Exact Forward for Finite-Difference Schemes on the Log-transformed Black-Scholes PDE

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(v1.0 released December 2016)

Abstract *This note shows that while the forward is exactly preserved on a direct discretisation of the Black-Scholes PDE, it is not anymore on the log-transformed PDE, which is very commonly used in practice. A remedy reminiscent of the exponential fitting technique is proposed.*

Key Words: Finite difference, Black-Scholes, Calibration

1. Exact forward in the direct discretisation

We will assume zero interest rates and no dividends on the asset S for clarity. The results can be easily generalized to the case with non-zero interest rates and dividends. Under those assumptions, the Black-Scholes PDE is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0. \tag{1}$$

An implicit Euler discretisation on a uniform grid in S of width h with linear boundary conditions (zero Gamma) leads to:

$$V_i^{k+1} - V_i^k = \frac{1}{2}\sigma^2 \Delta t S_i^2 \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{h^2}. \quad \text{for } i = 1, ..., m-1, \tag{2}$$

$$V_i^{k+1} - V_i^k = 0 \quad \text{for } i = 0, m. \tag{3}$$

This forms a linear system $M \cdot V^k = V^{k+1}$ with M a tridiagonal matrix where each of its rows sums to 1 exactly. Furthermore, the payoff corresponding to the forward price $V_i = S_i$ is exactly preserved as well by such a system as the discretized second derivative will be exactly zero. The scheme can be seen as preserving the zero-th and first moments.

As a consequence, by linearity, the put-call parity relationship will hold exactly (see Le Floc'h (2014) for a proof). Note that in between nodes, any interpolation used should also be consistent with the put-call parity for the result to be global. This result stays true for a non-uniform discretisation, and with other finite difference schemes.

ISSN: Print/ISSN Online/YY/issnum1stp–ppcnt© 2016 Unpublished
DOI:

2. The log-transformed problem

It is common to consider the log-transformed problem in $X = \ln(S)$ as the diffusion is constant then, and a uniform grid much more adapted to the process.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} - \frac{1}{2}\sigma^2 \frac{\partial V}{\partial X} = 0. \tag{4}$$

An implicit Euler discretisation on a uniform grid in X of width h with linear boundary conditions (zero Gamma in S) leads to:

$$V_i^{k+1} - V_i^k = \frac{1}{2}\sigma^2 \Delta t \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{h^2} - \frac{1}{2}\sigma^2 \Delta t \frac{V_{i+1}^k - V_{i-1}^k}{2h} \quad \text{for } i = 1, ..., m-1, \tag{5}$$

$$V_i^{k+1} - V_i^k = 0 \quad \text{for } i = 0, m. \tag{6}$$

Such a scheme will not preserve the forward price anymore. This is because now, the forward price is $V_i = e^{X_i}$. In particular, it is not linear in X.

3. A remedy

It is possible to preserve the forward by changing slightly the diffusion coefficient, very much as in the exponential fitting idea (Le Floc'h, 2016). The difference is that, here, we are not interested in handling a large drift (when compared to the diffusion) without oscillations, but merely to preserve the forward exactly. We want the adjusted volatility $\bar{\sigma}$ to solve

$$\frac{1}{2}\bar{\sigma}^2 \frac{e^h - 2 + e^{-h}}{h^2} - \frac{1}{2}\sigma^2 \frac{e^h - e^{-h}}{2h} = 0. \tag{7}$$

Note that the discretised drift does not change, only the discretised diffusion term. The solution is:

$$\bar{\sigma}^2 = \frac{\sigma^2 h}{2} \coth\left(\frac{h}{2}\right). \tag{8}$$

This needs to be applied only for $i = 1, ..., m-1$.

This is actually the same adjustment as the exponential fitting technique with a drift of zero. For a non-zero drift, the two adjustments would differ, as the exact forward adjustment will stay the same, along with an adjusted discrete drift.

References

Le Floc'h, F. (2014) Exact Forward and Put-Call Parity with TR-BDF2, *Available at SSRN https://papers.ssrn.com/abstract=2362969*.

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