

Partial Differential Equations

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Abstract

The most important results and ideas in classical PDEs.

1 Vector Calculus.

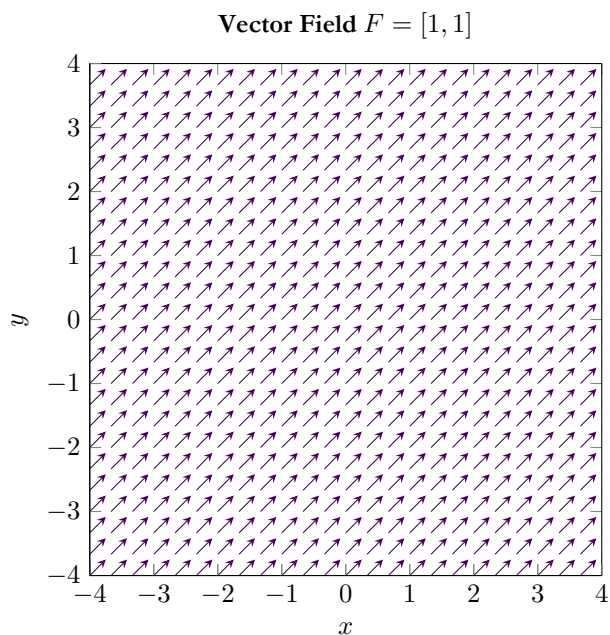
1.1 Vector Fields.

Definition 1.1. A vector field on \mathbf{R}^n is a mapping

$$\mathbf{F} : X \subseteq \mathbf{R}^n \rightarrow \mathbf{R}^n \tag{1.1}$$

We usually study vector fields on \mathbf{R}^2 or \mathbf{R}^3 . In such cases, we can think of a vector field as a function that assigns to each point $\mathbf{x} \in X$, a vector $\mathbf{F}(\mathbf{x})$ in \mathbf{R}^n , represented by a geometric arrow, whose tail is at the point \mathbf{x} . This allows us to visualize vector fields in a reasonable way.

Example 1.1. Suppose $\mathbf{F} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is defined by $\mathbf{F}(\mathbf{x}) = \mathbf{a}$ where \mathbf{a} is a constant vector. Then, \mathbf{F} assigns \mathbf{a} to each point of \mathbf{R}^2 . So, we can picture \mathbf{F} by drawing the same vector (parallel, translated of course) emanating from each point in the plane, as in the figure below.



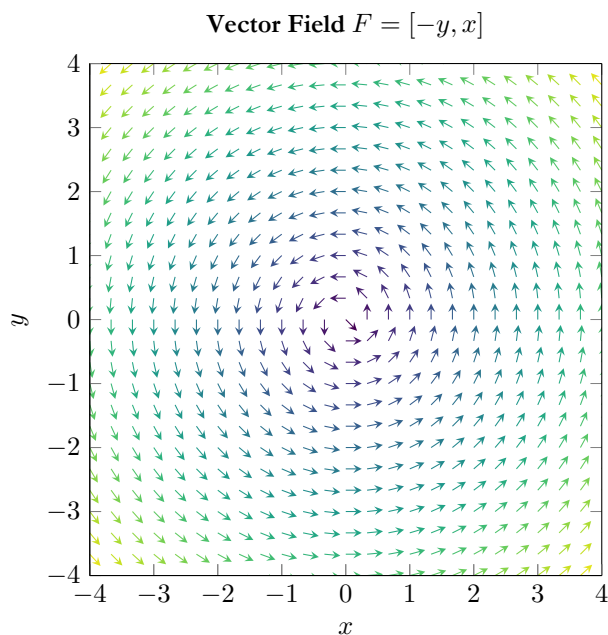
Example 1.2. Let's depict $\mathbf{G} : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, $\mathbf{G}(x, y) = y\mathbf{i} - x\mathbf{j}$. To understand \mathbf{G} better, we need to play around a bit. Note that:

$$\|\mathbf{G}(x, y)\| = \sqrt{y^2 + x^2} = \|\mathbf{r}\|$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, the position vector of the point (x, y) . From this observation, it follows that \mathbf{G} has a constant length a on the circle $x^2 + y^2 = a^2$. In addition, we have:

$$\mathbf{r} \cdot \mathbf{G}(x, y) = (x\mathbf{i} + y\mathbf{j}) \cdot (y\mathbf{j} - x\mathbf{i}) = 0$$

Hence, $\mathbf{G}(x, y)$ is always perpendicular to the position vector of the point (x, y) . These facts, together with a table like the preceding one, make it possible to see that \mathbf{G} looks like below.



Sometimes the scalar-valued function $f : X \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$ is called a scalar field. One thinks of a vector field on \mathbf{R}^n as attaching vector information (such as wind velocity) to each point and a scalar field as attaching real number information (such as temperature and pressure). We'll use the term scalar field only occasionally, but we don't want to shock you, when we do.

Example 1.3. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The so-called **inverse square** vector field in \mathbf{R}^3 is a function $\mathbf{F} : \mathbf{R}^3 - \{0\} \rightarrow \mathbf{R}^3$ given by:

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3} \mathbf{r}$$

where c is any non-zero constant. If the term “inverse square” seems inappropriate for you, we'll try to convince you otherwise. Set $\mathbf{u} = \frac{\mathbf{r}}{\|\mathbf{r}\|}$ so that $\mathbf{r} = \|\mathbf{r}\| \mathbf{u}$. Then, \mathbf{F} is given by:

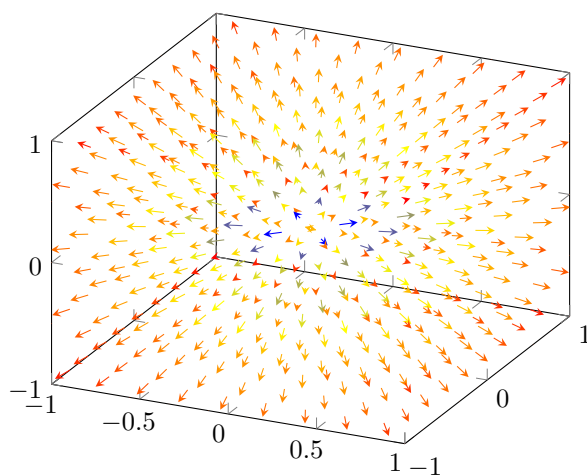
$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3} \mathbf{r} = \frac{c}{\|\mathbf{r}\|^2} \mathbf{u}$$

Thus, \mathbf{F} is a vector field whose direction at the point $P(x, y, z) \neq (0, 0, 0)$ is parallel to the vector from the origin to P and whose magnitude is inversely proportional to the square of the distance from the origin to P . Note, that \mathbf{F} points away from the origin if c is positive and towards the origin if c is negative.

We have seen an example of an inverse square field - namely, the Newtonian gravitational field between two bodies. If one of the bodies is at the origin and the other at the point (x, y, z) , then we have:

$$\mathbf{F} = -\frac{GMm}{\|\mathbf{r}\|^2} \mathbf{u}$$

In this case, the proportionality constant c is $-GMm$, which is negative. This means that the gravitational force is attractive. This vector field is drawn in the figure below.



1.2 Gradient fields and Potentials.

Inverse square fields are interesting not only for their origin in physics, but also because they are examples of gradient fields.

Definition 1.2. (Gradient Field) A gradient field on \mathbf{R}^n is a vector field $\mathbf{F} : X \subseteq \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that \mathbf{F} is the gradient of some differentiable scalar-valued function $f : X \rightarrow \mathbf{R}$. That is:

$$\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$$

at all \mathbf{x} in X . The function f is called a (scalar) **potential function** for the vector field \mathbf{F} . To see what this means in the case of the inverse square field, we write out the components of \mathbf{F} explicitly:

$$\mathbf{F} = \frac{c}{\|\mathbf{r}\|^2} \mathbf{u} = \frac{c}{(x^2 + y^2 + z^2)} \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

since the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$.

Let

$$f(x, y, z) = -\frac{c}{\sqrt{x^2 + y^2 + z^2}} = -\frac{c}{\|\mathbf{r}\|}$$

The gradient vector for the function f is:

$$\begin{aligned}\nabla f(x, y, z) &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix} \\ &= -c \begin{bmatrix} -\frac{1}{2} \cdot \frac{2x}{(x^2 + y^2 + z^2)^{3/2}} & -\frac{1}{2} \cdot \frac{2y}{(x^2 + y^2 + z^2)^{3/2}} & -\frac{1}{2} \cdot \frac{2z}{(x^2 + y^2 + z^2)^{3/2}} \end{bmatrix} \\ &= \frac{c}{(x^2 + y^2 + z^2)^{3/2}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= \mathbf{F}(\mathbf{x})\end{aligned}$$

Hence, $f(x, y, z)$ is the potential function for the field \mathbf{F} .

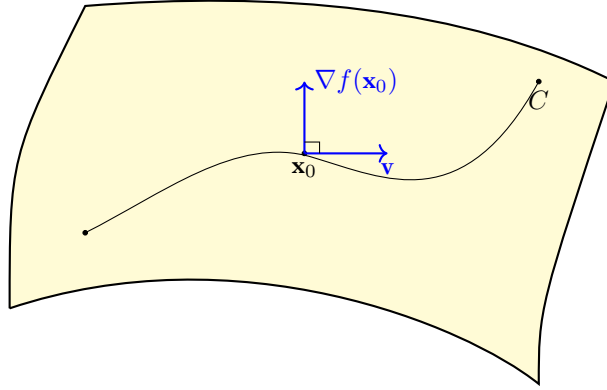
Remark. In Physics and Engineering, a negative sign is often introduced in the definition of a potential function (so that a potential function g for a vector field \mathbf{F} is one such that $\mathbf{F} = -\nabla g$). The motivation behind such a convention is that in physical applications, it is desirable to have the potential function represent potential energy in some sense. For example, in the case of the gravitational field $\mathbf{F} = -(GMm/\|\mathbf{r}\|^2)\mathbf{u}$, a physicist would take the potential function to be $U(\mathbf{x}) = -GMm/\|\mathbf{r}\|$. The advantage to the physicist in doing so is that the physicist's potential function increases with increasing $\|\mathbf{r}\|$. This corresponds to the notion that the greater the distance between the two bodies, the greater should be stored gravitational potential energy.

1.2.1 Gradient vector and the Level Sets.

Theorem 1.1. *Let $X \subseteq \mathbf{R}^n$ be an open set and $f : X \rightarrow \mathbf{R}$ be a function(surface) of class C^1 . If \mathbf{x}_0 is any point on the level set $S = \{\mathbf{x} | f(\mathbf{x}) = c\}$, then the gradient vector $\nabla f(\mathbf{x})$ is perpendicular to the level set S at the point \mathbf{x}_0 .*

Proof. We need to establish the following : if \mathbf{v} is any vector tangent to S at the point \mathbf{x}_0 , then $\nabla f(\mathbf{x}_0)$ is orthogonal to \mathbf{v} .

By a tangent vector to S at \mathbf{x}_0 , we mean that \mathbf{v} is the velocity vector of a curve C that lies in S and passes through \mathbf{x}_0 . Imagine that $f : X \subseteq \mathbf{R}^3 \rightarrow \mathbf{R}$ is defined as $w = f(x, y, z)$. Thus, the graph(plot) of f lives in \mathbf{R}^4 . The set $S = \{\mathbf{x} | f(\mathbf{x}) = c\}$ is a level surface of f that lives in \mathbf{R}^3 . The situation is pictured below:



Thus, let the curve C be given parametrically by $\mathbf{x}(t) = (x_1(t), x_2(t), \dots, x_n(t))$, where $a < t < b$ and $\mathbf{x}(t_0) = \mathbf{x}_0$. Since the curve C is contained in S , we have:

$$f(\mathbf{x}(t)) = f(x_1(t), x_2(t), \dots, x_n(t)) = c$$

Hence,

$$\frac{d}{dt}[f(\mathbf{x}(t))] = \frac{d}{dt}[c] = 0 \quad (1.2)$$

On the other hand, the chain rule applied to the composite function $f(\mathbf{x}(t)) : [a, b] \rightarrow \mathbf{R}$ tells us:

$$\frac{d}{dt}[f(\mathbf{x}(t))] = \nabla f(\mathbf{x}) \cdot \mathbf{x}'(t) \quad (1.3)$$

At $t = t_0$, we have:

$$\nabla f(\mathbf{x}_0) \cdot \mathbf{x}'(t_0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{v} = 0$$

But, the velocity vector \mathbf{v} of the curve C at \mathbf{x}_0 is always parallel to C . This closes the proof. \square

It follows that, if \mathbf{F} is a gradient vector field on \mathbf{R}^n , $\mathbf{F}(\mathbf{x})$ must be perpendicular to the level sets of the potential function of \mathbf{F} containing the point \mathbf{x} . If f is such a potential function, the level set $\{\mathbf{x} : f(\mathbf{x}) = c\}$ is called an **equipotential set** (or an equipotential surface) of the vector field \mathbf{F} . Everytime we look at weather-maps, the curves of constant barometric pressure (called isobars) or constant temperature (isotherms) are drawn. Perpendicular to the equipotential lines are associated gradient vector fields that point in the direction of the most rapid increase of pressure or temperature.

The gravitational pull of the earth is a force field. Its equipotential surface is a sphere.

1.2.2 Flow-lines of vector fields.

When we draw a sketch of a vector field in \mathbf{R}^2 or \mathbf{R}^3 , it is easy to imagine that the arrows represent the velocity of some fluid moving in space. It's natural to let the arrows blend into complete curves. What you're doing analytically, is drawing paths whose velocity vectors coincide with those of the vector field.

Definition 1.3. A **flow-line** of a vector field $\mathbf{F} : X \subseteq \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a differentiable path $\mathbf{x} : I \rightarrow \mathbf{R}^n$ such that:

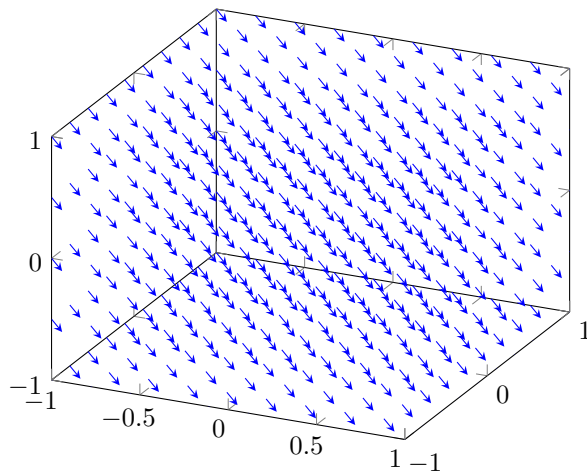
$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$$

That is, the velocity vector of \mathbf{x} at time t is given by the value of the vector field \mathbf{F} at the point on \mathbf{x} at time t .

Example 1.4. We calculate the flow lines of the constant vector field $\mathbf{F}(x, y, z) = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

A picture of this vector field below makes it easy to believe that the flow lines are indeed straight line paths. Indeed, if $\mathbf{x}(t) = (x(t), y(t), z(t))$ is a flow-line, then by definition, we must have:

$$\mathbf{x}'(t) = (x'(t), y'(t), z'(t)) = (2, -3, 1) = \mathbf{F}(\mathbf{x}(t))$$



Equating the components, we see that:

$$\begin{aligned} x'(t) &= 2 \\ y'(t) &= -3 \\ z'(t) &= 1 \end{aligned}$$

These differential equations are readily solved by integration; we obtain:

$$\begin{aligned}x(t) &= 2t + x_0 \\y(t) &= -3t + y_0 \\z(t) &= t + z_0\end{aligned}$$

where x_0, y_0, z_0 are arbitrary constants. Hence, as expected, we obtain the parametric equations for a straight line path passing through an arbitrary point (x_0, y_0, z_0) with the velocity vector $(2, -3, 1)$.

1.3 Gradient, Divergence, Curl and the Del Operator.

1.3.1 The Del Operator.

The del operator, denoted ∇ is an odd creature. It leads a double life as both a differential operator and a vector. In Cartesian coordinates on \mathbf{R}^3 , del is defined by the curious expression:

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \quad (1.4)$$

The empty partial derivatives are the components of a vector that awaits suitable scalar and vector fields on which to act.

For example, if $f : X \subseteq \mathbf{R}^3 \rightarrow \mathbf{R}$ is a differentiable function (scalar field), the gradient of f may be considered to be the result of multiplying the vector ∇ by the scalar f , except that when we *multiply* each component of ∇ by f , we actually compute the appropriate partial derivative:

$$\nabla f(x, y, z) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

The del operator can also be defined in \mathbf{R}^n for arbitrary n . If we take x_1, x_2, \dots, x_n to be the coordinates for \mathbf{R}^n , then del is simply:

$$\nabla = \frac{\partial}{\partial x_1} \mathbf{e}_1 + \frac{\partial}{\partial x_2} \mathbf{e}_2 + \dots + \frac{\partial}{\partial x_n} \mathbf{e}_n$$

1.3.2 The divergence of a vector field.

Whereas taking the gradient of a scalar field yields a vector field, the process of taking the divergence does just the opposite: it turns a vector field into a scalar field.

Definition 1.4. Let $\mathbf{F} : X \subseteq \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a differentiable vector field. Then, the **divergence** of \mathbf{F} , denoted $\text{div}\mathbf{F}$ or $\nabla \cdot \mathbf{F}$ (del dot \mathbf{F}) is the scalar field:

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n} \quad (1.5)$$

where x_1, x_2, \dots, x_n are Cartesian coordinates for \mathbf{R}^n and F_1, F_2, \dots, F_n are the component functions of \mathbf{F} .

It is essential that Cartesian coordinates be used in the formula of the definition (1.4).

Example 1.5. If $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}$, then:

$$\begin{aligned} \text{div}\mathbf{F} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xyz) \\ &= 2xy + xy \\ &= 3xy \end{aligned}$$

The notation for the divergence involving the dot product and the del operator is especially apt: if we write:

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + \dots + F_n\mathbf{e}_n$$

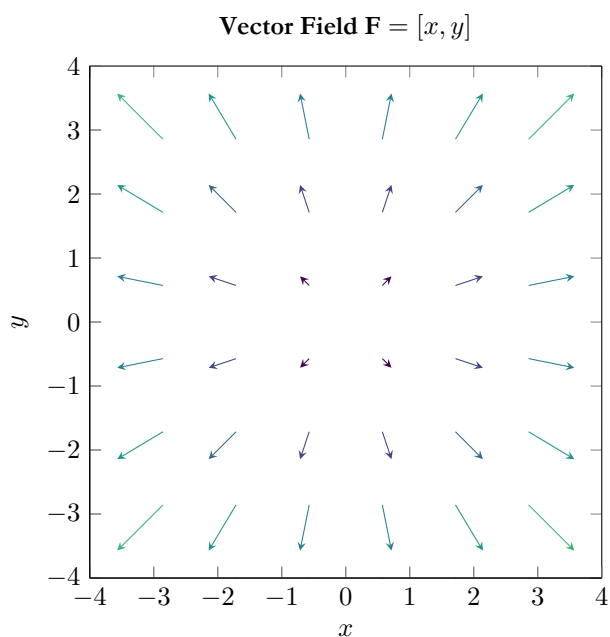
then,

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \left(\mathbf{e}_1 \frac{\partial}{\partial x_1} + \dots + \mathbf{e}_n \frac{\partial}{\partial x_n} \right) \cdot (F_1, \dots, F_n) \\ &= \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n} \end{aligned}$$

Intuitively, the value of the divergence of a vector field at a particular point gives a measure of the *net mass flow* or *flux density* of the vector field in to or out of that point. To understand what such a statement means, imagine that the vector field \mathbf{F} represents the velocity of a fluid. If $\nabla \cdot \mathbf{F}$ is zero at a point, then the rate at which fluid is flowing into that point is equal to the rate at which fluid is flowing out. Positive divergence at a point signifies more fluid flowing out than in, while negative divergence signifies just the opposite. We will make these assertions more precise, even prove them, when we will have some integral vector calculus at our disposal. For now, we can remark, that a vector field \mathbf{F} such that $\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = 0$ everywhere is called **incompressible** or **solenoidal**.

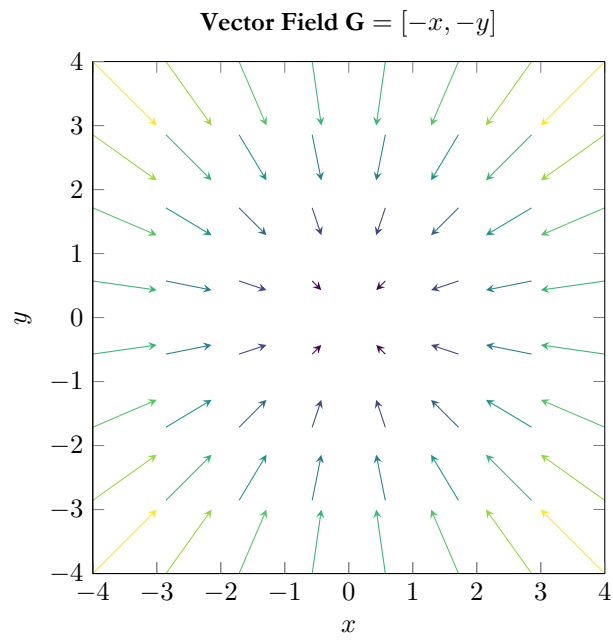
Example 1.6. The vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has:

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) \\ &= 2\end{aligned}$$



At any point in \mathbf{R}^2 , the arrow whose tail is at that point is longer than the arrow whose head is there (indicated by the color in the above plot). Hence, there is a greater flow *away* from each point, than into it; that is *diverging* at each point.

The vector field $\mathbf{G} = -x\mathbf{i} - y\mathbf{j}$ points in the direction opposite to the vector field \mathbf{F} , and it should be clear how \mathbf{G} 's divergence of -2 is reflected in the below plot.



Example 1.7. The constant vector field $\mathbf{F}(x, y, z) = \mathbf{a}$ is incompressible, since if $\mathbf{a} = (a_x, a_y, a_z)$, we have:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(a_x) + \frac{\partial}{\partial x}(a_y) + \frac{\partial}{\partial z}(a_z) \\ &= 0\end{aligned}$$