A short note on Black Scholes

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Abstract

A short note on Black-Scholes.

Definition. (Risk-free Asset) The price process $(B(t) : t \in [0, T])$ is the price of a risk-free asset if it has the dynamics:

$$dB(t) = r(t)B(t)dt \tag{1}$$

where $r(t)$ is any adapted process.

The defining property of a risk-free asset is thus that it has no driving $dW(t)$ term. We can also write the B-dynamics as:

$$\frac{dB(t)}{dt} = r(t)B(t) \tag{2}$$

We can integrate this using ODE cookbook methods, using separation of variables, so

$$B(t) = B(0)e^{\int_0^t r(s)ds} \tag{3}$$

and as notational convention, we put

$$B(0) = 1 \tag{4}$$

The natural interpretation of a risk-free asset is that it corresponds to a bank account with (possibly stochastic) short interest rate $r(t)$. Note that, the bank-account is **locally risk-free**, in the sense that, even if the short rate is a random process, the return $r(t)$ over an infinitesimal time-period dt is risk-free (that is deterministic, given the infomation available at time t). However, the return of B over a **longer interval** such as $[t, T]$ is typically stochastic.

Definition. The **Black-Scholes model** consists of two assets with dynamics given by:

$$dB(t) = rB(t)dt \tag{5}$$
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{6}$$

where r, μ and σ are deterministic constants.

Definition. (**Self-financing Trading Strategy**). Consider a trading strategy $h(t)$ of two-assets with relative weights $(w_B(t), w_S(t))$ at time t, with asset prices $(B(t), S(t))$. The value of the portfolio at time t is:

$$V^h(t) = w_B(t)B(t) + w_S(t)S(t) \tag{7}$$

The strategy h is called self-financing if:

$$\frac{dV^h(t)}{V^h(t)} = w_B(t)\frac{dB(t)}{B(t)} + w_S(t)\frac{dS(t)}{S(t)} \tag{8}$$

which implies that the change in the portfolio value from time t to $t + dt$ is only attributed to the changes in $S(t)$ and $B(t)$ and not to either infusion of extraction of funds.

Remark. For any sequence of real numbers (x_n) and (y_n), we have:

$$\Delta(xy)_n = x_n \Delta y_n + y_{n+1}\Delta x_n \tag{9}$$

Thus, in discrete time, the total change in the portfolio value is:

$$\Delta V_n = [w_B(n), w_S(n)] \cdot [B_{n+1} - B_n, S_{n+1} - S_n] + [B_{n+1}, S_{n+1}] \cdot [w_B(n+1) - w_B(n), w_S(n+1) - w_S(n)] \tag{10}$$

But, we are restricted from doing any exogenous infusion or extraction of cash. So, at time $n + 1$, we have the budget constraint:

$$\begin{aligned} w_B(n)B_{n+1} + w_S(n)S_{n+1} &= w_B(n+1)B_{n+1} + w_S(n+1)S_{n+1} \\ [B_{n+1}, S_{n+1}] \cdot [w_B(n+1) - w_B(n), w_S(n+1) - w_S(n)] &= 0 \end{aligned} \tag{11}$$

So, the second term on the right hand side of the expression for ΔV_n is 0. Hence,

$$\Delta V_n = w_B(n)\Delta B_n + w_S(n)\Delta S_n \tag{12}$$

Definition. (**Arbitrage Strategy**). An arbitrage strategy on a financial market is a self-financing portfolio h such that:

$$\begin{aligned} V^h(0) &= 0 \\ \mathbb{P}(V_T^h \geq 0) &= 1 \\ \mathbb{P}(V_T^h > 0) &> 0 \end{aligned} \tag{13}$$

The question is how do we identify an arbitrage strategy? There's a partial result that suffices for our purposes.

Proposition. *Suppose there exists a self-financing portfolio h, such that the value process V^h has the dynamics:*

$$dV^h(t) = k(t)V(t)dt \tag{14}$$

where $k(t)$ is an adapted process. Then, it must hold that $k(t) = r(t)$ for all t.

We sketch the argument, and assume for simplicity that k and r are constant and that $k > r$. Then, we can borrow \$1 from the bank account at $r(t)$ and invest it in the portfolio h for a small time period dt. The net value of investment is 0. The value of h at the end of this time period dt is $1 + k(t) \cdot dt$ and we have incurred a debt of $1 + r(t)dt$. So, we earn a risk-less profit $k(t) = r(t)$. Similarly, if $k < r$, then we can short-sell the portfolio h and deposit 1 dollar in the bank account. Again, we earn a riskless profit. Hence, if there is to be no-arbitrage, $k(t) = r(t)$.

The main point of the above is that if a portfolio has a value process that contains no driving brownian motion $W(t)$, that is a **locally risk-free portfolio**, then the rate of return of that portfolio must equal the short rate of interest.

The Black-Scholes Equation.

Consider a simple contingent claim χ of the form:

$$\chi = \Phi(S(t)) \tag{15}$$

and we assume that this claim can be traded on a market and that its price process has the form:

$$\Pi_t[\Phi] = F(t, S(t)) \tag{16}$$

1. We start by computing the dynamics of the derivative asset, and Ito's lemma applied to $V(t, S(t))$ gives:

$$\begin{aligned}
dF(t,S(t)) &= F_t(t,S_t)dt + F_x(t,S_t)dS_t + \frac{1}{2}F_{xx}(t,S_t) < S,S >_t \\
&= \frac{\partial F}{\partial t}(t,S(t)) + \frac{\partial F}{\partial x}(t,S_t)(\mu S(t)dt + \sigma S(t)dW(t)) + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 F}{\partial x^2}(t,S(t))dt \\
&= \left[\frac{\partial F}{\partial t}(t,S(t)) + \mu S(t)\frac{\partial F}{\partial x}(t,S(t)) + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 F}{\partial x^2}(t,S(t))\right]dt + \sigma S(t)\frac{\partial F}{\partial x}(t,S_t)dW(t) \qquad (17)
\end{aligned}$$

2. Let us now form a portfolio of two assets : the underlying stock and the derivative asset. Denote the relative portfolio by (w^S, w^F). We restrict this portfolio to be self-financing. Also, we suppress $(t, S(t))$ So, we must have:

$$\begin{aligned}
\frac{dV}{V} &= w^S\frac{dS}{S} + w^F\frac{dF}{F} \\
&= w^S(\mu dt + \sigma dW(t)) + w^F dF/F \\
&= \left[w^S\mu + \frac{w^F}{F}\left(\frac{\partial F}{\partial t} + \mu S\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2}\right)\right]dt \\
&+ \left[(w^S\sigma + w^F\frac{\sigma S}{F}\frac{\partial F}{\partial x})\right]dW(t) \qquad (18)
\end{aligned}$$

3. We define the relative portfolio by the linear system of equations:

$$w^S + w^F = 1 \tag{19}$$

$$w^S + w^F\frac{S}{F}F_x = 0 \tag{20}$$

for all $x \geq 0$ and $t \in [0, T)$.

We can solve for the relative portfolio weights:

$$w^F = \frac{F}{F - SF_x}, w^S = -\frac{SF_x}{F - SF_x} \tag{21}$$

Now, substituting (20) yields:

$$dV = V\left[w^S\mu + \frac{w^F}{F}\left(\frac{\partial F}{\partial t} + \mu S\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2}\right)\right]dt \tag{22}$$

4. In absence of arbitrage, by the proposition above,

$$
\begin{aligned}
w^S\mu + \frac{w^F}{F}\left(\frac{\partial F}{\partial t} + \mu S\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2}\right) &= r \\
\underbrace{w^S\mu + w^F\frac{\mu S}{F}\frac{\partial F}{\partial x}}_{\text{Equals } 0} + \frac{w^F}{F}\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2}\right) &= r \\
w^F\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2}\right) - rF &= 0 \\
\frac{F}{F - SF_x}\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2}\right) - rF &= 0 \\
\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2}\right) - r(F - SF_x) &= 0 \\
\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial x^2} &= rF \qquad (23)
\end{aligned}
$$

Thus, F satisfies the above PDE for all $x \geq 0$ and $t \in [0, T)$.