

Stoke's and Gauss's Theorem

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October 13, 2023

Exercise Problems.

1. Let $X : \mathbf{R}^2 \rightarrow \mathbf{R}^3$ be the parametrized surface given by

$$X(s, t) = (s^2 - t^2, s + t, s^2 + 3t)$$

(a) Determine a normal vector to this surface at the point

$$(3, 1, 1) = \mathbf{X}(2, -1)$$

Solution.

We have:

$$\mathbf{T}_s = (2s, 1, 2s)$$

$$\mathbf{T}_t = (-2t, 1, 3)$$

So, the standard normal vector at the point $\mathbf{X}(2, -1)$ is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s & 1 & 2s \\ -2t & 1 & 3 \end{vmatrix} \\ &= \mathbf{i}(3 - 2s) - \mathbf{j}(6s + 4st) + \mathbf{k}(2s + 2t) \\ &= \mathbf{i}(3 - 4) - \mathbf{j}(12 + 4(2)(-1)) + \mathbf{k}(4 - 2) \\ &= -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\end{aligned}$$

(b) Find an equation for the plane tangent to this surface at the point $(3, 1, 1)$.

Solution.

The tangent plane to this surface at the point $(3, 1, 1)$ is given by:

$$\begin{aligned}\mathbf{N} \cdot (\mathbf{x} - (3, 1, 1)) &= 0 \\ (-1, -4, 2) \cdot ((x, y, z) - (3, 1, 1)) &= 0 \\ -(x - 3) - 4(y - 1) + 2(z - 1) &= 0\end{aligned}$$

2. Find an equation for the plane tangent to the torus

$$\mathbf{X}(s, t) = ((5 + 2 \cos t) \cos s, (5 + 2 \cos t) \sin s, 2 \sin t)$$

at the point $((5 - \sqrt{3})/\sqrt{2}, (5 - \sqrt{3})/\sqrt{2}, 1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (-(5 + 2 \cos t) \sin s, (5 + 2 \cos t) \cos s, 0) \\ \mathbf{T}_t &= (-2 \sin t \cos s, -2 \sin t \sin s, 2 \cos t)\end{aligned}$$

The standard normal vector is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(5 + 2 \cos t) \sin s & (5 + 2 \cos t) \cos s & 0 \\ -2 \sin t \cos s & -2 \sin t \sin s & 2 \cos t \end{vmatrix} \\ &= \mathbf{i}(2(5 + 2 \cos t) \cos s \cos t) + \mathbf{j}(2(5 + 2 \cos t) \sin s \cos t) \\ &\quad + \mathbf{k}(2 \sin s \sin t(5 + 2 \cos t) + 2(5 + 2 \cos t) \sin t \cos^2 s) \\ &= 2(5 + 2 \cos t)(\cos s \cos t \mathbf{i} + \sin s \cos t \mathbf{j} + (\sin^2 s + \cos^2 s) \sin t \mathbf{k}) \\ &= 2(5 + 2 \cos t)(\cos s \cos t \mathbf{i} + \sin s \cos t \mathbf{j} + \sin t \mathbf{k})\end{aligned}$$

The point $((5 - \sqrt{3})/\sqrt{2}, (5 - \sqrt{3})/\sqrt{2}, 1) = ((5 + 2 \cos t) \cos s, (5 + 2 \cos t) \sin s, 2 \sin t)$ yields $\sin t = 1/2$, so $t_0 = \pi/6$ or $t_0 = 5\pi/6$.

Since $2 \cos t < 0$, $t_0 = 5\pi/6$. Then, we can see that :

$$\frac{5 - \sqrt{3}}{\sqrt{2}} = (5 - 2 \cdot \frac{\sqrt{3}}{2}) \sin s$$

So, $s_0 = \pi/4$.

Consequently, the equation of the tangent plane at $\mathbf{X}(\pi/4, 5\pi/6)$ is:

$$\begin{aligned}\mathbf{N} \cdot (\mathbf{x} - \mathbf{X}(s_0, t_0)) &= 0 \\ 2(5 - \sqrt{3})\left(-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{i} - \frac{\sqrt{3}}{2\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k}\right) \cdot \left((x, y, z) - \left(\frac{5 - \sqrt{3}}{\sqrt{2}}, \frac{5 - \sqrt{3}}{\sqrt{2}}, 1\right)\right) &= 0 \\ -\frac{\sqrt{3}}{\sqrt{2}}(x - (5 - \sqrt{3})/\sqrt{2}) - \frac{\sqrt{3}}{\sqrt{2}}(y - (5 - \sqrt{3})/\sqrt{2}) + (z - 1) &= 0 \\ -\sqrt{3}(x - (5 - \sqrt{3})/\sqrt{2}) - \sqrt{3}(y - (5 - \sqrt{3})/\sqrt{2}) + \sqrt{2}(z - 1) &= 0 \\ -\sqrt{3}x - \sqrt{3}y + \sqrt{2}z &= -2\sqrt{3}(5 - \sqrt{3})/\sqrt{2} + \sqrt{2} \\ &= -\sqrt{6}(5 - \sqrt{3}) + \sqrt{2} \\ &= -5\sqrt{6} + 3\sqrt{2} + \sqrt{2} \\ \sqrt{3}x + \sqrt{3}y - \sqrt{2}z &= 5\sqrt{6} - 4\sqrt{2}\end{aligned}$$

3. Find an equation of the plane tangent to the surface

$$x = e^s \quad y = t^2 e^{2s} \quad z = 2e^{-s} + t$$

at the point $(1, 4, 0)$.

Solution.

We have:

$$\mathbf{T}_s = (e^s, 2t^2 e^{2s}, -2e^{-s})$$

$$\mathbf{T}_t = (0, 2te^{2s}, 1)$$

The standard normal vector is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^s & 2t^2 e^{2s} & -2e^{-s} \\ 0 & 2te^{2s} & 1 \end{vmatrix} \\ &= \mathbf{i}(2t^2 e^{2s} + 4te^s) - \mathbf{j}(e^s) + \mathbf{k}(2te^{3s}) \\ &= e^s((2t^2 e^s + 4t)\mathbf{i} - \mathbf{j} + (2te^{2s})\mathbf{k})\end{aligned}$$

Since $e^s = 1$, $s = 0$. Also, as $4 = t^2 \cdot 1$, we have $t = \pm 2$. Moreover, $0 = 2 + t$, so $t = -2$. So, $\mathbf{N}(0, -2)$ is:

$$\mathbf{N}(0, -2) = -\mathbf{j} - 4\mathbf{k}$$

The equation of the tangent plane at $\mathbf{X}(0, -2)$ is:

$$\begin{aligned}\mathbf{N} \cdot (x - 1, y - 4, z) &= 0 \\ -(y - 4) - 4z &= 0 \\ y + 4z &= 4\end{aligned}$$

4. Let $\mathbf{X}(s, t) = (s^2 \cos t, s^2 \sin t, s)$, $-3 \leq s \leq 3$, $0 \leq t \leq 2\pi$.

(a) Find a normal vector at $(s, t) = (-1, 0)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (2s \cos t, 2s \sin t, 1) \\ \mathbf{T}_t &= (-s^2 \sin t, s^2 \cos t, 0)\end{aligned}$$

The standard normal vector \mathbf{N} is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s \cos t & 2s \sin t & 1 \\ -s^2 \sin t & s^2 \cos t & 0 \end{vmatrix} \\ &= \mathbf{i}(-s^2 \cos t) - \mathbf{j}(s^2 \sin t) + \mathbf{k}(2s^3 \cos^2 t + 2s^3 \sin^2 t) \\ &= -s^2 \cos t \mathbf{i} - s^2 \sin t \mathbf{j} + 2s^3 \mathbf{k}\end{aligned}$$

The vector at $(s, t) = (-1, 0)$ is:

$$\mathbf{N}(-1, 0) = -\mathbf{i} - 2\mathbf{k}$$

Hence, the equation of the tangent plane at $\mathbf{X}(-1, 0) = (1, 0, -1)$ is:

$$\begin{aligned}
(-\mathbf{i} - 2\mathbf{k}) \cdot ((x, y, z) - (1, 0, -1)) &= 0 \\
-(x - 1) - 2(z + 1) &= 0 \\
x - 1 + 2z + 2 &= 0 \\
x + 2z + 1 &= 0
\end{aligned}$$

(b) Find an equation for the image of \mathbf{X} in the form $F(x, y, z) = 0$.

Solution.

Let $x = s^2 \cos t$, $y = s^2 \sin t$. Then, $x^2 + y^2 = s^4(\cos^2 t + \sin^2 t) = s^4 = z^4$. So, $F(x, y, z) = x^2 + y^2 - z^4 = 0$.

5. Consider the parameterized surface $\mathbf{X}(s, t) = (s, s^2 + t, t^2)$.

(a) Graph the surface for $-2 \leq s \leq 2$, $-2 \leq t \leq 2$.

Solution.

The s -coordinate curve at $t = 0$ is:

$$\begin{aligned}
x &= s \\
y &= s^2 \\
z &= 0
\end{aligned}$$

This is the parabolic curve $y = x^2$ in the xy -plane.

The s -coordinate curve at $t = t_0$ is:

$$\begin{aligned}
x &= s \\
y &= s^2 + t_0 \\
z &= t_0^2
\end{aligned}$$

Thus, we get parabolas parallel to the xy -plane.

t	Curve	Center	z -plane
$t_0 = -2$	$y + 2 = x^2$	$(x, y) = (0, -2)$	$z = 4$
$t_0 = -1$	$y + 1 = x^2$	$(x, y) = (0, -1)$	$z = 1$
$t_0 = 0$	$y = x^2$	$(x, y) = (0, 0)$	$z = 0$
$t_0 = 1$	$y - 1 = x^2$	$(x, y) = (0, 1)$	$z = 1$
$t_0 = 2$	$y - 2 = x^2$	$(x, y) = (0, 2)$	$z = 4$

The t -coordinate curve at $s = 0$ is:

$$\begin{aligned}
x &= 0 \\
y &= t \\
z &= t^2
\end{aligned}$$

These are parabolas parallel to the yz -plane.

t	Curve	Center	x -plane
$s_0 = -2$	$z = (y - 4)^2$	$(y, z) = (2, 0)$	$x = -2$
$s_0 = -1$	$z = (y - 1)^2$	$(y, z) = (1, 0)$	$x = -1$
$s_0 = 0$	$z = y^2$	$(y, z) = (0, 0)$	$x = 0$
$s_0 = 1$	$z = (y - 1)^2$	$(y, z) = (1, 0)$	$x = 1$
$s_0 = 2$	$z = (y - 4)^2$	$(y, z) = (2, 0)$	$x = 2$

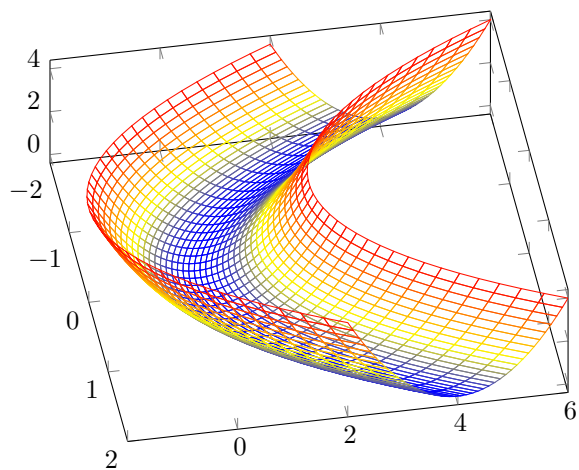


Figure. $\mathbf{X}(s, t) = (s, s^2 + t, t^2)$.

(b) Is the surface smooth?

Solution. The surface is smooth.

(c) Find an equation for the tangent plane at the point $(1, 0, 1)$.

Solution.

We have:

$$\mathbf{T}_s = (1, 2s, 0)$$

$$\mathbf{T}_t = (0, 1, 2t)$$

The standard normal vector \mathbf{N} is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2s & 0 \\ 0 & 1 & 2t \end{vmatrix} \\ &= 4sti - 2t\mathbf{j} + \mathbf{k} \end{aligned}$$

The point $(1, 0, 1)$ is $(s_0 = 1, t_0 = -1)$.

$$\mathbf{N}(1, -1) = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

The equation of the tangent plane at $(1, -1)$ is:

$$\begin{aligned} (-4, 2, 1) \cdot (x - 1, y, z - 1) &= 0 \\ -4(x - 1) + 2y + (z - 1) &= 0 \\ 4(x - 1) - 2y - (z - 1) &= 0 \\ 4x - 4 - 2y - z + 1 &= 0 \\ 4x - 2y - z &= 3 \end{aligned}$$

6. Describe the parameterized surface of exercise problem 1 by an equation of the form $z = f(x, y)$.

Solution.

The parametric surface $\mathbf{X}(s, t)$ is:

$$X(s, t) = (s^2 - t^2, s + t, s^2 + 3t)$$

In exercise (1), we see that $x = (s - t)(s + t) = y(s + t)$ so $s + t = x/y$ and $y = s - t$. This allows us to solve simultaneously for s and t . $2s = x/y + y$ and $2t = x/y - y$. This means that $z = s^2 + 3t$ can be written as $z = \frac{1}{4} \left(\frac{x}{y} + y \right)^2 + \frac{3}{2} \left(\frac{x}{y} - y \right)$.

7. Let S be the surface parameterized by:

$$x = s \cos t$$

$$y = s \sin t$$

$$z = s^2$$

where $s \geq 0, 0 \leq t \leq 2\pi$.

(a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$.

Solution.

The surface S is $x^2 + y^2 = z$. This is a paraboloid. It is smooth at all points.

We have:

$$\mathbf{T}_s = (\cos t, \sin t, 2s)$$

$$\mathbf{T}_t = (-s \sin t, s \cos t, 0)$$

The standard normal vector \mathbf{N} is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 2s \\ -s \sin t & s \cos t & 0 \end{vmatrix} \\ &= (-2s^2 \cos t)\mathbf{i} - (2s^2 \sin t)\mathbf{j} + (s \cos^2 t + s \sin^2 t)\mathbf{k} \\ &= (-2s^2 \cos t)\mathbf{i} - (2s^2 \sin t)\mathbf{j} + s\mathbf{k} \end{aligned}$$

At $s = 2, t = \pi/6$,

$$\mathbf{N}(2, \pi/6) = -4\sqrt{3}\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

The equation of the tangent plane at $\mathbf{X}(2, \pi/6)$ is:

$$\begin{aligned} (-4\sqrt{3}, -4, 2) \cdot (x - 1, y - \sqrt{3}, z - 4) &= 0 \\ 4\sqrt{3}(x - 1) + 4(y - \sqrt{3}) - 2(z - 4) &= 0 \\ 4\sqrt{3}x - 4\sqrt{3} + 4y - 4\sqrt{3} - 2z + 8 &= 0 \\ 4\sqrt{3}x + 4y - 2z &= 8(\sqrt{3} - 1) \\ 2\sqrt{3}x + 2y - z &= 4(\sqrt{3} - 1) \end{aligned}$$

(b) Sketch the graph of S . Can you recognize S as a familiar surface?

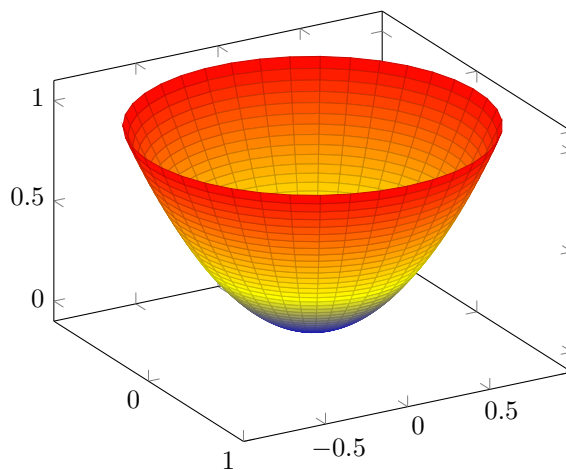


Figure. $\mathbf{X}(s, t) = (s \cos t, s \sin t, s^2)$

(c) Describe S by an equation of the form $z = f(x, y)$.

Solution. Again $z = x^2 + y^2$.

(d) Using your answer in part(c), discuss whether S has a tangent plane at every point.

Solution.

S has a tangent plane at every point and is smooth. Part (a) takes care of every point except the origin. At the origin $\mathbf{N} = (0, 0, 0)$. But, we easily see, that the tangent plane at the origin is the horizontal plane $z = 0$. Thus, smoothness as defined in the text, depends on both the parameterization and the geometry of the underlying surface.

8. Verify that the image of the parametrized surface

$$\mathbf{X}(s, t) = (2 \sin s \cos t, 3 \sin s \sin t, \cos s)$$

$0 \leq s \leq \pi$ and $0 \leq t \leq 2\pi$ is an ellipsoid.

Solution.

We can easily write:

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$$

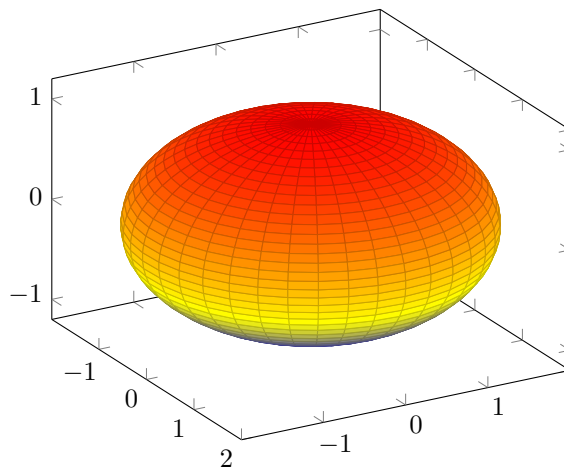


Figure. $\mathbf{X}(s, t) = (2 \sin s \cos t, 3 \sin s \sin t, \cos s)$

9. Verify that, for the torus of example 5, the s -coordinate curve, when $t = t_0$ is a circle of radius $a + b \cos t_0$.

Solution.

The parametric equations of the Torus in example 5 were:

$$\begin{aligned}x &= (a + b \cos t) \cos s \\y &= (a + b \cos t) \sin s \\z &= b \sin t\end{aligned}$$

The s -coordinate curve at $t = t_0$ is:

$$\begin{aligned}x &= (a + b \cos t_0) \cos s \\y &= (a + b \cos t_0) \sin s \\z &= b \sin t_0\end{aligned}$$

These are circles of radius $a + b \cos t_0$ in the plane $z = b \sin t_0$.

And they satisfy $x^2 + y^2 = (a + b \cos t_0)^2$.

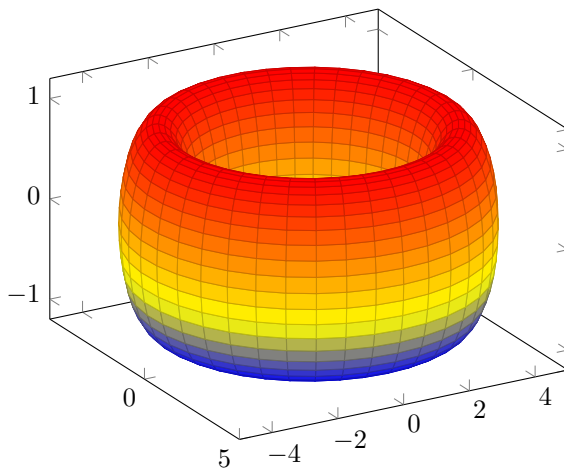


Figure. Torus $\mathbf{X}(s, t) = ((a + b \cos t) \cos s, (a + b \cos t) \sin s, b \sin t)$

9. The surface in \mathbf{R}^3 parametrized by:

$$\mathbf{X}(r, \theta) = (r \cos \theta, r \sin \theta, \theta)$$

where $r \geq 0$ and $-\infty < \theta < \infty$ is called a helicoid.

(a) Describe the r -coordinate curve when $\theta = \pi/3$. Give a general description of the r -coordinate curves.

Solution.

The r -coordinate curve when $\theta = \pi/3$ is:

$$\begin{aligned}x &= r/2 \\y &= \sqrt{3}r/2 \\z &= \pi/3\end{aligned}$$

It is the straight-line $y = \sqrt{3}x$ in the plane $z = \pi/3$.

The r -coordinate curve when $\theta = \theta_0$ is:

$$\begin{aligned}x &= r \cos \theta_0 \\y &= r \sin \theta_0 \\z &= \theta_0\end{aligned}$$

It is the straight line $y = (\tan \theta_0)x$ in the plane $z = \theta_0$.

(b) Describe the θ -coordinate curve when $r = 1$. Give a general description of the θ -coordinate curves.

Solution.

The θ -coordinate curve when $r = 1$ is:

$$\begin{aligned}x &= \cos \theta \\y &= \sin \theta \\z &= \theta\end{aligned}$$

This is a helix with parameter θ and radius 1.

The θ -coordinate curve when $r = r_0$ is:

$$\begin{aligned}x &= r_0 \cos \theta \\y &= r_0 \sin \theta \\z &= \theta\end{aligned}$$

These are helices of radius $r_0 \geq 0$.

(c) Sketch the graph of the helicoid using a computer for $0 \leq r \leq 1, 0 \leq \theta \leq 4\pi$. Can you see why the surface is called a helicoid?

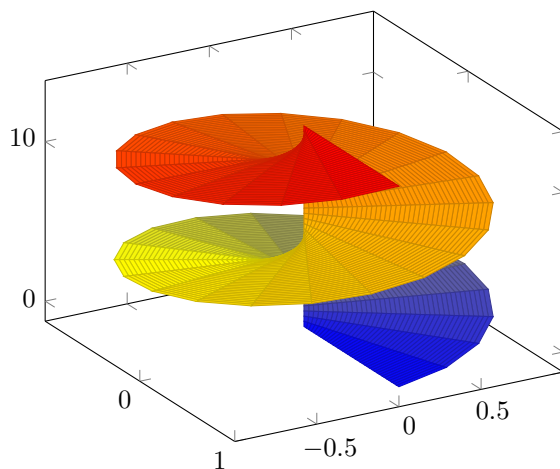


Figure. Helicoid $\mathbf{X}(r, \theta) = (r \cos \theta, r \sin \theta, \theta)$

11. Given a sphere of radius 2 centered at $(2, -1, 0)$, find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

(a) by consider the sphere as the graph of the function $f(x, y) = \sqrt{4 - (x - 2)^2 - (y + 1)^2}$.

Solution.

We have:

$$\begin{aligned}\text{grad } f &= \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right] \\&= \left[-\frac{(x-2)}{\sqrt{4-(x-2)^2-(y+1)^2}} \quad -\frac{(y+1)}{\sqrt{4-(x-2)^2-(y+1)^2}} \right]\end{aligned}$$

Thus,

$$\text{grad } f(1, 0) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

So, the tangent plane at the point $(1, 0, \sqrt{2})$ is:

$$\begin{aligned} z &= f(1, 0) + (\mathbf{x} - \mathbf{a}) \cdot \nabla f \\ &= \sqrt{2} + (x - 1, y) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \\ &= \sqrt{2} + \frac{x - 1}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\ \sqrt{2}z &= 2 + (x - 1) - y \\ x - y + 1 &= \sqrt{2}z \end{aligned}$$

(b) by considering the sphere as a level surface of the function

$$F(x, y, z) = (x - 2)^2 + (y + 1)^2 + z^2$$

The gradient ∇F is :

$$\begin{aligned} \nabla F &= \left[\frac{\partial F}{\partial x} \quad \frac{\partial F}{\partial y} \quad \frac{\partial F}{\partial z} \right] \\ &= \left[2(x - 2) \quad 2(y + 1) \quad 2z \right] \end{aligned}$$

If $\mathbf{x}_0 = (x_0, y_0, z_0)$ is a point on the level set $S = \{(x, y, z) : F(x, y, z) = c\}$, then the gradient vector $\nabla F(\mathbf{x})$ at the point \mathbf{x}_0 is perpendicular to S . $(1, 0, \sqrt{2})$ is point on the level set $S = \{(x, y, z) | (x - 2)^2 + (y + 1)^2 + z^2 = 4\}$. So, $\nabla F(1, 0, \sqrt{2}) = (-2, 2, 2\sqrt{2})$ is the normal vector to the sphere $F(x, y, z) = 4$ at the point $(1, 0, \sqrt{2})$.

If (x, y, z) is an arbitrary point in the tangent plane, we must have:

$$\begin{aligned} (x - 1, y, z - \sqrt{2}) \cdot (-2, 2, 2\sqrt{2}) &= 0 \\ (x - 1, y, z - \sqrt{2}) \cdot (-1, 1, \sqrt{2}) &= 0 \\ -(x - 1) + y + \sqrt{2}(z - \sqrt{2}) &= 0 \\ (x - 1) - y - \sqrt{2}(z - \sqrt{2}) &= 0 \\ x - y - \sqrt{2}z + 1 &= 0 \\ x - y + 1 &= \sqrt{2}z \end{aligned}$$

(c) By considering the sphere as the surface parametrized by :

$$\mathbf{X}(s, t) = (2 \sin s \cos t + 2, 2 \sin s \sin t - 1, 2 \cos s)$$

Solution.

We have:

$$\begin{aligned} \mathbf{T}_s &= (2 \cos s \cos t, 2 \cos s \sin t, -2 \sin s) \\ \mathbf{T}_t &= (-2 \sin s \sin t, 2 \sin s \cos t, 0) \end{aligned}$$

The standard normal vector \mathbf{N} is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 \cos s \cos t & 2 \cos s \sin t & -2 \sin s \\ -2 \sin s \sin t & 2 \sin s \cos t & 0 \end{vmatrix} \\ &= 4 \sin^2 s \cos t \mathbf{i} - (-4 \sin^2 s \sin t) \mathbf{j} + (4 \sin s \cos s \cos^2 t + 4 \sin s \cos s \sin^2 t) \mathbf{k} \\ &= 4 \sin^2 s \cos t \mathbf{i} + 4 \sin^2 s \sin t \mathbf{j} + 4 \sin s \cos s \mathbf{k}\end{aligned}$$

Now, $2 \cos s = \sqrt{2}$ so $\cos s = \frac{1}{\sqrt{2}}$ and thus $s = \pi/4$. Consequently, $\sqrt{2} \cos t + 2 = 1$ and therefore $\cos t = -\frac{1}{\sqrt{2}}$, which implies $t = 3\pi/4$.

The normal vector at $\mathbf{X}(\pi/4, 3\pi/4)$ is:

$$\begin{aligned}\mathbf{N}(\pi/4, 3\pi/4) &= 4 \cdot \frac{1}{2} \cdot \frac{-1}{\sqrt{2}} \mathbf{i} + 4 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \mathbf{j} + 2 \mathbf{k} \\ &= -\sqrt{2} \mathbf{i} + \sqrt{2} \mathbf{j} + 2 \mathbf{k}\end{aligned}$$

The equation of the tangent plane is:

$$\begin{aligned}(-\sqrt{2}, \sqrt{2}, 2) \cdot (x - 1, y, z - \sqrt{2}) &= 0 \\ (-1, 1, \sqrt{2}) \cdot (x - 1, y, z - \sqrt{2}) &= 0 \\ x - y + 1 &= \sqrt{2}z\end{aligned}$$

In exercises 12-15, represent the given surface as a piecewise smooth parameterized surface.

12. The lower hemisphere $x^2 + y^2 + z^2 = 9$ including the equatorial circle.

Solution.

We can parametrize the lower hemisphere of the sphere as $\mathbf{X}(s, t) = (s, t, -\sqrt{9 - (x^2 + y^2)})$. Alternatively, we may parametrize it as $\mathbf{X}(\phi, \theta)$:

$$\begin{aligned}x &= 3 \sin \phi \cos \theta \\ y &= 3 \sin \phi \sin \theta \\ z &= 3 \cos \phi\end{aligned}$$

where $0 \leq \theta \leq 2\pi$ and $\pi/2 \leq \phi \leq \pi$.

13. The part of the cylinder $x^2 + z^2 = 4$ lying between $y = -1$ and $y = 3$.

Solution.

We can parametrize the cylinder as:

$$\begin{aligned}x &= 2 \cos s \\ y &= t \\ z &= 2 \sin s\end{aligned}$$

where $0 \leq s \leq 2\pi$ and $-1 \leq t \leq 3$.

14. The closed triangular region in \mathbf{R}^3 with vertices $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 5)$.

Solution.

A parameterization of a plane can be written as :

$$\mathbf{x} = s\mathbf{a} + t\mathbf{b} + \mathbf{p}$$

where \mathbf{a} and \mathbf{b} are two vectors in the plane and \mathbf{p} is a point in the plane. To see why this is the case, suppose $\mathbf{x} = (x, y, z)$ is an arbitrary point in the plane and $\mathbf{p} = (x_0, y_0, z_0)$ is a known point. Then $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$ must be a linear combination of \mathbf{a} and \mathbf{b} . So, $\mathbf{x} - \mathbf{p} = s\mathbf{a} + t\mathbf{b}$.

We have four planes that are described by:

$$\mathbf{X}(s, t) = s(2, 0, -5) + t(0, 1, -5) + (0, 0, 5) = (2s, t, -5s - 5t + 5)$$

Since we are interested in the first octant of \mathbf{R}^3 all coordinates must be non-negative. So, $0 \leq 2s \leq 2$, that is $0 \leq s \leq 1$, $0 \leq t \leq 1$ and $-5s - 5t + 5 \geq 0$. In other words, $t \leq 1 - s$.

14. The hyperboloid $z^2 - x^2 - y^2 = 1$. (Hint: Use two maps to parametrize the surface)

Solution.

The equation of the hyperboloid as:

$$z = \pm \sqrt{1 + x^2 + y^2}$$

Therefore, the hyperboloid may be parameterized with two maps:

$$\begin{aligned}\mathbf{X}_1(s, t) &= (s, t, \sqrt{1 + x^2 + y^2}) \\ \mathbf{X}_2(s, t) &= (s, t, -\sqrt{1 + x^2 + y^2})\end{aligned}$$

16. This problem concerns the parameterized surface $\mathbf{X}(s, t) = (s^3, t^3, st)$.

(a) Find an equation of a plane tangent to this surface at the point $(1, -1, -1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (3s^2, 0, t) \\ \mathbf{T}_t &= (0, 3t^2, s)\end{aligned}$$

The standard normal vector \mathbf{N} is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3s^2 & 0 & t \\ 0 & 3t^2 & s \end{vmatrix} \\ &= -3t^3\mathbf{i} - 3s^3\mathbf{j} + 9s^2t^2\mathbf{k}\end{aligned}$$

We have $s_0 = 1, t_0 = -1$. So, $\mathbf{N}(1, -1) = (3, -3, 9)$. The equation of the tangent plane to the surface at $(1, -1, -1)$ is:

$$\begin{aligned}(3, -3, 9) \cdot (x - 1, y + 1, z + 1) &= 0 \\ (1, -1, 3) \cdot (x - 1, y + 1, z + 1) &= 0 \\ (x - 1) - (y + 1) + 3(z + 1) &= 0 \\ x - 1 - y - 1 + 3z + 3 &= 0 \\ x - y + 3z + 1 &= 0\end{aligned}$$

(b) Use a computer to graph this surface for $-1 \leq s \leq 1, -1 \leq t \leq 1$.

Solution.

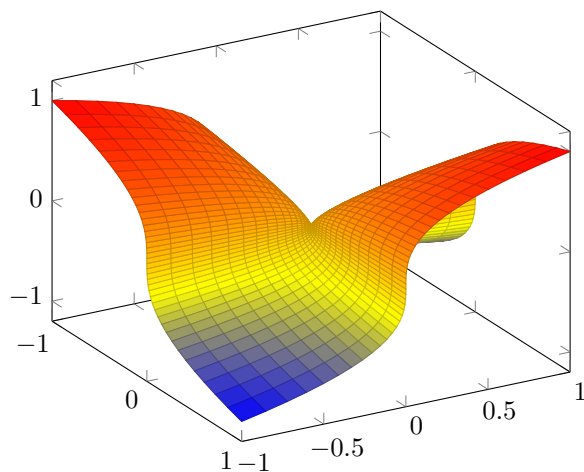


Figure. $\mathbf{X}(s, t) = (s^3, t^3, st)$

(c) Is the surface smooth?

Solution.

The normal vector $\mathbf{N} = 0$ at $(s_0, t_0) = (0, 0)$ that is at $(0, 0, 0)$. So, the surface fails to be smooth there.

17. The surface given parametrically by $\mathbf{X}(s, t) = (st, t, s^2)$ is known as **Whitney's umbrella**.

(a) Verify that this surface may also be described by the xyz -coordinate equation $y^2z = x^2$.

Solution.

Clearly, $y^2z = (t^2)(s^2) = (st)^2 = x^2$.

(b) Is \mathbf{X} smooth?

Solution.

We have:

$$\mathbf{T}_s = (t, 0, 2s)$$

$$\mathbf{T}_t = (s, 1, 0)$$

The standard normal vector is :

$$\begin{aligned} \mathbf{N} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 0 & 2s \\ s & 1 & 0 \end{vmatrix} \\ &= -2s\mathbf{i} + 2s^2\mathbf{j} + t\mathbf{k} \end{aligned}$$

The normal vector $\mathbf{N} = (0, 0, 0)$ at $(s, t) = (0, 0)$ that is at the point $(0, 0, 0)$. Hence, \mathbf{X} is not smooth at this point.

(c) Use a computer to graph this surface for $-2 \leq s \leq 2$, $-2 \leq t \leq 2$.

Solution.

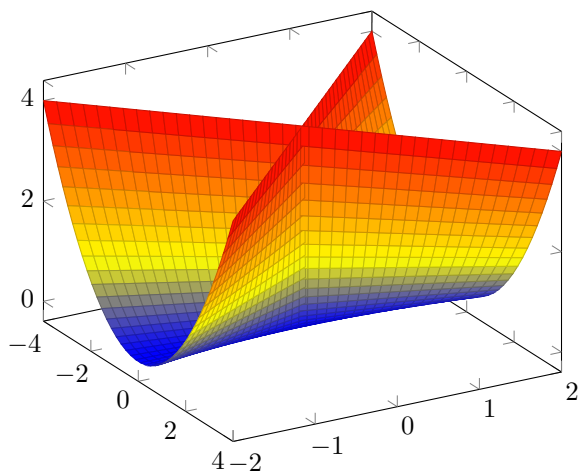


Figure. $\mathbf{X}(s, t) = (st, t, s^2)$

(d) Give an equation of the plane tangent to this surface at the point $(2, 1, 4)$.

Solution.

The standard normal vector at the point $\mathbf{X}(2, 1)$ is $(-4, 8, 1)$. Hence, the equation of the tangent plane to the surface at the point $\mathbf{X}(1, 2)$ is:

$$\begin{aligned} (-4, 8, 1) \cdot (x - 2, y - 1, z - 4) &= 0 \\ -4(x - 2) + 8(y - 1) + (z - 4) &= 0 \\ 4(x - 2) - 8(y - 1) - (z - 4) &= 0 \\ 4x - 8y - z &= -4 \end{aligned}$$

(e) Some points (x, y, z) of the surface do not correspond to a single parameter point (s, t) . Which ones? Explain how this relates to the graph?

Solution.

Let $\mathbf{X}(s_1, t_1) = \mathbf{X}(s_2, t_2)$. Then,

$$\begin{aligned} s_1 t_1 &= s_2 t_2 \\ t_1 &= t_2 \\ s_1^2 &= s_2^2 \end{aligned}$$

Thus, if $t_1 = t_2 = 0$ and $s_1 = \pm s_2$ we get the same image that is $\mathbf{X}(s, 0) = \mathbf{X}(-s, 0) = (0, 0, s^2)$. Thus, the positive z -axis does not correspond to a single point.

18. Let S be the surface defined as the graph of a function $f(x, y)$ of class C^1 . Then, example 4 shows that S is also a parametrized surface. Show that formula (5) for the tangent plane to S at $(a, b, f(a, b))$ agrees with formula (4) in section 2.3.

Solution.

We have:

$$\begin{aligned} \mathbf{T}_s &= (1, 0, f_s(s, t)) \\ \mathbf{T}_t &= (0, 1, f_t(s, t)) \end{aligned}$$

The standard normal vector \mathbf{N} at the point (s, t) is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_s(s, t) \\ 0 & 1 & f_t(s, t) \end{vmatrix} \\ &= -f_s(s, t)\mathbf{i} - f_t(s, t)\mathbf{j} + \mathbf{k} \end{aligned}$$

So, the equation of the tangent plane to the surface at $\mathbf{X}(s, t)$ is:

$$\begin{aligned} (-f_s, -f_t, 1) \cdot (x - s, y - t, z - f(s, t)) &= 0 \\ z - f(s, t) &= \frac{\partial f}{\partial s}(x - s) + \frac{\partial f}{\partial t}(y - t) \end{aligned}$$

19. (a) Write a formula for the tangent plane to the surface described by the equation $y = g(x, z)$.

Solution.

We have:

$$\mathbf{X}(s, t) = (s, g(s, t), t)$$

So,

$$\mathbf{T}_s = (1, g_s(s, t), 0)$$

$$\mathbf{T}_t = (0, g_t(s, t), 1)$$

The standard normal vector \mathbf{N} is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & g_s(s, t) & 0 \\ 0 & g_t(s, t) & 1 \end{vmatrix} \\ &= g_s(s, t)\mathbf{i} - \mathbf{j} + g_t(s, t)\mathbf{k} \end{aligned}$$

So, the equation of the tangent plane to the surface at the point $(x_0, g(x_0, z_0), z_0)$ is:

$$\begin{aligned} (g_s(s, t), -1, g_t(s, t)) \cdot (x - x_0, y - g(x_0, z_0), z - z_0) &= 0 \\ g_s(s, t)(x - x_0) - (y - g(x_0, z_0)) + g_t(z - z_0) &= 0 \\ y &= g(x_0, z_0) + g_s(x - x_0) + g_t(z - z_0) \end{aligned}$$

(b) Repeat part (a) for a surface described by the equation $x = h(y, z)$.

Solution.

The equation of the tangent plane to the surface at the point $(h(y_0, z_0), y_0, z_0)$ is:

$$x = h(y_0, z_0) + h_y(y - y_0) + h_z(z - z_0)$$

20. Suppose that $\mathbf{X} : D \rightarrow \mathbf{R}^3$ is a parameterized surface that is smooth at $\mathbf{X}(s_0, t_0)$. Show how the definition of the derivative $D\mathbf{X}(s_0, t_0)$ can be used to give vector parametric equations for the plane tangent to $S = \mathbf{X}(D)$ at the point $\mathbf{X}(s_0, t_0)$.

Solution.

Let $\mathbf{X}(s, t) = (x(s, t), y(s, t), z(s, t))$. The derivative $D\mathbf{X}(s, t)$ is:

$$\begin{aligned} D\mathbf{X}(s, t) &= \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} \\ &= [\mathbf{T}_s \quad \mathbf{T}_t] \end{aligned}$$

Hence, the equation of the tangent plane to surface $\mathbf{X}(s, t)$ at the point (s_0, t_0) is:

$$\begin{aligned} \mathbf{x}(s, t) &= \mathbf{X}(s_0, t_0) + D\mathbf{X}(s_0, t_0) \cdot \begin{bmatrix} s - s_0 \\ t - t_0 \end{bmatrix} \\ &= \mathbf{X}(s_0, t_0) + \mathbf{T}_s(s_0, t_0)(s - s_0) + \mathbf{T}_t(s_0, t_0)(t - t_0) \end{aligned}$$

Surface Integrals

Scalar Surface Integrals

Definition. Let $\mathbf{X} : D \rightarrow \mathbf{R}^3$ be a smooth parametrized surface whose domain $D \subset \mathbf{R}^2$ is a bounded region. Let f be a continuous function whose domain includes $S = \mathbf{X}(D)$. Then, the scalar surface integral of f along \mathbf{X} is:

$$\begin{aligned}\int \int_{\mathbf{X}} f \cdot dS &= \int \int_D f(\mathbf{X}(s, t)) \|\mathbf{T}_s \times \mathbf{T}_t\| ds dt \\ &= \int \int_D f(\mathbf{X}(s, t)) \|\mathbf{N}(s, t)\| ds dt\end{aligned}$$

Any rectangle with sides ds, dt and area $ds \cdot dt$ in D will be mapped to a parallelogram. The area of the parallelogram is $\|\mathbf{T}_s \times \mathbf{T}_t\| ds dt$. $\|\mathbf{T}_s \times \mathbf{T}_t\|$ is the scaling factor.

If f is identically 1 on all of $\mathbf{X}(D)$ then:

$$\int \int_{\mathbf{X}} f \cdot dS = \int \int_D 1 \|\mathbf{T}_s \times \mathbf{T}_t\| ds dt = \text{Surface area of } \mathbf{X}(D)$$

The scalar surface integral is thus a generalization of the integral we use to calculate the surface area. We can think of $\int \int_{\mathbf{X}} f \cdot dS$ as a limit of the weighted sum of the surface area pieces, the weightings given by f . If f represents the mass or electrical charge density, then $\int \int_{\mathbf{X}} f \cdot dS$ yields the total mass or the total charge on $\mathbf{X}(D)$.

For computational purposes, recall that if we write the components of \mathbf{X} as:

$$\mathbf{X}(s, t) = (x(s, t), y(s, t), z(s, t))$$

then

$$\begin{aligned}\mathbf{N}(s, t) &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} + \mathbf{k} \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \\ &= \frac{\partial(y, z)}{\partial(s, t)} \mathbf{i} - \frac{\partial(x, z)}{\partial(s, t)} \mathbf{j} + \frac{\partial(x, y)}{\partial(s, t)} \mathbf{k}\end{aligned}$$

So, we obtain:

$$\int \int_{\mathbf{X}} f \cdot dS = \int \int_D f(x(s, t), y(s, t), z(s, t)) \sqrt{\left(\frac{\partial(y, z)}{\partial(s, t)}\right)^2 + \left(\frac{\partial(x, z)}{\partial(s, t)}\right)^2 + \left(\frac{\partial(x, y)}{\partial(s, t)}\right)^2} ds dt$$

If the surface S is given by a graph of $z = g(x, y)$ where g is of class C^1 on some region D in \mathbf{R}^2 , then S is parameterized by $\mathbf{X}(x, y) = (x, y, g(x, y))$ with $(x, y) \in D$. Then, from example 13 in section 7.1,

$$\mathbf{N}(x, y) = -g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k}$$

so that:

$$\int \int_{\mathbf{X}} f \cdot dS = \int \int_D f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} dx dy$$

Vector Surface Integrals

We can develop a means to integrate vector fields along surfaces beginning with the definition.

Definition. Let $\mathbf{X} : D \rightarrow \mathbf{R}^3$ be a smooth parameterized surface, where D is a bounded region in the \mathbf{R}^2 plane, and let $\mathbf{F}(x, y, z)$ be a continuous vector field whose domain includes $S = \mathbf{X}(D)$. Then, the vector surface integral of \mathbf{F} along \mathbf{X} is:

$$\int \int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int \int_D \mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{N}(s, t) ds dt$$

where $\mathbf{N}(s, t) = \mathbf{T}_s \times \mathbf{T}_t$.

As with line integrals, we should be careful about the notation for surface integrals. In the vector surface integral $\int \int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, the differential term is considered to be a vector quantity, whereas in the scalar surface integral $\int \int_{\mathbf{X}} f \cdot dS$, the differential term is scalar quantity - the differential of the surface area.

Example. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z - 2y)\mathbf{k}$. We evaluate $\int \int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$ where \mathbf{X} is the helicoid

$$\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$$

where $0 \leq s \leq 1, 0 \leq t \leq 2\pi$.

Solution.

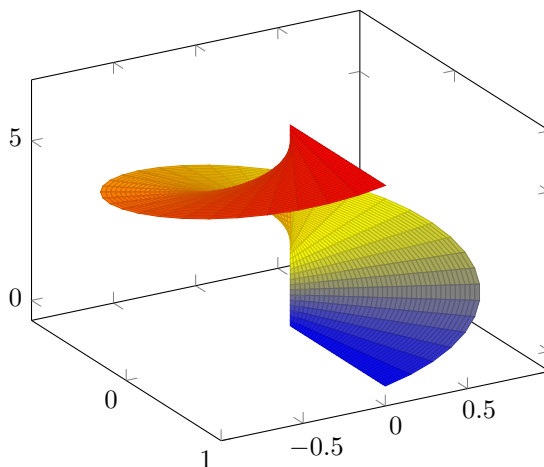


Figure. $\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$

We have:

$$\begin{aligned} \mathbf{T}_s &= (\cos t, \sin t, 0) \\ \mathbf{T}_t &= (-s \sin t, s \cos t, 1) \end{aligned}$$

The standard normal vector \mathbf{N} is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s \sin t & s \cos t & 1 \end{vmatrix} \\ &= \sin t \mathbf{i} - \cos t \mathbf{j} + s \mathbf{k} \end{aligned}$$

Hence:

$$\begin{aligned}
\int \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 (s \cos t, s \sin t, t - 2s \sin t) \cdot (\sin t, -\cos t, s) ds dt \\
&= \int_0^{2\pi} \int_0^1 (s \sin t \cos t - s \sin t \cos t + st - 2s^2 \sin t) ds dt \\
&= \int_0^{2\pi} \int_0^1 (st - 2s^2 \sin t) ds dt \\
&= \int_0^{2\pi} \left[\frac{s^2}{2} t - \frac{2s^3}{3} \sin t \right]_0^1 dt \\
&= \int_0^{2\pi} \left(\frac{t}{2} - \frac{2}{3} \sin t \right) dt \\
&= \left[\frac{t^2}{4} + \frac{2}{3} \cos t \right]_0^{2\pi} \\
&= \pi^2
\end{aligned}$$

Further Interpretations

As is the case for vector and scalar line integrals, there is a connection between vector and scalar surface integrals. Suppose $\mathbf{X} : D \rightarrow \mathbf{R}^3$ is a smooth parametrized surface and \mathbf{F} is continuous on $S = \mathbf{X}(D)$. Let $\mathbf{N}(s, t) = \mathbf{T}_s \times \mathbf{T}_t$ be the usual normal vector and let:

$$\mathbf{n}(s, t) = \frac{\mathbf{N}(s, t)}{\|\mathbf{N}(s, t)\|}$$

That is, \mathbf{n} is the unit vector pointing in the same direction as \mathbf{N} . In particular,

$$\mathbf{N}(s, t) = \|\mathbf{N}(s, t)\| \mathbf{n}(s, t)$$

Plugging this into the definition of the vector surface integral, we get:

$$\begin{aligned}
\int \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{S} &= \int \int_D \mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{N}(s, t) ds \cdot dt \\
&= \int \int_D \mathbf{F}(\mathbf{X}(D)) \cdot \|\mathbf{N}(s, t)\| \mathbf{n}(s, t) ds \cdot dt \\
&= \int \int_D (\mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{n}(s, t)) \|\mathbf{N}(s, t)\| ds \cdot dt \\
&= \int \int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{n}) dS
\end{aligned}$$

Since \mathbf{n} is a unit vector, the quantity $\mathbf{F} \cdot \mathbf{n}$ is precisely the component of the vector field \mathbf{F} in the direction of \mathbf{n} . In other words, the vector surface integral of \mathbf{F} along \mathbf{X} is the scalar surface integral of the component of \mathbf{F} normal to $S = \mathbf{X}(D)$. Recall that, the vector line integral of \mathbf{F} along a path \mathbf{x} is the scalar line integral of the component of \mathbf{F} tangent to the image curve. To summarize, we have the following results:

$$\text{Line Integrals : } \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^t (\mathbf{F} \cdot \mathbf{T}) ds \quad (1)$$

$$\text{Surface Integrals : } \int \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{S} = \int \int_D (\mathbf{F} \cdot \mathbf{n}) \cdot ds \cdot dt \quad (2)$$

The vector line integral $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ in equation (1) is called the flow-integral of \mathbf{F} along \mathbf{x} . The reason for this is the following.

Suppose \mathbf{F} represents the velocity vector field of a fluid. Consider the amount of fluid moved tangentially along a small segment of the path \mathbf{x} during a brief time interval $\Delta\tau$.

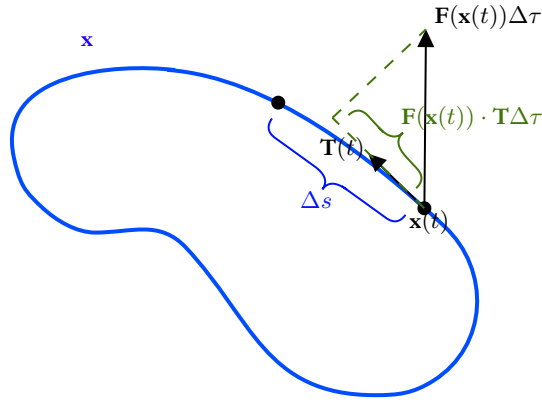


Figure. The amount of fluid transported tangentially along a segment of the closed path \mathbf{x} is approximately $(\mathbf{F}(\mathbf{x}(t))\Delta\tau \cdot \mathbf{T})\Delta s$

Since $\mathbf{F} \cdot \mathbf{T}$ gives the tangential component of the velocity vector \mathbf{F} , the rate of flow at the point $\mathbf{x}(t)$ is $\mathbf{F} \cdot \mathbf{T}$. So, the amount of fluid transported tangentially in $\Delta\tau$ time through the point $\mathbf{x}(t)$ is Rate of flow \times time = $(\mathbf{F}\Delta\tau) \cdot \mathbf{T}$. A line segment of the path $\mathbf{x}(t)$ of length Δs can be thought to be made up of Δs such points. So, the total amount of fluid transported tangentially in $\Delta\tau$ time along a segment of length Δs equals $(\mathbf{F}\Delta\tau) \cdot \mathbf{T}\Delta s$.

$$\text{Amount of fluid moved} \approx (\mathbf{F}(\mathbf{x}(t))\Delta\tau \cdot \mathbf{T})\Delta s$$

If we divide the above expression by time $\Delta\tau$, we get the average rate of transport of the fluid along the segment :

$$\frac{\Delta L}{\Delta\tau} \approx \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T} \cdot ds$$

$$\text{Instantaneous rate of fluid flow} = \lim_{\Delta\tau \rightarrow 0} \frac{\Delta L}{\Delta\tau} = \frac{dL}{d\tau} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T} \cdot ds$$

If \mathbf{x} is a closed path, the flow-integral is also called circulation.

Now, let's try to interpret the vector surface integral in equation (2).

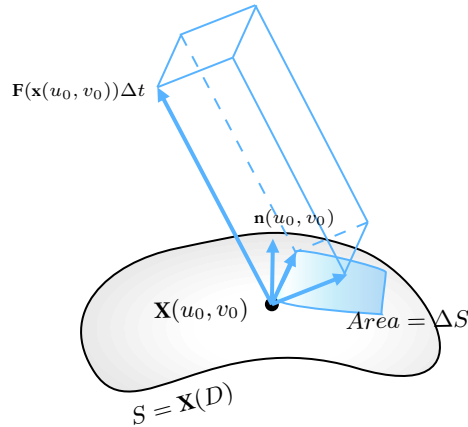


Figure. The amount of fluid transported across a small piece of S during a brief time interval Δt may be approximated by the volume of the parallelopiped.

Consider a small piece of S , having the area ΔS and the amount of fluid transported across it during a brief time interval Δt . This amount is the volume determined by \mathbf{F} during Δt . The above figure suggests that this volume can be approximated by the volume of an appropriate parallelopiped.

The height of the parallelopiped is the normal component of $\mathbf{F}\Delta t$, $\mathbf{F}\Delta t \cdot \mathbf{n}(u_0, v_0)$ and the area of the base is ΔS . Hence:

$$\begin{aligned}
\text{Amount of fluid displaced} &\approx \text{volume of parallelopiped} \\
&= (\text{height})(\text{area of base}) \\
&= \mathbf{F}(\mathbf{X}(u_0, v_0)) \Delta t \cdot \mathbf{n}(u_0, v_0) \Delta S
\end{aligned} \tag{3}$$

We obtain the average rate of transport across the surface piece during the time interval Δt by dividing (3) by Δt :

$$\text{Average rate of transport} \approx \mathbf{F}(\mathbf{X}(u_0, v_0)) \cdot \mathbf{n}(u_0, v_0) \Delta S \tag{4}$$

Now, we break up the entire surface $S = \mathbf{X}(D)$ into infinitely many such small partitions ΔS_{ij} and sum the corresponding contributions to the rate of transport in the form given (4). If we let all the pieces shrink, then, in the limit as all $\Delta S \rightarrow 0$, we have that the total average rate of transport of the fluid during Δt is approximately :

$$\frac{\Delta M}{\Delta t} \approx \int \int_{\mathbf{X}} (\mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{n}) dS \tag{5}$$

Passing to the limits as $\Delta t \rightarrow 0$, the instantaneous rate of fluid transport across \mathbf{X} can be defined as the flow integral:

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta M}{\Delta t} = \frac{dM}{dt} = \int \int_{\mathbf{X}} (\mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{n}) dS \tag{6}$$

Reparametrization of surfaces

Definition. Let $\mathbf{X} : D_1 \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3$ and $\mathbf{Y} : D_2 \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3$ be parametrized surfaces. \mathbf{Y} is said to be a reparametrization of \mathbf{X} , if there exists a one-to-one and onto function $\mathbf{H} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ such that $\mathbf{Y} = \mathbf{X}(\mathbf{H}(s, t))$.

Example. Suppose \mathbf{X} is a smooth parametrized surface. Let $\mathbf{Y}(s, t) = \mathbf{X}(u, v)$, where $u = t, v = s$. That is, $\mathbf{Y} = \mathbf{X}(\mathbf{H})$. Then, \mathbf{Y} is a smooth parametrization that appears to accomplish little. However, if we let $\mathbf{N}_{\mathbf{Y}}$ denote the usual normal vector $\mathbf{T}_s \times \mathbf{T}_t = \frac{\partial \mathbf{Y}}{\partial s} \times \frac{\partial \mathbf{Y}}{\partial t}$, then we have:

$$\frac{\partial \mathbf{Y}}{\partial s} = \frac{\partial \mathbf{X}}{\partial v} \quad \text{and} \quad \frac{\partial \mathbf{Y}}{\partial t} = \frac{\partial \mathbf{X}}{\partial u}$$

so that:

$$\begin{aligned}
\mathbf{N}_{\mathbf{Y}} &= \frac{\partial \mathbf{Y}}{\partial s} \times \frac{\partial \mathbf{Y}}{\partial t} \\
&= \frac{\partial \mathbf{X}}{\partial v} \times \frac{\partial \mathbf{X}}{\partial u} \\
&= -\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} \\
&= -\mathbf{N}_{\mathbf{X}}
\end{aligned}$$

The parametrized surface \mathbf{Y} is the same as \mathbf{X} , except that the standard normal vector arising from \mathbf{Y} points in the opposite direction to the one arising from \mathbf{X} .

The calculation in the above example thus generalizes. Suppose \mathbf{X} is a smooth parametrized surface and \mathbf{Y} is a smooth reparametrization of \mathbf{X} via \mathbf{H} , that is:

$$\mathbf{Y}(s, t) = \mathbf{X}(u, v) = \mathbf{X}(\mathbf{H}(s, t))$$

By the chain rule:

$$D\mathbf{Y}(s, t) = D\mathbf{X}(u, v) \cdot D\mathbf{H}(s, t)$$

$$\begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} u_s & u_t \\ v_s & v_t \end{bmatrix}$$

From the above, we see that:

$$\mathbf{T}_s = u_s \mathbf{T}_u + v_s \mathbf{T}_v$$

$$\mathbf{T}_t = u_t \mathbf{T}_u + v_t \mathbf{T}_v$$

So,

$$\begin{aligned} \mathbf{T}_s \times \mathbf{T}_t &= (u_s \mathbf{T}_u + v_s \mathbf{T}_v) \times (u_t \mathbf{T}_u + v_t \mathbf{T}_v) \\ &= u_s u_t \mathbf{T}_u \times \mathbf{T}_u + u_s v_t \mathbf{T}_u \times \mathbf{T}_v + v_s u_t \mathbf{T}_v \times \mathbf{T}_u + v_s v_t \mathbf{T}_v \times \mathbf{T}_v \\ &= (u_s v_t - u_t v_s) \mathbf{T}_u \times \mathbf{T}_v \\ \mathbf{N}_Y &= \frac{\partial(u, v)}{\partial(s, t)} \mathbf{N}_X \end{aligned}$$

Thus, \mathbf{N}_Y is always a scalar multiple of \mathbf{N}_X . In addition, since \mathbf{H} is invertible and both \mathbf{H} and \mathbf{H}^{-1} are of class C^1 , it follows that the jacobian (determinant) is always positive or negative. Hence, the standard normal vector \mathbf{N}_Y either always points in the same direction as \mathbf{N}_X or else always points in the opposite direction. Under these assumptions, we say that both \mathbf{H} and \mathbf{Y} are **orientation-preserving** if the Jacobian $\frac{\partial(u, v)}{\partial(s, t)}$ is positive, **orientation-reversing** if $\partial(u, v)/\partial(s, t)$ is negative.

Theorem. Let $\mathbf{X} : D_1 \rightarrow \mathbf{R}^3$ be a smooth parametrized surface and f any continuous function whose domain includes $\mathbf{X}(D_1)$. If $\mathbf{Y} : D_2 \rightarrow \mathbf{R}^3$ is any smooth reparametrization of \mathbf{X} , then:

$$\int \int_Y f \cdot dS = \int \int_X f \cdot dS$$

Proof.

We have: