Stoke's and Gauss's Theorem

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Exercise Problems.

1. Let $X : \mathbf{R}^2 \to \mathbf{R}^3$ be the parametrized surface given by

$$X(s,t) = (s^2 - t^2, s+t, s^2 + 3t)$$

(a) Determine a normal vector to this surface at the point

$$(3,1,1) = \mathbf{X}(2,-1)$$

Solution.

We have:

$$\begin{aligned} \mathbf{T}_s &= (2s, 1, 2s) \\ \mathbf{T}_t &= (-2t, 1, 3) \end{aligned}$$

So, the standard normal vector at the point $\mathbf{X}(2,-1)$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s & 1 & 2s \\ -2t & 1 & 3 \end{vmatrix} \\ &= \mathbf{i}(3-2s) - \mathbf{j}(6s + 4st) + \mathbf{k}(2s + 2t) \\ &= \mathbf{i}(3-4) - \mathbf{j}(12 + 4(2)(-1)) + \mathbf{k}(4-2) \\ &= -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \end{aligned}$$

(b) Find an equation for the plane tangent to this surface at the point $(3,1,1)$.

Solution.

The tangent plane to this surface at the point $(3,1,1)$ is given by:

$$\begin{aligned} \mathbf{N} \cdot (\mathbf{x} - (3,1,1)) &= 0 \\ (-1,-4,2) \cdot ((x,y,z) - (3,1,1)) &= 0 \\ -(x-3) - 4(y-1) + 2(z-1) &= 0 \end{aligned}$$

2. Find an equation for the plane tangent to the torus

$$\mathbf{X}(s,t) = ((5 + 2\cos t)\cos s, (5 + 2\cos t)\sin s, 2\sin t)$$

at the point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (-(5+2\cos t)\sin s,(5+2\cos t)\cos s,0)\\ \mathbf{T}_t &= (-2\sin t\cos s,-2\sin t\sin s,2\cos t)\end{aligned}$$

The standard normal vector is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s\times\mathbf{T}_t\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -(5+2\cos t)\sin s & (5+2\cos t)\cos s & 0\\ -2\sin t\cos s & -2\sin t\sin s & 2\cos t\end{vmatrix}\\ &= \mathbf{i}(2(5+2\cos t)\cos s\cos t))+\mathbf{j}(2(5+2\cos t)\sin s\cos t)\\ &+\mathbf{k}(2\sin s\sin t(5+2\cos t)+2(5+2\cos t)\sin t\cos^2 s)\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i}+\sin s\cos t\mathbf{j}+(\sin^2 s+\cos^2 s)\sin t\mathbf{k})\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i}+\sin s\cos t\mathbf{j}+\sin t\mathbf{k})\end{aligned}$$

The point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1)=((5+2\cos t)\cos s,(5+2\cos t)\sin s,2\sin t)$ yields $\sin t=1/2$, so $t_0=\pi/6$ or $t_0=5\pi/6$. Since $2\cos t<0$, $t_0=5\pi/6$. Then, we can see that :

$$\frac{5-\sqrt{3}}{\sqrt{2}}=(5-2\cdot\frac{\sqrt{3}}{2})\sin s$$

So, $s_0=\pi/4$.

Consequently, the equation of the tangent plane at $\mathbf{X}(\pi/4,5\pi/6)$ is:

$$\begin{aligned}\mathbf{N}\cdot(\mathbf{x}-\mathbf{X}(s_0,t_0)) &= 0\\ 2(5-\sqrt{3})(-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{i}-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{j}+\frac{1}{2}\mathbf{k})((x,y,z)-(\frac{5-\sqrt{3}}{\sqrt{2}},\frac{5-\sqrt{3}}{\sqrt{2}},1)) &= 0\\ -\frac{\sqrt{3}}{\sqrt{2}}(x-(5-\sqrt{3})/\sqrt{2})-\frac{\sqrt{3}}{\sqrt{2}}(y-(5-\sqrt{3})/\sqrt{2})+(z-1) &= 0\\ -\sqrt{3}(x-(5-\sqrt{3})/\sqrt{2})-\sqrt{3}(y-(5-\sqrt{3})/\sqrt{2})+\sqrt{2}(z-1) &= 0\\ -\sqrt{3}x-\sqrt{3}y+\sqrt{2}z &= -2\sqrt{3}(5-\sqrt{3})/\sqrt{2}+\sqrt{2}\\ &= -\sqrt{6}(5-\sqrt{3})+\sqrt{2}\\ &= -5\sqrt{6}+3\sqrt{2}+\sqrt{2}\\ \sqrt{3}x+\sqrt{3}y-\sqrt{2}z &= 5\sqrt{6}-4\sqrt{2}\end{aligned}$$

3. Find an equation of the plane tangent to the surface

$$x=e^s\quad y=t^2e^{2s}\quad z=2e^{-s}+t$$

at the point $(1,4,0)$.

Solution.

We have:

$$
\begin{aligned}
\mathbf{T}_s &= (e^s, 2t^2e^{2s}, -2e^{-s}) \\
\mathbf{T}_t &= (0, 2te^{2s}, 1)
\end{aligned}
$$

The standard normal vector is:

$$
\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^s & 2t^2e^{2s} & -2e^{-s} \\ 0 & 2te^{2s} & 1 \end{vmatrix} \\
&= \mathbf{i}(2t^2e^{2s} + 4te^s) - \mathbf{j}(e^s) + \mathbf{k}(2te^{3s}) \\
&= e^s((2t^2e^s + 4t)\mathbf{i} - \mathbf{j} + (2te^{2s})\mathbf{k})
\end{aligned}
$$

Since $e^s = 1$, $s = 0$. Also, as $4 = t^2 \cdot 1$, we have $t = \pm 2$. Moreover, $0 = 2 + t$, so $t = -2$. So, $\mathbf{N}(0, -2)$ is:

$$
\mathbf{N}(0, -2) = -\mathbf{j} - 4\mathbf{k}
$$

The equation of the tangent plane at $\mathbf{X}(0, -2)$ is:

$$
\begin{aligned}
\mathbf{N} \cdot (x - 1, y - 4, z) &= 0 \\
-(y - 4) - 4z &= 0 \\
y + 4z &= 4
\end{aligned}
$$

4. Let $\mathbf{X}(s, t) = (s^2 \cos t, s^2 \sin t, s)$, $-3 \le s \le 3, 0 \le t \le 2\pi$.

(a) Find a normal vector at $(s, t) = (-1, 0)$.

Solution.

We have:

$$
\begin{aligned}
\mathbf{T}_s &= (2s \cos t, 2s \sin t, 1) \\
\mathbf{T}_t &= (-s^2 \sin t, s^2 \cos t, 0)
\end{aligned}
$$

The standard normal vector $\mathbf{N}$ is:

$$
\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s \cos t & 2s \sin t & 1 \\ -s^2 \sin t & s^2 \cos t & 0 \end{vmatrix} \\
&= \mathbf{i}(-s^2 \cos t) - \mathbf{j}(s^2 \sin t) + \mathbf{k}(2s^3 \cos^2 t + 2s^3 \sin^2 t) \\
&= -s^2 \cos t\mathbf{i} - s^2 \sin t\mathbf{j} + 2s^3\mathbf{k}
\end{aligned}
$$

The vector at $(s, t) = (-1, 0)$ is:

$$
\mathbf{N}(-1, 0) = -\mathbf{i} - 2\mathbf{k}
$$

Hence, the equation of the tangent plane at $\mathbf{X}(-1, 0) = (1, 0, -1)$ is:

$$
\begin{aligned}
(-\mathbf{i}-2\mathbf{k})\cdot((x,y,z)-(1,0,-1)) &= 0\\
-(x-1)-2(z+1) &= 0\\
x-1+2z+2 &= 0\\
x+2z+1 &= 0
\end{aligned}
$$

(b) Find an equation for the image of $\mathbf{X}$ in the form $F(x,y,z)=0$.

Solution.

Let $x = s^2\cos t$, $y = s^2 \sin t$. Then, $x^2+y^2 = s^4(\cos^2 t + \sin^2 t) = s^4 = z^4$. So, $F(x,y,z) = x^2+y^2-z^4 = 0$.

5. Consider the parameterized surface $\mathbf{X}(s,t) = (s, s^2+t, t^2)$.

(a) Graph the surface for $-2 \le s \le 2$, $-2 \le t \le 2$.

Solution.

The s-coordinate curve at $t=0$ is:

$$
\begin{aligned}
x &= s\\
y &= s^2\\
z &= 0
\end{aligned}
$$

This is the parabolic curve $y = x^2$ in the xy-plane.

The s-coordinate curve at $t = t_0$ is:

$$
\begin{aligned}
x &= s\\
y &= s^2 + t_0\\
z &= t_0^2
\end{aligned}
$$

Thus, we get parabolas parallel to the xy-plane.

t	Curve	Center	z-plane
$t_0=-2$	$y+2=x^2$	$(x,y)=(0,-2)$	$z=4$
$t_0=-1$	$y+1=x^2$	$(x,y)=(0,-1)$	$z=1$
$t_0=0$	$y=x^2$	$(x,y)=(0,0)$	$z=0$
$t_0=1$	$y-1=x^2$	$(x,y)=(0,1)$	$z=1$
$t_0=2$	$y-2=x^2$	$(x,y)=(0,2)$	$z=4$

The $t-$coordinate curve at $s=0$ is:

$$
\begin{aligned}
x &= 0\\
y &= t\\
z &= t^2
\end{aligned}
$$

These are parabolas parallel to the yz-plane.

t	Curve	Center	x-plane
$s_0=-2$	$z=(y-4)^2$	$(y,z)=(2,0)$	$x=-2$
$s_0=-1$	$z=(y-1)^2$	$(y,z)=(1,0)$	$x=-1$
$s_0=0$	$z=y^2$	$(y,z)=(0,0)$	$x=0$
$s_0=1$	$z=(y-1)^2$	$(y,z)=(1,0)$	$x=1$
$s_0=2$	$z=(y-4)^2$	$(y,z)=(2,0)$	$x=2$

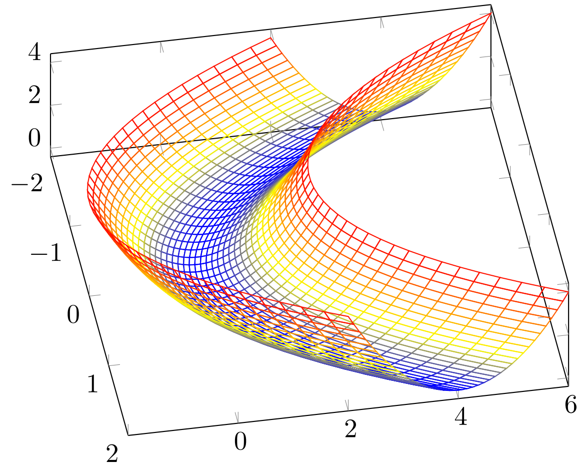

Figure. $\mathbf{X}(s,t) = (s, s^2 + t, t^2)$.

(b) Is the surface smooth?

Solution. The surface is smooth.

(c) Find an equation for the tangent plane at the point $(1, 0, 1)$.

Solution.

We have:

$$\begin{aligned} \mathbf{T}_s &= (1, 2s, 0) \\ \mathbf{T}_t &= (0, 1, 2t) \end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2s & 0 \\ 0 & 1 & 2t \end{vmatrix} \\ &= 4st\mathbf{i} - 2t\mathbf{j} + \mathbf{k} \end{aligned}$$

The point $(1, 0, 1)$ is $(s_0 = 1, t_0 = -1)$.

$$\mathbf{N}(1, -1) = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

The equation of the tangent plane at $(1, -1)$ is:

$$\begin{aligned} (-4, 2, 1) \cdot (x - 1, y, z - 1) &= 0 \\ -4(x - 1) + 2y + (z - 1) &= 0 \\ 4(x - 1) - 2y - (z - 1) &= 0 \\ 4x - 4 - 2y - z + 1 &= 0 \\ 4x - 2y - z &= 3 \end{aligned}$$

6. Describe the parameterized surface of exercise problem 1 by an equation of the form $z = f(x, y)$.

Solution.

The parametric surface $\mathbf{X}(s,t)$ is:

$$X(s,t) = (s^2 - t^2, s+t, s^2+3t)$$

In exercise (1), we see that $x = (s-t)(s+t) = y(s+t)$ so $s+t = x/y$ and $y = s-t$. This allows us to solve simultaneously for s and t. $2s = x/y + y$ and $2t = x/y - y$. This means that $z = s^2 + 3t$ can be written as $z = \frac{1}{4}\left(\frac{x}{y} + y\right)^2 + \frac{3}{2}\left(\frac{x}{y} - y\right)$.

7. Let S be the surface parameterized by:

$$\begin{aligned} x &= s \cos t \\ y &= s \sin t \\ z &= s^2 \end{aligned}$$

where $s \geq 0, 0 \leq t \leq 2\pi$.

(a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$.

Solution.

The surface S is $x^2 + y^2 = z$. This is a paraboloid. It is smooth at all points.

We have:

$$\begin{aligned} \mathbf{T}_s &= (\cos t, \sin t, 2s) \\ \mathbf{T}_t &= (-s \sin t, s \cos t, 0) \end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 2s \\ -s \sin t & s \cos t & 0 \end{vmatrix} \\ &= (-2s^2 \cos t)\mathbf{i} - (2s^2 \sin t)\mathbf{j} + (s \cos^2 + s \sin^2 t)\mathbf{k} \\ &= (-2s^2 \cos t)\mathbf{i} - (2s^2 \sin t)\mathbf{j} + s\mathbf{k} \end{aligned}$$

At $s = 2, t = \pi/6$,

$$\mathbf{N}(2, \pi/6) = -4\sqrt{3}\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

The equation of the tangent plane at $\mathbf{X}(2, \pi/6)$ is:

$$\begin{aligned} (-4\sqrt{3}, -4, 2) \cdot (x - 1, y - \sqrt{3}, z - 4) &= 0 \\ 4\sqrt{3}(x-1) + 4(y - \sqrt{3}) - 2(z-4) &= 0 \\ 4\sqrt{3}x - 4\sqrt{3} + 4y - 4\sqrt{3} - 2z + 8 &= 0 \\ 4\sqrt{3}x + 4y - 2z &= 8(\sqrt{3} - 1) \\ 2\sqrt{3}x + 2y - z &= 4(\sqrt{3} - 1) \end{aligned}$$

(b) Sketch the graph of S. Can you recognize S as a familiar surface?

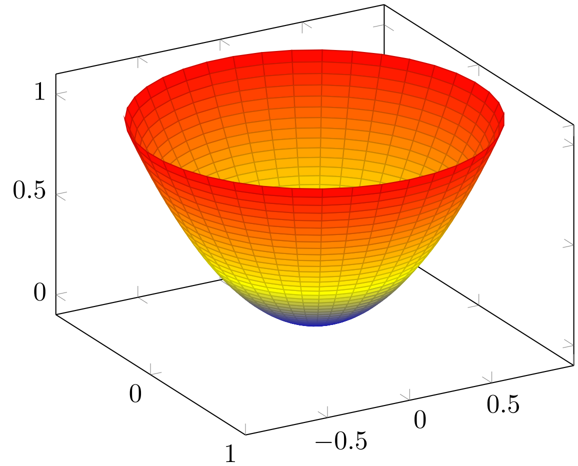

Figure. $\mathbf{X}(s,t) = (s\cos t, s\sin t, s^2)$

(c) Describe S by an equation of the form $z = f(x, y)$.

Solution. Again $z = x^2 + y^2$.

(d) Using your answer in part(c), discuss whether S has a tangent plane at every point.

Solution.

S has a tangent plane at every point and is smooth. Part (a) takes care of every point except the origin. At the origin $\mathbf{N} = (0, 0, 0)$. But, we easily see, that the tangent plane at the origin is the horizontal plane $z = 0$. Thus, smoothness as defined in the text, depends on both the parameterization and the geometry of the underlying surface.

8. Verify that the image of the parametrized surface

$$\mathbf{X}(s,t) = (2\sin s\cos t, 3\sin s\sin t, \cos s)$$

$0 \le s \le \pi$ and $0 \le t \le 2\pi$ is an ellipsoid.

Solution.

We can easily write:

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$$

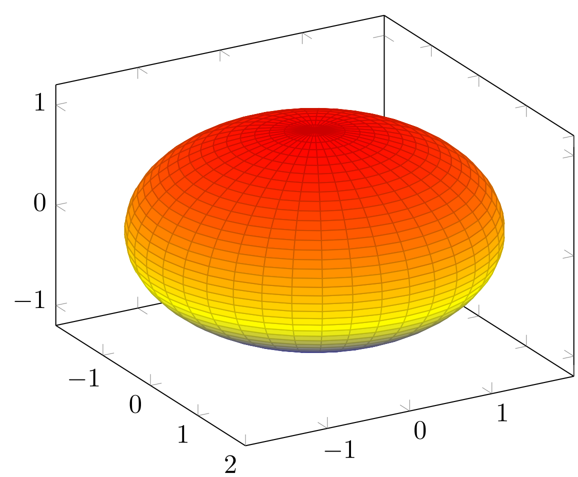

Figure. $\mathbf{X}(s,t) = (2\sin s\cos t, 3\sin s\sin t, \cos s)$

9. Verify that, for the torus of example 5, the s-coordinate curve, when $t = t_0$ is a circle of radius $a + b\cos t_0$.

Solution.

The parametric equations of the Torus in example 5 were:

$$\begin{aligned} x &= (a + b\cos t)\cos s \\ y &= (a + b\cos t)\sin s \\ z &= b\sin t \end{aligned}$$

The s-coordinate curve at $t = t_0$ is:

$$\begin{aligned} x &= (a + b\cos t_0)\cos s \\ y &= (a + b\cos t_0)\sin s \\ z &= b\sin t_0 \end{aligned}$$

These are circles of radius $a + b\cos t_0$ in the plane $z = b\sin t_0$.

And they satisfy $x^2 + y^2 = (a + b\cos t_0)^2$.

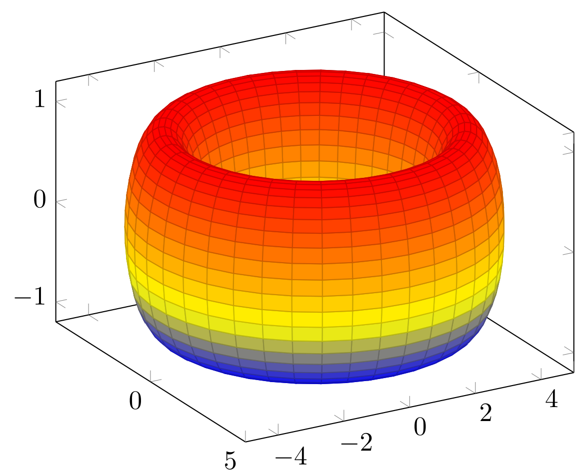

Figure. Torus $\mathbf{X}(s,t) = ((a + b\cos t)\cos s, (a + b\cos t)\sin s, b\sin t)$

9. The surface in $\mathbf{R}^3$ parametrized by:

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

where $r \geq 0$ and $-\infty < \theta < \infty$ is called a helicoid.

(a) Describe the r-coordinate curve when $\theta = \pi/3$. Give a general description of the r-coordinate curves.

Solution.

The r-coordinate curve when $\theta = \pi/3$ is:

$$\begin{aligned} x &= r/2 \\ y &= \sqrt{3}r/2 \\ z &= \pi/3 \end{aligned}$$

It is the straight-line $y = \sqrt{3}x$ in the plane $z = \pi/3$.

The r-coordinate curve when $\theta = \theta_0$ is:

$$
\begin{aligned}
x &= r\cos\theta_0 \\
y &= r\sin\theta_0 \\
z &= \theta_0
\end{aligned}
$$

It is the straight line $y = (\tan\theta_0)x$ in the plane $z = \theta_0$.

(b) Describe the θ-coordinate curve when $r = 1$. Give a general description of the θ-coordinate curves.

Solution.

The θ-coordinate curve when $r = 1$ is:

$$
\begin{aligned}
x &= \cos\theta \\
y &= \sin\theta \\
z &= \theta
\end{aligned}
$$

This is a helix with parameter θ and radius 1.

The θ-coordinate curve when $r = r_0$ is:

$$
\begin{aligned}
x &= r_0\cos\theta \\
y &= r_0\sin\theta \\
z &= \theta
\end{aligned}
$$

These are helixes of radius $r_0 \geq 0$.

(c) Sketch the graph of the helicoid using a computer for $0 \leq r \leq 1, 0 \leq \theta \leq 4\pi$. Can you see why the surface is called a helicoid?

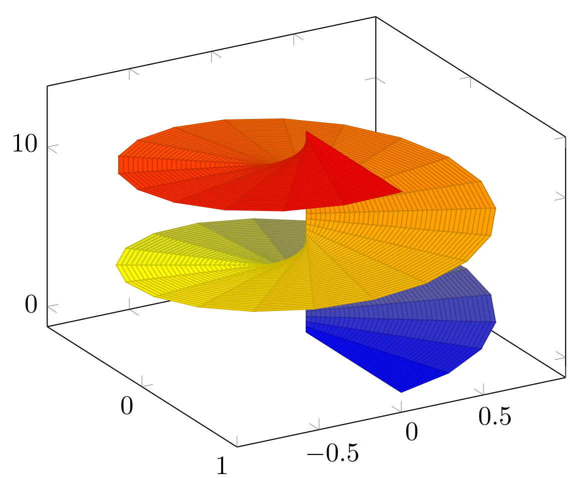

Figure. Helicoid $\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$

11. Given a sphere of radius 2 centered at $(2, -1, 0)$, find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

(a) by consider the sphere as the graph of the function $f(x, y) = \sqrt{4 - (x-2)^2 - (y+1)^2}$.

Solution.

We have:

$$
\begin{aligned}
\text{grad } f &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{(x-2)}{\sqrt{4-(x-2)^2-(y+1)^2}} & -\frac{(y+1)}{\sqrt{4-(x-2)^2-(y+1)^2}} \end{bmatrix}
\end{aligned}
$$

Thus,

$$\text{grad } f(1,0) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

So, the tangent plane at the point $(1, 0, \sqrt{2})$ is:

$$\begin{aligned} z &= f(1,0) + (\mathbf{x} - \mathbf{a}) \cdot \nabla f \\ &= \sqrt{2} + (x-1, y) \cdot ((1/\sqrt{2}), -(1/\sqrt{2})) \\ &= \sqrt{2} + \frac{x-1}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\ \sqrt{2}z &= 2 + (x-1) - y \\ x - y + 1 &= \sqrt{2}z \end{aligned}$$

(b) by considering the sphere as a level surface of the function

$$F(x,y,z) = (x-2)^2 + (y+1)^2 + z^2$$

The gradient ∇F is :

$$\begin{aligned} \nabla F &= \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} 2(x-2) & 2(y+1) & 2z \end{bmatrix} \end{aligned}$$

If $\mathbf{x}_0 = (x_0, y_0, z_0)$ is a point on the level set $S = \{(x,y,z) : F(x,y,z) = c\}$, then the gradient vector $\nabla F(\mathbf{x})$ at the point $\mathbf{x}_0$ is perpendicular to S. $(1, 0, \sqrt{2})$ is point on the level set $S = \{(x,y,z) | (x-2)^2 + (y+1)^2 + z^2 = 4\}$. So, $\nabla F(1, 0, \sqrt{2}) = (-2, 2, 2\sqrt{2})$ is the normal vector to the sphere $F(x,y,z) = 4$ at the point $(1, 0, \sqrt{2})$.

If (x, y, z) is an arbitrary point in the tangent plane, we must have:

$$\begin{aligned} (x-1, y, z-\sqrt{2}) \cdot (-2, 2, 2\sqrt{2}) &= 0 \\ (x-1, y, z-\sqrt{2}) \cdot (-1, 1, \sqrt{2}) &= 0 \\ -(x-1) + y + \sqrt{2}(z-\sqrt{2}) &= 0 \\ (x-1) - y - \sqrt{2}(z-\sqrt{2}) &= 0 \\ x - y - \sqrt{2}z + 1 &= 0 \\ x - y + 1 &= \sqrt{2}z \end{aligned}$$

(c) By considering the sphere as the surface parametrized by :

$$\mathbf{X}(s,t) = (2\sin s\cos t + 2, 2\sin s\sin t - 1, 2\cos s)$$

Solution.

We have:

$$\begin{aligned} \mathbf{T}_s &= (2\cos s\cos t, 2\cos s\sin t, -2\sin s) \\ \mathbf{T}_t &= (-2\sin s\sin t, 2\sin s\cos t, 0) \end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos s\cos t & 2\cos s\sin t & -2\sin s \\ -2\sin s\sin t & 2\sin s\cos t & 0 \end{vmatrix} \\ &= 4\sin^2 s\cos t\mathbf{i} - (-4\sin^2 s\sin t)\mathbf{j} + (4\sin s\cos s\cos^2 t + 4\sin s\cos s\sin^2 t)\mathbf{k} \\ &= 4\sin^2 s\cos t\mathbf{i} + 4\sin^2 s\sin t\mathbf{j} + 4\sin s\cos s\mathbf{k}\end{aligned}$$

Now, $2\cos s = \sqrt{2}$ so $\cos s = \frac{1}{\sqrt{2}}$ and thus $s = \pi/4$. Consequently, $\sqrt{2}\cos t + 2 = 1$ and therefore $\cos t = -\frac{1}{\sqrt{2}}$, which implies $t = 3\pi/4$. The normal vector at $\mathbf{X}(\pi/4, 3\pi/4)$ is:

$$\begin{aligned}\mathbf{N}(\pi/4, 3\pi/4) &= 4\cdot\frac{1}{2}\cdot\frac{-1}{\sqrt{2}}\mathbf{i} + 4\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\mathbf{j} + 2\mathbf{k} \\ &= -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 2\mathbf{k}\end{aligned}$$

The equation of the tangent plane is:

$$\begin{aligned}(-\sqrt{2}, \sqrt{2}, 2)\cdot(x-1, y, z-\sqrt{2}) &= 0 \\ (-1, 1, \sqrt{2})\cdot(x-1, y, z-\sqrt{2}) &= 0 \\ x - y + 1 &= \sqrt{2}z\end{aligned}$$

In exercises 12-15, represent the given surface as a piecewise smooth parameterized surface.

12. The lower hemisphere $x^2 + y^2 + z^2 = 9$ including the equatorial circle.

Solution.

We can parametrize the lower hemisphere of the sphere as $\mathbf{X}(s,t) = (s, t, -\sqrt{9-(x^2+y^2)})$. Alternatively, we may parametrize it as $\mathbf{X}(\phi, \theta)$:

$$\begin{aligned}x &= 3\sin\phi\cos\theta \\ y &= 3\sin\phi\sin\theta \\ z &= 3\cos\phi\end{aligned}$$

where $0 \le \theta \le 2\pi$ and $\pi/2 \le \phi \le \pi$.

13. The part of the cylinder $x^2 + z^2 = 4$ lying between $y = -1$ and $y = 3$.

Solution.

We can parametrize the cylinder as:

$$\begin{aligned}x &= 2\cos s \\ y &= t \\ z &= 2\sin s\end{aligned}$$

where $0 \le s \le 2\pi$ and $-1 \le t \le 3$.

14. The closed triangular region in $\mathbf{R}^3$ with vertices $(2,0,0)$, $(0,1,0)$ and $(0,0,5)$.

Solution.

A parameterization of a plane can be written as :

$$\mathbf{x} = s\mathbf{a} + t\mathbf{b} + \mathbf{p}$$

where $\mathbf{a}$ and $\mathbf{b}$ are two vectors in the plane and $\mathbf{p}$ is a point in the plane. To see why this is the case, suppose $\mathbf{x} = (x, y, z)$ is an arbitrary point in the plane and $\mathbf{p} = (x_0, y_0, z_0)$ is a known point. Then $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$ must be a linear combination of $\mathbf{a}$ and $\mathbf{b}$. So, $\mathbf{x} - \mathbf{p} = s\mathbf{a} + t\mathbf{b}$.

We have four planes that are described by:

$$\mathbf{X}(s,t) = s(2,0,-5) + t(0,1,-5) + (0,0,5) = (2s, t, -5s - 5t + 5)$$

Since we are interested in the first octant of $\mathbf{R}^3$ all coordinates must be non-negative. So, $0 \leq 2s \leq 2$, that is $0 \leq s \leq 1$, $0 \leq t \leq 1$ and $-5s - 5t + 5 \geq 0$. In other words, $t \leq 1 - s$.

14. The hyperboloid $z^2 - x^2 - y^2 = 1$. (Hint: Use two maps to parametrize the surface)

Solution.

The equation of the hyperboloid as:

$$z = \pm\sqrt{1 + x^2 + y^2}$$

Therefore, the hyperboloid may be paramterized with two maps:

$$\begin{aligned}\mathbf{X}_1(s,t) &= (s, t, \sqrt{1 + x^2 + y^2}) \\ \mathbf{X}_2(s,t) &= (s, t, -\sqrt{1 + x^2 + y^2})\end{aligned}$$

16. This problem concerns the parameterized surface $\mathbf{X}(s,t) = (s^3, t^3, st)$.

(a) Find an equation of a plane tangent to this surface at the point $(1, -1, -1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (3s^2, 0, t) \\ \mathbf{T}_t &= (0, 3t^2, s)\end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3s^2 & 0 & t \\ 0 & 3t^2 & s \end{vmatrix} \\ &= -3t^3\mathbf{i} - 3s^3\mathbf{j} + 9s^2t^2\mathbf{k}\end{aligned}$$

We have $s_0 = 1, t_0 = -1$. So, $\mathbf{N}(1, -1) = (3, -3, 9)$. The equation of the tangent plane to the surface at $(1, -1, -1)$ is:

$$\begin{aligned}(3, -3, 9) \cdot (x - 1, y + 1, z + 1) &= 0 \\ (1, -1, 3) \cdot (x - 1, y + 1, z + 1) &= 0 \\ (x - 1) - (y + 1) + 3(z + 1) &= 0 \\ x - 1 - y - 1 + 3z + 3 &= 0 \\ x - y + 3z + 1 &= 0\end{aligned}$$

(b) Use a computer to graph this surface for $-1 \leq s \leq 1, -1 \leq t \leq 1$.

Solution.

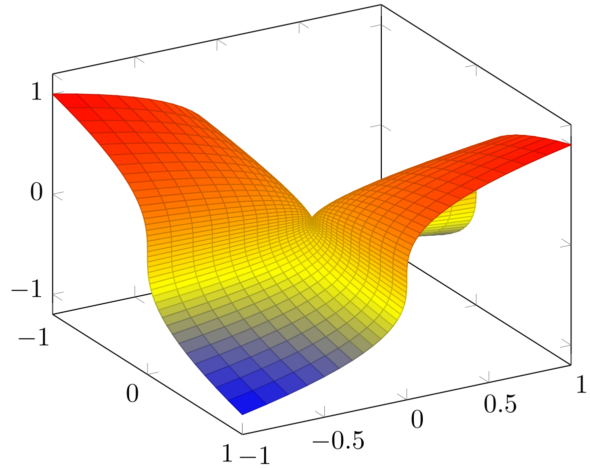

Figure. $\mathbf{X}(s,t) = (s^3, t^3, st)$

(c) Is the surface smooth?

Solution.

The normal vector $\mathbf{N} = 0$ at $(s_0, t_0) = (0,0)$ that is at $(0,0,0)$. So, the surface fails to be smooth there.

17. The surface given parametrically by $\mathbf{X}(s,t) = (st, t, s^2)$ is known as **Whitney's umbrella**.

(a) Verify that this surface may also be described by the xyz-coordinate equation $y^2 z = x^2$.

Solution.

Clearly, $y^2 z = (t^2)(s^2) = (st)^2 = x^2$.

(b) Is $\mathbf{X}$ smooth?

Solution.

We have:

$$\mathbf{T}_s = (t, 0, 2s)$$
$$\mathbf{T}_t = (s, 1, 0)$$

The standard normal vector is :

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 0 & 2s \\ s & 1 & 0 \end{vmatrix}$$
$$= -2s\mathbf{i} + 2s^2\mathbf{j} + t\mathbf{k}$$

The normal vector $\mathbf{N} = (0,0,0)$ at $(s,t) = (0,0)$ that is at the point $(0,0,0)$. Hence, $\mathbf{X}$ is not smooth at this point.

(c) Use a computer to graph this surface for $-2 \le s \le 2$, $-2 \le t \le 2$.

Solution.

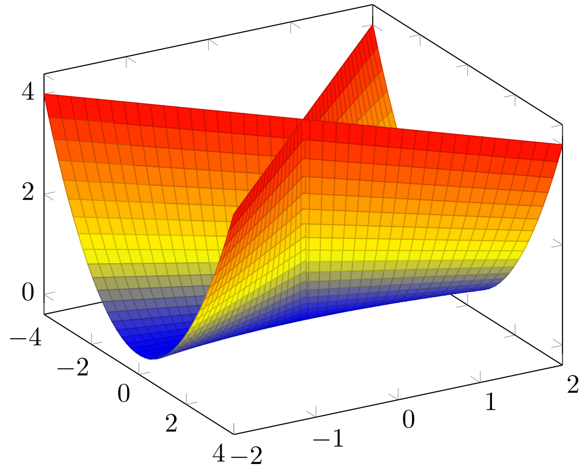

Figure. $\mathbf{X}(s,t) = (st, t, s^2)$

(d) Give an equation of the plane tangent to this surface at the point $(2,1,4)$.

Solution.

The standard normal vector at the point $\mathbf{X}(2,1)$ is $(-4,8,1)$. Hence, the equation of the tangent plane to the surface at the point $\mathbf{X}(1,2)$ is:

$$\begin{aligned}(-4,8,1)\cdot(x-2,y-1,z-4) &= 0\\ -4(x-2)+8(y-1)+(z-4) &= 0\\ 4(x-2)-8(y-1)-(z-4) &= 0\\ 4x-8y-z &= -4\end{aligned}$$

(e) Some points (x,y,z) of the surface do not correspond to a single parameter point (s,t). Which ones? Explain how this relates to the graph?

Solution.

Let $\mathbf{X}(s_1,t_1) = \mathbf{X}(s_2,t_2)$. Then,

$$\begin{aligned}s_1t_1 &= s_2t_2\\ t_1 &= t_2\\ s_1^2 &= s_2^2\end{aligned}$$

Thus, if $t_1 = t_2 = 0$ and $s_1 = \pm s_2$ we get the same image that is $\mathbf{X}(s,0) = \mathbf{X}(-s,0) = (0,0,s^2)$. Thus, the positive z-axis does not corespond to a single point.

18. Let S be the surface defined as the graph of a function $f(x,y)$ of class C^1. Then, example 4 shows that S is also a parametrized surface. Show that formula (5) for the tangent plane to S at $(a,b,f(a,b))$ agrees with formula (4) in section 2.3.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (1,0,f_s(s,t))\\ \mathbf{T}_t &= (0,1,f_t(s,t))\end{aligned}$$

The standard normal vector $\mathbf{N}$at the point (s,t) is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & f_s(s,t)\\ 0 & 1 & f_t(s,t)\end{vmatrix}\\ &= -f_s(s,t)\mathbf{i} - f_t(s,t)\mathbf{j} + \mathbf{k}\end{aligned}$$

So, the equation of the tangent plane to the surface at $\mathbf{X}(s,t)$ is:

$$
\begin{aligned}
(-f_s, -f_t, 1) \cdot (x-s, y-t, z-f(s,t)) = 0 \\
z - f(s,t) = \frac{\partial f}{\partial s}(x-s) + \frac{\partial f}{\partial t}(y-t)
\end{aligned}
$$

19. (a) Write a formula for the tangent plane to the surface described by the equation $y = g(x,z)$.

Solution.

We have:

$$\mathbf{X}(s,t) = (s, g(s,t), t)$$

So,

$$
\begin{aligned}
\mathbf{T}_s &= (1, g_s(s,t), 0) \\
\mathbf{T}_t &= (0, g_t(s,t), 1)
\end{aligned}
$$

The standard normal vector $\mathbf{N}$ is:

$$
\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & g_s(s,t) & 0 \\ 0 & g_t(s,t) & 1 \end{vmatrix} \\
&= g_s(s,t)\mathbf{i} - \mathbf{j} + g_t(s,t)\mathbf{k}
\end{aligned}
$$

So, the equation of the tangent plane to the surface at the point $(x_0, g(x_0,z_0), z_0)$ is:

$$
\begin{aligned}
(g_s(s,t), -1, g_t(s,t)) \cdot (x-x_0, y-g(x_0,z_0), z-z_0) = 0 \\
g_s(s,t)(x-x_0) - (y - g(x_0,z_0)) + g_t(z-z_0) = 0 \\
y = g(x_0,z_0) + g_s(x-x_0) + g_t(z-z_0)
\end{aligned}
$$

(b) Repeat part (a) for a surface described by the equation $x = h(y,z)$.

Solution.

The equation of the tangent plane to the suface at the point $(h(y_0,z_0), y_0, z_0)$ is:

$$x = h(y_0,z_0) + h_y(y-y_0) + h_z(z-z_0)$$

20. Suppose that $\mathbf{X} : D \to \mathbf{R}^3$ is a parameterized surface that is smooth at $\mathbf{X}(s_0,t_0)$. Show how the definition of the derivative $D\mathbf{X}(s_0,t_0)$ can be used to give vector parametric equations for the plane tangent to $S = \mathbf{X}(D)$ at the point $\mathbf{X}(s_0,t_0)$.

Solution.

Let $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$. The derivative $D\mathbf{X}(s,t)$ is:

$$
\begin{aligned}
D\mathbf{X}(s,t) &= \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{T}_s & \mathbf{T}_t \end{bmatrix}
\end{aligned}
$$

Hence, the equation of the tangent plane to surface $\mathbf{X}(s,t)$ at the point (s_0,t_0) is:

$$
\begin{aligned}
\mathbf{x}(s,t) &= \mathbf{X}(s_0,t_0) + D\mathbf{X}(s_0,t_0) \cdot \begin{bmatrix} s-s_0 \\ t-t_0 \end{bmatrix} \\
&= \mathbf{X}(s_0,t_0) + \mathbf{T}_s(s_0,t_0)(s-s_0) + \mathbf{T}_t(s_0,t_0)(t-t_0)
\end{aligned}
$$

Surface Integrals

Scalar Surface Integrals

Definition. Let $\mathbf{X} : D \to \mathbf{R}^3$ be a smooth parametrized surface whose domain $D \subset \mathbf{R}^2$ is a bounded region. Let f be a continuous function whose domain includes $S = \mathbf{X}(D)$. Then, the scalar surface integral of f along $\mathbf{X}$ is:

$$\begin{aligned}\int\int_{\mathbf{X}} f \cdot dS &= \int\int_D f(\mathbf{X}(s,t))\,\|\mathbf{T}_s \times \mathbf{T}_t\|\,ds\,dt \\ &= \int\int_D f(\mathbf{X}(s,t))\,\|\mathbf{N}(s,t)\|\,ds\,dt\end{aligned}$$

$\mathbf{X}$ maps any rectangle with sides ds, dt and area $ds \cdot dt$ in D to a parallelogram in $\mathbf{X}(D)$ with sides $ds\mathbf{T}_s$ and $dt\mathbf{T}_t$. The area of the parallelogram is the cross-product $\|\mathbf{T}_s \times \mathbf{T}_t\|\,ds\,dt$. Thus,$\|\mathbf{T}_s \times \mathbf{T}_t\|$ is the scaling factor.

If f is identically 1 on all of $\mathbf{X}(D)$ then:

$$\int\int_{\mathbf{X}} f \cdot dS = \int\int_D 1\,\|\mathbf{T}_s \times \mathbf{T}_t\|\,ds\,dt = \text{Surface area of } \mathbf{X}(D)$$

The scalar surface integral is thus a generalization of the integral we use to calculate the surface area. We can think of $\int\int_{\mathbf{X}} f \cdot dS$ as a limit of the weighted sum of the surface area pieces, the weightings given by f. If f represents the mass or electrical charge density, then $\int\int_{\mathbf{X}} f \cdot dS$ yields the total mass or the total charge on $\mathbf{X}(D)$.

For computational purposes, recall that if we write the components of $\mathbf{X}$ as:

$$\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$$

then

$$\begin{aligned}\mathbf{N}(s,t) &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \\ &= \mathbf{i}\begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \\ &= \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k}\end{aligned}$$

So, we obtain:

$$\int\int_{\mathbf{X}} f \cdot dS = \int\int_D f(x(s,t), y(s,t), z(s,t))\sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\,ds\,dt$$

If the surface S is given by a graph of $z = g(x,y)$ where g is of class C^1 on some region D in $\mathbf{R}^2$, then S is parameterized by $\mathbf{X}(x,y) = (x, y, g(x,y))$ with $(x,y) \in D$. Then, from example 13 in section 7.1,

$$\mathbf{N}(x,y) = -g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}$$

so that:

$$\int\int_{\mathbf{X}} f \cdot dS = \int\int_D f(x, y, g(x,y))\sqrt{1 + g_x^2 + g_y^2}\,dx\,dy$$

Vector Surface Integrals

We can develop a means to integrate vector fields along surfaces beginning with the definition.

Definition. Let $\mathbf{X} : D \to \mathbf{R}^3$ be a smooth parameterized surface, where D is a bounded region in the $\mathbf{R}^2$ plane, and let $\mathbf{F}(x, y, z)$ be a continuous vector field whose domain includes $S = \mathbf{X}(D)$. Then, the vector surface integral of $\mathbf{F}$ along $\mathbf{X}$ is:

$$\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{D} \mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{N}(s, t) ds\ dt$$

where $\mathbf{N}(s, t) = \mathbf{T}_s \times \mathbf{T}_t$.

As with line integrals, we should be careful about the notation for surface integrals. In the vector surface integral $\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, the differential term is considered to be a vector quantity, whereas in the scalar surface integral $\int\int_{\mathbf{X}} f \cdot dS$, the differential term is scalar quantity - the differential of the surface area.

Example. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z - 2y)\mathbf{k}$. We evaluate $\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{X}$ is the helicoid

$$\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$$

where $0 \leq s \leq 1, 0 \leq t \leq 2\pi$.

Solution.

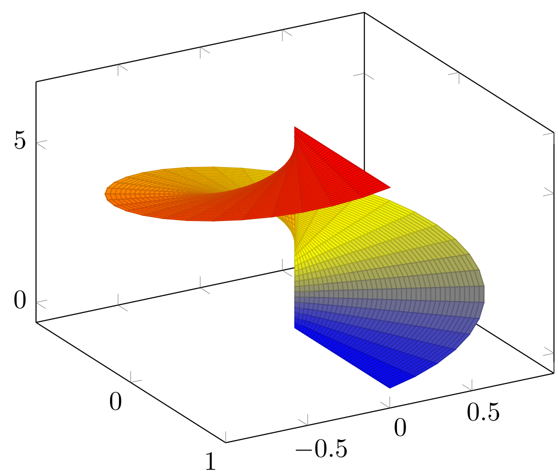

Figure. $\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$

We have:

$$\mathbf{T}_s = (\cos t, \sin t, 0)$$
$$\mathbf{T}_t = (-s \sin t, s \cos t, 1)$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s \sin t & s \cos t & 1 \end{vmatrix} \\
&= \sin t\mathbf{i} - \cos t\mathbf{j} + s\mathbf{k}
\end{aligned}$$

Hence:

$$\begin{aligned}
\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 (s\cos t, s\sin t, t - 2s\sin t) \cdot (\sin t, -\cos t, s) ds dt \\
&= \int_0^{2\pi} \int_0^1 (s \sin t \cos t - s \sin t \cos t + st - 2s^2 \sin t) ds dt \\
&= \int_0^{2\pi} \int_0^1 (st - 2s^2 \sin t) ds dt \\
&= \int_0^{2\pi} [\frac{s^2}{2} t - \frac{2s^3}{3} \sin t]_0^1 dt \\
&= \int_0^{2\pi} (\frac{t}{2} - \frac{2}{3} \sin t) dt \\
&= \left[\frac{t^2}{4} + \frac{2}{3} \cos t\right]_0^{2\pi} \\
&= \pi^2
\end{aligned}$$

Further Interpretations

As is the case for vector and scalar line integrals, there is a connection between vector and scalar surface integrals. Suppose $\mathbf{X} : D \to \mathbf{R}^3$ is a smooth parametrized surface and $\mathbf{F}$ is continuous on $S = \mathbf{X}(D)$. Let $\mathbf{N}(s,t) = \mathbf{T}_s \times \mathbf{T}_t$ be the usual normal vector and let:

$$\mathbf{n}(s,t) = \frac{\mathbf{N}(s,t)}{\|\mathbf{N}(s,t)\|}$$

That is, $\mathbf{n}$ is the unit vector pointing in the same direction as $\mathbf{N}$. In particular,

$$\mathbf{N}(s,t) = \|\mathbf{N}(s,t)\| \, \mathbf{n}(s,t)$$

Plugging this into the definition of the vector surface integral, we get:

$$\begin{aligned}
\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{N}(s,t) ds \cdot dt \\
&= \int\int_D \mathbf{F}(\mathbf{X}(D)) \cdot \|\mathbf{N}(s,t)\| \, \mathbf{n}(s,t) ds \cdot dt \\
&= \int\int_D (\mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{n}(s,t)) \, \|\mathbf{N}(s,t)\| \, ds \cdot dt \\
&= \int\int_{\mathbf{X}} (\mathbf{F} \cdot \mathbf{n}) dS
\end{aligned}$$

Since $\mathbf{n}$ is a unit vector, the quantity $\mathbf{F} \cdot \mathbf{n}$ is precisely the component of the vector field $\mathbf{F}$ in the direction of $\mathbf{n}$. In other words, the vector surface integral of $\mathbf{F}$ along $\mathbf{X}$ is the scalar surface integral of the component of $\mathbf{F}$ normal to $S = \mathbf{X}(D)$. Recall that, the vector line integral of $\mathbf{F}$ along a path $\mathbf{x}$ is the scalar line integral of the component of $\mathbf{F}$ tangent to the image curve. To summarize, we have the following results:

$$\text{Line Integrals :} \quad \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \quad = \int_0^t (\mathbf{F} \cdot \mathbf{T}) ds \tag{1}$$

$$\text{Surface Integrals :} \int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int\int_D (\mathbf{F} \cdot \mathbf{n}) \cdot ds \cdot dt \tag{2}$$

The vector line integral $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ in equation (1) is called the flow-integral of $\mathbf{F}$ along $\mathbf{x}$. The reason for this is the following.

Suppose $\mathbf{F}$ represents the velocity vector field of a fluid. Consider the amount of fluid moved tangentially along a small segment of the path $\mathbf{x}$ during a brief time interval $\Delta\tau$.

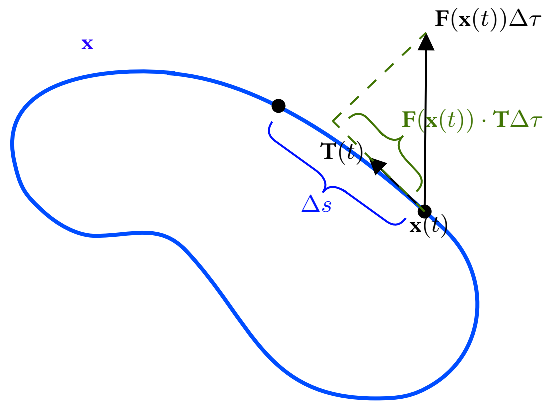

Figure. The amount of fluid transported tangentially along a segment of the closed path $\mathbf{x}$ is approximately $(\mathbf{F}(\mathbf{x}(t))\Delta\tau \cdot \mathbf{T})\Delta s$

Since $\mathbf{F} \cdot \mathbf{T}$ gives the tangential component of the velocity vector $\mathbf{F}$, the rate of flow at the point $\mathbf{x}(t)$ is $\mathbf{F} \cdot \mathbf{T}$. So, the amount of fluid transported tangentially in $\Delta\tau$ time through the point $\mathbf{x}(t)$ is Rate of flow $\times$ time $= (\mathbf{F}\Delta\tau) \cdot \mathbf{T}$. A line segment of the path $\mathbf{x}(t)$ of length Δs can be thought to be made up of Δs such points. So, the total amount of fluid transported tangentially in $\Delta\tau$ time along a segment of length Δs equals $(\mathbf{F}\Delta\tau) \cdot \mathbf{T}\Delta s$.

$$\text{Amount of fluid moved} \approx (\mathbf{F}(\mathbf{x}(t))\Delta\tau \cdot \mathbf{T})\Delta s$$

If we divide the above expression by time $\Delta\tau$, we get the average rate of transport of the fluid along the segment :

$$\frac{\Delta L}{\Delta\tau} \approx \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T} \cdot ds$$

$$\text{Instantaneous rate of fluid flow} = \lim_{\Delta\tau \to 0} \frac{\Delta L}{\Delta\tau} = \frac{dL}{d\tau} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T} \cdot ds$$

If $\mathbf{x}$ is a closed path, the flow-integral is also called circulation.

Now, let's try to interpret the vector surface integral in equation (2).

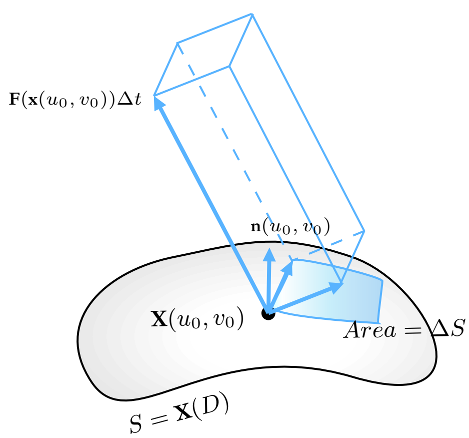

Figure. The amount of fluid transported across a small piece of S during a brief time interval Δt may be approximated by the volume of the parallelopiped.

Consider a small piece of S, having the area ΔS and the amount of fluid transported across it during a brief time interval Δt. This amount is the volume determined by $\mathbf{F}$ during Δt. The above figure suggests that this volume can be approximated by the volume of an appropriate parallelopiped.

The height of the parallelopiped is the normal component of $\mathbf{F}\Delta t$, $\mathbf{F}\Delta t \cdot \mathbf{n}(u_0, v_0)$ and the area of the base is ΔS. Hence:

$$\begin{aligned}\text{Amount of fluid displaced} &\approx \text{volume of parallelopiped}\\ &= (\text{height})(\text{area of base})\\ &= \mathbf{F}(\mathbf{X}(u_0,v_0))\Delta t\cdot \mathbf{n}(u_0,v_0)\Delta S \end{aligned} \tag{3}$$

We obtain the average rate of transport across the surface piece during the time interval Δt by dividing (3) by Δt:

$$\text{Average rate of transport} \approx \mathbf{F}(\mathbf{X}(u_0,v_0))\cdot \mathbf{n}(u_0,v_0)\Delta S \tag{4}$$

Now, we break up the entire surface $S = \mathbf{X}(D)$ into infinitely many such small partitions ΔS_{ij} and sum the corresponding contributions to the rate of transport in the form given (4). If we let all the pieces shrink, then, in the limit as all $\Delta S \to 0$, we have that the total average rate of transport of the fluid during Δt is approximately :

$$\frac{\Delta M}{\Delta t} \approx \int\int_{\mathbf{X}} (\mathbf{F}(\mathbf{X}(D))\cdot \mathbf{n})dS \tag{5}$$

Passing to the limits as $\Delta t \to 0$, the instantaneous rate of fluid transport across $\mathbf{X}$ can be defined as the flow integral:

$$\lim_{\Delta t\to 0}\frac{\Delta M}{\Delta t} = \frac{dM}{dt} = \int\int_{\mathbf{X}} (\mathbf{F}(\mathbf{X}(D))\cdot \mathbf{n})dS \tag{6}$$

Reparametrization of surfaces

Definition. Let $\mathbf{X} : D_1 \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ and $\mathbf{Y} : D_2 \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ be parametrized surfaces. $\mathbf{Y}$ is said to be a reparametrization of $\mathbf{X}$, if there exists a one-to-one and onto function $\mathbf{H} : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathbf{Y} = \mathbf{X}(\mathbf{H}(s,t))$.

Example. Suppose $\mathbf{X}$ is a smooth parametrized surface. Let $\mathbf{Y}(s,t) = \mathbf{X}(u,v)$, where $u = t, v = s$. That is, $\mathbf{Y} = \mathbf{X}(\mathbf{H})$. Then, $\mathbf{Y}$ is a smooth parametrization that appears to accomplish little. However, if we let $\mathbf{N_Y}$ denote the usual normal vector $\mathbf{T}_s \times \mathbf{T}_t = \frac{\partial \mathbf{Y}}{\partial s} \times \frac{\partial \mathbf{Y}}{\partial t}$, then we have:

$$\frac{\partial \mathbf{Y}}{\partial s} = \frac{\partial \mathbf{X}}{\partial v} \quad \text{and} \quad \frac{\partial \mathbf{Y}}{\partial t} = \frac{\partial \mathbf{X}}{\partial u}$$

so that:

$$\begin{aligned}\mathbf{N_Y} &= \frac{\partial \mathbf{Y}}{\partial s}\times\frac{\partial \mathbf{Y}}{\partial t}\\ &= \frac{\partial \mathbf{X}}{\partial v}\times\frac{\partial \mathbf{X}}{\partial u}\\ &= -\frac{\partial \mathbf{X}}{\partial u}\times\frac{\partial \mathbf{X}}{\partial v}\\ &= -\mathbf{N_X}\end{aligned}$$

The parametrized surface $\mathbf{Y}$ is the same as $\mathbf{X}$, except that the standard normal vector arising from $\mathbf{Y}$ points in the opposite direction to the one arising from $\mathbf{X}$.

The calculation in the above example thus generalizes. Suppose $\mathbf{X}$ is a smooth parametrized surface and $\mathbf{Y}$ is a smooth reparametrization of $\mathbf{X}$ via $\mathbf{H}$, that is:

$$\mathbf{Y}(s,t) = \mathbf{X}(u,v) = \mathbf{X}(\mathbf{H}(s,t))$$

By the chain rule:

$$
\begin{aligned}
D\mathbf{Y}(s,t) &= D\mathbf{X}(u,v)\cdot D\mathbf{H}(s,t)\\
\begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} &= \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}\begin{bmatrix} u_s & u_t \\ v_s & v_t \end{bmatrix}
\end{aligned}
$$

From the above, we see that:

$$
\begin{aligned}
\mathbf{T}_s &= u_s\mathbf{T}_u + v_s\mathbf{T}_v\\
\mathbf{T}_t &= u_t\mathbf{T}_u + v_t\mathbf{T}_v
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathbf{T}_s \times \mathbf{T}_t &= (u_s\mathbf{T}_u + v_s\mathbf{T}_v) \times (u_t\mathbf{T}_u + v_t\mathbf{T}_v)\\
&= u_s u_t \mathbf{T}_u \times \mathbf{T}_u + u_s v_t \mathbf{T}_u \times \mathbf{T}_v + v_s u_t \mathbf{T}_v \times \mathbf{T}_u + v_s v_t \mathbf{T}_v \times \mathbf{T}_v\\
&= (u_s v_t - u_t v_s)\mathbf{T}_u \times \mathbf{T}_v\\
\mathbf{N}_\mathbf{Y} &= \frac{\partial(u,v)}{\partial(s,t)}\mathbf{N}_\mathbf{X}
\end{aligned}
$$

Thus, $\mathbf{N}_\mathbf{Y}$ is always a scalar multiple of $\mathbf{N}_\mathbf{X}$. In addition, since $\mathbf{H}$ is invertible and both $\mathbf{H}$ and $\mathbf{H}^{-1}$ are of class C^1, it follows that the jacobian (determinant) is always positive or negative. Hence, the standard normal vector $\mathbf{N}_\mathbf{Y}$ either always points in the same direction as $\mathbf{N}_\mathbf{X}$or else always points in the opposite direction. Under thse assumptions, we say that both $\mathbf{H}$ and $\mathbf{Y}$ are **orientiation-preserving** if the Jacobian $\frac{\partial(u,v)}{\partial(s,t)}$ is positive, **orientation-reversing** if $\partial(u,v)/\partial(s,t)$ is negative.

Theorem. *Let $\mathbf{X} : D_1 \to \mathbf{R}^3$ be a smooth parametrized surface and f any continuous function whose domain includes $\mathbf{X}(D_1)$. If $\mathbf{Y} : D_2 \to \mathbf{R}^3$ is any smooth reparametrization of $\mathbf{X}$, then:*

$$
\int\int_\mathbf{Y} f \cdot dS = \int\int_\mathbf{X} f \cdot dS
$$

Proof.

We have:

$$
\begin{aligned}
\int\int_\mathbf{Y} f \cdot dS &= \int\int_{(s,t)\in D_2} f(\mathbf{Y}(s,t))\,\|\mathbf{N}_\mathbf{Y}(s,t)\|\,ds\,dt\\
&= \int\int_{(s,t)\in D_2} f(\mathbf{X}(u,v))\,\|\mathbf{N}_\mathbf{X}(u,v)\|\left|\frac{\partial(u,v)}{\partial(s,t)}\right| ds\,dt
\end{aligned}
$$

Since $u = u(s,t), v = v(s,t)$, by the change of variables theorem, $du \cdot dv = \left|\frac{\partial(u,v)}{\partial(s,t)}\right| ds \cdot dt$. So, we have:

$$
\begin{aligned}
\int\int_\mathbf{Y} f \cdot dS &= \int\int_{(u,v)\in D_1} f(\mathbf{X}(u,v))\,\|\mathbf{N}_\mathbf{X}(u,v)\|\,du\,dv\\
&= \int\int_\mathbf{X} f \cdot dS
\end{aligned}
$$

Theorem. *Let $\mathbf{X} : D_1 \to \mathbf{R}^3$ be a smooth parametrized surface and $\mathbf{F}$ be any continuous vector field whose domain includes $\mathbf{X}(D_1)$. If $\mathbf{Y} : D_2 \to \mathbf{R}^3$ is any smooth reparametrization of $\mathbf{X}$, then:*

(1) If $\mathbf{Y}$is orientation-preserving, we have:

$$\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$

(2) If $\mathbf{Y}$ *is orientiation-reversing, we have:*

$$\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = -\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$

Proof.

This result can be established along the lines of the previous proof. Beginning with the definition and using the lemma just established, we have:

$$\begin{aligned}\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{D_2} \mathbf{F}(\mathbf{Y}(s,t)) \cdot \mathbf{N}_{\mathbf{Y}}(s,t) ds\ dt \\ &= \int\int_{D_2} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \frac{\partial(u,v)}{\partial(x,y)} ds\ dt\end{aligned}$$

If $\mathbf{Y}$is orientation preserving, then $\partial(u,v)/\partial(x,y) > 0$, so $|\partial(u,v)/\partial(x,y)| = \partial(u,v)/\partial(x,y)$. Then, by change of variables theorem:

$$\begin{aligned}\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{(s,t)\in D_2} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| ds\ dt \\ &= \int\int_{(u,v)\in D_1} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \cdot du \cdot dv \\ &= \int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}\end{aligned}$$

If $\mathbf{Y}$ is orientation reversing, then $\partial(u,v)/\partial(x,y) < 0$. So, $|\partial(u,v)/\partial(x,y)| = -\partial(u,v)/\partial(x,y)$. Thus,

$$\begin{aligned}\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} &= -\int\int_{(s,t)\in D_2} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| ds\ dt \\ &= -\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}\end{aligned}$$

Given a smooth surface, we need to choose an orientation for it. This is akin to orienting a curve, but perhaps surprisingly, it is not always possible, even for a well-behaved, smooth parametrized surface, as the next example illustrates.

Definition. A smooth, connected surface S is **orientable** (or **two-sided**) if it is possible to define a single unit normal vector at each point of S so that the collection of these normal vectors varies continuously over S. (In particular, this means that unit normal vectors must point to the same side of S.) Otherwise, S is called as non-orientable.

Example. Consider the surface parametrized by

$$\begin{aligned}x &= (1 + t\cos(s/2))\cos s \\ y &= (1 + t\cos(s/2))\sin s \\ z &= t \sin s/2\end{aligned}$$

where $0 \le s \le 2\pi$ and $-1/2 \le t \le t$. This is called a **Mobius strip**. It may be visualized as follows : The t-coordinate curve at $s = s_0$ is:

$$\begin{aligned}x &= \cos(s_0/2)\cos(s_0)t + \cos(s_0) \\ y &= \cos(s_0/2)\sin(s_0)t + \sin(s_0) \\ z &= t(\sin s_0/2)\end{aligned}$$

If we isolate t in the above three equations, we find:

$$\frac{x - \cos s_0}{\cos(s_0/2)\cos s_0} = \frac{y - \sin s_0}{\cos(s_0/2)\sin s_0} = \frac{z - 0}{\sin(s_0/2)}$$

This is a plane that passes through the point $(\cos s_0, \sin s_0, 0)$ and parallel to the vector:

$$\mathbf{a} = (\cos(s_0/2)\cos s_0, \cos(s_0/2)\sin s_0, \sin s_0/2)$$

Consider a few such coordinate curves:

s	Parallel Vector	Passes through the Point	Plane
$s_0 = 0$	$\mathbf{a} = (1, 0, 0)$	$(1, 0, 0)$	$\mathbf{x} = t(1, 0, 0) + (1, 0, 0)$
$s_0 = \pi/2$	$\mathbf{a} = (0, 1/\sqrt{2}, 1/\sqrt{2})$	$(0, 1, 0)$	$\mathbf{x} = t(0, 1/\sqrt{2}, 1/\sqrt{2}) + (0, 1, 0)$
$s_0 = \pi$	$\mathbf{a} = (0, 0, 1)$	$(-1, 0, 0)$	$\mathbf{x} = t(0, 0, 1) + (-1, 0, 0)$
$s_0 = 3\pi/2$	$\mathbf{a} = (0, 1/\sqrt{2}, 1/\sqrt{2})$	$(0, -1, 0)$	$\mathbf{x} = t(0, 1/\sqrt{2}, 1/\sqrt{2}) + (0, -1, 0)$
$s_0 = 2\pi$	$\mathbf{a} = (-1, 0, 0)$	$(1, 0, 0)$	$\mathbf{x} = t(-1, 0, 0) + (1, 0, 0)$

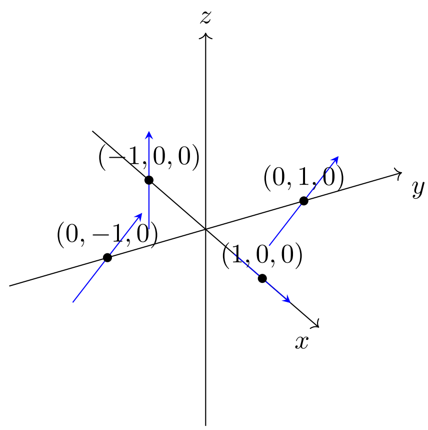

Figure. t-coordinate curves

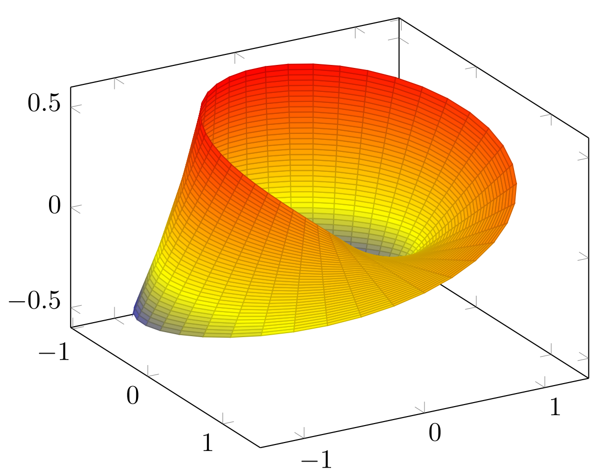

Figure. Mobius Strip

We see that the Mobius strip is generated by a moving line segment that begins at $s = 0$ lying along the positive x-axis, rises to a vertical position with the center at $(-1, 0, 0)$ when $s = \pi$ and then falls back to the center with direction reversal at $s = 2\pi$. The s-coordinate curve at $t = 0$ is parametrized by:

$$
\begin{aligned}
x &= \cos s \\
y &= \sin s \\
z &= 0
\end{aligned}
$$

and so is a circle in the xy-plane. The full Mobius strip is shown in the figure above. You can make a physical model by taking a strip of paper, giving it a half-twist, and joining the short-ends.

You can understand the gluing process analytically by noting that the map :

$$
\mathbf{X} : [0, 2\pi] \times [-\frac{1}{2}, -\frac{1}{2}] \to \mathbf{R}^3
$$

defining the Mobius strip as a parametrized surface has the property that $\mathbf{X}(0,t) = \mathbf{X}(2\pi, -t)$ but is otherwise one to one. Therefore, every point $(0,t)$ on the left edge of the domain rectangle $[0, 2\pi] \times [-\frac{1}{2}, \frac{1}{2}]$ is mapped to the point $(1+t, 0, 0)$ of the Mobius strip, as is the point $(2\pi, -t)$ on the right edge of the rectangle.

Now, let's investigate the orientability of the Mobius strip. Firstly,

$$
\begin{aligned}
x &= (1 + t\cos(s/2))\cos s \\
x_s &= -\sin s + t(\cos s/2 \cdot \cos s)_s \\
&= -\sin s + t(-\frac{1}{2}\sin s/2\cos s - \cos s/2 \sin s) \\
&= -\sin s - t((1/2)\sin s/2 \cos s + \cos s/2 \sin s) \\
x_t &= \cos(s/2)\cos s
\end{aligned}
$$

$$
\begin{aligned}
y &= (1 + t\cos(s/2))\sin s \\
&= \sin s + t\cos(s/2)\sin s \\
y_s &= \cos s + t(-1/2\sin s/2 \sin s + \cos s/2 \cos s) \\
&= \cos s + t(\cos s/2 \cos s - (1/2)\sin s/2 \sin s) \\
y_t &= \cos s/2 \sin s
\end{aligned}
$$

$$
\begin{aligned}
z &= t\sin s/2 \\
z_s &= t/2 \cos s/2 \\
z_t &= \sin s/2
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\frac{\partial(y,z)}{\partial(s,t)} &= y_s z_t - z_s y_t \\
&= (\cos s + t(\cos s/2 \cos s - (1/2)\sin s/2 \sin s))(\sin s/2) - (t/2\cos s/2)(\cos s/2 \sin s) \\
&= \cos s \sin s/2 + t\sin s/2 \cos s/2 \cos s - t/2 \sin^2 s/2 \sin s - t/2 \sin s \cos^2 s/2 \\
&= \cos s \sin s/2 + t\sin s/2 \cos s/2 \cos s - t/2 \sin s(\sin^2 s/2 + \cos^2 s/2) \\
&= \cos s \sin s/2 + t\sin s/2 \cos s/2 \cos s - t\sin s/2 \cos s/2 \\
&= \sin s/2(\cos s - t\cos s/2(1 - \cos s)) \\
&= \sin s/2(\cos s - 2t\cos s/2(\sin^2 s/2)) \\
&= \sin s/2(\cos s - 2t\cos s/2(1 - \cos^2 s/2)) \\
&= \sin s/2(\cos s - 2t(\cos s/2 - \cos^3 s/2))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(x,z)}{\partial(s,t)} &= x_s z_t - x_t z_s \\
&= (-\sin s - t((1/2)\sin s/2 \cos s + \cos s/2 \sin s)\sin s/2 - (\cos(s/2)\cos s)(t/2 \cos s/2) \\
&= -\sin s \sin s/2 - t/2 \sin^2 s/2 \cos s - t \sin s/2 \cos s/2 \sin s - t/2 \cos s \cos^2 s/2 \\
&= -\sin s \sin s/2 - t/2 \cos s - t/2 \sin^2 s \\
&= -2\sin^2 s/2 \cos s/2 - t/2 \cos s - t/2(1-\cos^2 s) \\
&= -\frac{1}{2}(4(1-\cos^2 s/2)\cos s/2 + t\cos s + t - t\cos^2 s) \\
&= -\frac{1}{2}(4\cos s/2 - 4\cos^3 s/2 + t(1+\cos s - \cos^2 s))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(x,y)}{\partial(s,t)} &= x_s y_t - x_t y_s \\
&= (-\sin s - t((1/2)\sin s/2 \cos s + \cos s/2 \sin s))(\cos s/2 \sin s) - (\cos(s/2)\cos s)(\cos s + t(\cos s/2 \cos s - (1/2)\sin s/2 \sin s)) \\
&= -\sin^2 s \cos s/2 - \cancel{t/2 \sin s/2 \cos s/2 \sin s \cos s} - t\cos^2 s/2(\sin^2 s + \cos^2 s) - \cos s/2 \cos^2 s + \cancel{t/2 \sin s/2 \cos s/2 \sin s \cos s} \\
&= -\cos s/2(\sin^2 s + \cos^2 s) - t\cos^2 s/2 \\
&= -\cos s/2(1 + t\cos s/2)
\end{aligned}
$$

The standard normal vector is:

$$
\begin{aligned}
\mathbf{N}(s,t) &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k} \\
&= \sin s/2(\cos s - 2t(\cos s/2 - \cos^3 s/2))\mathbf{i} \\
&+ \frac{1}{2}(4\cos s/2 - 4\cos^3 s/2 + t(1+\cos s - \cos^2 s))\mathbf{j} \\
&- \cos s/2(1 + t\cos s/2)\mathbf{k}
\end{aligned}
$$

We have:

$$
\mathbf{N}(0,t) = \frac{t}{2}\mathbf{j} - (1+t)\mathbf{k}
$$

and

$$
\mathbf{N}(2\pi,-t) = -\frac{t}{2}\mathbf{j} + (1+t)\mathbf{k} = -\mathbf{N}(0,t)
$$

Therefore, a uniquely determined normal vector has not been defined. More vividly, imagine travelling around the Mobius strip via the $s-$coordinate path at $t=0$, that is along the circular path

$$
\mathbf{x}(s) = \mathbf{X}(s,0) = (\cos s, \sin s, 0), \quad 0 \le s \le 2\pi
$$

Follow the standard normal $\mathbf{N}$. At $s=0$, it is $\mathbf{N}(0,0) = -\mathbf{k}$. But, by the time we close the loop, it is $\mathbf{N}(2\pi,0) = \mathbf{k}$. This apparent reversal of the normal vector means that the strip is not orientable at all.

A smooth orientable surface together with an explicit choice of orientation for it, is called an **oriented** surface. If S is a smooth oriented surface, then we define the vector surface integral of $\mathbf{F}$ along S by finding a smooth parametrization of $\mathbf{X}$ of S, such that the unit normal vector $\mathbf{n} = \mathbf{N}(s,t)/||\mathbf{N}(s,t)||$ arising from the parameterization agrees with the choice of the orientation normal. We take the surface integral to be

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$

By the theorem above, if $\mathbf{Y}$ is any orientation preserving reparametrization of $\mathbf{X}$, the value of $\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S}$ is the same as $\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, and so this notion of a surface integral over the underlying oriented surface S is well-defined. Even though we may perfectly well calculate $\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{X}$ is the parametrized Mobius strip of the previous example, it does not make sense to consider the surface integral over the underlying Mobius strip, since there is no way to orient it. Similarly, the interpreation of the vector surface integral as the total flux of $\mathbf{F}$ across the surface S only makes sense once an orientation of the surface is chosen. Then, the flux measures the flow rate, positive or negative, depending on the choice of orientation.

Another reason for de-emphasizing the role of parameterization in surface integrals is that we can often exploit the geometry of the underlying surface and vector field when making calculations. If S is a smooth, orientable surface and $\mathbf{n}$ a unit normal that gives an orientation of S (so, in particular $\mathbf{n}$ is understood to vary continuously with the points of S), then for a continuous vector field $\mathbf{F}$ defined on S, we have:

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_S \mathbf{F} \cdot \mathbf{n} dS$$

If we can determine a continuously varying unit normal vector at each point of S (for example, S is the graph of a function $f(x, y)$ of two variables or the graph of a level set $f(x, y, z) = c$ of a function of three variables), then there is a good chance that the surface integral can be evaluated readily.

Example. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a radial vector field and suppose S is the sphere of radius a with equation $x^2 + y^2 + z^2 = a^2$. Orient S by outwardpointing unit normal vectors as shown in the figure. We calculate the flux of $\mathbf{F}$ across S in two ways : (1) by means of parametrization of S and (2) via geometric considerations, that is, without resorting to explicit parametrization of the sphere.

Solution.

For approach (1), use the usual parametrization $\mathbf{X}$ of the sphere:

$$\begin{aligned} x &= a \cos s \sin t \\ y &= a \sin s \sin t \\ z &= a \cos t \end{aligned}$$

where $0 \le s \le 2\pi$ and $0 \le t \le \pi$.

The standard normal vector for this parametrization is given by:

$$\begin{aligned} \mathbf{T}_s &= (-a \sin s \sin t, a \cos s \sin t, 0) \\ \mathbf{T}_t &= (a \cos s \cos t, a \sin s \cos t, -a \sin t) \\ \mathbf{N}(s,t) &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin s \sin t & a \cos s \sin t & 0 \\ a \cos s \cos t & a \sin s \cos t & -a \sin t \end{vmatrix} \\ &= -a^2 \cos s \sin^2 t\mathbf{i} - a^2 \sin s \sin^2 t\mathbf{j} - a^2 \sin t \cos t(\sin^2 s + \cos^2 s)\mathbf{k} \\ &= -a^2 \sin t(\cos s \sin t\mathbf{i} + \sin s \sin t\mathbf{j} + \cos t\mathbf{k}) \end{aligned}$$

If we normalize $\mathbf{N}$, we find that:

$$\begin{aligned} \mathbf{n} &= \frac{\mathbf{N}(s,t)}{||\mathbf{N}(s,t)||} \\ &= -\frac{1}{\sin^2 t(\cos^2 s + \sin^2 s) + \cos^2 t}(\cos s \sin t, \sin s \sin t, \cos t) \\ &= -\frac{1}{\sin^2 t \cdot 1 + \cos^2 t}(\cos s \sin t, \sin s \sin t, \cos t) \\ &= -(\cos s \sin t, \sin s \sin t, \cos t) \end{aligned}$$

Thus, **n** is inward-pointing at every point on the sphere. Therefore, we must make a sign-change when we evaluate the vector surface integral, if we use the parameterization just given. Hence, we have:

$$
\begin{aligned}
\int\int_S \mathbf{F}\cdot d\mathbf{S} &= -\int\int_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} \\
&= -\int_0^{\pi}\int_0^{2\pi} \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{N}(s,t)dsdt \\
&= a^2\int_0^{\pi}\int_0^{2\pi} (a\cos s\sin t, a\sin s\sin t, a\cos t)\cdot \sin t(\cos s\sin t, \sin s\sin t, \cos t)dsdt \\
&= a^3\int_0^{\pi}\int_0^{2\pi} \sin t(\cos^2 s\sin^2 t + \sin^2 s\sin^2 t + \cos^2 t)dsdt \\
&= a^3\int_0^{\pi}\int_0^{2\pi} \sin t(\sin^2 t(\cos^2 s + \sin^2 s) + \cos^2 t)dsdt \\
&= a^3\int_0^{\pi}\int_0^{2\pi} \sin t(\sin^2 t + \cos^2 t)dsdt \\
&= a^3\int_0^{\pi}\int_0^{2\pi} \sin t dsdt \\
&= 2\pi a^3\int_0^{\pi} \sin t dt \\
&= 2\pi a^3[-\cos t]_0^{\pi} \\
&= 4\pi a^3
\end{aligned}
$$

Now, reconsider the calculation along the lines of the approach (2). Since S is defined as a level set of the function $f(x,y,z) = x^2+y^2+z^2$, normal vectors can be obtained from the gradient :

$$\nabla f(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

If we normalize the gradient, then we have the normal unit vectors:

$$
\begin{aligned}
\mathbf{n} &= \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} \\
&= \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} \\
&= \frac{(x,y,z)}{a}
\end{aligned}
$$

because $x^2+y^2+z^2 = a^2$ at all points on S. Note, that **n** is always outward pointing.

Therefore,

$$
\begin{aligned}
\int\int_S \mathbf{F}\cdot d\mathbf{S} &= \int\int_S \mathbf{F}\cdot\mathbf{n}dS \\
&= \frac{1}{a}\int\int_S (x,y,z)\cdot(x,y,z)dS \\
&= \frac{1}{a}\int\int_S (x^2+y^2+z^2)dS \\
&= \frac{1}{a}\int\int_S a^2 dS \\
&= a\int\int_S dS \\
&= a(4\pi a^2) \\
&= 4\pi a^3
\end{aligned}
$$

Example. We evaluate $\int\int_S (x^3\mathbf{i} + y^3\mathbf{j}) \cdot d\mathbf{S}$ where S is the closed cylinder bounded laterally by $x^2 + y^2 = 4$ and on bottom and top by the planes $z = 0$ and $z = 5$ oriented by outward normal vectors.

Solution.