Stoke's and Gauss's Theorem

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Exercise Problems.

1. Let $X : \mathbf{R}^2 \to \mathbf{R}^3$ be the parametrized surface given by

$$X(s,t) = (s^2 - t^2, s + t, s^2 + 3t)$$

(a) Determine a normal vector to this surface at the point

$$(3,1,1) = \mathbf{X}(2,-1)$$

Solution.

We have:

$$\begin{aligned} \mathbf{T}_s &= (2s, 1, 2s) \\ \mathbf{T}_t &= (-2t, 1, 3) \end{aligned}$$

So, the standard normal vector at the point $\mathbf{X}(2,-1)$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s & 1 & 2s \\ -2t & 1 & 3 \end{vmatrix} \\ &= \mathbf{i}(3 - 2s) - \mathbf{j}(6s + 4st) + \mathbf{k}(2s + 2t) \\ &= \mathbf{i}(3 - 4) - \mathbf{j}(12 + 4(2)(-1)) + \mathbf{k}(4 - 2) \\ &= -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \end{aligned}$$

(b) Find an equation for the plane tangent to this surface at the point $(3,1,1)$.

Solution.

The tangent plane to this surface at the point $(3,1,1)$ is given by:

$$\begin{aligned} \mathbf{N} \cdot (\mathbf{x} - (3,1,1)) &= 0 \\ (-1,-4,2) \cdot ((x,y,z) - (3,1,1)) &= 0 \\ -(x-3) - 4(y-1) + 2(z-1) &= 0 \end{aligned}$$

2. Find an equation for the plane tangent to the torus

$$\mathbf{X}(s,t) = ((5 + 2\cos t)\cos s, (5 + 2\cos t)\sin s, 2\sin t)$$

at the point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (-(5+2\cos t)\sin s,(5+2\cos t)\cos s,0)\\ \mathbf{T}_t &= (-2\sin t\cos s,-2\sin t\sin s,2\cos t)\end{aligned}$$

The standard normal vector is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s\times\mathbf{T}_t\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -(5+2\cos t)\sin s & (5+2\cos t)\cos s & 0\\ -2\sin t\cos s & -2\sin t\sin s & 2\cos t\end{vmatrix}\\ &= \mathbf{i}(2(5+2\cos t)\cos s\cos t))+\mathbf{j}(2(5+2\cos t)\sin s\cos t)\\ &+\mathbf{k}(2\sin s\sin t(5+2\cos t)+2(5+2\cos t)\sin t\cos^2 s)\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i}+\sin s\cos t\mathbf{j}+(\sin^2 s+\cos^2 s)\sin t\mathbf{k})\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i}+\sin s\cos t\mathbf{j}+\sin t\mathbf{k})\end{aligned}$$

The point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1)=((5+2\cos t)\cos s,(5+2\cos t)\sin s,2\sin t)$ yields $\sin t=1/2$, so $t_0=\pi/6$ or $t_0=5\pi/6$. Since $2\cos t<0$, $t_0=5\pi/6$. Then, we can see that :

$$\frac{5-\sqrt{3}}{\sqrt{2}}=(5-2\cdot\frac{\sqrt{3}}{2})\sin s$$

So, $s_0=\pi/4$.

Consequently, the equation of the tangent plane at $\mathbf{X}(\pi/4,5\pi/6)$ is:

$$\begin{aligned}\mathbf{N}\cdot(\mathbf{x}-\mathbf{X}(s_0,t_0)) &= 0\\ 2(5-\sqrt{3})(-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{i}-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{j}+\frac{1}{2}\mathbf{k})((x,y,z)-(\frac{5-\sqrt{3}}{\sqrt{2}},\frac{5-\sqrt{3}}{\sqrt{2}},1)) &= 0\\ -\frac{\sqrt{3}}{\sqrt{2}}(x-(5-\sqrt{3})/\sqrt{2})-\frac{\sqrt{3}}{\sqrt{2}}(y-(5-\sqrt{3})/\sqrt{2})+(z-1) &= 0\\ -\sqrt{3}(x-(5-\sqrt{3})/\sqrt{2})-\sqrt{3}(y-(5-\sqrt{3})/\sqrt{2})+\sqrt{2}(z-1) &= 0\\ -\sqrt{3}x-\sqrt{3}y+\sqrt{2}z &= -2\sqrt{3}(5-\sqrt{3})/\sqrt{2}+\sqrt{2}\\ &= -\sqrt{6}(5-\sqrt{3})+\sqrt{2}\\ &= -5\sqrt{6}+3\sqrt{2}+\sqrt{2}\\ \sqrt{3}x+\sqrt{3}y-\sqrt{2}z &= 5\sqrt{6}-4\sqrt{2}\end{aligned}$$

3. Find an equation of the plane tangent to the surface