Asymptotic bounds of the normal volatility

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ABSTRACT *Roger Lee prooved that the Black-Scholes implied variance can not grow faster than linearly in log-moneyness. This paper offers an alternate simple explanation of this fact starting from the Dupire equation. We then investigate what happens in the Bachelier (or Normal) implied volatility world, making sure to cover the various aspects of vanilla option arbitrages.*

KEY WORDS: implied volatility, b.p. vol, Bachelier

1. Introduction

In (Lee, 2004), Roger Lee prooves that the Black-Scholes implied volatility asymptotics can not be arbitrary: the implied distribution does not have finite moments if the square of the total implied variance $w(y,T) = \sigma^2(y,T)$ grows faster than $2|y|$ where y is the log-moneyness $y = \ln \frac{K}{F}$ where K is the option strike, F the underlying asset forward to expiry T and σ is the Black-Scholes implied volatility for the given moneyness and maturity. As a consequence, any implied variance extrapolation must be at most linear in the wings, a fact of great practical interest.

With the advent of low or negative interest rates, practitioners have moved towards the use of the Bachelier (or normal) model. Then, a natural question that arises is: what kind of extrapolation is acceptable in the normal model?

We will start with an interpretation of the Black-Scholes implied volatility asymptotics in terms of the Dupire local volatility. We will then explore the asymptotic behavior of the Dupire equation corresponding to the Bachelier (or normal) model. We will continue with a more direct analysis of the no-arbitrage conditions on vanilla option slopes for the Bachelier model, and finish with the bounds related to the direct option prices.

2. Black-Scholes asymptotics from the Dupire equation

Before Roger Lee, Jim Gatheral derived the same rule from the butterfly no-arbitrage condition in (Gatheral *et al.*, 2000). We will show here that the rule can also be guessed from the Dupire local volatility equation.

In the Black local volatility model, the forward price F of an asset follows the stochastic

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differential equation:

$$dF = \sigma^\star(F,T)FdW \tag{1}$$

with W a Brownian motion and $\sigma^\star$ the local volatility.

The classic Dupire local volatility formula relating vanilla option prices to the local volatility is Dupire *et al.* (1994):

$$\sigma^{\star 2}(K,T) = \frac{1}{2}\frac{\frac{\partial C_0}{\partial T}}{K^2\frac{\partial^2 C_0}{\partial K^2}} \tag{2}$$

where $C_0(K,T)$ is an undiscounted call option price of strike K and maturity T expressed in terms of forward $F(0,T)$.

It can be expressed in terms of total implied variance $w(y,T)$:

$$\sigma^{\star 2}(y,T) = \frac{\frac{\partial w}{\partial T}}{1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}}. \tag{3}$$

Calendar spread arbitrage is avoided if the numerator is positive, and Butterfly spread arbitrage is avoided if the denominator is positive. The relations can also be derived directly, without the local volatility analogy as in Carr (2004); Roper (2010).

We now assume that the implied variance is $w(y,T) = A|y|^\beta T$ in order to analyze the conditions on the leading order of the implied variance. When $|y| \to \infty$, the denominator tends towards:

$$1 - \beta + \frac{\beta^2}{4}\left(-\frac{1}{4}A^2T^2|y|^{2\beta-2} - AT|y|^{\beta-2} + 1\right) + \frac{\beta(\beta-1)}{2}AT|y|^{\beta-2} \tag{4}$$

for $\beta \notin \{0,1\}$. If $\beta = 1$, the last term is 0.

When $\beta > 0$, the dominant term $-\frac{1}{16}A^2T^2|y|^{2\beta-2}$ must stay positive. It will grow to $-\infty$ unless $\beta \leq 1$.

If $\beta = 1$ and $|y| \to \infty$ the positive denominator condition translates to

$$\frac{1}{4}\left(-\frac{1}{4}A^2T^2 + 1\right) > 0, \tag{5}$$

or equivalently

$$AT < 2. \tag{6}$$

We find back that the slope of the total implied variance is at most 2. Furthermore the local volatility behaves then as $\sigma^{\star 2}(y,T) \propto y$, or in plain words, it is linear in the wings as well.

3. Bachelier asymptotics

3.1 Asymptotics bounds from Dupire

In the Bachelier local volatility model, the forward price F of an asset follows the stochastic differential equation:

$$dF = \sigma_N^\star(F,T)dW \tag{7}$$

with W a Brownian motion and $\sigma_N^\star$ the local volatility.

Under the Bachelier model, the normal Dupire equation is:

$$\sigma^{\star 2}(K,T) = \frac{1}{2}\frac{\frac{\partial C_0}{\partial T}}{\frac{\partial^2 C_0}{\partial K^2}} \tag{8}$$

Viorel Costeanu and Dan Pirjol derived the equivalent formulation in terms of implied normal variance $w = \sigma_N^2 T$ as a function of moneyness $y = K - F$ for an option of strike K, maturity T, with normal implied volatility σ_N in Costeanu and Pirjol (2011):

$$\sigma_N^{\star 2}(y+F,T) = \frac{\frac{\partial w(y+F,T)}{\partial T}}{1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}}. \tag{9}$$

Let us assume that $w(y,T) = AT|y|^\beta$. When $|y| \to \infty$ the denominator tends towards

$$1 - \beta + \frac{\beta^2}{4}\left(-AT|y|^{\beta-2} + 1\right) + \frac{\beta(\beta-1)}{2}AT|y|^{\beta-2} = \left(1 - \frac{\beta}{2}\right)^2 + \frac{\beta^2 - 2\beta}{4}AT|y|^{\beta-2} \tag{10}$$

It is positive for $A \geq 0$ and any $\beta > 2$. For $\beta < 2$, the dominant term $\left(1 - \frac{\beta}{2}\right)^2$ is also positive and the normal local volatility behaves as $\sigma_N^{\star 2}(y+F,T) \sim |y|^\beta$. For $\beta = 2$, the denominator is zero.

It would be however wrong to believe that the denominator is always positive in the limit $y \to \infty$. In deed, Henry-Labordère (2008) shows the following relationship when $\ln K \to \infty$:

$$\sigma(K) \sim \sigma_N \frac{\ln \frac{K}{F}}{K}. \tag{11}$$

As we know that the Black volatility σ is bounded by $\sqrt{2 \ln \frac{K}{F}}$, we must have an equivalent boundary for σ_N. In deed, if we know assume that $w(K - F) = \frac{A(K-F)^2}{(\ln \frac{K}{F})^\beta}$, the denominator becomes

$$-\frac{A\beta}{2(\ln \frac{K}{F})^{\beta+1}} + \frac{A\beta^2 + 2A\beta}{4(\ln \frac{K}{F})^{\beta+2}} + \frac{\beta^2}{4(\ln \frac{K}{F})^2} \tag{12}$$

The denominator is guaranteed to become negative for $\beta < 1$ when K is sufficiently large. For $\beta = 1$, the condition for positivity is

$$-\frac{A}{2} + \frac{1}{4} > 0 \tag{13}$$

or equivalently

$$A < \frac{1}{2} \tag{14}$$

It appears then that the normal volatility σ_N is bounded by $\frac{K-F}{\sqrt{2T \ln \frac{K}{F}}}$. We will look for a more rigorous proof in the following sections.

3.2 Asymptotics bounds from option price slopes

In addition to the non negative butterfly spread, the call options prices must be decreasing with strike, and the put options prices must be increasing with strike. Let $C(K,T)$ and $P(K,T)$ be respectively an undiscounted call and an undiscounted put option price of strike K and maturity T, we must have

$$-1 \leq \frac{\partial C}{\partial K} \leq 0, \tag{15}$$

$$0 \leq \frac{\partial P}{\partial K} \leq 1. \tag{16}$$

Under the Bachelier model, the undiscounted prices of vanilla options are:

$$C(y,v) = -yN\left(-\frac{y}{v}\right) + vn\left(\frac{y}{v}\right), \tag{17}$$

$$P(y,v) = yN\left(\frac{y}{v}\right) + vn\left(\frac{y}{v}\right) \tag{18}$$

with $y = K - F$ and $v = \sigma_N\sqrt{T}$ where $\sigma_N(y,T)$ is the Bachelier normal volatility. n and N are respectively the normal density function and the cumulative normal density function.

The partial derivatives in term of moneyness y and deviation v are:

$$\frac{\partial C}{\partial y} = -N\left(-\frac{y}{v}\right), \frac{\partial C}{\partial v} = n\left(\frac{y}{v}\right), \tag{19}$$

$$\frac{\partial P}{\partial y} = N\left(\frac{y}{v}\right), \frac{\partial P}{\partial v} = n\left(\frac{y}{v}\right). \tag{20}$$

By chain rule calculus, the conditions on the price slopes in terms of the variables y and v are:

$$-1 \leq -N\left(-\frac{y}{v}\right) + n\left(\frac{y}{v}\right)v'(y) \leq 0, \tag{21}$$

$$0 \leq N\left(\frac{y}{v}\right) + n\left(\frac{y}{v}\right)v'(y) \leq 1 \tag{22}$$

where $v' = \frac{\partial v}{\partial y}$. The conditions on the scaled normal volatility slope are:

$$-\frac{N\left(\frac{y}{v}\right)}{n\left(\frac{y}{v}\right)} \leq v'(y) \leq \frac{N\left(-\frac{y}{v}\right)}{n\left(\frac{y}{v}\right)} \tag{23}$$

As in Hodges (1996), we recall that the cumulative normal distribution behaves in the limiting cases as

$$N(x) = \begin{cases} 1 - \frac{n(x)}{x} + \frac{n(x)}{x^3} + ... & \text{when } x >> 1 \\ \frac{1}{2} + \frac{x}{\sqrt{2\pi}} + ... & \text{when } x << 1 \\ -\frac{n(x)}{x} + \frac{n(x)}{x^3} + ... & \text{when } x << -1 \end{cases} \tag{24}$$

We split the analysis of the limiting behavior of the scaled normal volatility slope in two different cases.

When $\frac{y}{v} >> 1$, equation 23 translates to an upper boundary on the scaled normal volatility slope

$$v'(y) \leq \frac{v}{y} \tag{25}$$

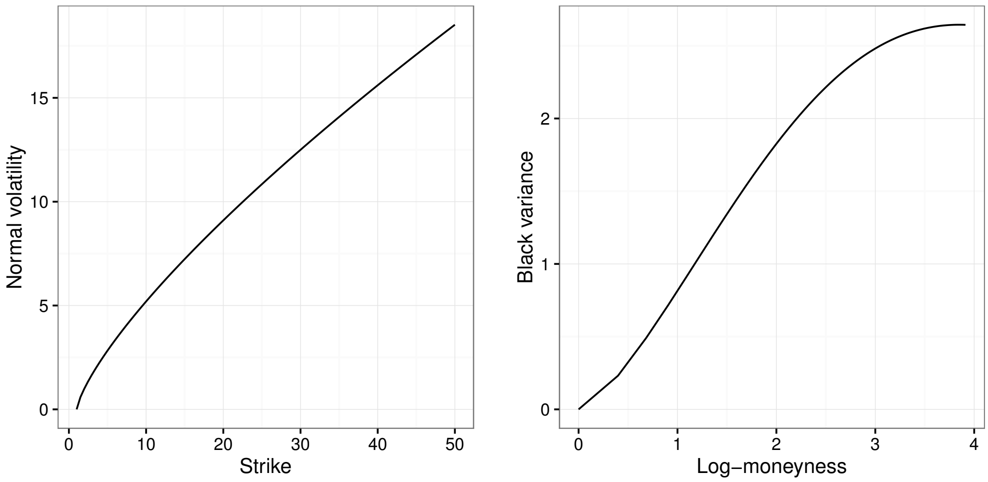
or $(\ln v)'(y) \leq \frac{1}{y}$. By the mean value theorem and the monotonicity of the logarithm, if there exists a so that $v(a) \leq a$, then we must have for all $y \geq a$, $v(y) \leq y$, the scaled normal volatility is bounded by the line of slope 1. In particular, if $v = Ay^\beta$, we must have $\beta \leq 1$. When $y \to \infty$, this is consistent with the initial hypothesis $\frac{y}{v} >> 1$.

When $\frac{y}{v} << 1$, we have then

$$-\sqrt{\frac{\pi}{2}} \leq v'(y) \leq \sqrt{\frac{\pi}{2}}. \tag{26}$$

The slope at $y = 0$ is limited. We are however more interested in the asymptotic behavior when $y \to \infty$. In particular, if $v = Ay^\beta$, the bounds on the slope imply $\beta \leq 1$, which contradicts the initial hypothesis $\frac{y}{v} << 1$. This suggests that the scaled normal volatility can not grow faster than linearly.

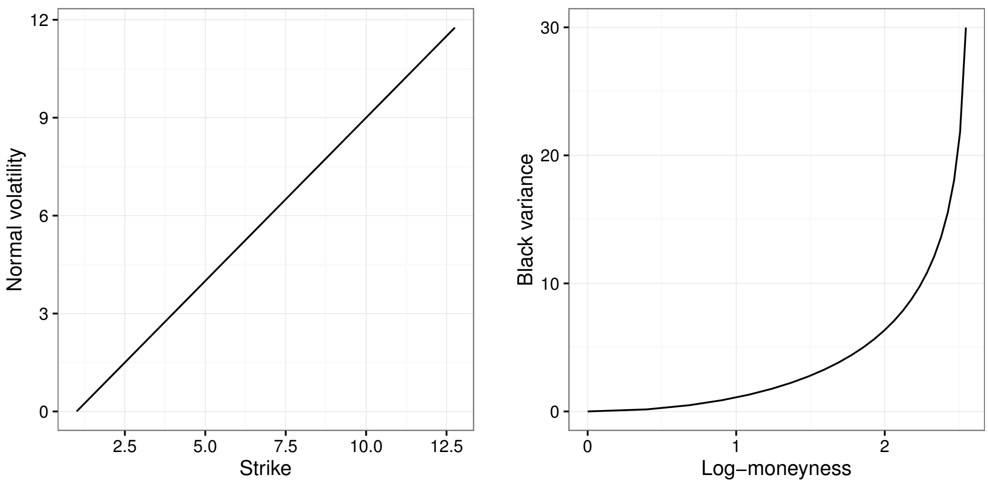
Figure 1 shows that the Black implied volatility will flatten in general for large strikes when the Bachelier implied volatility is of the form $\sigma_N = (K - f)^\beta$ with $\beta < 1$.



(a) Bachelier (b) Black

Figure 1.: Black variance corresponding to the Bachelier normal volatility smile $\sigma_N = (K - f)^{\frac{3}{4}}$.

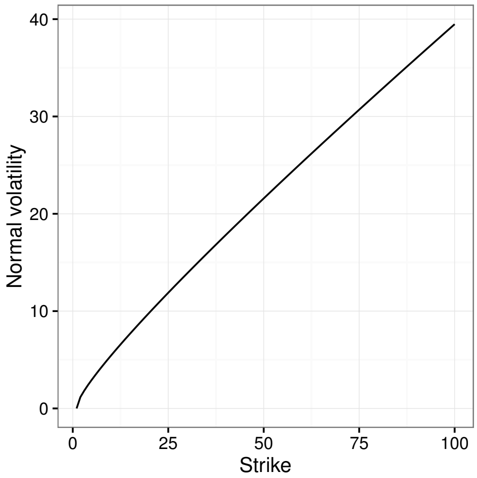
A linear normal volatility can lead to super-linear Black variance as in Figure 2. The limit



(a) Bachelier (b) Black

Figure 2.: Black variance corresponding to the Bachelier normal volatility smile $\sigma_N = (K - f)$.

case for Black volatilities, a linear Black variance looks almost linear (but is not) in the wings in terms of normal volatility.



(a) Bachelier

(b) Black

Figure 3.: Bachelier normal volatility corresponding to the Black variance $w = \ln \frac{K}{F}$.

3.3 Asymptotic bounds from option prices

We must have $\lim_{K\to\infty} C(K, v(K)) = \lim_{K\to\infty} \mathbb{E}[|F-K|^+] = 0$. If we let $v(K) = (K-F)\sqrt{\frac{b}{\ln\frac{K}{F}}}$, we have

$$\frac{K-F}{v} = \sqrt{\frac{\ln\frac{K}{F}}{b}}, \tag{27}$$

$$n\left(\frac{K-F}{v}\right) = \sqrt{\frac{F^{\frac{1}{b}}}{2\pi K^{\frac{1}{b}}}} \tag{28}$$

when $K \to \infty$, the cumulative normal distribution expansion leads to:

$$C(K, v(K)) = \frac{v^3(K)}{(K-F)^2} n\left(\frac{K-F}{v}\right) + ... \tag{29}$$

$$= b^{\frac{3}{2}} \frac{K-F}{(\ln\frac{K}{F})^{\frac{3}{2}}} \sqrt{\frac{F^{\frac{1}{b}}}{2\pi K^{\frac{1}{b}}}} + ... \tag{30}$$

For $b > \frac{1}{2}$, $C(K, v(K)) \to \infty$. As $C(K, v)$ is monotone in its second argument, v is bounded by $v(K) = (K-F)\sqrt{\frac{b}{\ln\frac{K}{F}}}$ when $K \to \infty$ for all $b > \frac{1}{2}$.

A close estimate for the boundary can be solved numerically by letting $C(K, v(K)) = -(K-F)N\left(-\frac{K-F}{v(K)}\right) + v(K)n\left(\frac{K-F}{v(K)}\right) = \epsilon$ with a small ϵ (see Figure 4). This corresponds to solving the Bachelier implied volatility corresponding to the price ϵ for the moneyness $K-F$. Numerically, the problem can be considered as a simple closed form function evaluation thanks to the machine precision rational function representation of Le Floc'h (2014).

If we want to map Bachelier prices to Black-Scholes prices, we might want to add as additional condition

$$\lim_{K\to 0} P(K, v(K)) = 0. \tag{31}$$

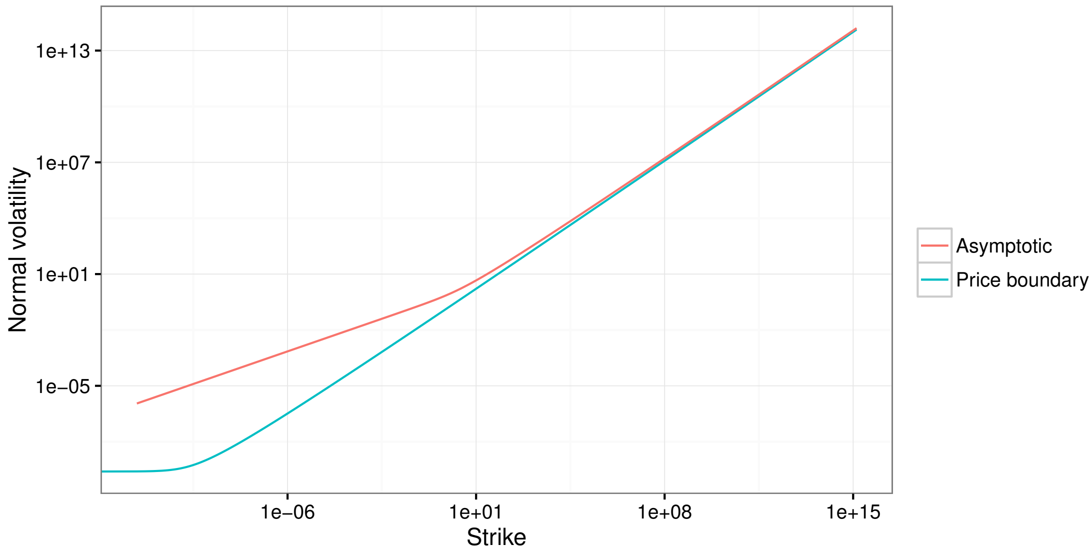


Figure 4.: Boundary of the scaled normal volatility v corresponding to $C(K,v) = 10^{-10}$ compared to the upper asymptotic bound $v(K) = \frac{K-F}{\sqrt{2\ln\frac{K}{F}}}$.

This is only possible if the normal implied volatility is bounded near $K = 0$. In deed, if $\lim_{K\to 0} v(K) = +\infty$, we have $P(K) \sim \frac{v(K)}{\sqrt{2\pi}}$. In particular, when $\lim_{K\to 0} v(K) = 0$, then $\lim_{K\to 0} P(K) = 0$ both terms of the put option price tend to zero. Furthermore, for all $v > 0$, we have $P(0,v) > 0$ as P is strictly monotone in its second argument therefore we must have in all cases $\lim_{K\to 0} v(K) = 0$ to ensure $P(0, v(0)) = 0$.

4. A remark on the interpolation of normal volatilities

When written in terms of normal volatility, the Dupire local volatility formula of Costeanu and Pirjol (2011) shows that as long as the normal volatility function is convex, the implied density will stay positive.

This can be interesting to interpolate a set of option prices without introducing spurious arbitrage. In the Black-Scholes world, the necessary conditions are not as simple in terms of Black volatility. Still, this presupposes that the reference normal volatilities at given strikes are convex, which is not necessarily true in practice.

The condition of no-arbitrage for the SVI parameterisation of Gatheral (2004) appear to be much simpler in the Bachelier world than in the Black-Scholes world (Sun *et al.*, 2016).

5. Conclusion

We have shown that the function $\sigma_N(K) = \frac{K-F}{\sqrt{2T\ln\frac{K}{F}}}$ constitutes an upper asymptotic bound of the implied volatility under the Bachelier model to guarantee the absence of arbitrages.

Appendix A. Asymptotic expansions relating Black and Bachelier volatilities

Although the problem of converting Black volatilities (also called basis point volatilities or Bachelier volatilities) to normal volatilities can be solved quite efficiently by relying on inverting the Black-Scholes price through the nearly closed form representation of Le Floc'h (2014), or if the reverse is needed, through a good Black implied volatility solver such as the one of Jäckel (2013), the behavior around the money is well described by the following simple and fast expansions:

- Hagan *et al.* (2002) simple second order approximation

$$\sigma_N = \sigma_B \frac{f-K}{\ln\frac{f}{K}}\left(1+\frac{1}{24}\sigma_B^2 T+...\right) \tag{A1}$$

- Hagan *et al.* (2002) fourth order approximation

$$\sigma_N = \sigma_B\sqrt{fK}\frac{1+\frac{1}{24}\ln^2\frac{f}{K}+\frac{1}{1920}\ln^4\frac{f}{K}+...}{1+\frac{1}{24}\left(1-\frac{1}{120}\ln^2\frac{f}{K}\right)\sigma_B^2 T+\frac{\sigma_B^4 T^2}{5760}+...} \tag{A2}$$

- The small time $O(T^2\ln T)$ expansion of Grunspan (2011)

$$\sigma_N = \sigma_B\frac{f-K}{\ln\frac{f}{K}}\left(1-\frac{\ln\left(\frac{1}{\sqrt{Kf}}\frac{f-K}{\ln\frac{f}{K}}\right)}{\ln^2\frac{f}{K}}\sigma_B^2 T\right)+O(T^2\ln T) \tag{A3}$$

Following their paper derivation, we can also find the inverse expression.

- The third order Black implied volatility expansion of Lorig *et al.* (2015)

$$\sigma_B = \frac{\sigma_N}{f}\left[1-\frac{1}{2}\ln\frac{K}{f}+\frac{1}{96}\left(8\ln^2\frac{K}{f}+\left(\frac{\sigma_N}{f}\right)^2 T(4-\left(\frac{\sigma_N}{f}\right)^2 T)\right)\right. \tag{A4}$$

$$\left.+\frac{1}{192}\ln\frac{K}{f}\left(\frac{\sigma_N}{f}\right)^2 T(-12+5\left(\frac{\sigma_N}{f}\right)^2 T)\right] \tag{A5}$$

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