Stoke's and Gauss's Theorem

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Exercise Problems.

1. Let $X : \mathbf{R}^2 \to \mathbf{R}^3$ be the parametrized surface given by

$$X(s,t) = (s^2 - t^2, s + t, s^2 + 3t)$$

(a) Determine a normal vector to this surface at the point

$$(3,1,1) = \mathbf{X}(2,-1)$$

Solution.

We have:

$$\begin{aligned}
\mathbf{T}_s &= (2s, 1, 2s) \\
\mathbf{T}_t &= (-2t, 1, 3)
\end{aligned}$$

So, the standard normal vector at the point $\mathbf{X}(2,-1)$ is:

$$\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s & 1 & 2s \\ -2t & 1 & 3 \end{vmatrix} \\
&= \mathbf{i}(3 - 2s) - \mathbf{j}(6s + 4st) + \mathbf{k}(2s + 2t) \\
&= \mathbf{i}(3 - 4) - \mathbf{j}(12 + 4(2)(-1)) + \mathbf{k}(4 - 2) \\
&= -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

(b) Find an equation for the plane tangent to this surface at the point $(3,1,1)$.

Solution.

The tangent plane to this surface at the point $(3,1,1)$ is given by:

$$\begin{aligned}
\mathbf{N} \cdot (\mathbf{x} - (3,1,1)) &= 0 \\
(-1,-4,2) \cdot ((x,y,z) - (3,1,1)) &= 0 \\
-(x-3) - 4(y-1) + 2(z-1) &= 0
\end{aligned}$$

2. Find an equation for the plane tangent to the torus

$$\mathbf{X}(s,t) = ((5 + 2\cos t)\cos s, (5 + 2\cos t)\sin s, 2\sin t)$$

at the point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (-(5+2\cos t)\sin s, (5+2\cos t)\cos s, 0)\\ \mathbf{T}_t &= (-2\sin t\cos s, -2\sin t\sin s, 2\cos t)\end{aligned}$$

The standard normal vector is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ -(5+2\cos t)\sin s & (5+2\cos t)\cos s & 0\\ -2\sin t\cos s & -2\sin t\sin s & 2\cos t\end{vmatrix}\\ &= \mathbf{i}(2(5+2\cos t)\cos s\cos t)) + \mathbf{j}(2(5+2\cos t)\sin s\cos t)\\ &+ \mathbf{k}(2\sin s\sin t(5+2\cos t) + 2(5+2\cos t)\sin t\cos^2 s)\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i} + \sin s\cos t\mathbf{j} + (\sin^2 s+\cos^2 s)\sin t\mathbf{k})\\ &= 2(5+2\cos t)(\cos s\cos t\mathbf{i} + \sin s\cos t\mathbf{j} + \sin t\mathbf{k})\end{aligned}$$

The point $((5-\sqrt{3})/\sqrt{2},(5-\sqrt{3})/\sqrt{2},1) = ((5+2\cos t)\cos s,(5+2\cos t)\sin s, 2\sin t)$ yields $\sin t = 1/2$, so $t_0 = \pi/6$ or $t_0 = 5\pi/6$. Since $2\cos t < 0$, $t_0 = 5\pi/6$. Then, we can see that :

$$\frac{5-\sqrt{3}}{\sqrt{2}} = (5 - 2\cdot\frac{\sqrt{3}}{2})\sin s$$

So, $s_0 = \pi/4$.

Consequently, the equation of the tangent plane at $\mathbf{X}(\pi/4, 5\pi/6)$ is:

$$\begin{aligned}\mathbf{N}\cdot(\mathbf{x}-\mathbf{X}(s_0,t_0)) &= 0\\ 2(5-\sqrt{3})(-\frac{\sqrt{3}}{2\sqrt{2}}\mathbf{i} - \frac{\sqrt{3}}{2\sqrt{2}}\mathbf{j} + \frac{1}{2}\mathbf{k})((x,y,z) - (\frac{5-\sqrt{3}}{\sqrt{2}}, \frac{5-\sqrt{3}}{\sqrt{2}}, 1)) &= 0\\ -\frac{\sqrt{3}}{\sqrt{2}}(x-(5-\sqrt{3})/\sqrt{2}) - \frac{\sqrt{3}}{\sqrt{2}}(y-(5-\sqrt{3})/\sqrt{2}) + (z-1) &= 0\\ -\sqrt{3}(x-(5-\sqrt{3})/\sqrt{2}) - \sqrt{3}(y-(5-\sqrt{3})/\sqrt{2}) + \sqrt{2}(z-1) &= 0\\ -\sqrt{3}x - \sqrt{3}y + \sqrt{2}z &= -2\sqrt{3}(5-\sqrt{3})/\sqrt{2} + \sqrt{2}\\ &= -\sqrt{6}(5-\sqrt{3}) + \sqrt{2}\\ &= -5\sqrt{6} + 3\sqrt{2} + \sqrt{2}\\ \sqrt{3}x + \sqrt{3}y - \sqrt{2}z &= 5\sqrt{6} - 4\sqrt{2}\end{aligned}$$

3. Find an equation of the plane tangent to the surface

$$x = e^s \quad y = t^2 e^{2s} \quad z = 2e^{-s} + t$$

at the point $(1,4,0)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (e^s, 2t^2e^{2s}, -2e^{-s})\\ \mathbf{T}_t &= (0, 2te^{2s}, 1)\end{aligned}$$

The standard normal vector is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ e^s & 2t^2e^{2s} & -2e^{-s}\\ 0 & 2te^{2s} & 1\end{vmatrix}\\ &= \mathbf{i}(2t^2e^{2s} + 4te^s) - \mathbf{j}(e^s) + \mathbf{k}(2te^{3s})\\ &= e^s((2t^2e^s + 4t)\mathbf{i} - \mathbf{j} + (2te^{2s})\mathbf{k})\end{aligned}$$

Since $e^s = 1$, $s = 0$. Also, as $4 = t^2 \cdot 1$, we have $t = \pm 2$. Moreover, $0 = 2 + t$, so $t = -2$. So, $\mathbf{N}(0, -2)$ is:

$$\mathbf{N}(0, -2) = -\mathbf{j} - 4\mathbf{k}$$

The equation of the tangent plane at $\mathbf{X}(0, -2)$ is:

$$\begin{aligned}\mathbf{N} \cdot (x - 1, y - 4, z) &= 0\\ -(y - 4) - 4z &= 0\\ y + 4z &= 4\end{aligned}$$

4. Let $\mathbf{X}(s, t) = (s^2 \cos t, s^2 \sin t, s)$, $-3 \leq s \leq 3, 0 \leq t \leq 2\pi$.

(a) Find a normal vector at $(s, t) = (-1, 0)$.

Solution.

We have:

$$\begin{aligned}\mathbf{T}_s &= (2s\cos t, 2s\sin t, 1)\\ \mathbf{T}_t &= (-s^2 \sin t, s^2 \cos t, 0)\end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned}\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2s\cos t & 2s\sin t & 1\\ -s^2\sin t & s^2\cos t & 0\end{vmatrix}\\ &= \mathbf{i}(-s^2\cos t) - \mathbf{j}(s^2\sin t) + \mathbf{k}(2s^3\cos^2 t + 2s^3\sin^2 t)\\ &= -s^2\cos t\mathbf{i} - s^2\sin t\mathbf{j} + 2s^3\mathbf{k}\end{aligned}$$

The vector at $(s, t) = (-1, 0)$ is:

$$\mathbf{N}(-1, 0) = -\mathbf{i} - 2\mathbf{k}$$

Hence, the equation of the tangent plane at $\mathbf{X}(-1, 0) = (1, 0, -1)$ is:

$$
\begin{aligned}
(-\mathbf{i} - 2\mathbf{k}) \cdot ((x, y, z) - (1, 0, -1)) &= 0 \\
-(x-1) - 2(z+1) &= 0 \\
x - 1 + 2z + 2 &= 0 \\
x + 2z + 1 &= 0
\end{aligned}
$$

(b) Find an equation for the image of $\mathbf{X}$ in the form $F(x, y, z) = 0$.

Solution.

Let $x = s^2 \cos t$, $y = s^2 \sin t$. Then, $x^2 + y^2 = s^4(\cos^2 t + \sin^2 t) = s^4 = z^4$. So, $F(x, y, z) = x^2 + y^2 - z^4 = 0$.

5. Consider the parameterized surface $\mathbf{X}(s, t) = (s, s^2 + t, t^2)$.

(a) Graph the surface for $-2 \leq s \leq 2$, $-2 \leq t \leq 2$.

Solution.

The s-coordinate curve at $t = 0$ is:

$$
\begin{aligned}
x &= s \\
y &= s^2 \\
z &= 0
\end{aligned}
$$

This is the parabolic curve $y = x^2$ in the xy-plane.

The s-coordinate curve at $t = t_0$ is:

$$
\begin{aligned}
x &= s \\
y &= s^2 + t_0 \\
z &= t_0^2
\end{aligned}
$$

Thus, we get parabolas parallel to the xy-plane.

t	Curve	Center	z-plane
$t_0 = -2$	$y + 2 = x^2$	$(x, y) = (0, -2)$	$z = 4$
$t_0 = -1$	$y + 1 = x^2$	$(x, y) = (0, -1)$	$z = 1$
$t_0 = 0$	$y = x^2$	$(x, y) = (0, 0)$	$z = 0$
$t_0 = 1$	$y - 1 = x^2$	$(x, y) = (0, 1)$	$z = 1$
$t_0 = 2$	$y - 2 = x^2$	$(x, y) = (0, 2)$	$z = 4$

The $t-$coordinate curve at $s = 0$ is:

$$
\begin{aligned}
x &= 0 \\
y &= t \\
z &= t^2
\end{aligned}
$$

These are parabolas parallel to the yz-plane.

t	Curve	Center	x-plane
$s_0 = -2$	$z = (y-4)^2$	$(y, z) = (2, 0)$	$x = -2$
$s_0 = -1$	$z = (y-1)^2$	$(y, z) = (1, 0)$	$x = -1$
$s_0 = 0$	$z = y^2$	$(y, z) = (0, 0)$	$x = 0$
$s_0 = 1$	$z = (y-1)^2$	$(y, z) = (1, 0)$	$x = 1$
$s_0 = 2$	$z = (y-4)^2$	$(y, z) = (2, 0)$	$x = 2$

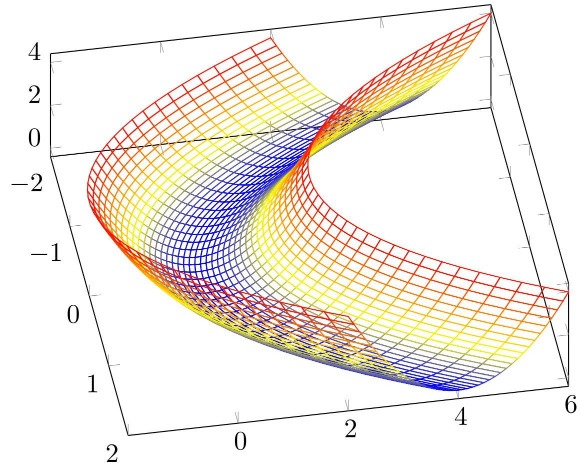

Figure. $\mathbf{X}(s,t) = (s, s^2+t, t^2)$.

(b) Is the surface smooth?

Solution. The surface is smooth.

(c) Find an equation for the tangent plane at the point $(1,0,1)$.

Solution.

We have:

$$\mathbf{T}_s = (1, 2s, 0)$$
$$\mathbf{T}_t = (0, 1, 2t)$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2s & 0 \\ 0 & 1 & 2t \end{vmatrix} \\ &= 4st\mathbf{i} - 2t\mathbf{j} + \mathbf{k} \end{aligned}$$

The point $(1,0,1)$ is $(s_0 = 1, t_0 = -1)$.

$$\mathbf{N}(1,-1) = -4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

The equation of the tangent plane at $(1,-1)$ is:

$$\begin{aligned} (-4,2,1)\cdot(x-1, y, z-1) &= 0 \\ -4(x-1) + 2y + (z-1) &= 0 \\ 4(x-1) - 2y - (z-1) &= 0 \\ 4x - 4 - 2y - z + 1 &= 0 \\ 4x - 2y - z &= 3 \end{aligned}$$

6. Describe the parameterized surface of exercise problem 1 by an equation of the form $z = f(x,y)$.

Solution.

The parametric surface $\mathbf{X}(s,t)$ is:

$$X(s,t) = (s^2 - t^2, s+t, s^2+3t)$$

In exercise (1), we see that $x = (s-t)(s+t) = y(s+t)$ so $s+t = x/y$ and $y = s-t$. This allows us to solve simultaneously for s and t. $2s = x/y + y$ and $2t = x/y - y$. This means that $z = s^2 + 3t$ can be written as $z = \frac{1}{4}\left(\frac{x}{y}+y\right)^2 + \frac{3}{2}\left(\frac{x}{y}-y\right)$.

7. Let S be the surface parameterized by:

$$\begin{aligned} x &= s\cos t \\ y &= s\sin t \\ z &= s^2 \end{aligned}$$

where $s \geq 0, 0 \leq t \leq 2\pi$.

(a) At what points is S smooth? Find an equation for the tangent plane at the point $(1, \sqrt{3}, 4)$.

Solution.

The surface S is $x^2 + y^2 = z$. This is a paraboloid. It is smooth at all points.

We have:

$$\begin{aligned} \mathbf{T}_s &= (\cos t, \sin t, 2s) \\ \mathbf{T}_t &= (-s\sin t, s\cos t, 0) \end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 2s \\ -s\sin t & s\cos t & 0 \end{vmatrix} \\ &= (-2s^2\cos t)\mathbf{i} - (2s^2\sin t)\mathbf{j} + (s\cos^2 + s\sin^2 t)\mathbf{k} \\ &= (-2s^2\cos t)\mathbf{i} - (2s^2\sin t)\mathbf{j} + s\mathbf{k} \end{aligned}$$

At $s = 2, t = \pi/6$,

$$\mathbf{N}(2, \pi/6) = -4\sqrt{3}\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

The equation of the tangent plane at $\mathbf{X}(2, \pi/6)$ is:

$$\begin{aligned} (-4\sqrt{3}, -4, 2)\cdot(x-1, y-\sqrt{3}, z-4) &= 0 \\ 4\sqrt{3}(x-1) + 4(y-\sqrt{3}) - 2(z-4) &= 0 \\ 4\sqrt{3}x - 4\sqrt{3} + 4y - 4\sqrt{3} - 2z + 8 &= 0 \\ 4\sqrt{3}x + 4y - 2z &= 8(\sqrt{3}-1) \\ 2\sqrt{3}x + 2y - z &= 4(\sqrt{3}-1) \end{aligned}$$

(b) Sketch the graph of S. Can you recognize S as a familiar surface?

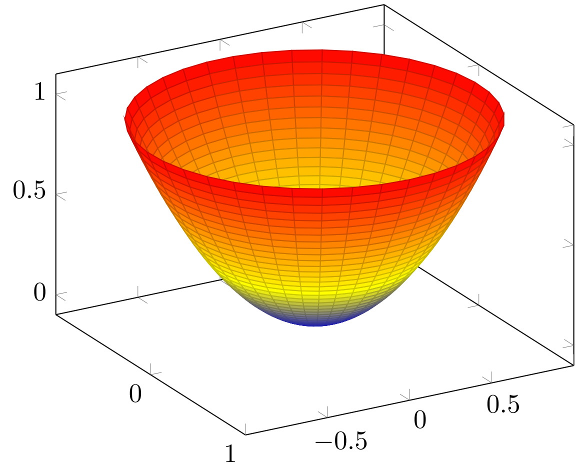

Figure. $\mathbf{X}(s,t) = (s\cos t, s\sin t, s^2)$

(c) Describe S by an equation of the form $z = f(x,y)$.

Solution. Again $z = x^2 + y^2$.

(d) Using your answer in part(c), discuss whether S has a tangent plane at every point.

Solution.

S has a tangent plane at every point and is smooth. Part (a) takes care of every point except the origin. At the origin $\mathbf{N} = (0,0,0)$. But, we easily see, that the tangent plane at the origin is the horizontal plane $z = 0$. Thus, smoothness as defined in the text, depends on both the parameterization and the geometry of the underlying surface.

8. Verify that the image of the parametrized surface

$$\mathbf{X}(s,t) = (2\sin s\cos t, 3\sin s\sin t, \cos s)$$

$0 \le s \le \pi$ and $0 \le t \le 2\pi$ is an ellipsoid.

Solution.

We can easily write:

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$$

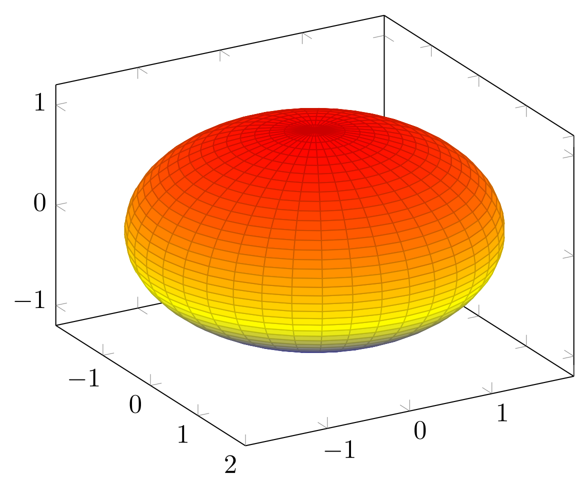

Figure. $\mathbf{X}(s,t) = (2\sin s\cos t, 3\sin s\sin t, \cos s)$

9. Verify that, for the torus of example 5, the s-coordinate curve, when $t = t_0$ is a circle of radius $a + b\cos t_0$.

Solution.

The parametric equations of the Torus in example 5 were:

$$\begin{aligned} x &= (a + b\cos t)\cos s \\ y &= (a + b\cos t)\sin s \\ z &= b\sin t \end{aligned}$$

The s-coordinate curve at $t = t_0$ is:

$$\begin{aligned} x &= (a + b\cos t_0)\cos s \\ y &= (a + b\cos t_0)\sin s \\ z &= b\sin t_0 \end{aligned}$$

These are circles of radius $a + b\cos t_0$ in the plane $z = b\sin t_0$.

And they satisfy $x^2 + y^2 = (a + b\cos t_0)^2$.

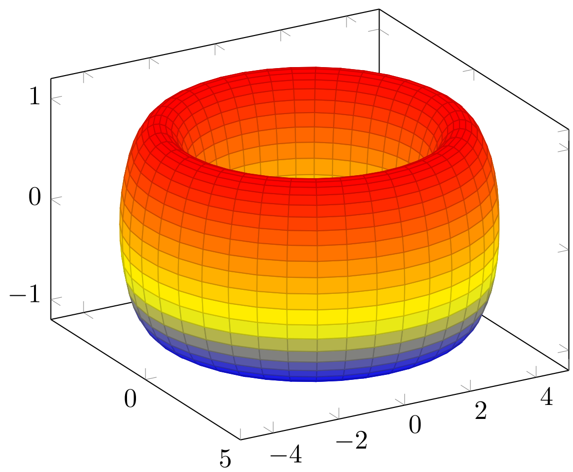

Figure. Torus $\mathbf{X}(s,t) = ((a + b\cos t)\cos s, (a + b\cos t)\sin s, b\sin t)$

9. The surface in $\mathbf{R}^3$ parametrized by:

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

where $r \geq 0$ and $-\infty < \theta < \infty$ is called a helicoid.

(a) Describe the r-coordinate curve when $\theta = \pi/3$. Give a general description of the r-coordinate curves.

Solution.

The r-coordinate curve when $\theta = \pi/3$ is:

$$\begin{aligned} x &= r/2 \\ y &= \sqrt{3}r/2 \\ z &= \pi/3 \end{aligned}$$

It is the straight-line $y = \sqrt{3}x$ in the plane $z = \pi/3$.

The r-coordinate curve when $\theta = \theta_0$ is:

$$\begin{aligned} x &= r\cos\theta_0 \\ y &= r\sin\theta_0 \\ z &= \theta_0 \end{aligned}$$

It is the straight line $y = (\tan\theta_0)x$ in the plane $z = \theta_0$.

(b) Describe the θ-coordinate curve when $r = 1$. Give a general description of the θ-coordinate curves.

Solution.

The θ-coordinate curve when $r = 1$ is:

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \\ z &= \theta \end{aligned}$$

This is a helix with parameter θ and radius 1.

The θ-coordinate curve when $r = r_0$ is:

$$\begin{aligned} x &= r_0\cos\theta \\ y &= r_0\sin\theta \\ z &= \theta \end{aligned}$$

These are helixes of radius $r_0 \geq 0$.

(c) Sketch the graph of the helicoid using a computer for $0 \leq r \leq 1, 0 \leq \theta \leq 4\pi$. Can you see why the surface is called a helicoid?

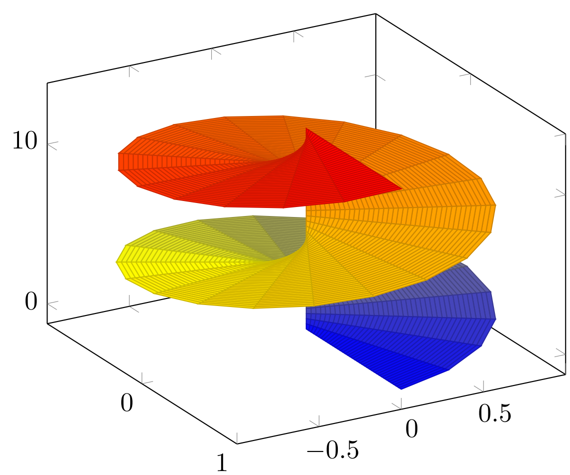

Figure. Helicoid $\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$

11. Given a sphere of radius 2 centered at $(2, -1, 0)$, find an equation for the plane tangent to it at the point $(1, 0, \sqrt{2})$ in three ways:

(a) by consider the sphere as the graph of the function $f(x, y) = \sqrt{4 - (x-2)^2 - (y+1)^2}$.

Solution.

We have:

$$\begin{aligned} \text{grad } f &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(x-2)}{\sqrt{4-(x-2)^2-(y+1)^2}} & -\frac{(y+1)}{\sqrt{4-(x-2)^2-(y+1)^2}} \end{bmatrix} \end{aligned}$$

Thus,

$$\text{grad } f(1,0) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

So, the tangent plane at the point $(1, 0, \sqrt{2})$ is:

$$\begin{aligned}
z &= f(1,0) + (\mathbf{x} - \mathbf{a}) \cdot \nabla f \\
&= \sqrt{2} + (x-1, y) \cdot ((1/\sqrt{2}), -(1/\sqrt{2})) \\
&= \sqrt{2} + \frac{x-1}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\
\sqrt{2}z &= 2 + (x-1) - y \\
x - y + 1 &= \sqrt{2}z
\end{aligned}$$

(b) by considering the sphere as a level surface of the function

$$F(x,y,z) = (x-2)^2 + (y+1)^2 + z^2$$

The gradient ∇F is :

$$\begin{aligned}
\nabla F &= \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix} \\
&= \begin{bmatrix} 2(x-2) & 2(y+1) & 2z \end{bmatrix}
\end{aligned}$$

If $\mathbf{x}_0 = (x_0, y_0, z_0)$ is a point on the level set $S = \{(x,y,z) : F(x,y,z) = c\}$, then the gradient vector $\nabla F(\mathbf{x})$ at the point $\mathbf{x}_0$ is perpendicular to S. $(1, 0, \sqrt{2})$ is point on the level set $S = \{(x,y,z) | (x-2)^2 + (y+1)^2 + z^2 = 4\}$. So, $\nabla F(1, 0, \sqrt{2}) = (-2, 2, 2\sqrt{2})$ is the normal vector to the sphere $F(x,y,z) = 4$ at the point $(1, 0, \sqrt{2})$.

If (x,y,z) is an arbitrary point in the tangent plane, we must have:

$$\begin{aligned}
(x-1, y, z-\sqrt{2}) \cdot (-2, 2, 2\sqrt{2}) &= 0 \\
(x-1, y, z-\sqrt{2}) \cdot (-1, 1, \sqrt{2}) &= 0 \\
-(x-1) + y + \sqrt{2}(z-\sqrt{2}) &= 0 \\
(x-1) - y - \sqrt{2}(z-\sqrt{2}) &= 0 \\
x - y - \sqrt{2}z + 1 &= 0 \\
x - y + 1 &= \sqrt{2}z
\end{aligned}$$

(c) By considering the sphere as the surface parametrized by :

$$\mathbf{X}(s,t) = (2\sin s \cos t + 2, 2\sin s \sin t - 1, 2\cos s)$$

Solution.

We have:

$$\begin{aligned}
\mathbf{T}_s &= (2\cos s \cos t, 2\cos s \sin t, -2\sin s) \\
\mathbf{T}_t &= (-2\sin s \sin t, 2\sin s \cos t, 0)
\end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos s\cos t & 2\cos s\sin t & -2\sin s \\ -2\sin s\sin t & 2\sin s\cos t & 0 \end{vmatrix} \\
&= 4\sin^2 s\cos t\mathbf{i} - (-4\sin^2 s\sin t)\mathbf{j} + (4\sin s\cos s\cos^2 t + 4\sin s\cos s\sin^2 t)\mathbf{k} \\
&= 4\sin^2 s\cos t\mathbf{i} + 4\sin^2 s\sin t\mathbf{j} + 4\sin s\cos s\mathbf{k}
\end{aligned}$$

Now, $2\cos s = \sqrt{2}$ so $\cos s = \frac{1}{\sqrt{2}}$ and thus $s = \pi/4$. Consequently, $\sqrt{2}\cos t + 2 = 1$ and therefore $\cos t = -\frac{1}{\sqrt{2}}$, which implies $t = 3\pi/4$. The normal vector at $\mathbf{X}(\pi/4, 3\pi/4)$ is:

$$\begin{aligned}
\mathbf{N}(\pi/4, 3\pi/4) &= 4 \cdot \frac{1}{2} \cdot \frac{-1}{\sqrt{2}}\mathbf{i} + 4 \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\mathbf{j} + 2\mathbf{k} \\
&= -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 2\mathbf{k}
\end{aligned}$$

The equation of the tangent plane is:

$$\begin{aligned}
(-\sqrt{2}, \sqrt{2}, 2) \cdot (x-1, y, z-\sqrt{2}) &= 0 \\
(-1, 1, \sqrt{2}) \cdot (x-1, y, z-\sqrt{2}) &= 0 \\
x - y + 1 &= \sqrt{2}z
\end{aligned}$$

In exercises 12-15, represent the given surface as a piecewise smooth parameterized surface.

12. The lower hemisphere $x^2 + y^2 + z^2 = 9$ including the equatorial circle.

Solution.

We can parametrize the lower hemisphere of the sphere as $\mathbf{X}(s,t) = (s, t, -\sqrt{9 - (x^2 + y^2)})$. Alternatively, we may parametrize it as $\mathbf{X}(\phi, \theta)$:

$$\begin{aligned}
x &= 3\sin\phi\cos\theta \\
y &= 3\sin\phi\sin\theta \\
z &= 3\cos\phi
\end{aligned}$$

where $0 \le \theta \le 2\pi$ and $\pi/2 \le \phi \le \pi$.

13. The part of the cylinder $x^2 + z^2 = 4$ lying between $y = -1$ and $y = 3$.

Solution.

We can parametrize the cylinder as:

$$\begin{aligned}
x &= 2\cos s \\
y &= t \\
z &= 2\sin s
\end{aligned}$$

where $0 \le s \le 2\pi$ and $-1 \le t \le 3$.

14. The closed triangular region in $\mathbf{R}^3$ with vertices $(2,0,0)$, $(0,1,0)$ and $(0,0,5)$.

Solution.

A parameterization of a plane can be written as :

$$\mathbf{x} = s\mathbf{a} + t\mathbf{b} + \mathbf{p}$$

where **a** and **b** are two vectors in the plane and **p** is a point in the plane. To see why this is the case, suppose $\mathbf{x} = (x, y, z)$ is an arbitrary point in the plane and $\mathbf{p} = (x_0, y_0, z_0)$ is a known point. Then $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$ must be a linear combination of **a** and **b**. So, $\mathbf{x} - \mathbf{p} = s\mathbf{a} + t\mathbf{b}$.

We have four planes that are described by:

$$\mathbf{X}(s, t) = s(2, 0, -5) + t(0, 1, -5) + (0, 0, 5) = (2s, t, -5s - 5t + 5)$$

Since we are interested in the first octant of $\mathbf{R}^3$ all coordinates must be non-negative. So, $0 \leq 2s \leq 2$, that is $0 \leq s \leq 1$, $0 \leq t \leq 1$ and $-5s - 5t + 5 \geq 0$. In other words, $t \leq 1 - s$.

14. The hyperboloid $z^2 - x^2 - y^2 = 1$. (Hint: Use two maps to parametrize the surface)

Solution.

The equation of the hyperboloid as:

$$z = \pm\sqrt{1 + x^2 + y^2}$$

Therefore, the hyperboloid may be paramterized with two maps:

$$\begin{aligned} \mathbf{X}_1(s, t) &= (s, t, \sqrt{1 + x^2 + y^2}) \\ \mathbf{X}_2(s, t) &= (s, t, -\sqrt{1 + x^2 + y^2}) \end{aligned}$$

16. This problem concerns the parameterized surface $\mathbf{X}(s, t) = (s^3, t^3, st)$.

(a) Find an equation of a plane tangent to this surface at the point $(1, -1, -1)$.

Solution.

We have:

$$\begin{aligned} \mathbf{T}_s &= (3s^2, 0, t) \\ \mathbf{T}_t &= (0, 3t^2, s) \end{aligned}$$

The standard normal vector **N** is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3s^2 & 0 & t \\ 0 & 3t^2 & s \end{vmatrix} \\ &= -3t^3\mathbf{i} - 3s^3\mathbf{j} + 9s^2t^2\mathbf{k} \end{aligned}$$

We have $s_0 = 1, t_0 = -1$. So, $\mathbf{N}(1, -1) = (3, -3, 9)$. The equation of the tangent plane to the surface at $(1, -1, -1)$ is:

$$\begin{aligned} (3, -3, 9) \cdot (x - 1, y + 1, z + 1) &= 0 \\ (1, -1, 3) \cdot (x - 1, y + 1, z + 1) &= 0 \\ (x - 1) - (y + 1) + 3(z + 1) &= 0 \\ x - 1 - y - 1 + 3z + 3 &= 0 \\ x - y + 3z + 1 &= 0 \end{aligned}$$

(b) Use a computer to graph this surface for $-1 \leq s \leq 1, -1 \leq t \leq 1$.

Solution.

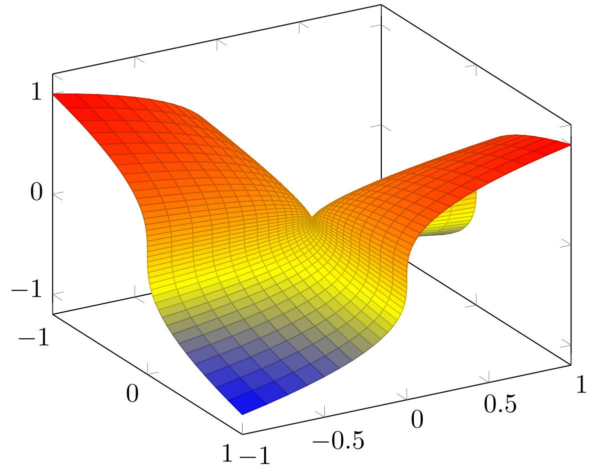

Figure. $\mathbf{X}(s, t) = (s^3, t^3, st)$

(c) Is the surface smooth?

Solution.

The normal vector $\mathbf{N} = 0$ at $(s_0, t_0) = (0, 0)$ that is at $(0, 0, 0)$. So, the surface fails to be smooth there.

17. The surface given parametrically by $\mathbf{X}(s, t) = (st, t, s^2)$ is known as **Whitney's umbrella**.

(a) Verify that this surface may also be described by the xyz-coordinate equation $y^2 z = x^2$.

Solution.

Clearly, $y^2 z = (t^2)(s^2) = (st)^2 = x^2$.

(b) Is $\mathbf{X}$ smooth?

Solution.

We have:

$$\mathbf{T}_s = (t, 0, 2s)$$
$$\mathbf{T}_t = (s, 1, 0)$$

The standard normal vector is :

$$\begin{aligned} \mathbf{N} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 0 & 2s \\ s & 1 & 0 \end{vmatrix} \\ &= -2s\mathbf{i} + 2s^2\mathbf{j} + t\mathbf{k} \end{aligned}$$

The normal vector $\mathbf{N} = (0, 0, 0)$ at $(s, t) = (0, 0)$ that is at the point $(0, 0, 0)$. Hence, $\mathbf{X}$ is not smooth at this point.

(c) Use a computer to graph this surface for $-2 \leq s \leq 2$, $-2 \leq t \leq 2$.

Solution.

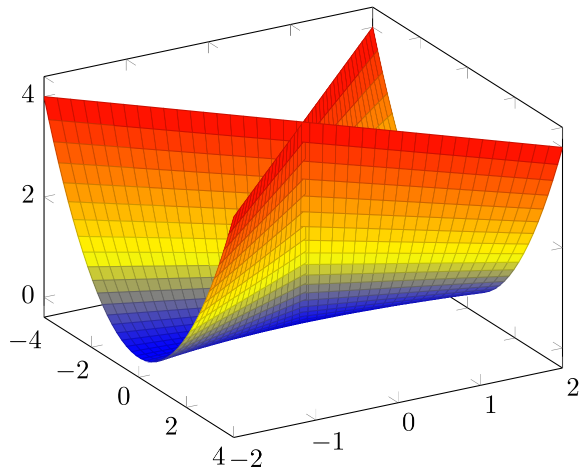

Figure. $\mathbf{X}(s,t) = (st, t, s^2)$

(d) Give an equation of the plane tangent to this surface at the point $(2, 1, 4)$.

Solution.

The standard normal vector at the point $\mathbf{X}(2,1)$ is $(-4, 8, 1)$. Hence, the equation of the tangent plane to the surface at the point $\mathbf{X}(1,2)$ is:

$$(-4, 8, 1) \cdot (x-2, y-1, z-4) = 0$$
$$-4(x-2) + 8(y-1) + (z-4) = 0$$
$$4(x-2) - 8(y-1) - (z-4) = 0$$
$$4x - 8y - z = -4$$

(e) Some points (x, y, z) of the surface do not correspond to a single parameter point (s, t). Which ones? Explain how this relates to the graph?

Solution.

Let $\mathbf{X}(s_1, t_1) = \mathbf{X}(s_2, t_2)$. Then,

$$s_1 t_1 = s_2 t_2$$
$$t_1 = t_2$$
$$s_1^2 = s_2^2$$

Thus, if $t_1 = t_2 = 0$ and $s_1 = \pm s_2$ we get the same image that is $\mathbf{X}(s,0) = \mathbf{X}(-s,0) = (0,0,s^2)$. Thus, the positive z-axis does not corespond to a single point.

18. Let S be the surface defined as the graph of a function $f(x,y)$ of class C^1. Then, example 4 shows that S is also a parametrized surface. Show that formula (5) for the tangent plane to S at $(a, b, f(a,b))$ agrees with formula (4) in section 2.3.

Solution.

We have:

$$\mathbf{T}_s = (1, 0, f_s(s,t))$$
$$\mathbf{T}_t = (0, 1, f_t(s,t))$$

The standard normal vector $\mathbf{N}$at the point (s,t) is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_s(s,t) \\ 0 & 1 & f_t(s,t) \end{vmatrix} \\ &= -f_s(s,t)\mathbf{i} - f_t(s,t)\mathbf{j} + \mathbf{k} \end{aligned}$$

So, the equation of the tangent plane to the surface at $\mathbf{X}(s,t)$ is:

$$
\begin{aligned}
(-f_s, -f_t, 1) \cdot (x - s, y - t, z - f(s,t)) = 0 \\
z - f(s,t) = \frac{\partial f}{\partial s}(x - s) + \frac{\partial f}{\partial t}(y - t)
\end{aligned}
$$

19. (a) Write a formula for the tangent plane to the surface described by the equation $y = g(x,z)$.

Solution.

We have:

$$\mathbf{X}(s,t) = (s, g(s,t), t)$$

So,

$$
\begin{aligned}
\mathbf{T}_s &= (1, g_s(s,t), 0) \\
\mathbf{T}_t &= (0, g_t(s,t), 1)
\end{aligned}
$$

The standard normal vector $\mathbf{N}$ is:

$$
\begin{aligned}
\mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & g_s(s,t) & 0 \\ 0 & g_t(s,t) & 1 \end{vmatrix} \\
&= g_s(s,t)\mathbf{i} - \mathbf{j} + g_t(s,t)\mathbf{k}
\end{aligned}
$$

So, the equation of the tangent plane to the surface at the point $(x_0, g(x_0, z_0), z_0)$ is:

$$
\begin{aligned}
(g_s(s,t), -1, g_t(s,t)) \cdot (x - x_0, y - g(x_0, z_0), z - z_0) = 0 \\
g_s(s,t)(x - x_0) - (y - g(x_0, z_0)) + g_t(z - z_0) = 0 \\
y = g(x_0, z_0) + g_s(x - x_0) + g_t(z - z_0)
\end{aligned}
$$

(b) Repeat part (a) for a surface described by the equation $x = h(y,z)$.

Solution.

The equation of the tangent plane to the suface at the point $(h(y_0, z_0), y_0, z_0)$ is:

$$x = h(y_0, z_0) + h_y(y - y_0) + h_z(z - z_0)$$

20. Suppose that $\mathbf{X} : D \to \mathbf{R}^3$ is a parameterized surface that is smooth at $\mathbf{X}(s_0, t_0)$. Show how the definition of the derivative $D\mathbf{X}(s_0, t_0)$ can be used to give vector parametric equations for the plane tangent to $S = \mathbf{X}(D)$ at the point $\mathbf{X}(s_0, t_0)$.

Solution.

Let $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$. The derivative $D\mathbf{X}(s,t)$ is:

$$
\begin{aligned}
D\mathbf{X}(s,t) &= \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{T}_s & \mathbf{T}_t \end{bmatrix}
\end{aligned}
$$

Hence, the equation of the tangent plane to surface $\mathbf{X}(s,t)$ at the point (s_0, t_0) is:

$$
\begin{aligned}
\mathbf{x}(s,t) &= \mathbf{X}(s_0, t_0) + D\mathbf{X}(s_0, t_0) \cdot \begin{bmatrix} s - s_0 \\ t - t_0 \end{bmatrix} \\
&= \mathbf{X}(s_0, t_0) + \mathbf{T}_s(s_0, t_0)(s - s_0) + \mathbf{T}_t(s_0, t_0)(t - t_0)
\end{aligned}
$$

Surface Integrals

Scalar Surface Integrals

Definition. Let $\mathbf{X} : D \to \mathbf{R}^3$ be a smooth parametrized surface whose domain $D \subset \mathbf{R}^2$ is a bounded region. Let f be a continuous function whose domain includes $S = \mathbf{X}(D)$. Then, the scalar surface integral of f along $\mathbf{X}$ is:

$$\begin{aligned}\int\int_{\mathbf{X}} f \cdot dS &= \int\int_D f(\mathbf{X}(s,t))\, \|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt \\ &= \int\int_D f(\mathbf{X}(s,t))\, \|\mathbf{N}(s,t)\|\, ds\, dt\end{aligned}$$

$\mathbf{X}$ maps any rectangle with sides ds, dt and area $ds \cdot dt$ in D to a parallelogram in $\mathbf{X}(D)$ with sides $ds\mathbf{T}_s$ and $dt\mathbf{T}_t$. The area of the parallelogram is the cross-product $\|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt$. Thus,$\|\mathbf{T}_s \times \mathbf{T}_t\|$ is the scaling factor.

If f is identically 1 on all of $\mathbf{X}(D)$ then:

$$\int\int_{\mathbf{X}} f \cdot dS = \int\int_D 1\, \|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt = \text{Surface area of } \mathbf{X}(D)$$

The scalar surface integral is thus a generalization of the integral we use to calculate the surface area. We can think of $\int\int_{\mathbf{X}} f \cdot dS$ as a limit of the weighted sum of the surface area pieces, the weightings given by f. If f represents the mass or electrical charge density, then $\int\int_{\mathbf{X}} f \cdot dS$ yields the total mass or the total charge on $\mathbf{X}(D)$.

For computational purposes, recall that if we write the components of $\mathbf{X}$ as:

$$\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$$

then

$$\begin{aligned}\mathbf{N}(s,t) &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \\ &= \mathbf{i}\begin{vmatrix} \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \\ &= \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k}\end{aligned}$$

So, we obtain:

$$\int\int_{\mathbf{X}} f \cdot dS = \int\int_D f(x(s,t), y(s,t), z(s,t))\sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\, ds\, dt$$

If the surface S is given by a graph of $z = g(x,y)$ where g is of class C^1 on some region D in $\mathbf{R}^2$, then S is parameterized by $\mathbf{X}(x,y) = (x, y, g(x,y))$ with $(x,y) \in D$. Then, from example 13 in section 7.1,

$$\mathbf{N}(x,y) = -g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}$$

so that:

$$\int\int_{\mathbf{X}} f \cdot dS = \int\int_D f(x, y, g(x,y))\sqrt{1 + g_x^2 + g_y^2}\, dx\, dy$$

Vector Surface Integrals

We can develop a means to integrate vector fields along surfaces beginning with the definition.

Definition. Let $\mathbf{X} : D \to \mathbf{R}^3$ be a smooth parameterized surface, where D is a bounded region in the $\mathbf{R}^2$ plane, and let $\mathbf{F}(x, y, z)$ be a continuous vector field whose domain includes $S = \mathbf{X}(D)$. Then, the vector surface integral of $\mathbf{F}$ along $\mathbf{X}$ is:

$$\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{D} \mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{N}(s, t) ds\, dt$$

where $\mathbf{N}(s, t) = \mathbf{T}_s \times \mathbf{T}_t$.

As with line integrals, we should be careful about the notation for surface integrals. In the vector surface integral $\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, the differential term is considered to be a vector quantity, whereas in the scalar surface integral $\int\int_{\mathbf{X}} f \cdot dS$, the differential term is scalar quantity - the differential of the surface area.

Example. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (z - 2y)\mathbf{k}$. We evaluate $\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{X}$ is the helicoid

$$\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$$

where $0 \le s \le 1, 0 \le t \le 2\pi$.

Solution.

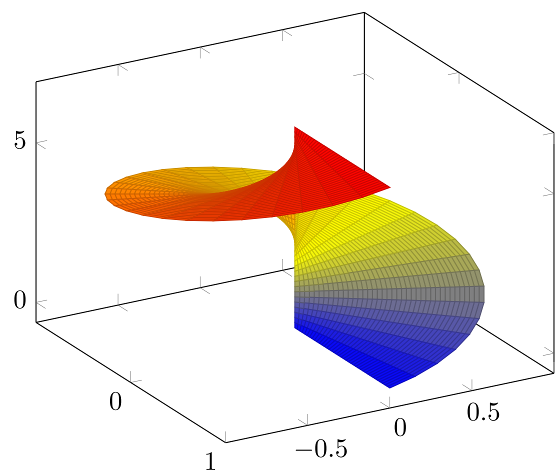

Figure. $\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$

We have:

$$\begin{aligned} \mathbf{T}_s &= (\cos t, \sin t, 0) \\ \mathbf{T}_t &= (-s \sin t, s \cos t, 1) \end{aligned}$$

The standard normal vector $\mathbf{N}$ is:

$$\begin{aligned} \mathbf{N} &= \mathbf{T}_s \times \mathbf{T}_t \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s \sin t & s \cos t & 1 \end{vmatrix} \\ &= \sin t\mathbf{i} - \cos t\mathbf{j} + s\mathbf{k} \end{aligned}$$

Hence:

$$\begin{aligned}
\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^1 (s\cos t, s\sin t, t - 2s\sin t) \cdot (\sin t, -\cos t, s) ds dt \\
&= \int_0^{2\pi} \int_0^1 (s\sin t\cos t - s\sin t\cos t + st - 2s^2 \sin t) ds dt \\
&= \int_0^{2\pi} \int_0^1 (st - 2s^2 \sin t) ds dt \\
&= \int_0^{2\pi} [\frac{s^2}{2}t - \frac{2s^3}{3}\sin t]_0^1 dt \\
&= \int_0^{2\pi} (\frac{t}{2} - \frac{2}{3}\sin t) dt \\
&= \left[\frac{t^2}{4} + \frac{2}{3}\cos t\right]_0^{2\pi} \\
&= \pi^2
\end{aligned}$$

Further Interpretations

As is the case for vector and scalar line integrals, there is a connection between vector and scalar surface integrals. Suppose $\mathbf{X} : D \to \mathbf{R}^3$ is a smooth parametrized surface and $\mathbf{F}$ is continuous on $S = \mathbf{X}(D)$. Let $\mathbf{N}(s,t) = \mathbf{T}_s \times \mathbf{T}_t$ be the usual normal vector and let:

$$\mathbf{n}(s,t) = \frac{\mathbf{N}(s,t)}{\|\mathbf{N}(s,t)\|}$$

That is, $\mathbf{n}$ is the unit vector pointing in the same direction as $\mathbf{N}$. In particular,

$$\mathbf{N}(s,t) = \|\mathbf{N}(s,t)\|\, \mathbf{n}(s,t)$$

Plugging this into the definition of the vector surface integral, we get:

$$\begin{aligned}
\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \int\int_D \mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{N}(s,t) ds \cdot dt \\
&= \int\int_D \mathbf{F}(\mathbf{X}(D)) \cdot \|\mathbf{N}(s,t)\|\, \mathbf{n}(s,t) ds \cdot dt \\
&= \int\int_D (\mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{n}(s,t))\, \|\mathbf{N}(s,t)\|\, ds \cdot dt \\
&= \int\int_{\mathbf{X}} (\mathbf{F} \cdot \mathbf{n}) dS
\end{aligned}$$

Since $\mathbf{n}$ is a unit vector, the quantity $\mathbf{F} \cdot \mathbf{n}$ is precisely the component of the vector field $\mathbf{F}$ in the direction of $\mathbf{n}$. In other words, the vector surface integral of $\mathbf{F}$ along $\mathbf{X}$ is the scalar surface integral of the component of $\mathbf{F}$ normal to $S = \mathbf{X}(D)$. Recall that, the vector line integral of $\mathbf{F}$ along a path $\mathbf{x}$ is the scalar line integral of the component of $\mathbf{F}$ tangent to the image curve. To summarize, we have the following results:

$$\text{Line Integrals :} \quad \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \quad = \int_0^t (\mathbf{F} \cdot \mathbf{T}) ds \tag{1}$$

$$\text{Surface Integrals :} \int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int\int_D (\mathbf{F} \cdot \mathbf{n}) \cdot ds \cdot dt \tag{2}$$

The vector line integral $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ in equation (1) is called the flow-integral of $\mathbf{F}$ along $\mathbf{x}$. The reason for this is the following.

Suppose $\mathbf{F}$ represents the velocity vector field of a fluid. Consider the amount of fluid moved tangentially along a small segment of the path $\mathbf{x}$ during a brief time interval $\Delta\tau$.

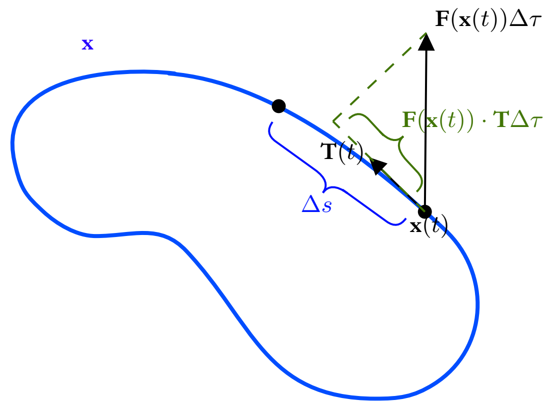

Figure. The amount of fluid transported tangentially along a segment of the closed path $\mathbf{x}$ is approximately $(\mathbf{F}(\mathbf{x}(t))\Delta\tau \cdot \mathbf{T})\Delta s$

Since $\mathbf{F} \cdot \mathbf{T}$ gives the tangential component of the velocity vector $\mathbf{F}$, the rate of flow at the point $\mathbf{x}(t)$ is $\mathbf{F} \cdot \mathbf{T}$. So, the amount of fluid transported tangentially in $\Delta\tau$ time through the point $\mathbf{x}(t)$ is Rate of flow $\times$ time $= (\mathbf{F}\Delta\tau) \cdot \mathbf{T}$. A line segment of the path $\mathbf{x}(t)$ of length Δs can be thought to be made up of Δs such points. So, the total amount of fluid transported tangentially in $\Delta\tau$ time along a segment of length Δs equals $(\mathbf{F}\Delta\tau) \cdot \mathbf{T}\Delta s$.

$$\text{Amount of fluid moved} \approx (\mathbf{F}(\mathbf{x}(t))\Delta\tau \cdot \mathbf{T})\Delta s$$

If we divide the above expression by time $\Delta\tau$, we get the average rate of transport of the fluid along the segment :

$$\frac{\Delta L}{\Delta\tau} \approx \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T} \cdot ds$$

$$\text{Instantaneous rate of fluid flow} = \lim_{\Delta\tau \to 0} \frac{\Delta L}{\Delta\tau} = \frac{dL}{d\tau} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T} \cdot ds$$

If $\mathbf{x}$ is a closed path, the flow-integral is also called circulation.

Now, let's try to interpret the vector surface integral in equation (2).

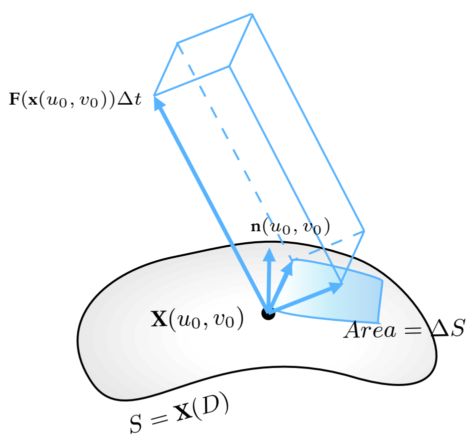

Figure. The amount of fluid transported across a small piece of S during a brief time interval Δt may be approximated by the volume of the parallelopiped.

Consider a small piece of S, having the area ΔS and the amount of fluid transported across it during a brief time interval Δt. This amount is the volume determined by $\mathbf{F}$ during Δt. The above figure suggests that this volume can be approximated by the volume of an appropriate parallelopiped.

The height of the parallelopiped is the normal component of $\mathbf{F}\Delta t$, $\mathbf{F}\Delta t \cdot \mathbf{n}(u_0, v_0)$ and the area of the base is ΔS. Hence:

$$\begin{aligned}\text{Amount of fluid displaced} &\approx \text{volume of parallelopiped}\\ &= (\text{height})(\text{area of base})\\ &= \mathbf{F}(\mathbf{X}(u_0, v_0))\Delta t \cdot \mathbf{n}(u_0, v_0)\Delta S \end{aligned} \tag{3}$$

We obtain the average rate of transport across the surface piece during the time interval Δt by dividing (3) by Δt:

$$\text{Average rate of transport} \approx \mathbf{F}(\mathbf{X}(u_0, v_0)) \cdot \mathbf{n}(u_0, v_0)\Delta S \tag{4}$$

Now, we break up the entire surface $S = \mathbf{X}(D)$ into infinitely many such small partitions ΔS_{ij} and sum the corresponding contributions to the rate of transport in the form given (4). If we let all the pieces shrink, then, in the limit as all $\Delta S \to 0$, we have that the total average rate of transport of the fluid during Δt is approximately :

$$\frac{\Delta M}{\Delta t} \approx \int\int_{\mathbf{X}} (\mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{n}) dS \tag{5}$$

Passing to the limits as $\Delta t \to 0$, the instantaneous rate of fluid transport across $\mathbf{X}$ can be defined as the flow integral:

$$\lim_{\Delta t \to 0} \frac{\Delta M}{\Delta t} = \frac{dM}{dt} = \int\int_{\mathbf{X}} (\mathbf{F}(\mathbf{X}(D)) \cdot \mathbf{n}) dS \tag{6}$$

Reparametrization of surfaces

Definition. Let $\mathbf{X} : D_1 \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ and $\mathbf{Y} : D_2 \subseteq \mathbf{R}^2 \to \mathbf{R}^3$ be parametrized surfaces. $\mathbf{Y}$ is said to be a reparametrization of $\mathbf{X}$, if there exists a one-to-one and onto function $\mathbf{H} : \mathbf{R}^2 \to \mathbf{R}^2$ such that $\mathbf{Y} = \mathbf{X}(\mathbf{H}(s, t))$.

Example. Suppose $\mathbf{X}$ is a smooth parametrized surface. Let $\mathbf{Y}(s, t) = \mathbf{X}(u, v)$, where $u = t, v = s$. That is, $\mathbf{Y} = \mathbf{X}(\mathbf{H})$. Then, $\mathbf{Y}$ is a smooth parametrization that appears to accomplish little. However, if we let $\mathbf{N_Y}$ denote the usual normal vector $\mathbf{T}_s \times \mathbf{T}_t = \frac{\partial \mathbf{Y}}{\partial s} \times \frac{\partial \mathbf{Y}}{\partial t}$, then we have:

$$\frac{\partial \mathbf{Y}}{\partial s} = \frac{\partial \mathbf{X}}{\partial v} \quad \text{and} \ \frac{\partial \mathbf{Y}}{\partial t} = \frac{\partial \mathbf{X}}{\partial u}$$

so that:

$$\begin{aligned}\mathbf{N_Y} &= \frac{\partial \mathbf{Y}}{\partial s} \times \frac{\partial \mathbf{Y}}{\partial t}\\ &= \frac{\partial \mathbf{X}}{\partial v} \times \frac{\partial \mathbf{X}}{\partial u}\\ &= -\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v}\\ &= -\mathbf{N_X}\end{aligned}$$

The parametrized surface $\mathbf{Y}$ is the same as $\mathbf{X}$, except that the standard normal vector arising from $\mathbf{Y}$ points in the opposite direction to the one arising from $\mathbf{X}$.

The calculation in the above example thus generalizes. Suppose $\mathbf{X}$ is a smooth parametrized surface and $\mathbf{Y}$ is a smooth reparametrization of $\mathbf{X}$ via $\mathbf{H}$, that is:

$$\mathbf{Y}(s, t) = \mathbf{X}(u, v) = \mathbf{X}(\mathbf{H}(s, t))$$

By the chain rule:

$$
\begin{gathered}
D\mathbf{Y}(s,t) = D\mathbf{X}(u,v) \cdot D\mathbf{H}(s,t) \\
\begin{bmatrix} x_s & x_t \\ y_s & y_t \\ z_s & z_t \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} u_s & u_t \\ v_s & v_t \end{bmatrix}
\end{gathered}
$$

From the above, we see that:

$$
\begin{aligned}
\mathbf{T}_s &= u_s \mathbf{T}_u + v_s \mathbf{T}_v \\
\mathbf{T}_t &= u_t \mathbf{T}_u + v_t \mathbf{T}_v
\end{aligned}
$$

So,

$$
\begin{aligned}
\mathbf{T}_s \times \mathbf{T}_t &= (u_s \mathbf{T}_u + v_s \mathbf{T}_v) \times (u_t \mathbf{T}_u + v_t \mathbf{T}_v) \\
&= u_s u_t \mathbf{T}_u \times \mathbf{T}_u + u_s v_t \mathbf{T}_u \times \mathbf{T}_v + v_s u_t \mathbf{T}_v \times \mathbf{T}_u + v_s v_t \mathbf{T}_v \times \mathbf{T}_v \\
&= (u_s v_t - u_t v_s) \mathbf{T}_u \times \mathbf{T}_v \\
\mathbf{N}_{\mathbf{Y}} &= \frac{\partial(u,v)}{\partial(s,t)} \mathbf{N}_{\mathbf{X}}
\end{aligned}
$$

Thus, $\mathbf{N}_{\mathbf{Y}}$ is always a scalar multiple of $\mathbf{N}_{\mathbf{X}}$. In addition, since $\mathbf{H}$ is invertible and both $\mathbf{H}$ and $\mathbf{H}^{-1}$ are of class C^1, it follows that the jacobian (determinant) is always positive or negative. Hence, the standard normal vector $\mathbf{N}_{\mathbf{Y}}$ either always points in the same direction as $\mathbf{N}_{\mathbf{X}}$or else always points in the opposite direction. Under thse assumptions, we say that both $\mathbf{H}$ and $\mathbf{Y}$ are **orientiation-preserving** if the Jacobian $\frac{\partial(u,v)}{\partial(s,t)}$ is positive, **orientation-reversing** if $\partial(u,v)/\partial(s,t)$ is negative.

Theorem. *Let* $\mathbf{X} : D_1 \to \mathbf{R}^3$ *be a smooth parametrized surface and* f *any continuous function whose domain includes* $\mathbf{X}(D_1)$*. If* $\mathbf{Y} : D_2 \to \mathbf{R}^3$ *is any smooth reparametrization of* $\mathbf{X}$*, then:*

$$
\int\!\!\int_{\mathbf{Y}} f \cdot dS = \int\!\!\int_{\mathbf{X}} f \cdot dS
$$

Proof.

We have:

$$
\begin{aligned}
\int\!\!\int_{\mathbf{Y}} f \cdot dS &= \int\!\!\int_{(s,t)\in D_2} f(\mathbf{Y}(s,t)) \left\| \mathbf{N}_{\mathbf{Y}}(s,t) \right\| ds\, dt \\
&= \int\!\!\int_{(s,t)\in D_2} f(\mathbf{X}(u,v)) \left\| \mathbf{N}_{\mathbf{X}}(u,v) \right\| \left| \frac{\partial(u,v)}{\partial(s,t)} \right| ds\, dt
\end{aligned}
$$

Since $u = u(s,t), v = v(s,t)$, by the change of variables theorem, $du \cdot dv = \left| \frac{\partial(u,v)}{\partial(s,t)} \right| ds \cdot dt$. So, we have:

$$
\begin{aligned}
\int\!\!\int_{\mathbf{Y}} f \cdot dS &= \int\!\!\int_{(u,v)\in D_1} f(\mathbf{X}(u,v)) \left\| \mathbf{N}_{\mathbf{X}}(u,v) \right\| du\, dv \\
&= \int\!\!\int_{\mathbf{X}} f \cdot dS
\end{aligned}
$$

Theorem. *Let* $\mathbf{X} : D_1 \to \mathbf{R}^3$ *be a smooth parametrized surface and* $\mathbf{F}$ *be any continuous vector field whose domain includes* $\mathbf{X}(D_1)$*. If* $\mathbf{Y} : D_2 \to \mathbf{R}^3$ *is any smooth reparametrization of* $\mathbf{X}$*, then:*

(1) If $\mathbf{Y}$*is orientation-preserving, we have:*

$$\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$

(2) If $\mathbf{Y}$ *is orientiation-reversing, we have:*

$$\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = -\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$

Proof.

This result can be established along the lines of the previous proof. Beginning with the definition and using the lemma just established, we have:

$$\begin{aligned}\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{D_2} \mathbf{F}(\mathbf{Y}(s,t)) \cdot \mathbf{N}_{\mathbf{Y}}(s,t) ds\ dt \\ &= \int\int_{D_2} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \frac{\partial(u,v)}{\partial(x,y)} ds\ dt\end{aligned}$$

If $\mathbf{Y}$is orientation preserving, then $\partial(u,v)/\partial(x,y) > 0$, so $|\partial(u,v)/\partial(x,y)| = \partial(u,v)/\partial(x,y)$. Then, by change of variables theorem:

$$\begin{aligned}\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} &= \int\int_{(s,t)\in D_2} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| ds\ dt \\ &= \int\int_{(u,v)\in D_1} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \cdot du \cdot dv \\ &= \int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}\end{aligned}$$

If $\mathbf{Y}$ is orientation reversing, then $\partial(u,v)/\partial(x,y) < 0$. So, $|\partial(u,v)/\partial(x,y)| = -\partial(u,v)/\partial(x,y)$. Thus,

$$\begin{aligned}\int\int_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} &= -\int\int_{(s,t)\in D_2} \mathbf{F}(\mathbf{X}(u,v)) \cdot \mathbf{N}_{\mathbf{X}}(u,v) \left|\frac{\partial(u,v)}{\partial(x,y)}\right| ds\ dt \\ &= -\int\int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}\end{aligned}$$

Given a smooth surface, we need to choose an orientation for it. This is akin to orienting a curve, but perhaps surprisingly, it is not always possible, even for a well-behaved, smooth parametrized surface, as the next example illustrates.

Definition. A smooth, connected surface S is **orientable** (or **two-sided**) if it is possible to define a single unit normal vector at each point of S so that the collection of these normal vectors varies continuously over S. (In particular, this means that unit normal vectors must point to the same side of S.) Otherwise, S is called as non-orientable.

Example. Consider the surface parametrized by

$$\begin{aligned}x &= (1 + t\cos(s/2))\cos s \\ y &= (1 + t\cos(s/2))\sin s \\ z &= t\sin s/2\end{aligned}$$

where $0 \le s \le 2\pi$ and $-1/2 \le t \le t$. This is called a **Mobius strip**. It may be visualized as follows : The t-coordinate curve at $s = s_0$ is:

$$\begin{aligned}x &= \cos(s_0/2)\cos(s_0)t + \cos(s_0) \\ y &= \cos(s_0/2)\sin(s_0)t + \sin(s_0) \\ z &= t(\sin s_0/2)\end{aligned}$$

If we isolate t in the above three equations, we find:

$$\frac{x - \cos s_0}{\cos(s_0/2)\cos s_0} = \frac{y - \sin s_0}{\cos(s_0/2)\sin s_0} = \frac{z - 0}{\sin(s_0/2)}$$

This is a plane that passes through the point $(\cos s_0, \sin s_0, 0)$ and parallel to the vector:

$$\mathbf{a} = (\cos(s_0/2)\cos s_0, \cos(s_0/2)\sin s_0, \sin s_0/2)$$

Consider a few such coordinate curves:

s	Parallel Vector	Passes through the Point	Plane
$s_0 = 0$	$\mathbf{a} = (1, 0, 0)$	$(1, 0, 0)$	$\mathbf{x} = t(1, 0, 0) + (1, 0, 0)$
$s_0 = \pi/2$	$\mathbf{a} = (0, 1/\sqrt{2}, 1/\sqrt{2})$	$(0, 1, 0)$	$\mathbf{x} = t(0, 1/\sqrt{2}, 1/\sqrt{2}) + (0, 1, 0)$
$s_0 = \pi$	$\mathbf{a} = (0, 0, 1)$	$(-1, 0, 0)$	$\mathbf{x} = t(0, 0, 1) + (-1, 0, 0)$
$s_0 = 3\pi/2$	$\mathbf{a} = (0, 1/\sqrt{2}, 1/\sqrt{2})$	$(0, -1, 0)$	$\mathbf{x} = t(0, 1/\sqrt{2}, 1/\sqrt{2}) + (0, -1, 0)$
$s_0 = 2\pi$	$\mathbf{a} = (-1, 0, 0)$	$(1, 0, 0)$	$\mathbf{x} = t(-1, 0, 0) + (1, 0, 0)$

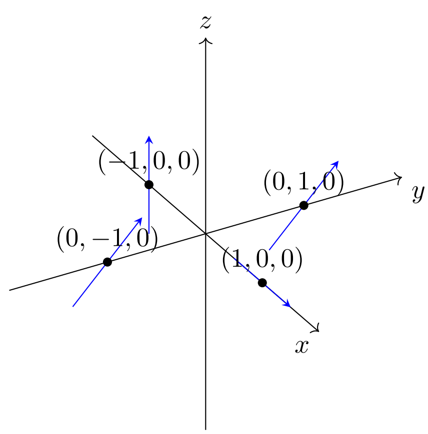

Figure. t-coordinate curves

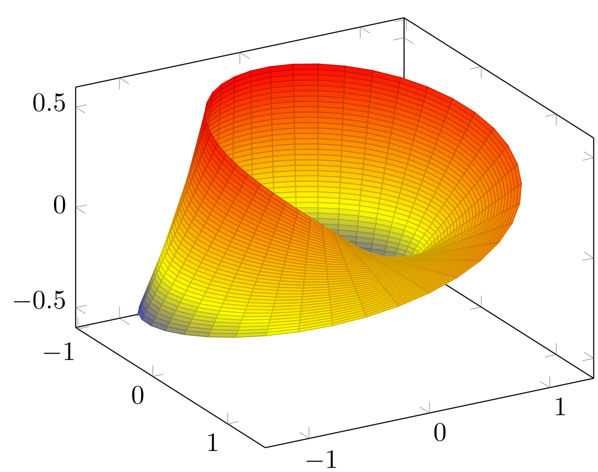

Figure. Mobius Strip

We see that the Mobius strip is generated by a moving line segment that begins at $s = 0$ lying along the positive x-axis, rises to a vertical position with the center at $(-1, 0, 0)$ when $s = \pi$ and then falls back to the center with direction reversal at $s = 2\pi$. The s-coordinate curve at $t = 0$ is parametrized by:

$$
\begin{aligned}
x &= \cos s \\
y &= \sin s \\
z &= 0
\end{aligned}
$$

and so is a circle in the xy-plane. The full Mobius strip is shown in the figure above. You can make a physical model by taking a strip of paper, giving it a half-twist, and joining the short-ends.

You can understand the gluing process analytically by noting that the map :

$$
\mathbf{X} : [0, 2\pi] \times [-\frac{1}{2}, -\frac{1}{2}] \to \mathbf{R}^3
$$

defining the Mobius strip as a parametrized surface has the property that $\mathbf{X}(0, t) = \mathbf{X}(2\pi, -t)$ but is otherwise one to one. Therefore, every point $(0, t)$ on the left edge of the domain rectangle $[0, 2\pi] \times [-\frac{1}{2}, \frac{1}{2}]$ is mapped to the point $(1 + t, 0, 0)$ of the Mobius strip, as is the point $(2\pi, -t)$ on the right edge of the rectangle.

Now, let's investigate the orientability of the Mobius strip. Firstly,

$$
\begin{aligned}
x &= (1 + t\cos(s/2))\cos s \\
x_s &= -\sin s + t(\cos s/2 \cdot \cos s)_s \\
&= -\sin s + t(-\frac{1}{2}\sin s/2\cos s - \cos s/2\sin s) \\
&= -\sin s - t((1/2)\sin s/2\cos s + \cos s/2\sin s) \\
x_t &= \cos(s/2)\cos s
\end{aligned}
$$

$$
\begin{aligned}
y &= (1 + t\cos(s/2))\sin s \\
&= \sin s + t\cos(s/2)\sin s \\
y_s &= \cos s + t(-1/2\sin s/2\sin s + \cos s/2\cos s) \\
&= \cos s + t(\cos s/2\cos s - (1/2)\sin s/2\sin s) \\
y_t &= \cos s/2\sin s
\end{aligned}
$$

$$
\begin{aligned}
z &= t\sin s/2 \\
z_s &= t/2\cos s/2 \\
z_t &= \sin s/2
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\frac{\partial(y, z)}{\partial(s, t)} &= y_s z_t - z_s y_t \\
&= (\cos s + t(\cos s/2\cos s - (1/2)\sin s/2\sin s))(\sin s/2) - (t/2\cos s/2)(\cos s/2\sin s) \\
&= \cos s\sin s/2 + t\sin s/2\cos s/2\cos s - t/2\sin^2 s/2\sin s - t/2\sin s\cos^2 s/2 \\
&= \cos s\sin s/2 + t\sin s/2\cos s/2\cos s - t/2\sin s(\sin^2 s/2 + \cos^2 s/2) \\
&= \cos s\sin s/2 + t\sin s/2\cos s/2\cos s - t\sin s/2\cos s/2 \\
&= \sin s/2(\cos s - t\cos s/2(1 - \cos s)) \\
&= \sin s/2(\cos s - 2t\cos s/2(\sin^2 s/2)) \\
&= \sin s/2(\cos s - 2t\cos s/2(1 - \cos^2 s/2)) \\
&= \sin s/2(\cos s - 2t(\cos s/2 - \cos^3 s/2))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial(x,z)}{\partial(s,t)} &= x_s z_t - x_t z_s \\
&= (-\sin s - t((1/2)\sin s/2\cos s + \cos s/2\sin s)\sin s/2 - (\cos(s/2)\cos s)(t/2\cos s/2) \\
&= -\sin s\sin s/2 - t/2\sin^2 s/2\cos s - t\sin s/2\cos s/2\sin s - t/2\cos s\cos^2 s/2 \\
&= -\sin s\sin s/2 - t/2\cos s - t/2\sin^2 s \\
&= -2\sin^2 s/2\cos s/2 - t/2\cos s - t/2(1-\cos^2 s) \\
&= -\frac{1}{2}(4(1-\cos^2 s/2)\cos s/2 + t\cos s + t - t\cos^2 s) \\
&= -\frac{1}{2}(4\cos s/2 - 4\cos^3 s/2 + t(1+\cos s - \cos^2 s))
\end{aligned}
$$

$$
\frac{\partial(x,y)}{\partial(s,t)} =
$$

The standard normal vector is:

$$
\begin{aligned}
\mathbf{N}(s,t) &= \mathbf{T}_s \times \mathbf{T}_t \\
&= \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k}
\end{aligned}
$$