Stoke's and Gauss's Theorem

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Exercise Problems.

1. Let $X : \mathbf{R}^2 \to \mathbf{R}^3$ be the parametrized surface given by

$$X(s,t) = (s^2 - t^2, s + t, s^2 + 3t)$$

(a) Determine a normal vector to this surface at the point

$$(3,1,1) = \mathbf{X}(2,-1)$$

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (2s, 1, 2s)\\ \mathbf{T}_t &= (-2t, 1, 3)\end{aligned}$$

So, the standard normal vector at the point $\mathbf{X}(2,-1)$ is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2s & 1 & 2s\\ -2t & 1 & 3 \end{vmatrix}\\ &= \mathbf{i}(3-2s) - \mathbf{j}(6s+4st) + \mathbf{k}(2s+2t)\\ &= \mathbf{i}(3-4) - \mathbf{j}(12+4(2)(-1)) + \mathbf{k}(4-2)\\ &= -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\end{aligned}$$

(b) Find an equation for the plane tangent to this surface at the point $(3,1,1)$.

Solution.

The tangent plane to this surface at the point $(3,1,1)$ is given by:

$$\begin{aligned}\mathbf{N}\cdot(\mathbf{x} - (3,1,1)) &= 0\\ (-1,-4,2)\cdot((x,y,z) - (3,1,1)) &= 0\\ -(x-3) - 4(y-1) + 2(z-1) &= 0\end{aligned}$$

2. Find an equation for the plane tangent to the torus

$$\mathbf{X}(s,t) = ((5 + 2\cos t)\cos s, (5 + 2\cos t)\sin s, 2\sin t)$$

at the point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (-(5+2\cos t)\sin s,(5+2\cos t)\cos s,0)\\ \mathbf{T}_t &= (-2\sin t\cos s,-2\sin t\sin s,2\cos t)\end{aligned}$$

The standard normal vector is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s\times\mathbf{T}_t\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -(5+2\cos t)\sin s & (5+2\cos t)\cos s & 0\\ -2\sin t\cos s & -2\sin t\sin s & 2\cos t\end{vmatrix}\\ &= \mathbf{i}(2(5+2\cos t)\cos s\cos t))+\mathbf{j}(2(5+2\cos t)\sin s\cos t)\\ &+\mathbf{k}(2\sin s\sin t(5+2\cos t)+2(5+2\cos t)\sin t\cos^2 s)\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i}+\sin s\cos t\mathbf{j}+(\sin^2 s+\cos^2 s)\sin t\mathbf{k})\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i}+\sin s\cos t\mathbf{j}+\sin t\mathbf{k})\end{aligned}$$

The point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1)=((5+2\cos t)\cos s,(5+2\cos t)\sin s,2\sin t)$ yields $\sin t=1/2$, so $t_0=\pi/6$ or $t_0=5\pi/6$. Since $2\cos t<0$, $t_0=5\pi/6$. Then, we can see that :

$$\frac{5-\sqrt{3}}{\sqrt{2}}=(5-2\cdot\frac{\sqrt{3}}{2})\sin s$$

So, $s_0=\pi/4$.

Consequently, the equation of the tangent plane at $\mathbf{X}(\pi/4,5\pi/6)$ is:

$$\begin{aligned}\mathbf{N}\cdot(\mathbf{x}-\mathbf{X}(s_0,t_0)) &= 0\\ 2(5-\sqrt{3})(-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{i}-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{j}+\frac{1}{2}\mathbf{k})((x,y,z)-(\frac{5-\sqrt{3}}{\sqrt{2}},\frac{5-\sqrt{3}}{\sqrt{2}},1)) &= 0\\ -\frac{\sqrt{3}}{\sqrt{2}}(x-(5-\sqrt{3})/\sqrt{2})-\frac{\sqrt{3}}{\sqrt{2}}(y-(5-\sqrt{3})/\sqrt{2})+(z-1) &= 0\\ -\sqrt{3}(x-(5-\sqrt{3})/\sqrt{2})-\sqrt{3}(y-(5-\sqrt{3})/\sqrt{2})+\sqrt{2}(z-1) &= 0\\ -\sqrt{3}x-\sqrt{3}y+\sqrt{2}z &= -2\sqrt{3}(5-\sqrt{3})/\sqrt{2}+\sqrt{2}\\ &= -\sqrt{6}(5-\sqrt{3})+\sqrt{2}\\ &= -5\sqrt{6}+3\sqrt{2}+\sqrt{2}\\ \sqrt{3}x+\sqrt{3}y-\sqrt{2}z &= 5\sqrt{6}-4\sqrt{2}\end{aligned}$$

3. Find an equation of the plane tangent to the surface

$$x=e^s\quad y=t^2e^{2s}\quad z=2e^{-s}+t$$

at the point $(1,4,0)$.

Solution.

We have:

$$
\begin{aligned}
\mathbf{T}_s &= (e^s, 2t^2e^{2s}, -2e^{-s}) \\
\mathbf{T}_t &= (0, 2te^{2s}, 1)
\end{aligned}
$$

The standard normal vector is:

$$
\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^s & 2t^2e^{2s} & -2e^{-s} \\ 0 & 2te^{2s} & 1 \end{vmatrix} \\
&= \mathbf{i}(2t^2e^{2s} + 4te^s) - \mathbf{j}(e^s) + \mathbf{k}(2te^{3s}) \\
&= e^s((2t^2e^s + 4t)\mathbf{i} - \mathbf{j} + (2te^{2s})\mathbf{k})
\end{aligned}
$$

Since $e^s = 1$, $s = 0$. Also, as $4 = t^2 \cdot 1$, we have $t = \pm 2$. Moreover, $0 = 2 + t$, so $t = -2$. So, $\mathbf{N}(0, -2)$ is:

$$
\mathbf{N}(0, -2) = -\mathbf{j} - 4\mathbf{k}
$$

The equation of the tangent plane at $\mathbf{X}(0, -2)$ is:

$$
\begin{aligned}
\mathbf{N} \cdot (x - 1, y - 4, z) &= 0 \\
-(y - 4) - 4z &= 0 \\
y + 4z &= 4
\end{aligned}
$$

4. Let $\mathbf{X}(s, t) = (s^2 \cos t, s^2 \sin t, s)$, $-3 \le s \le 3, 0 \le t \le 2\pi$.

(a) Find a normal vector at $(s, t) = (-1, 0)$.

Solution.

We have:

$$
\begin{aligned}
\mathbf{T}_s &= (2s \cos t, 2s \sin t, 1) \\
\mathbf{T}_t &= (-s^2 \sin t, s^2 \cos t, 0)
\end{aligned}
$$

The standard normal vector $\mathbf{N}$ is:

$$
\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s \cos t & 2s \sin t & 1 \\ -s^2 \sin t & s^2 \cos t & 0 \end{vmatrix} \\
&= \mathbf{i}(-s^2 \cos t) - \mathbf{j}(s^2 \sin t) + \mathbf{k}(2s^3 \cos^2 t + 2s^3 \sin^2 t) \\
&= -s^2 \cos t\mathbf{i} - s^2 \sin t\mathbf{j} + 2s^3\mathbf{k}
\end{aligned}
$$

The vector at $(s, t) = (-1, 0)$ is:

$$
\mathbf{N}(-1, 0) = -\mathbf{i} - 2\mathbf{k}
$$

Hence, the equation of the tangent plane at $\mathbf{X}(-1, 0) = (1, 0, -1)$ is:

$$\begin{aligned}(-\mathbf{i}-2\mathbf{k})\cdot((x,y,z)-(1,0,-1)) &= 0\\ -(x-1)-2(z+1) &= 0\\ x-1+2z+2 &= 0\\ x+2z+1 &= 0\end{aligned}$$

(b) Find an equation for the image of $\mathbf{X}$ in the form $F(x,y,z)=0$.

Solution.

Let $x=s^2\cos t$, $y=s^2\sin t$. Then, $x^2+y^2=s^4(\cos^2 t+\sin^2 t)=s^4=z^4$. So, $F(x,y,z)=x^2+y^2-z^4=0$.

5. Consider the parameterized surface $\mathbf{X}(s,t)=(s,s^2+t,t^2)$.

(a) Graph the surface for $-2\le s\le 2$, $-2\le t\le 2$.

Solution.

The s-coordinate curve at $t=0$ is:

$$\begin{aligned}x&=s\\ y&=s^2\\ z&=0\end{aligned}$$

This is the parabolic curve $y=x^2$ in the xy-plane.

The s-coordinate curve at $t=t_0$ is:

$$\begin{aligned}x&=s\\ y&=s^2+t_0\\ z&=t_0^2\end{aligned}$$

Thus, we get parabolas parallel to the xy-plane.

t	Curve	Center	z-plane
$t_0=-2$	$y+2=x^2$	$(x,y)=(0,-2)$	$z=4$
$t_0=-1$	$y+1=x^2$	$(x,y)=(0,-1)$	$z=1$
$t_0=0$	$y=x^2$	$(x,y)=(0,0)$	$z=0$
$t_0=1$	$y-1=x^2$	$(x,y)=(0,1)$	$z=1$
$t_0=2$	$y-2=x^2$	$(x,y)=(0,2)$	$z=4$

The $t-$coordinate curve at $s=0$ is:

$$\begin{aligned}x&=0\\ y&=t\\ z&=t^2\end{aligned}$$

These are parabolas parallel to the yz-plane.

t	Curve	Center	x-plane
$s_0=-2$	$z=(y-4)^2$	$(y,z)=(2,0)$	$x=-2$
$s_0=-1$	$z=(y-1)^2$	$(y,z)=(1,0)$	$x=-1$
$s_0=0$	$z=y^2$	$(y,z)=(0,0)$	$x=0$
$s_0=1$	$z=(y-1)^2$	$(y,z)=(1,0)$	$x=1$
$s_0=2$	$z=(y-4)^2$	$(y,z)=(2,0)$	$x=2$

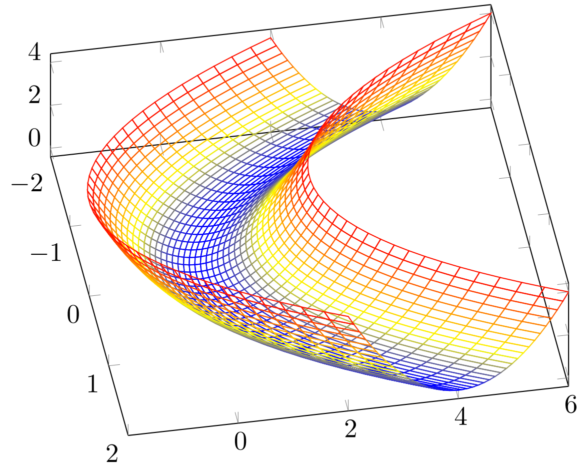

Figure. $\mathbf{X}(s,t) = (s, s^2 + t, t^2)$.

(b) Is the surface smooth?

Solution. The surface is smooth.

(c) Find an equation for the tangent plane at the point $(1, 0, 1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (1, 2s, 0)\\ \mathbf{T}_t &= (0, 1, 2t)\end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2s & 0\\ 0 & 1 & 2t \end{vmatrix}\\ &= 4st\mathbf{i} - 2t\mathbf{j} + \mathbf{k}\end{aligned}$$

The point $(1, 0, 1)$ is $(s_0 = 1, t_0 = -1)$.

$$\mathbf{N}(1, -1) = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

The equation of the tangent plane at $(1, -1)$ is:

$$\begin{aligned}(-4, 2, 1) \cdot (x - 1, y, z - 1) &= 0\\ -4(x - 1) + 2y + (z - 1) &= 0\\ 4(x - 1) - 2y - (z - 1) &= 0\\ 4x - 4 - 2y - z + 1 &= 0\\ 4x - 2y - z &= 3\end{aligned}$$

6. Describe the parameterized surface of exercise problem 1 by an equation of the form $z = f(x, y)$.

Solution.

The parametric surface $\mathbf{X}(s,t)$ is:

$$X(s,t) = (s^2 - t^2, s + t, s^2 + 3t)$$

In exercise (1), we see that $x = (s-t)(s+t) = y(s+t)$ so $s+t = x/y$ and $y = s-t$. This allows us to solve simultaneously for s and t. $2s = x/y + y$ and $2t = x/y - y$. This means that $z = s^2 + 3t$ can be written as $z = \frac{1}{4}\left(\frac{x}{y} + y\right)^2 + \frac{3}{2}\left(\frac{x}{y} - y\right)$.

7. Let S be the surface parameterized by:

$$\begin{aligned} x &= s\cos t \\ y &= s\sin t \\ z &= s^2 \end{aligned}$$

where $s \geq 0, 0 \leq t \leq 2\pi$.

(a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$.

Solution.

The surface S is $x^2 + y^2 = z$. This is a paraboloid. It is smooth at all points.

We have:

$$\begin{aligned} \mathbf{T}_s &= (\cos t, \sin t, 2s) \\ \mathbf{T}_t &= (-s\sin t, s\cos t, 0) \end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 2s \\ -s\sin t & s\cos t & 0 \end{vmatrix} \\ &= (-2s^2\cos t)\mathbf{i} - (2s^2\sin t)\mathbf{j} + (s\cos^2 + s\sin^2 t)\mathbf{k} \\ &= (-2s^2\cos t)\mathbf{i} - (2s^2\sin t)\mathbf{j} + s\mathbf{k} \end{aligned}$$

At $s = 2, t = \pi/6$,

$$\mathbf{N}(2, \pi/6) = -4\sqrt{3}\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

The equation of the tangent plane at $\mathbf{X}(2, \pi/6)$ is:

$$\begin{aligned} (-4\sqrt{3}, -4, 2) \cdot (x - 1, y - \sqrt{3}, z - 4) &= 0 \\ 4\sqrt{3}(x-1) + 4(y - \sqrt{3}) - 2(z-4) &= 0 \\ 4\sqrt{3}x - 4\sqrt{3} + 4y - 4\sqrt{3} - 2z + 8 &= 0 \\ 4\sqrt{3}x + 4y - 2z &= 8(\sqrt{3} - 1) \\ 2\sqrt{3}x + 2y - z &= 4(\sqrt{3} - 1) \end{aligned}$$

(b) Sketch the graph of S. Can you recognize S as a familiar surface?

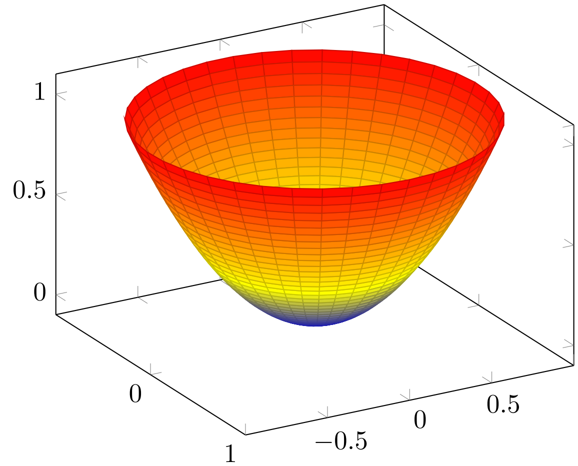

Figure. $\mathbf{X}(s,t) = (s\cos t, s\sin t, s^2)$

(c) Describe S by an equation of the form $z = f(x,y)$.

Solution. Again $z = x^2 + y^2$.

(d) Using your answer in part(c), discuss whether S has a tangent plane at every point.

Solution.

S has a tangent plane at every point and is smooth. Part (a) takes care of every point except the origin. At the origin $\mathbf{N} = (0,0,0)$. But, we easily see, that the tangent plane at the origin is the horizontal plane $z = 0$. Thus, smoothness as defined in the text, depends on both the parameterization and the geometry of the underlying surface.

8. Verify that the image of the parametrized surface

$$\mathbf{X}(s,t) = (2\sin s\cos t, 3\sin s\sin t, \cos s)$$

$0 \le s \le \pi$ and $0 \le t \le 2\pi$ is an ellipsoid.

Solution.

We can easily write:

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$$

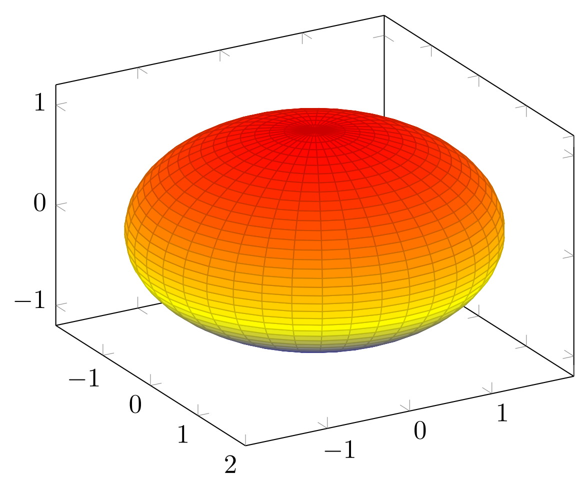

Figure. $\mathbf{X}(s,t) = (2\sin s\cos t, 3\sin s\sin t, \cos s)$

9. Verify that, for the torus of example 5, the s-coordinate curve, when $t = t_0$ is a circle of radius $a + b\cos t_0$.

Solution.

The parametric equations of the Torus in example 5 were:

$$
\begin{aligned}
x &= (a + b\cos t)\cos s \\
y &= (a + b\cos t)\sin s \\
z &= b\sin t
\end{aligned}
$$

The s-coordinate curve at $t = t_0$ is:

$$
\begin{aligned}
x &= (a + b\cos t_0)\cos s \\
y &= (a + b\cos t_0)\sin s \\
z &= b\sin t_0
\end{aligned}
$$

These are circles of radius $a + b\cos t_0$ in the plane $z = b\sin t_0$.

And they satisfy $x^2 + y^2 = (a + b\cos t_0)^2$.

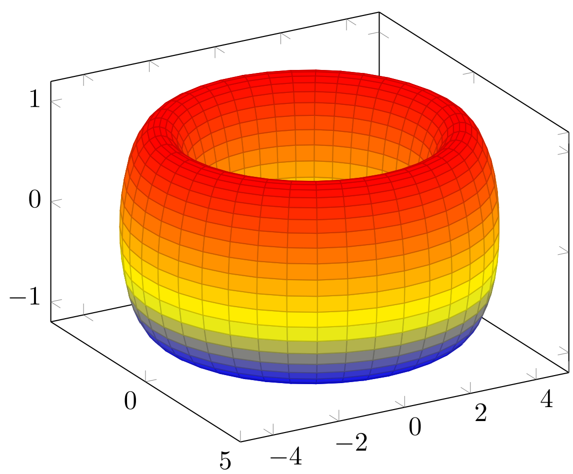

Figure. Torus $\mathbf{X}(s,t) = ((a + b\cos t)\cos s, (a + b\cos t)\sin s, b\sin t)$

9. The surface in $\mathbf{R}^3$ parametrized by:

$$\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$$

where $r \geq 0$ and $-\infty < \theta < \infty$ is called a helicoid.

(a) Describe the r-coordinate curve when $\theta = \pi/3$. Give a general description of the r-coordinate curves.

Solution.

The r-coordinate curve when $\theta = \pi/3$ is:

$$
\begin{aligned}
x &= r/2 \\
y &= \sqrt{3}r/2 \\
z &= \pi/3
\end{aligned}
$$

It is the straight-line $y = \sqrt{3}x$ in the plane $z = \pi/3$.

The r-coordinate curve when $\theta = \theta_0$ is:

$$\begin{aligned} x &= r\cos\theta_0 \\ y &= r\sin\theta_0 \\ z &= \theta_0 \end{aligned}$$

It is the straight line $y = (\tan\theta_0)x$ in the plane $z = \theta_0$.

(b) Describe the θ-coordinate curve when $r = 1$. Give a general description of the θ-coordinate curves.

Solution.

The θ-coordinate curve when $r = 1$ is:

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \\ z &= \theta \end{aligned}$$

This is a helix with parameter θ and radius 1.

The θ-coordinate curve when $r = r_0$ is:

$$\begin{aligned} x &= r_0\cos\theta \\ y &= r_0\sin\theta \\ z &= \theta \end{aligned}$$

These are helixes of radius $r_0 \geq 0$.

(c) Sketch the graph of the helicoid using a computer for $0 \leq r \leq 1, 0 \leq \theta \leq 4\pi$. Can you see why the surface is called a helicoid?

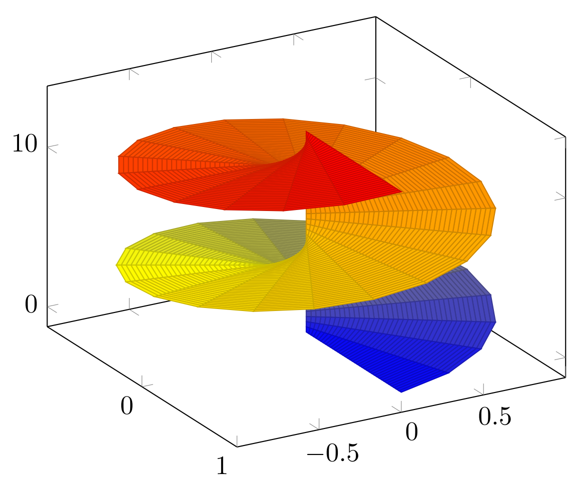

Figure. Helicoid $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$

11. Given a sphere of radius 2 centered at $(2, -1, 0)$, find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

(a) by consider the sphere as the graph of the function $f(x,y) = \sqrt{4 - (x-2)^2 - (y+1)^2}$.

Solution.

We have:

$$\begin{aligned} \text{grad } f &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(x-2)}{\sqrt{4-(x-2)^2-(y+1)^2}} & -\frac{(y+1)}{\sqrt{4-(x-2)^2-(y+1)^2}} \end{bmatrix} \end{aligned}$$

Thus,

$$\text{grad } f(1,0) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

So, the tangent plane at the point $(1, 0, \sqrt{2})$ is:

$$\begin{aligned}
z &= f(1,0) + (\mathbf{x} - \mathbf{a}) \cdot \nabla f \\
&= \sqrt{2} + (x-1, y) \cdot ((1/\sqrt{2}), -(1/\sqrt{2})) \\
&= \sqrt{2} + \frac{x-1}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\
\sqrt{2}z &= 2 + (x-1) - y \\
x - y + 1 &= \sqrt{2}z
\end{aligned}$$

(b) by considering the sphere as a level surface of the function

$$F(x,y,z) = (x-2)^2 + (y+1)^2 + z^2$$