Math 16:642:623 – Homework Assignment 1[*] – Spring 2010 © Paul Feehan

Practice or *advanced* problems are optional and are not graded: *practice problems* are intended as drill problems and aides to exam preparation while *advanced problems* are intended for students with additional mathematics background. Please consult the *Homework Submission Requirements* before commencing work on this assignment.

1. Reading Assignments and Sample Code

1.1. **Reading assignments.** The primary texts and readings for the course are Glasserman, Joshi, Wilmott, and Achdou and Pironneau. You may find the suggested alternative readings helpful.

(a) Achdou and Pironneau, sections 1.1, 1.2, 1.3, and 1.4.
(b) Brandimarte, sections 4.2 and 4.4
(c) Glasserman, sections 3.1, 3.2.1, 3.2.3, 6.1, 6.2.1, and 6.2.2.
(d) Joshi, chapter 1.
(e) Seydel, sections 3.1, 3.2, 3.3, 3.5.1, and 3.5.2.
(f) Tavella, chapter 4, pages 77-93 and 100-102.

1.2. **Sample code and libraries.**

(a) The main programs, `SimpleMCMainN.cpp`, where $N = 1, \dots, 4$, and their include files available from `www.markjoshi.com/design/index.htm`.
(b) GNU Scientific Library Manual, `www.gnu.org/software/gsl/`; a high-quality numerical methods C/C++ library, `gsl` is freely available for Cygwin, Linux, or Unix.
(c) Numerical Recipes, `www.nrbook.com/a/bookcpdf.php`, free C library; selected C++ code will be made available for download from Sakai.
(d) Ohio State University, Computational Physics 780.20, for sample Makefiles and GSL usage, `www.physics.ohio-state.edu/~ntg/780/`.

2. Programming and Written Assignments

1. Write a C++ program to price a European-style call or put option using each of the methods described below; the option price style and computational method are user-defined. For the program output test, assume that the stock price process is geometric Brownian motion with volatility $\sigma = 0.3$, initial asset price $S(0) = 100$, constant risk-free interest rate $r = 0.05$, dividend yield $d = 0.02$, option type call, strike $K = 110$, and maturity $T = 1$ year.

You may modify any one of the main programs (and their include files), `SimpleMCMainN.cpp`, $N = 1, \dots, 4$, together with the code (`Normals.cpp`, `Normals.h`, `Random1.cpp`, and `Random1.h`), authored by Joshi.

The code `Random1.cpp` provides the Box-Muller method to generate samples from the univariate normal distribution using the `rand` function in `<cstdlib>`; see, for example, Glasserman, p. 65, for details of the Box-Muller method. The code `Normals.cpp` provides the Hastings approximation to the cumulative normal distribution, as modified in Abromowitz and Stegun; see, for example, Glasserman, p. 67, for details of the Hastings approximation. Later in the course we shall discuss methods of generating sequences of pseudo-random and quasi-random numbers and samples from different probability distributions.

(a) Use the closed-form formulae to compute the prices for a European-style call and put. Your submitted program should output the result as

```
Option price using closed-form formula =
```

(b) Use the analytical solution, $S(T)$, to $dS(t) = S(t)((r-d)\,dt + \sigma\,dW(t))$ and Monte Carlo simulation to compute the prices for a European-style call and put. Choose one time step and $I = 10,000$ paths. Your submitted program should output the result as

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`Option price using single-step exact SDE solution =`

(c) Use the Euler numerical solution to $dS(t) = S(t)((r-d)\,dt + \sigma\,dW(t))$ and Monte Carlo simulation to compute the prices for a European-style call and put. Choose I time steps using 252 time steps per year and $J = 10,000$ paths. Your submitted program should output the result as

`Option price using Euler numerical solution of SDE for spot =`

(d) Use the Euler numerical solution to $d\log S(t) = (r - d - \sigma^2/2)\,dt + \sigma\,dW(t)$, and Monte Carlo simulation to compute the prices for a European-style call and put. Choose I time steps using 252 time steps per year and $J = 10,000$ paths. Your submitted program should output the result as

`Option price using Euler numerical solution of SDE for log spot =`

(e) Use the Milstein numerical solution to $dS(t) = S(t)((r-d)\,dt + \sigma\,dW(t))$, and Monte Carlo simulation to compute the prices for a European-style call and put. Choose I time steps using 252 time steps per year and $J = 10,000$ paths. Your submitted program should output the result as

`Option price using Milstein numerical solution of SDE for spot =`

(f) Write a report with a short explanation of the algorithms and their implementation and an analysis of your results, including a comparison of your results with those from the Excel-VBA spreadsheets of Haug, Back, or Rouah-Vainberg, or the MATLAB functions of Brandimarte or Mathworks' toolboxes. Is it necessary to simulate entire paths, $\{S(t)\}_{t\in[0,T]}$, in order to compute the option price when the payoff is $(S(T)-K)^+$? What about when the payoff is that of a continuously monitored European-style Asian call option, $(\bar{S}(T) - K)^+$ where $\bar{S}(T) = T^{-1} \in_0^T S(t)\,dt$, or a discretely monitored European-style Asian call option, when $\bar{S}(T) = m^{-1}\sum_{i=1}^{m} S(t_i)$?

3. Practice Exercises

2. Modify your C++ code so that the Monte Carlo simulation uses the GNU Scientific Library Gaussian random number generator and functions to compute the cumulative normal distributions, rather than the functions supplied in the sample code of Joshi (see `Normals.cpp` and `Random1.cpp`).

3. Modify your C++ code to use Cholesky decomposition and the method of sampling from the I-dimensional joint distribution of $W(t_i)$, for $i = 1, \ldots, I$, to obtain approximate solutions, $S^j(t_i)$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, to $dS(t) = S(t)((r-d)\,dt + \sigma\,dW(t))$, and hence compute the call and put option prices. In general this method may involve intensive linear algebra calculation if I is large, but in this example the correlation matrix for $W(t_1), \ldots, W(t_I)$ has a simple structure and Cholesky decomposition; any required linear algebra functions may be drawn from GNU Scientific Library or Numerical Recipes.

4. Lecture Summary

Suppose $(\Omega, \mathscr{F}, \mathbb{Q}, \mathbb{F})$ is a filtered probability space, with filtration $\mathbb{F} = \{\mathscr{F}(t)\}_{t\geq 0}$ generated by $\mathbb{R}^d$-valued Brownian motion, $\{W(t)\}_{t\geq 0}$. Let $T > 0$ and let H be an $\mathscr{F}(T)$-measurable random variable. The solution to an option pricing problem requires the numerical evaluation

$$\mathbb{E}_{\mathbb{Q}}[H|\mathscr{F}(t)], \quad 0 \leq t \leq T.$$

For example, suppose a financial market consists of $p+1$ assets with price processes, $\{S_0(t), S_1(t, \ldots, S_p(t)\}$, where $S_0(t) = N(t)$ is a numéraire asset, with associated discount process $D(t) = N(t)^{-1}$. Though not the only choice, $D(t)$ can be defined by an interest short rate process, $R(t)$, so that

$$D(t) = \exp\left(-\int_0^t R(u)\, du\right).$$

The market is assumed to be arbitrage free, with $\mathbb{Q}$ being a martingale (that is, risk-neutral) measure. Our option payoff, H, is usually a deterministic function $h(\cdot)$ of the terminal values of the asset price processes, $S_i(T)$, or path-dependent quantities, such as $\max_{t\in[0,T]} S_i(t)$ or $T^{-1}\int_0^T S_i(t)\, dt$. The solution to an option pricing problem thus requires the computation of

$$V(t) := D(t)^{-1}\mathbb{E}_{\mathbb{Q}}[D(T)H|\mathscr{F}(t)], \quad 0 \leq t \leq T. \tag{1}$$

The asset price vector $S(t) = (S_1(t), \ldots, S_p(t))$ and interest rate process, $R(t)$, may be modeled by Itô processes (more generally, semi-martingale processes allowing for jumps) such as

$$dX(t) = b(t)\, dt + a(t)\, dW(t),$$

where $a(t)$ and $b(t)$ are adapted processes. We shall often assume that the process $X(t)$ is Markov, so that

$$dX(t) = b(t, X(t))\, dt + a(t, X(t))\, dW(t), \tag{2}$$

where $a(t, x)$ and $b(t, x)$ are deterministic functions. Recall that a stochastic process, $\{X(t)\}_{t\geq 0}$, is a function $X : [0, \infty) \times \Omega \to \mathbb{R}^m$ such that $X(t, \cdot) : \Omega \to \mathbb{R}^m$ is an $\mathscr{F}$-measurable function (that is, a random variable on $(\Omega, \mathscr{F})$).

Example 4.1 (Path-independent payoff)**.** For $K > 0$ and $T > 0$ we wish to compute

$$V = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[(S(T) - K)^+\right].$$

If $dS(t) = S(t)(r\, dt + \sigma\, dW(t))$, one-dimensional geometric Brownian motion with constant volatility σ and constant risk-free interest rate r, this SDE has an exact solution

$$S(T) = S(0)\exp\left((r - \sigma^2/2)T + \sigma W(T)\right),$$

where $W(T) = \sqrt{T}Z$ and Z is a standard normal random variable. If $Z^1, Z^2, \ldots$ is a sequence of standard normal random variables and $n \in \mathbb{N}$, consider the estimator

$$\hat{V}_n := \frac{1}{n}\sum_{j=1}^n V^j,$$

where $S^j(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}Z^j)$ and $V^j = e^{-rT}(S^j(T) - K)^+$. For any $n \geq 1$, the estimator $\hat{V}_n$ is *unbiased*,

$$\mathbb{E}_{\mathbb{Q}}[\hat{V}_n] = V,$$

and *strongly consistent*,

$$\mathbb{Q}\left[\lim_{n\to\infty} \hat{V}_n = V\right] = 1.$$

Let s_V denote the sample standard deviation of $\hat{V}_1, \ldots, \hat{V}_n$,

$$s_V^2 = \frac{1}{n-1}\sum_{j=1}^n (V^j - \hat{V}_n)^2,$$

and let z_δ denote the $1-\delta$ quantile of the standard normal distribution, that is, $\Phi(z_\delta) = 1-\delta$. Then,

$$\hat{V}_n \pm z_{\delta/2}\frac{s_C}{\sqrt{n}}$$

is an asymptotically valid $1-\delta$ confidence interval for V (that is, the probability that the interval includes V approaches $1-\delta$ as $n \to \infty$).

Example 4.2 (Path-dependent payoff)**.** See notes for an Asian option.

4.1. **Generating paths of Brownian motion.**

- Generating discretely-sampled paths of one-dimensional Brownian motion;
  - Random walk generation of $W := (W(t_1), \dots, W(t_m))$:
    - Recursion using $W(t_{i+1}) = W(t_i) + \sqrt{t_i - t_{i+1}}Z_i$, $i = 0, \dots, m-1$ and sequence of *iid* $N(0,1)$ random variables, $Z_1, \dots, Z_m$;
    - As in previous method but using a single m-dimensional, $N(0, I)$ random variable, $Z = (Z_1, \dots, Z_m)$, where I is the $m \times m$ identity matrix;
    - Cholesky decomposition, $C = AA^*$, of $C = (C_{ij})$, where $C_{ij} := \min\{t_i, t_j\}$, and $W = AZ$, to generate the samples from the joint distribution of $W(t_1), \dots, W(t_m)$;
  - Brownian bridge interpolation;
  - Principal components simulation of discretely-sampled Brownian motion and Karhunen-Loève expansion for continuously-sampled Brownian motion.
- Generating sample paths of d-dimensional Brownian motion.

4.2. **Numerical solution of stochastic differential equations (SDEs).**

4.2.1. *Exact solution.* This is always preferred. In practice, this will mean that one can generate exact values, $\{X(t_i)\}_{i\in\mathbb{N}}$, in terms of values of Brownian motion, $\{W(t_j)\}_{j\in\mathbb{N}}$. Some examples are (see Glasserman §3.2 and §3.3):

- Scalar geometric Brownian motion,
$$dS(t) = S(t)(\mu\, dt + \sigma\, dW(t)).$$
- Multidimensional geometric Brownian motion,
$$dS_i(t) = S_i(t)(\mu_i\, dt + \sigma_i\, dB_i(t)), \quad i = 1, \dots, d,$$
where each $B_i(t)$ is a standard one-dimensional Brownian motion and $[B_i, B_j](t) = \rho_{ij}\, dt$.
- Vasicek short rate model,
$$dR(t) = \alpha(b - R(t))\, dt + \sigma\, dW(t),$$
where α, b, σ are positive constants.
- Multidimensional linear Gaussian processes,
$$dX(t) = C(b - X(t))\, dt + D\, dW(t),$$
where C, D are $d \times d$ matrices, and b, $W(t)$, and $X(t)$ are in $\mathbb{R}^d$.

A simple example where one does not have an exact closed-form solution is the square root or Cox-Ingersoll-Ross process; see §3.4 in Glasserman.

4.2.2. *First-order finite difference methods.* Given a discretization $0 = t_0 < t_1 < \cdots < t_m = T$ of the interval $[0, T]$, let $\{\hat{X}(t_i)\}_{i\geq 0}$ denote the discrete time approximation of the continuous time process $\{X(t)\}_{t\geq 0}$. The *Euler* method (§6.1.1 in Glasserman) uses the relation

$$\hat{X}(t_{i+1}) = \hat{X}(t_i) + a(t_i, \hat{X}(t_i))(t_{i+1} - t_i) + b(t_i, \hat{X}(t_i))(W(t_{i+1}) - W(t_i)), \quad i = 0, \dots, m-1, \tag{3}$$

with initial condition $\hat{X}(0) = X(0)$, where the Brownian motion increments may be generated using

$$W(t_{i+1}) - W(t_i) = \sqrt{t_{i+1} - t_i}Z_{i+1}.$$

The *Milstein* method (§6.1.1 in Glasserman) uses

$$\begin{aligned}\hat{X}(t_{i+1}) &= \hat{X}(t_i) + a(t_i, \hat{X}(t_i))(t_{i+1} - t_i) + b(t_i, \hat{X}(t_i))(W(t_{i+1}) - W(t_i)) \\ &\quad + \frac{1}{2} b_x(t_i, \hat{X}(t_i)) b(t_i, \hat{X}(t_i)) \Big((W(t_{i+1}) - W(t_i))^2 - (t_{i+1} - t_i) \Big) \quad i = 0, \ldots, m-1,\end{aligned} \tag{4}$$

4.2.3. *Second-order finite difference methods.* This and higher order methods will be discussed briefly in Lecture 2.

4.2.4. *Finite difference methods for vector-valued processes.* Complications which arise when one attempts to generalize methods for scalar SDEs to vector SDEs will be discussed in Lecture 2.

4.2.5. *Convergence order.* See §6.1.2 in Glasserman for details. Let $\{\hat{X}(kh)\}_{k\in\mathbb{N}}$ be any discrete time approximation of an $\mathbb{R}^d$-valued continuous time process $\{X(t)\}_{t\geq 0}$, let $T > 0$, and $n = \lfloor T/h \rfloor$.

Definition 4.3 (Strong order of convergence)**.** A discretization has *strong order of convergence* $\beta > 0$ if

$$\mathbb{E}\left[|\hat{X}(nh) - X(T)| \right] \leq ch^{\beta},$$

for some constant c, all sufficiently small h, and vector norm $|\cdot|$ on $\mathbb{R}^d$.

Definition 4.4 (Weak order of convergence)**.** A discretization has *weak order of convergence* $\beta > 0$ if

$$\Big| \mathbb{E}[f(\hat{X}(nh))] - \mathbb{E}[f(X(T))] \Big| \leq ch^{\beta},$$

for some constant c and all sufficiently small h, for all f in the set $C_P^{2\beta+2}$ of functions $f : \mathbb{R}^d \to \mathbb{R}$ whose derivatives of order $0, 1, \ldots, 2\beta + 2$ are polynomially bounded; the constant c may depend on f.

A function $g : \mathbb{R}^d \to \mathbb{R}$ is *polynomially bounded* if

$$|g(x)| \leq k\left(1 + |x|^q\right),$$

for some constants k and q and all $x \in \mathbb{R}^d$.

Example 4.5 (Convergence order for Euler's scheme)**.** Assume that the the coefficients in Equation (2) obey the conditions of Theorem 5.1 for existence and uniqueness of a strong solution, $\{X(t)\}_{t\in[0,T]}$, together with

$$\mathbb{E}\left[|X(0) - \hat{X}(0)|^2 \right] \leq K\sqrt{h}, \tag{5}$$

and

$$|a(t,x) - a(s,x)| + |b(t,x) - b(s,x)| \leq K(1 + |x|)\sqrt{|t-s|}, \quad x \in \mathbb{R}^d, \quad t \in [0,T], \tag{6}$$

The one can show (see Glasserman pp. 345-346) that the Euler scheme has strong order of convergence 1/2. The scheme has weak order of convergence 1 if a and b are at least C^4 in the space variable with polynomially bounded derivatives. See Kloeden-Platen for detailed proofs and results.

Example 4.6 (Convergence order for Milstein's scheme)**.** One can show (see Glasserman pp. 347-348) that the Milstein scheme has both strong and weak order of convergence 1, under additional conditions on the coefficients a and b. See Kloeden-Platen for detailed proofs and results.

5. Supplementary Notes

5.1. **Options on stocks with dividends.** We are given a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = \{\mathscr{F}(t)\}_{t\geq 0}$ of $\mathscr{F}$ and Brownian motion $W(t)$, an Itô process representing the price of a stock, $S(t)$, which pays dividends at rate $A(t)$ (an adapted process) per share per unit time, an interest rate process, $R(t)$, a maturity date $T > 0$, and an option payoff represented by an $\mathscr{F}(T)$-measurable random variable H. Setting $Q(t) = \exp(\int_0^t A(u)\,du)$, the auxiliary process, $\widetilde{S}(t) = Q(t)S(t)$, is the price of one share of the stock plus accumulated dividends or, equivalently, the price of one share of the stock when dividends are reinvested.

Suppose one can find a replicating portfolio for this payoff, H, with value process

$$X(t) = \Delta(t)S(t) + \underbrace{(X(t) - \Delta(t)S(t))}_{\text{Cash}}, \quad 0 \leq t \leq T.$$

We assume that dividends are reinvested (either to purchase additional shares of the stock or money market account), so the self-financing condition should be written

$$dX(t) = \Delta(t)dS(t) + A(t)\Delta(t)S(t)\,dt + R(t)\left(X(t) - \Delta(t)S(t)\right)\,dt. \tag{7}$$

Noting that

$$d\left(Q(t)S(t)\right) = Q(t)dS(t) + A(t)Q(t)S(t)\,dt,$$

we obtain

$$\begin{aligned} dX(t) &= \Delta(t)Q(t)^{-1}\left(Q(t)dS(t) + A(t)Q(t)S(t)\,dt\right) + R(t)\left(X(t) - \Delta(t)S(t)\right)\,dt \\ &= \Delta(t)Q(t)^{-1}d\widetilde{S}(t) + R(t)\left(X(t) - \Delta(t)S(t)\right)\,dt \end{aligned}$$

Rearranging, we obtain

$$dX(t) - R(t)X(t)\,dt = \Delta(t)Q(t)^{-1}(d\widetilde{S}(t) - R(t)\widetilde{S}(t)\,dt).$$

Using

$$d(D(t)X(t)) = D(t)\left(dX(t) - R(t)X(t)\,dt\right),$$

and similarly for $d(D(t)\widetilde{S}(t))$, we see that

$$d(D(t)X(t)) = \Delta(t)Q(t)^{-1}d(D(t)\widetilde{S}(t)).$$

For this market, $\mathbb{Q}$ will be a martingale measure (equivalent to $\mathbb{P}$) if the process $D(t)\widetilde{S}(t)$ is a $\mathbb{Q}$-martingale with respect to $\mathbb{F}$; if $\mathbb{Q}$ exists, then the market will be arbitrage free and $D(t)X(t)$ will be a $\mathbb{Q}$-martingale with respect to $\mathbb{F}$ as well. Assume $\mathbb{Q}$ exists and let $V(t)$ be the no-arbitrage price, $t \in [0, T]$, of the derivative security with payoff H at time T; the Law of One Price ensures that $V(t) = X(t)$ for $t \in [0, T]$ since $X(T) = H = V(T)$. Then $D(t)V(t)$ is a martingale (since it is equal to $D(t)X(t)$) and so we obtain the risk-neutral pricing formula familiar from the no-dividend discussion,

$$V(t) = D(t)^{-1}\mathbb{E}_{\mathbb{Q}}\left[D(T)V(T)|\mathscr{F}(t)\right].$$

To see when $\mathbb{Q}$ exists, observe that

$$\begin{aligned} d(D(t)Q(t)S(t)) &= D(t)Q(t)S(t)\left((\alpha(t) - R(t))\,dt + \sigma(t)\,dW(t)\right) \\ &= D(t)Q(t)S(t)\sigma(t)\left(\Theta(t)\,dt + dW(t)\right) \end{aligned}$$

if we write the evolution equation of $S(t)$ as

$$dS(t) = S(t)\left((\alpha(t) - A(t))\,dt + \sigma(t)\,dW(t)\right),$$

and define $\Theta(t) := (\alpha(t) - R(t))/\sigma(t)$. Hence, just as in the no-dividend case, Girsanov's Theorem ensures that a risk-neutral measure, $\mathbb{Q}$, exists and is unique when $\sigma(t) > 0$ for $t \in [0, T]$, such that

$\widetilde{W}(t) := \int_0^t \Theta(u)\,du + W(t)$ is $\mathbb{Q}$-Brownian motion. Moreover,

$$
\begin{aligned}
d(D(t)Q(t)S(t)) &= D(t)Q(t)S(t)\sigma(t)\,d\widetilde{W}(t),\\
d(Q(t)S(t)) &= Q(t)S(t)\left(R(t)\,dt + \sigma(t)\,d\widetilde{W}(t)\right),\\
dS(t) &= S(t)\left((R(t)-A(t))\;dt + \sigma(t)\,d\widetilde{W}(t)\right).
\end{aligned}
$$

Observe that $\widetilde{S}(t) = Q(t)S(t)$ is a $\mathbb{Q}$-martingale, but $S(t)$ is not.

5.2. **Stochastic differential equations.** Let $W(t) := (W_1(t), \ldots, W_d(t))$ be d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, with filtration $\mathbb{F} = \{\mathscr{F}(t)\}_{t\geq 0}$. A *stochastic differential equation* for a process $X(u) := (X_1(u), \ldots, X_m(u))$ is an equation of the form (see Equation (6.2.1) in Shreve for the case $d = m = 1$)

$$
dX(u) = \beta(u, X(u))\,du + \gamma(u, X(u))\,dW(u), \quad u \geq t \tag{8}
$$

where $\beta(u,x) := (\beta_1(u,x), \ldots, \beta_m(u,x))$ and $\gamma(u,x) := (\gamma_{ij}(u,x))_{m\times d}$, and the $\beta_i(u,x)$ and $\gamma_{ij}(u,x)$ are Borel measurable deterministic functions. Given $X(t)$, a process $X(T)$, $T \geq t$, is called a *strong solution* to Equation (8) if $X(t)$ is a random variable and, for all $T \geq t$,

$$
X(T) = X(t) + \int_t^T \beta(u, X(u))\,du + \int_t^T \gamma(u, X(u))\,dW(u). \tag{9}
$$

5.3. **Existence and uniqueness of solutions to stochastic differential equations.** From Øksendal (Theorem 5.2.1) or Glasserman (Theorem B.2.1):

Theorem 5.1. *Suppose* $\mathbb{E}\left[|X(t)|^2\right] < \infty$ *and* $\beta : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ *and* $\gamma : [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ *are Borel measurable functions satisfying the growth condition,*

$$
|\beta(t,x)| + |\gamma(t,x)| \leq C(1+|x|), \quad x \in \mathbb{R}^m, \quad t \in [0,T], \tag{10}
$$

for some constant C*, and the Lipschitz condition,*

$$
|\beta(t,x) - \beta(t,y)| + |\gamma(t,x) - \gamma(t,y)| \leq D|x-y|, \quad x, y \in \mathbb{R}^m, \quad t \in [0,T], \tag{11}
$$

for some constant D*. Then Equation* (8) *has a unique strong solution,* $X(u)$*, which is continuous with respect to* u*, such that* $\{X(u)\}_{u\in[t,T]}$ *is adapted to* $\mathbb{F}$ *and* $\mathbb{E}\left[\int_t^T |X(u)|^2\,du\right] < \infty$.

The solution is unique in the sense that if $\{\tilde{X}(t)\}_{t\in[0,T]}$ is also a solution, then

$$
\mathbb{P}\left[X(t) = \tilde{X}(t), \forall t \in [0,T]\right] = 1.
$$