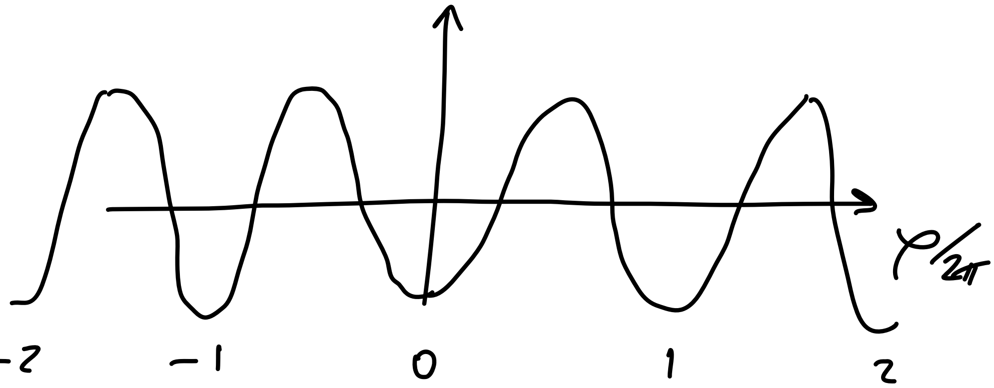
- idea: encode qubit in continuous d.o.f.
  - → improved robustness against perturbations
  - → basis for bosonic codes
- consider cont. d.o.f.

$[\varphi, q] = i$

imagine potential:



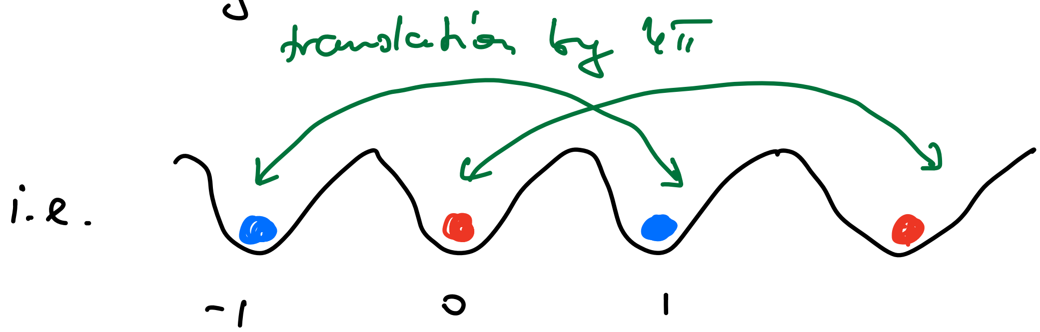
def. stabilizers:

$S_\varphi := e^{+i\varphi}$

code space: $S_\varphi |\psi_L\rangle = +|\psi_L\rangle \Rightarrow \varphi = 2\pi k$

↳ need another state for qubit:

e.g. consider even vs odd minima



⇒ second stabilizer:

$S_q := e^{i 4\pi q}$

same state $|1\rangle$, $|0\rangle$

check: $[S_\varphi, S_q] = e^{i\varphi} e^{i4\pi q} - e^{i4\pi q} e^{i\varphi} = 0$:

recall BCH: $e^{iX} e^{iY} = e^{iX + iY - \frac{1}{2}[X,Y] - \frac{i}{12}[X,[X,Y]] + \dots}$

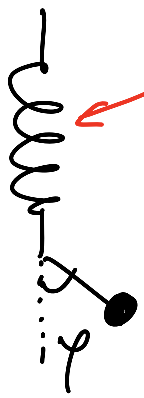
for $[\varphi, q] = i \Rightarrow$ all higher order comm. vanish

$$\Rightarrow e^{i\varphi} e^{i4\pi q} = e^{i(\varphi + 4\pi q) - \frac{4\pi i}{2}} = e^{i(\varphi + 4\pi q)}$$

$$e^{i4\pi q} e^{i\varphi} = e^{i(4\pi q + \varphi) + \frac{4\pi i}{2}} = e^{i(\varphi + 4\pi q)}$$

4π is smallest possible #

<u>intuition</u> : torsion pendulum (cavendish experiment)

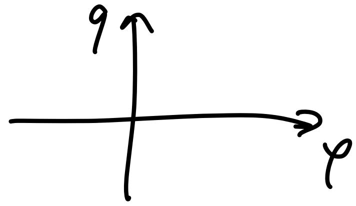


spring angle can tell winding!
$\rightarrow$ decompactifies φ

pendulum bob : position def. mod 2π
can observe bob position (not spring angle)

$\Rightarrow$ all observables $f(\varphi)$: periodic w/ 2π

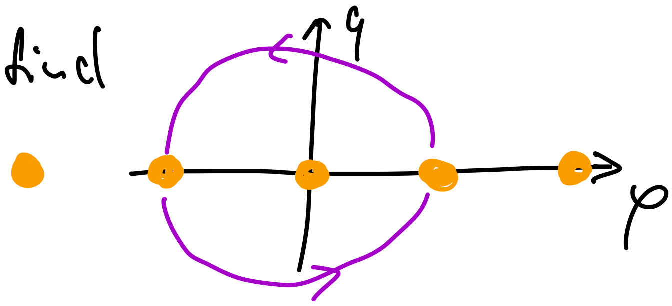
phase space:



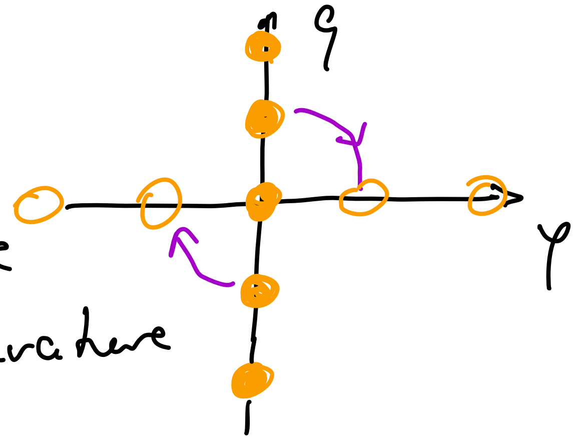
harmonic osc. [ignoring man. inertia & torque, etc.]

$$\begin{cases} \dot{\varphi} = \frac{1}{m} q \\ \dot{q} = -k\varphi \end{cases} \qquad \omega = \sqrt{k/m} \quad \Rightarrow \qquad \begin{aligned} \varphi(t) &= \sin \omega t \\ q(t) &= m\omega \cos \omega t \end{aligned}$$

- <u>phase variable</u>:
- at times $t_n = n\frac{T}{2}$ we find the same state
  $\Rightarrow$ "stabilization" of φ



- momentum variable:



- wait $t = T/4$
  → momentum quadrature turns into phase quadrature
- then measure phase

want: measurement of momentum from meas. of phase

but: $q(t) = \underbrace{m\omega}\, \varphi(t - T/4)$

↑ amplitude mismatch, want to set to unity

but: stabilizer: extra 4π

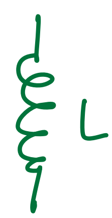
$$\Rightarrow 4\pi\, q(t) = 4\pi m\omega\, \varphi(t - T/4) \overset{!}{=} \varphi(t - T/4)$$

$$\Rightarrow 4\pi m\omega \overset{!}{=} 1$$

⇒ measurements only in φ can stabilize periodic structure in φ & q

- relation to LC circuits:

inductor:



voltage: $V = L \dfrac{dI}{dt}$

inductance

(*)

Josephson relation: for phase φ of SC

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar} V(t) = \frac{2eL}{\hbar}\frac{dI}{dt}$$

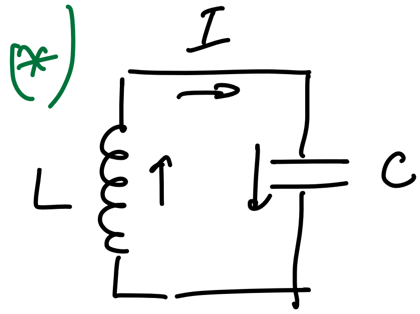
$$\Rightarrow \varphi = \frac{2eL}{\hbar} I = \frac{2eL}{\hbar}\frac{dq}{dt}$$

charge

capacitor: $\dashv\vdash\; C$

$q = C V$ => $\frac{d\varphi}{dt} = \frac{2e}{\hbar C}\, q$ (*)

capacitance   Josephson



relation to HO: LC circuit

Kirchhoff's laws: $V_L + V_c = 0$ (for closed loops)

$$I_L = I_c$$

$$\left.\begin{array}{l} V_L = L \dot{I}_L \\ I_c = C \dot{V}_c \end{array}\right\} \Rightarrow \begin{array}{c} L \ddot{I}_L + \frac{1}{C} I_c = 0 \\ I_c = I_L \end{array} \Rightarrow \ddot{I} = -\frac{1}{LC} I$$

$$\Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

analogy: $m \hat{=} C$
$k \hat{=} 1/L$

QM: charge q & phase φ are conjugate:

$$[\varphi, q] = i\, 2e$$ ← charge of Cooper pair

=> stabilizers:

$$S_\varphi = e^{i\varphi}, \quad S_q = e^{i 4\pi \frac{q}{2e}}$$

GKP code as a SC LC circuit

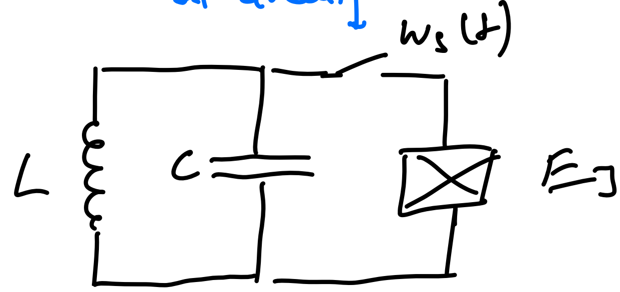
condition: (*) => $q = \frac{C \hbar\omega}{2e} \varphi \Rightarrow \frac{4\pi}{2e} C \frac{\hbar\omega}{2e} \stackrel{!}{=} 1$

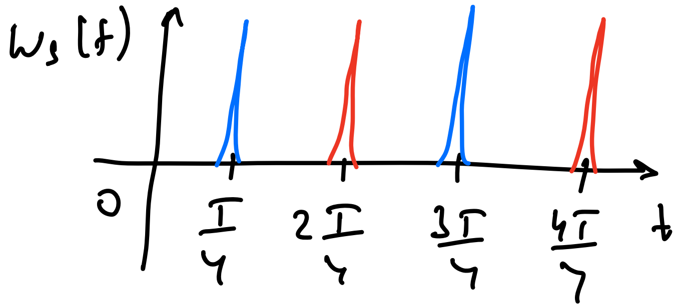
meas. of φ stabilizes q => $\sqrt{\frac{L}{C}} \stackrel{!}{=} 2 \frac{h}{(2e)^2} = 2 R_Q$

resistance quantum

measurements: connect circuit to Josephson element

applies $E(\varphi)$ pot. & projects state of circuit onto it





quadrature: φ

quadrature: $4\pi \frac{q}{2e}$

$$E(\varphi) = -E_J \cos \varphi$$

$$E_J = \frac{\Phi_0 I_c}{2\pi} \; ; \; \Phi_0 = \frac{h}{2e}$$

flux quantum

- inductor can register how many times current I winds (similar to spring in torsion pendulum)
  $\Rightarrow$ each potential valley is physical (no cpct variable)
  $\rightarrow$ fluxonium qubit

potential of inductor $\propto \varphi^2$

$\rightarrow$ Floquet problem
- requires a very precise "clock"

$\hookrightarrow$ Gill & Frederik's paper

PRX Quantum 2, 020101 (2021)

- "encoding a qubit in an oscillator"
  PRA $\underline{64}$, 01310 (2001): Gottesman, Kitaev, Preskill
  → correct small shifts in position & momentum

- realizations:
  1) trapped ions: Nature $\underline{566}$, 513 (2019)
  2) microwave cavity & circuit QED:
     Nature $\underline{584}$, 368 (2020)

- broad idea: robust encoding of q'info using bosonic dof

- concretely: encode k logical qudits into n bosonic modes

simplest GKP code: $k = 1 = n$ & $d = 2$ (qubit)

- requirement for implementation:
  - ctrl of high-quality (Q) HO mode
  - strong & high-quality ancilla to ctrl HO mode

cQED: HO ↔ microwave cavity field
ancilla ↔ transmon (2LS)

trapped ions: HO ↔ single trapped ion
ancilla ↔ atomic pseudospin states

- real systems:
  - GKP codes can correct for small quadrature in oscillator
  - still, incorrectible errors present on exp. platform limit logical error rate

$\Rightarrow$ realistic approach: use GKP as building block for surface code

- need for a GKP code:
  (i) logical state preparation
  (ii) s.p. & entangling gates b/w 2 GKP qubits
  (iii) measurements

<u>structure of GKP code</u> $(n = k = 1,\ d = 2)$:

bosonic mode $[a, a^\dagger] = 1$

$$q = \frac{1}{\sqrt{2}}(a + a^\dagger)\ ,\ p = \frac{-i}{\sqrt{2}}(a - a^\dagger)$$

coherent state: $a|\alpha\rangle = \alpha|\alpha\rangle\ ,\ \alpha \in \mathbb{C}$

$$|\alpha\rangle = D(\alpha)|0\rangle$$

<u>def</u>: displacement operator: $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$

$$D(\beta)D(\alpha) \underset{\substack{\uparrow \\ BCH}}{=} e^{\frac{1}{2}(\beta\alpha^* - \beta^*\alpha)} D(\alpha + \beta)$$

$$D(\alpha)D(\beta) = e^{-\frac{1}{2}(\beta\alpha^* - \beta^*\alpha)} D(\alpha+\beta)$$

$$\Rightarrow D(\beta)D(\alpha) = e^{\beta\alpha^* - \beta^*\alpha} D(\alpha)D(\beta) \quad (*)$$

for $\beta\alpha^* - \beta^*\alpha = i\pi \Rightarrow [D(\alpha), D(\beta)]_+ = 0$

$\beta\alpha^* - \beta^*\alpha = 2i\pi \Rightarrow [D(\alpha), D(\beta)] = 0 \quad (*)$

– logical operators: $\bar{X}$, $\bar{Z}$

pick α, β s.t. $\beta\alpha^* - \beta^*\alpha = i\pi$

& def $\bar{X} := D(\alpha)$, $\bar{Z} := D(\beta)$, $\bar{Y} := i\bar{X}\bar{Z}$

$\Rightarrow [\bar{X}, \bar{Z}]_+ = 0$ ✓

need: $\bar{X}^2 = \bar{Z}^2 = \bar{Y}^2 = \mathbb{1}$ on code space

$\Rightarrow$ define stabilizers as:

$S_x = \bar{X}^2 = D(2\alpha)$, $S_z = \bar{Z}^2 = D(2\beta)$

$(*) \Rightarrow [S_x, S_z] = 0$ ✓

$[S_x, \bar{Z}] = [D(2\alpha), D(\beta)] \overset{(*)}{=} 0$

β, α locked to π

$\Rightarrow \beta, 2\alpha$ locked to 2π

& $[S_z, \bar{X}] = 0$

stabilizer group for GKP code:

$$\mathcal{S} = \{ S_X^k , S_Z^\ell : k,\ell \in \mathbb{Z} \}$$

code words (logical states): write in terms of sums of quadratures:

$$Q := \frac{i}{\sqrt{\pi}}(\beta^* a - \beta a^\dagger)$$

$$P := -\frac{i}{\sqrt{\pi}}(\alpha^* a - \alpha a^\dagger)$$

$$[Q,P] = \frac{1}{\pi}[\beta^* a - \beta a^\dagger , \alpha^* a - \alpha a^\dagger]$$

$$= \frac{1}{\pi}\left(-\beta^*\alpha [a,a^\dagger] - \beta\alpha^*[a^\dagger , a]\right)$$

$$= \frac{1}{\pi}\left(\beta\alpha^* - \beta^*\alpha\right) = 1 \quad \checkmark$$

$$\Rightarrow \bar{X} = e^{-i\sqrt{\pi}P}, \quad \bar{Z} = e^{i\sqrt{\pi}Q}, \quad \bar{Y} = e^{i\sqrt{\pi}(Q-P)}$$

notation: $|\alpha\rangle_A$: $A|\alpha\rangle_A = \alpha|\alpha\rangle_A$

think of as generalization of coh. state

logical $\bar{Z}$-basis:

$$|0_L\rangle := \sum_{\ell=-\infty}^{\infty} |2\ell\sqrt{\pi}\rangle_Q$$

$$|1_L\rangle := \sum_{\ell=-\infty}^{\infty} |(2\ell+1)\sqrt{\pi}\rangle_Q$$

check: $\bar{Z}|0_L\rangle = \sum_\ell e^{i\sqrt{\pi}Q}|2\ell\sqrt{\pi}\rangle_Q = \sum_\ell \underbrace{e^{2i\pi\ell}}_{=1}|2\ell\sqrt{\pi}\rangle_Q \quad \checkmark$

& similarly: $\bar{Z}|1_L\rangle = -|1_L\rangle$

$$\bar{X}|0_L\rangle = \sum_\ell e^{-i\sqrt{\pi}P}|2\ell\sqrt{\pi}\rangle_Q \overset{\overset{\text{P translates Q e'states}}{\downarrow}}{=} \sum_\ell |2\ell\sqrt{\pi}+\sqrt{\pi}\rangle_Q = |1_L\rangle$$

also: $S_x|0_L\rangle = \bar{X}^2|0_L\rangle = \bar{X}|1_L\rangle = +|0_L\rangle$ ✓

& $S_z|0_L\rangle = +|0_L\rangle$ ✓, etc.

– alternative def. of logical states using coh. states:

observe: $|0_L\rangle$ is $+1$ e'state of $\bar{Z}, S_x, S_z = \bar{Z}^2$

$$\Rightarrow |0_L\rangle = \underbrace{\tfrac{1}{2}(\mathbb{1}+\bar{Z})}_{\text{projects within code space}} \underbrace{\prod_{k'} \tfrac{1}{2}(\mathbb{1}+S_x^{k'}) \prod_{\ell'} \tfrac{1}{2}(\mathbb{1}+S_z^{\ell'})}_{\text{projects onto code space}} \overset{\overset{\text{arbitrary}}{\downarrow}}{|\psi\rangle}$$

$$\underset{\text{not normalizable}}{\propto} \sum_{k,\ell\in\mathbb{Z}} S_x^k \bar{Z}^\ell |\psi\rangle \quad \text{for any } |\psi\rangle \text{, s.t. } \langle 0_L|\psi\rangle \neq 0$$

$k=\ell=0 \Leftrightarrow \mathbb{1}$

set $|\psi\rangle = |0\rangle$ vacuum:

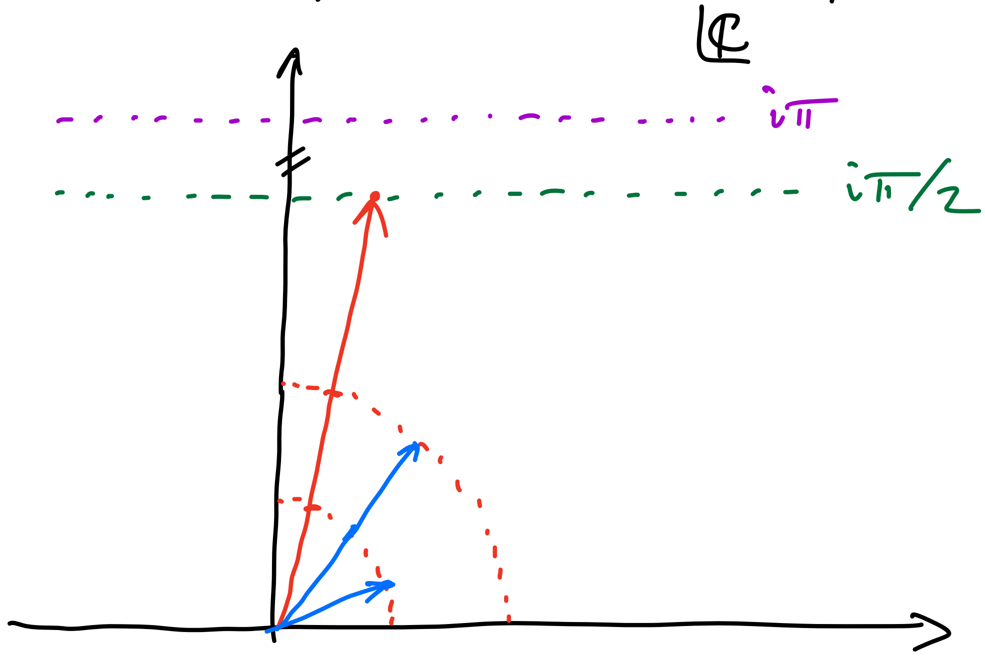
$$\Rightarrow |0_L\rangle = \sum_{k,\ell\in\mathbb{Z}} \underbrace{D(2k\alpha)\,D(\ell\beta)}_{\overset{(*)}{=} e^{\frac{\cancel{2}k\ell}{\cancel{2}}\underbrace{(\alpha\beta^*-\alpha^*\beta)}_{=-i\pi}} * D(\alpha+\beta)} |0\rangle$$

$$= \sum_{k,l} e^{-i\pi l k} |2k\alpha + l\beta\rangle \quad (*)$$

$$\Rightarrow |1_L\rangle = \sum_{k,l} e^{-i\pi(kl + l/2)} |(2k+1)\alpha + l\beta\rangle$$

- <u>recall</u>: $\beta\alpha^* - \beta^*\alpha = 2i\,\mathrm{Im}(\beta\alpha^*) \overset{!}{=} i\pi$

$\Rightarrow$



$\Rightarrow$ infinitely many pairs $\alpha, \beta \in \mathbb{C}$

$\rightarrow$ ∞-many GKP codes!

- 3 most common GKP codes:

(1) square: $\alpha = \sqrt{\frac{\pi}{2}}$, $\beta = i\sqrt{\frac{\pi}{2}}$

$$\Rightarrow Q_\square = q = \frac{1}{\sqrt{2}}(a + a^\dagger) \; ; \; P_\square = p = \frac{-i}{\sqrt{2}}(a - a^\dagger)$$

(2) rectangle: $\alpha = \lambda\sqrt{\frac{\pi}{2}}$, $\beta = i\frac{1}{\lambda}\sqrt{\frac{\pi}{2}}$, $\lambda > 0$

(3) hexagonal: $\alpha = \sqrt{\frac{\pi}{\sqrt{3}}}$, $\beta = e^{\frac{2i\pi}{3}}\sqrt{\frac{\pi}{\sqrt{3}}}$

approximate codewords:

issues: • $|0_L\rangle$, $|1_L\rangle$ not normalizable

• no physical process can prepare them

$\rightarrow$ need approximation

idea: take any pair of normalized states

$|\tilde{\mu}_L\rangle$, $\mu = 0, 1$, s.t.

$S_x |\tilde{\mu}_L\rangle \longrightarrow + |\mu_L\rangle$

$S_z |\tilde{\mu}_L\rangle \longrightarrow + |\mu_L\rangle$

$\bar{Z} |\tilde{\mu}_L\rangle \longrightarrow (-1)^\mu |\mu_L\rangle$

$\uparrow$

some meaningful limit

approach 1: def. $|\tilde{\mu}_L\rangle \propto e^{-\Delta^2 a^\dagger a} |\mu_L\rangle$

ideal limit: $\Delta \rightarrow 0$

• Gaussian envelope for coherent states in (*)

$$|\zeta\rangle \longrightarrow e^{-\frac{|\zeta|^2}{2}(1-e^{-2\Delta^2})} |e^{-\Delta^2}\zeta\rangle$$

$$\approx e^{-\Delta^2|\zeta|^2} |e^{-\Delta^2}\zeta\rangle$$

• aternative view: "normalize" stabilizers codewords $|\tilde{\mu}_L\rangle$ are stabilized by

$$S^{\Delta}_{x,z} := e^{-\Delta^2 a^\dagger a} S_{x,z}\, e^{+\Delta^2 a^\dagger a}$$

& logical op's: $\bar{P}^{\Delta} = e^{-\Delta^2 a^\dagger a} \bar{P} e^{+\Delta^2 a^\dagger a}$, $\bar{P} = \bar{X}, \bar{Z}$

approach 2: apply weighted displacements to ideal codewords

– metric to quantify how close an arbitrary state ρ is from ideal GKP state:

def: modular squeezing parameter (for stabilizer)

$$\Delta_x := \frac{1}{2|\alpha|}\sqrt{-\log|\mathrm{tr}[S_x, \rho]|^2}$$

$$\Delta_z := \frac{1}{2|\beta|}\sqrt{-\log|\mathrm{tr}[S_z, \rho]|^2}$$

$\Delta_{x,z} \geq 0$ & $\Delta_{x,z} = 0$ iff ρ is e'state of S_p

for ρ = approx. GKP codeword: $\Delta_{x,z} = \Delta \to 0$

often: modular squeezing measured in dB

$$\$_{x,z} := -10 \log_{10}(\Delta^2_{x,z})$$

often: $\Delta_x \approx \Delta_z = \Delta$