that give the dominant contributions in the kinematic regime of interest (e.g., $q_T^2 \ll Q^2$). The goal is to show that contributions from these regions can be systematically organized into a factorized form. This involves techniques like identifying pinch singularities and using Ward identities.

2. **Effective Field Theories (EFTs):** Modern approaches often utilize EFTs like Soft-Collinear Effective Theory (SCET) or variations tailored for TMDs. In an EFT framework, one constructs an effective Lagrangian that describes the interactions of relevant degrees of freedom (e.g., soft, collinear, and hard modes) at the appropriate scales. Factorization then emerges more systematically from the structure of the effective theory and the decoupling of different modes.

3. **Power Counting:** A crucial element is power counting in the small ratio q_T/Q (or Λ_{QCD}/Q). One needs to demonstrate that the proposed factorized structure captures the leading-power terms in this expansion and that any deviations or additional terms are power-suppressed.

4. **Universality Checks:** The TMDs defined within the factorization scheme for one process should, after accounting for the specific Wilson line structures, be applicable to other processes. Consistency across different processes is a strong check of the factorization framework.

5. **Explicit Calculations:** Factorization theorems are often established and refined through explicit perturbative calculations at one or more loop orders. These calculations help to identify the structure of the hard functions, the evolution kernels for the TMDs, and any potential issues or subtleties.

For a new process, establishing TMD factorization is a significant theoretical task. It involves showing that the cross-section can be written in the desired convoluted form, identifying the correct operator definitions for the TMDs (including their gauge links), and demonstrating that the hard coefficients are calculable and infrared safe. The Jefferson Lab PDF (qiu_sidis_tmd.txt, slides 23-25, 28-29) provides some insights into the general arguments for factorization in Drell-Yan and SIDIS, emphasizing the suppression of quantum interference between short and long distances and the role of long-lived parton states.

In summary, TMD factorization is a powerful tool, but its applicability is restricted to specific kinematic regimes and requires careful theoretical justification based on the underlying dynamics of QCD.

3 Reputable Academic Sources

Below is a list of reputable academic sources that provide foundational knowledge and recent developments in the field of Transverse Momentum Distributions (TMDs) and their application in processes like Semi-Inclusive Deep Inelastic Scattering (SIDIS). These sources include textbooks, review articles, and key research papers.

1. **Foundations of Perturbative QCD by John Collins.**
   - Description: This is a standard textbook that provides a comprehensive and rigorous treatment of perturbative QCD, including detailed discussions on factorization, parton distribution functions, and the theoretical foundations relevant to TMDs.
   - Publisher: Cambridge University Press.
   - Availability: Can be found through university libraries or purchased from booksellers. (e.g., Cambridge University Press, Amazon)

2. **"Introduction to QCD" by Jianwei Qiu (Lecture Notes).**
   - Description: These lecture notes from the 2021 CFNS Summer School on the Physics of the Electron-Ion Collider provide an excellent pedagogical introduction to QCD,

factorization, PDFs, and concepts leading to TMDs. The document processed earlier in this research was based on these notes.

- Link: JLab Theory Files

3. **"A short review on recent developments in TMD factorization and implementation" by Ignazio Scimemi.**
   - Description: This review provides an overview of recent theoretical and phenomenological advances in TMD factorization and evolution, aimed at a broad audience including those not strictly experts in the field.
   - arXiv Link: arXiv:1901.08398
   - Published in: Advances in High Energy Physics, vol. 2019, Article ID 3142510.
   - DOI Link: 10.1155/2019/3142510
4. **"TMD Handbook" (Various Authors, coordinated by TMD Collaboration).**
   - Description: A comprehensive handbook reviewing transverse-momentum-dependent parton distribution functions and fragmentation functions, covering theoretical formalism, phenomenology, and experimental aspects.
   - Inspire HEP Link: TMD Handbook (This link might lead to the record; specific versions or chapters might be found through the collaboration's resources.)
5. **"Jet definition and TMD factorisation in SIDIS" by Paul Caucal, Edmond Iancu, A. H. Mueller, Feng Yuan.**
   - Description: A recent research paper discussing TMD factorization in the context of jet production in SIDIS, which is a frontier topic.
   - arXiv Link: arXiv:2408.03129 (Note: The year 2408 is a placeholder from the search result, actual preprint would be from current/recent years, e.g. 2024. The link provided was 'https://arxiv.org/html/2408.03129v1' which is likely a typo in the search result for the year. A more realistic recent paper would be sought, but I will use the link as provided by the search for now, assuming it's a very recent preprint.)
6. **"TMD and spin asymmetries in SIDIS" by Andrea Bressan.**
   - Description: A review of measurements of leading twist TMDs and fragmentation functions from HERMES, COMPASS, and JLab experiments, focusing on spin asymmetries in SIDIS.
   - Journal: EPJ Web of Conferences 85, 01007 (2015).
   - DOI Link: 10.1051/epjconf/20158501007
   - Direct Link: EPJ Web of Conferences
7. **"An Overview of Transverse Momentum Dependent Factorization and Evolution" by John Collins.**
   - Description: A concise overview of TMD factorization and evolution theorems, emphasizing the Collins-Soper-Sterman (CSS) formalism.
   - Conference Proceeding: Int.J.Mod.Phys.Conf.Ser. 37 (2015) 1560021.
   - arXiv Link: arXiv:1509.04766

8. **“QCD Factorization for Semi-inclusive Deep Inelastic Scattering” by Feng Yuan (Slides/Notes).**
    - Description: Presentation slides discussing QCD factorization for SIDIS, often covering TMD aspects. While slides are less formal, they can provide good summaries.
    - Example Link (from earlier search, specific content may vary): WISC Agenda

This list provides a starting point for a deep dive into TMDs. Many more specific research articles and phenomenological studies can be found by following citations from these key resources and searching on platforms like INSPIRE-HEP and arXiv.org.

4 Advanced Topics (To be detailed based on new requests)

4.1 Visualizing Feynman Diagrams for SIDIS

4.2 The Concept of Twist

4.3 Operators Defining Unpolarized TMDs

4.4 Visualizing Wilson Lines in SIDIS