(BRIEF) REVIEW OF QLDPC CODES

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Logical and Physical Qubits

Physical Qubits Physical qubits are highly error prone due to phenomenon of "decoherence" - which is loss of information to the environment

Logical Qubits Abstract qubits upon which a quantum algorithm is executed

Many physical qubits result in a single logical qubit

Bit-Flip and Phase-Flip Errors

Bit Flip Error A physical qubit has an X gate acting on that qubit

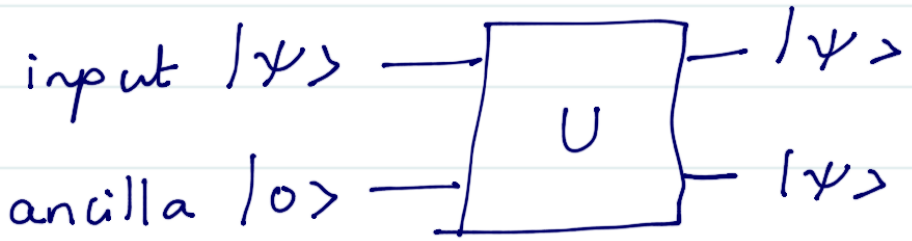
Phase Flip Error A physical qubit has a Z gate acting on that qubit.

To correct these errors and get higher "fidelity" for the logical qubits,

"QUANTUM ERROR CORRECTION" is needed

No-cloning Theorem

There is no Unitary gate U that operates as shown.

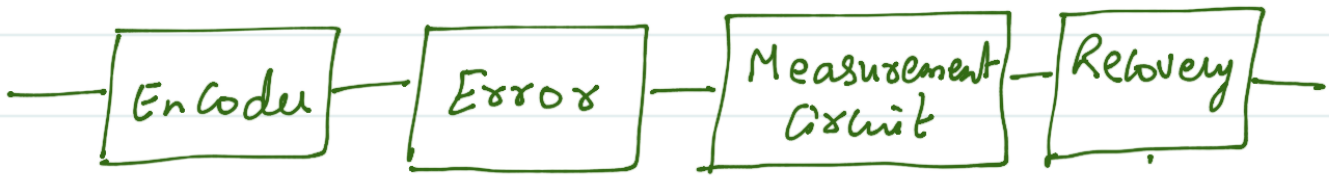


No quantum gate can create an exact replica of the input qubit of an arbitrary state

TAKEAWAY: We cannot have 'natural' analog of classical repetition code.

3 - Qubit Single Bit-Flip Correcting Code

Block Diagram

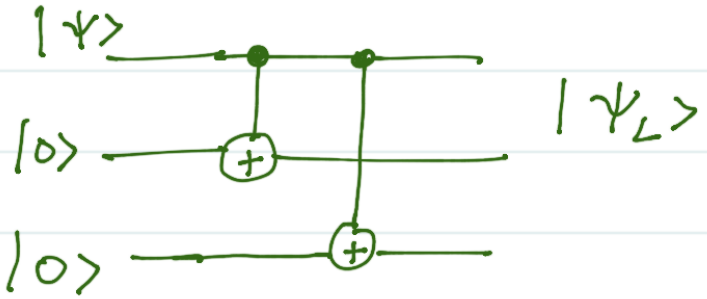


- Note that each operation is reversible except measurement

- In this code, one logical qubit is encoded within three physical qubits

- The code can correct a bit-flip (x) error in atmost one of the three physical qubits

Encoder of 3-qubit Bit-flip QECC



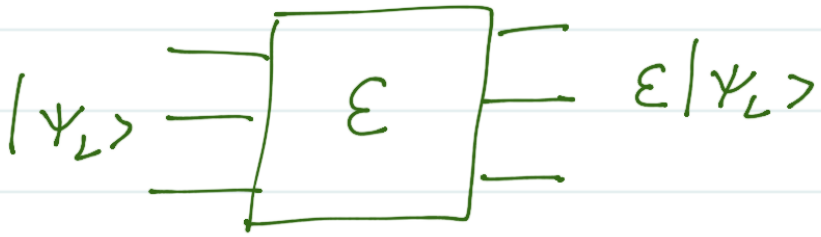
$$|0\rangle \rightarrow |000\rangle := |0_L\rangle$$

$$|1\rangle \rightarrow |111\rangle := |1_L\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \longmapsto \alpha|000\rangle + \beta|111\rangle$$

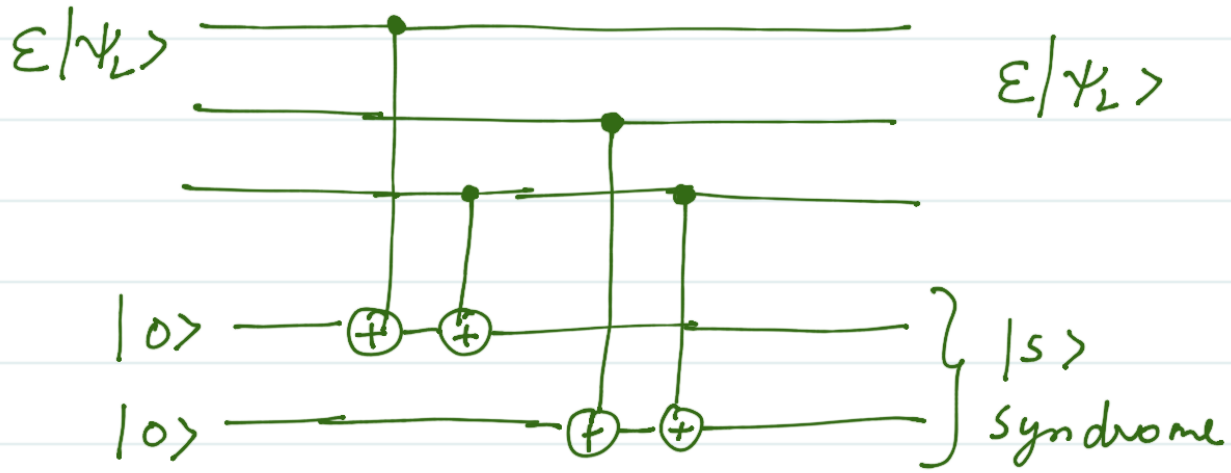
$$|\psi\rangle \longrightarrow |\psi_L\rangle$$

Bit-flip Error



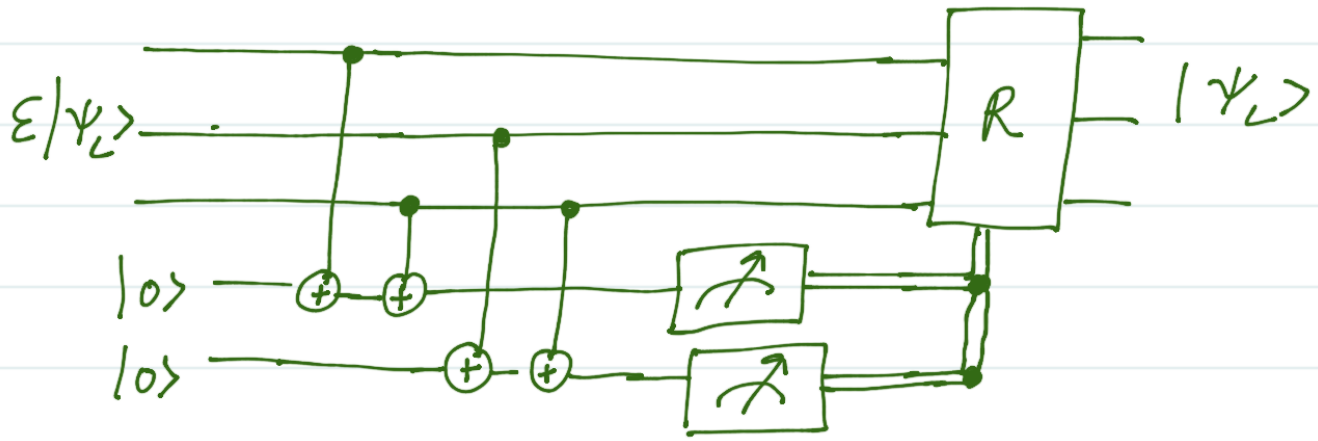
$\mathcal{E}$	$\mathcal{E}\|\psi_L\rangle$
$I \otimes I \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
$X \otimes I \otimes I$	$\alpha\|100\rangle + \beta\|011\rangle$
$I \otimes X \otimes I$	$\alpha\|010\rangle + \beta\|101\rangle$
$I \otimes I \otimes X$	$\alpha\|001\rangle + \beta\|110\rangle$

Syndrome Measurement



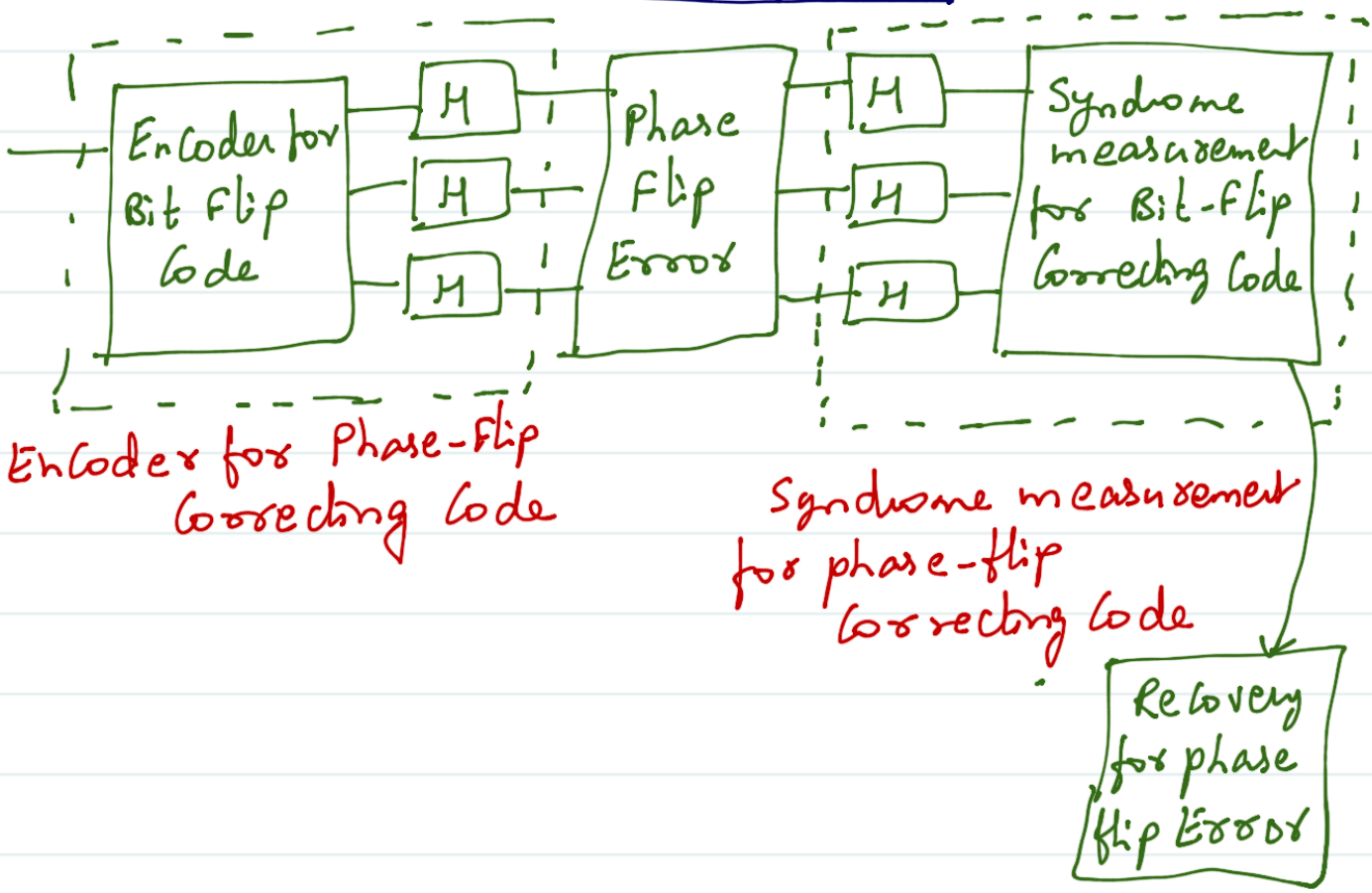
$\mathcal{E}$	$\mathcal{E}\lvert\psi_L\rangle$	Syndrome $\lvert s\rangle$
$I \otimes I \otimes I$	$\alpha\lvert 000\rangle + \beta\lvert 111\rangle$	$\lvert 00\rangle$
$X \otimes I \otimes I$	$\alpha\lvert 100\rangle + \beta\lvert 011\rangle$	$\lvert 10\rangle$
$I \otimes X \otimes I$	$\alpha\lvert 010\rangle + \beta\lvert 101\rangle$	$\lvert 01\rangle$
$I \otimes I \otimes X$	$\alpha\lvert 001\rangle + \beta\lvert 110\rangle$	$\lvert 11\rangle$

Syndrome Measurement and Recovery



$\mathcal{E}$	$\mathcal{E}\|\psi_L\rangle$	s	R
$I\otimes I\otimes I$	$\alpha\|000\rangle+\beta\|111\rangle$	00	$I\otimes I\otimes I$
$X\otimes I\otimes I$	$\alpha\|100\rangle+\beta\|011\rangle$	10	$X\otimes I\otimes I$
$I\otimes X\otimes I$	$\alpha\|010\rangle+\beta\|101\rangle$	01	$I\otimes X\otimes I$
$I\otimes I\otimes X$	$\alpha\|001\rangle+\beta\|110\rangle$	11	$I\otimes I\otimes X$

3-Qubit Single Phase-Flip Correcting Code



- One logical qubit is encoded within three physical qubits
- Can correct a phase-flip (Z) error in atmost one of the three physical qubits.

Shor's 9-qubit Code (1995)

- obtained by concatenating the 3-qubit phase-flip code with 3-qubit bit-flip code

- Outer Code: 3-qubit phase-flip code

$$|0\rangle \to |+++\rangle, \quad |1\rangle \to |---\rangle$$

- Inner Code: 3-qubit bit-flip code

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \to \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \to \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

Shor's 9-qubit Code

- Resulting code encodes one logical qubit into 9 physical qubits

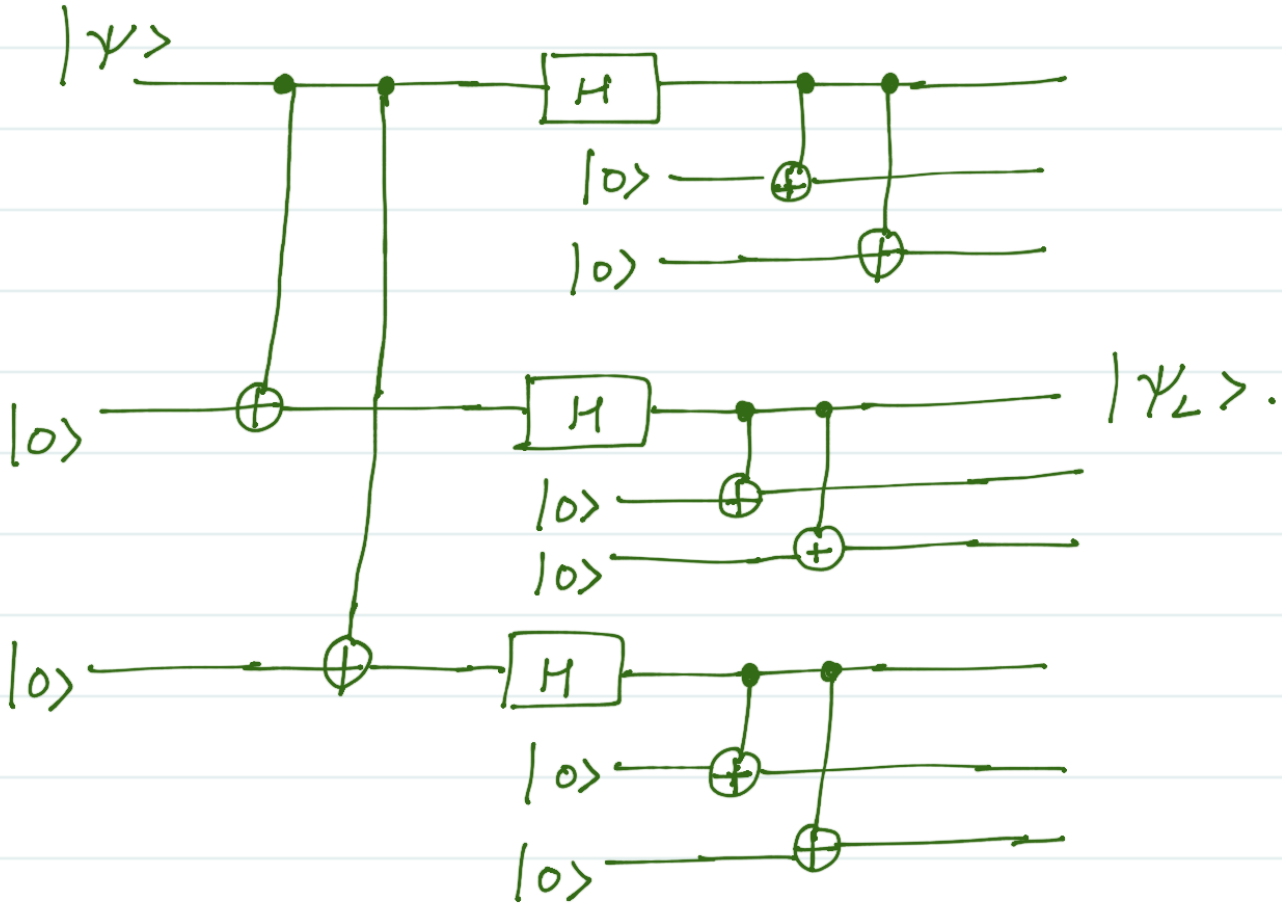
$$|0\rangle \rightarrow |0_L\rangle := \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$$

$$|1\rangle \rightarrow |1_L\rangle := \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$$

- This code is capable of correcting an "arbitrary" unitary error on any of the 9 physical qubits

- Illustrates the principle of discretization of errors

Encoder of Shor's 9-qubit Code



Bit-Flip and Phase-Flip Errors

- Encoded $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:

$$\alpha\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)$$
$$+\beta\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)$$

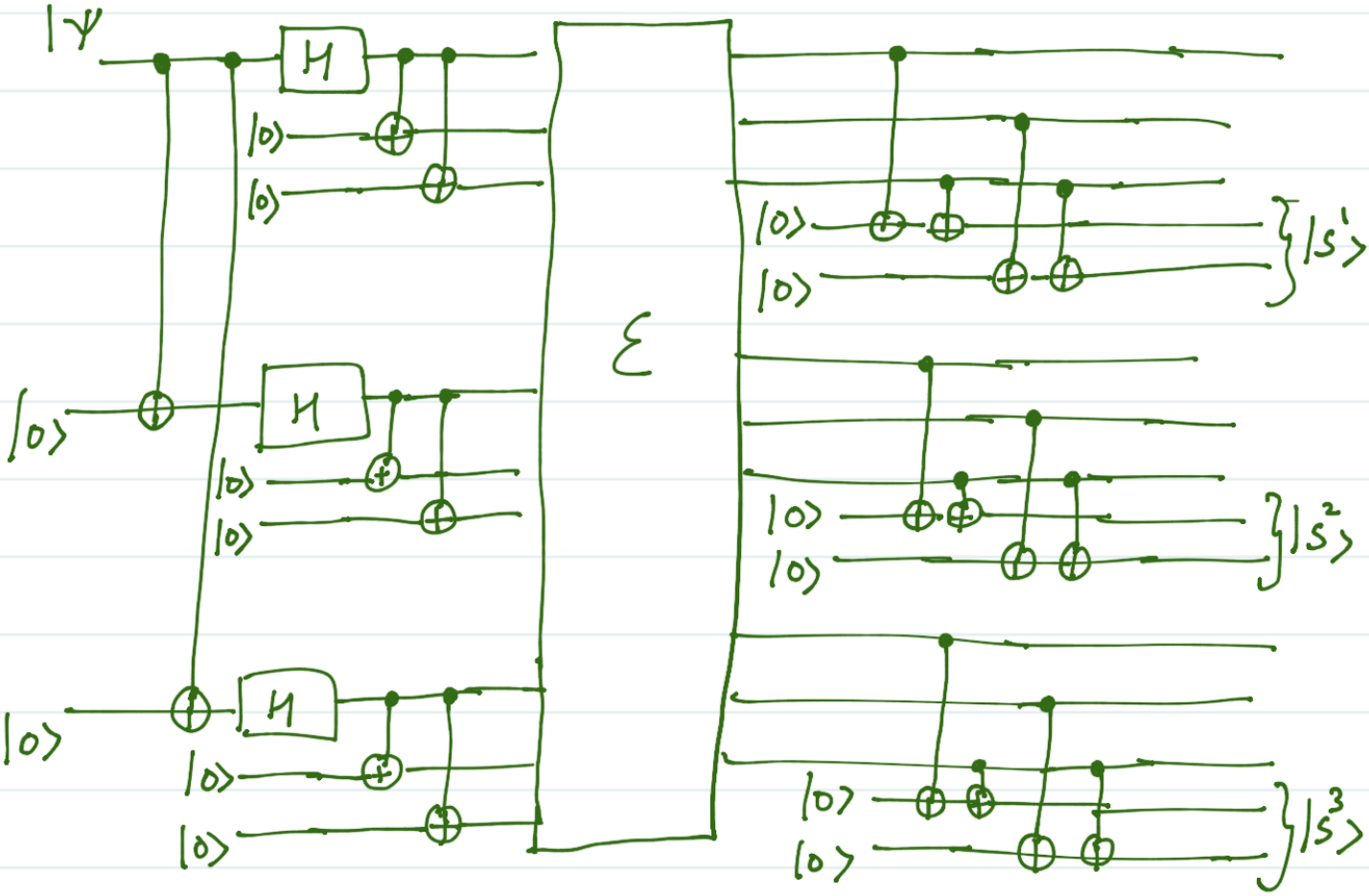
- Bit-Flip Error in the 5th qubit:

$$\alpha\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|010\rangle+|101\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)$$
$$+\beta\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|010\rangle-|101\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)$$

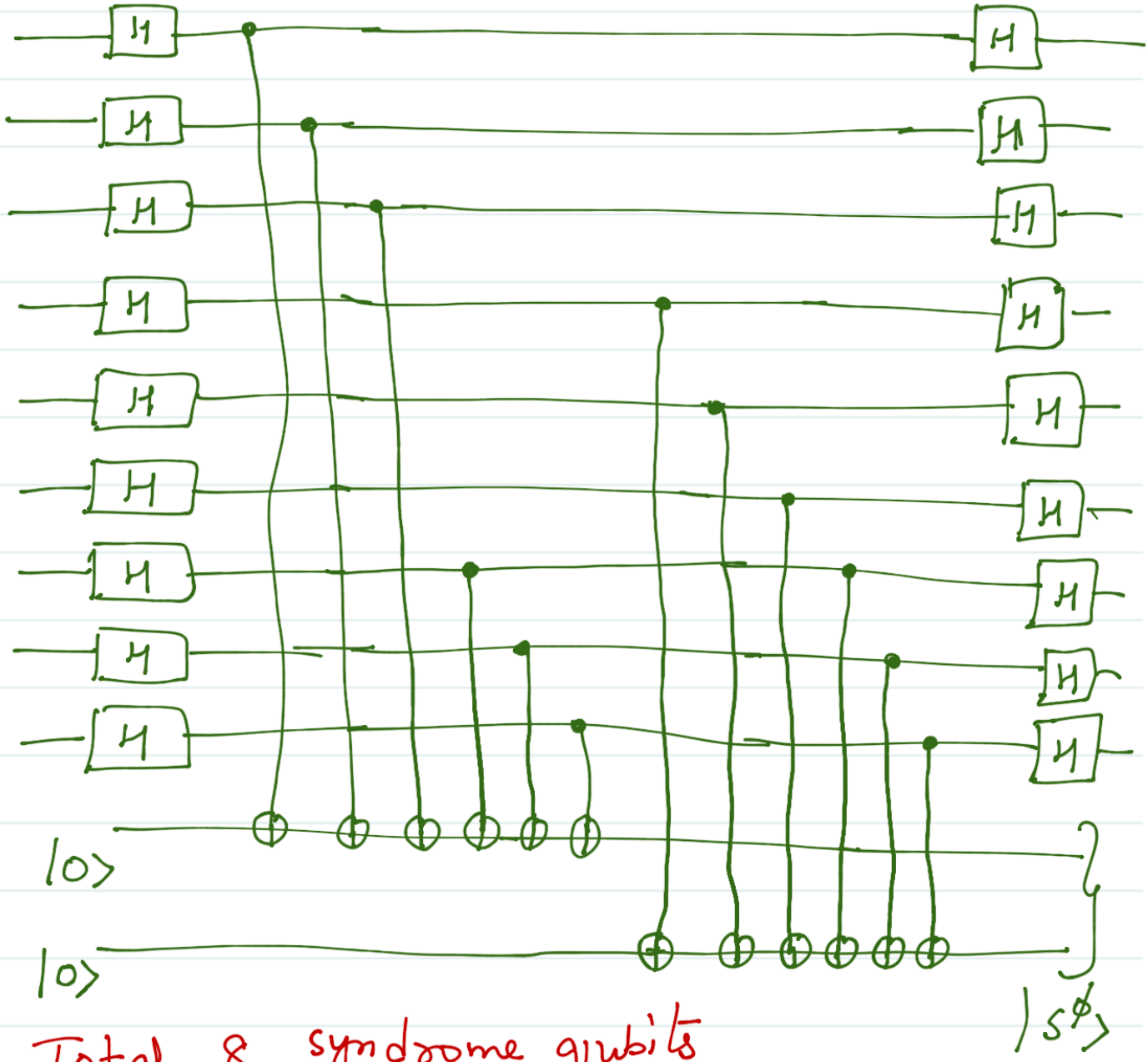
- Phase-Flip Error in the 7th qubit:

$$\alpha\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)$$
$$+\beta\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle-|111\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|000\rangle+|111\rangle}{\sqrt{2}}\right)$$

Syndrome Measurement for Bit-Flip Error Correction



Syndrome Measurement for Phase-Flip Error Correction



Total 8 syndrome qubits
- 6 for bit-flip errors
- 3 for phase-flip errors.

Pauli Group P_n

$$P_n = \left\{ i^l \cdot M_1 \otimes \cdots \otimes M_n,\ M_j \in \{I, X, Y, Z\},\ l \in \{0,1,2,3\} \right\}$$

- Multiplicative subgroup of the set of all $2^n \times 2^n$ unitary matrices.

- Non-Abelian, any two elements either commute or anti-commute

Stabilizer Codes

- S be a abelian subgroup of P_n s.t. $-I \notin S$

- $Q_S = \{ |\psi\rangle : M|\psi\rangle = |\psi\rangle \ \forall M \in S \}$

  Q_S is a subspace of $\mathbb{C}^N$.

  Q_S - common $+1$-eigenspace of all $M \in S$.

- Q_S is termed a "quantum stabilizer code" stabilized by S.

- Dimension of quantum stabilizer code

$$\dim Q_S = \frac{2^n}{|S|}.$$

Symplectic Notation for Pauli Operators

$M = M_1 \otimes \cdots \otimes M_n$ can be represented as

$$[\underline{a} \mid \underline{b}] = [a_1 a_2 \ldots a_n \mid b_1 \ldots b_n]$$

$$(a_j, b_j) = \begin{cases} (0,0) & \text{if } M_j = I \\ (1,0) & \text{if } M_j = X \\ (0,1) & \text{if } M_j = Z \\ (1,1) & \text{if } M_j = Y = iXZ \end{cases}$$

$$M = X(\underline{a})\, Z(\underline{b})$$

- Product of Pauli operators results in binary sum in the symplectic notation. with a global phase

- Say $M = X(\underline{a})Z(\underline{b})$, $M' = X(\underline{a}')Z(\underline{b}')$

$$MM' = (-1)^{\underline{a}' \cdot \underline{b}}\, X(\underline{a} \oplus \underline{a}')\, Z(\underline{b} \oplus \underline{b}')$$

$$M'M = (-1)^{\underline{a} \cdot \underline{b}'}\, X(\underline{a} \oplus \underline{a}')\, Z(\underline{b} \oplus \underline{b}')$$

- M & M' commute iff $\underline{a}' \cdot \underline{b} = \underline{a} \cdot \underline{b}' \bmod 2$.

$\underline{a}' \cdot \underline{b} - \underline{a} \cdot \underline{b}' \bmod 2$ referred to as symplectic inner product

CSS (Calderbank-Shor-Steane) Construction

- Recipe for constructing quantum codes from classical codes.

- Let C_1 & C_2 be binary classical linear codes with parameters (n_1, k_1) & (n_2, k_2)

- H_1, H_2 be parity check matrices

- Consider the matrix

$$H = \left[\begin{array}{c|c} H_1 & 0 \\ \hline 0 & H_2 \end{array}\right]$$

H_1 - X-type stabilizers

H_2 - Z-type stabilizers

- X & Z-type stabilizers commute iff symplectic inner product of any two rows in H is zero

- In particular, $H_1 H_2^T = 0 \iff C_2^{\perp} \subseteq C_1$ & $C_1^{\perp} \subseteq C_2$

Dimension of CSS Code

$$= \frac{2^n}{|S|} = \frac{2^n}{2^{n-k_1} \cdot 2^{n-k_2}} = 2^{k_1+k_2-n}$$

Shor Code as CSS Code

$$H_1 = \begin{bmatrix} 1&1&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&1 \end{bmatrix} \leftarrow \text{X-type stabilizers}$$

$$H_2 = \begin{bmatrix} 1&1&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&1 \end{bmatrix} \leftarrow \text{Z-type stabilizers}$$

Check $H_1 H_2^T = 0$

& dim of Shor Code $= 2^{k_1+k_2-n}$

$$= 2^{7+3-9} = 2$$

Example 2 (Steane Code)

- Let $C_1 = C_2 = (7,4)$ binary Hamming Code with parity-check matrix

$$H_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- check $H_1 H_1^T = 0$
- CSS will give $H = \left[\begin{array}{c|c} H_1 & 0 \\ \hline 0 & H_1 \end{array}\right]$

- Dimension of the Code

$$2^{k_1+k_2-n} = 2^{4+4-7} = 2$$

In terms of $[[n,k]]$ notation for CSS Code,
shor Code is $[[9,1]]$
and steane Code is $[[7,1]]$.

Logical Operators

- Elements in P_n which commute with all stabilizer elements but themselves not part of the stabilizer group.

- Minimum distance is the minimum weight of a logical operator.

- Logical operators are like undetectable errors.

Minimum distance of CSS Codes

$d_{min}(Q) = \min\{d_1, d_2)$ where

$$d_1 = \min\{w(\underline{c}) : \underline{c} \in C_1 \setminus C_2^{\perp}\}$$

$$d_2 = \min\{w(\underline{c}) : \underline{c} \in C_2 \setminus C_1^{\perp}\}$$

- For Hamming Code C, it can be verified that

$$\min\{w(\underline{c}) : \underline{c} \in C \setminus C^{\perp}\} = 3.$$

Steane Code is a $[[7, 1, 3]]_2$ quantum CSS Code.

Error Correction via Syndromes

- $S = \langle g_1, g_2 \ldots, g_{n-k} \rangle$ with $Q_S - [[n,k]]_2$ stabilizer code

$$|\psi\rangle \longrightarrow \boxed{E_1} \longrightarrow |\psi'\rangle.$$

- Syndrome of E_1 is $\underline{s} = [s_1, \ldots, s_{n-k}]$ given by

$$s_\ell = \begin{cases} 0 & \text{if } E_1 \text{ commutes with } g_\ell \\ 1 & \text{if } E_1 \text{ anticommutes with } g_\ell \end{cases}$$

for error correction

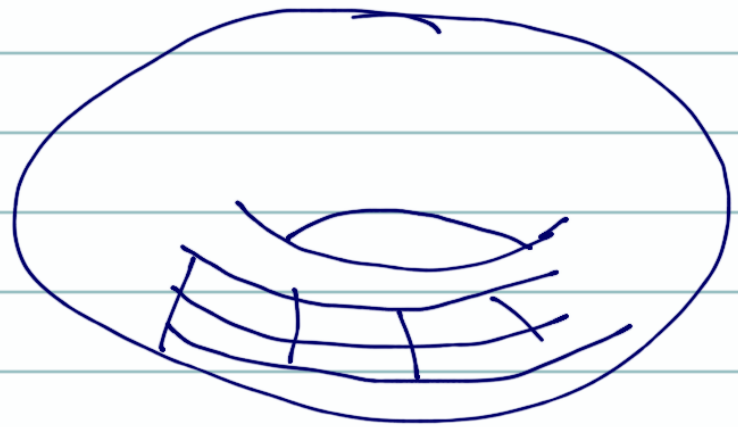
① Determine syndrome $\underline{s}$ by measuring $|\psi'\rangle$ in each of the observables g_ℓ.

② Identify E_2 which has syndrome equal to $\underline{s}$

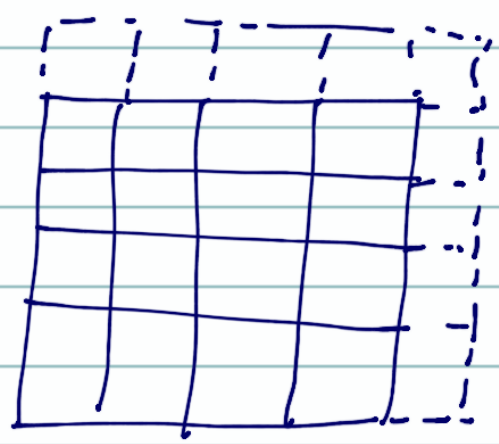
③ Apply $E_2^\dagger$ to $|\psi'\rangle$

Toric Codes

- CSS Codes based on algebraic topology



- $L \times L$ grid on a torus
- Cut open the torus and spread on 2D
- Grid of size $(L+1) \times (L+1)$



- Top row is folded onto bottom row
- Left column is folded onto the right column.

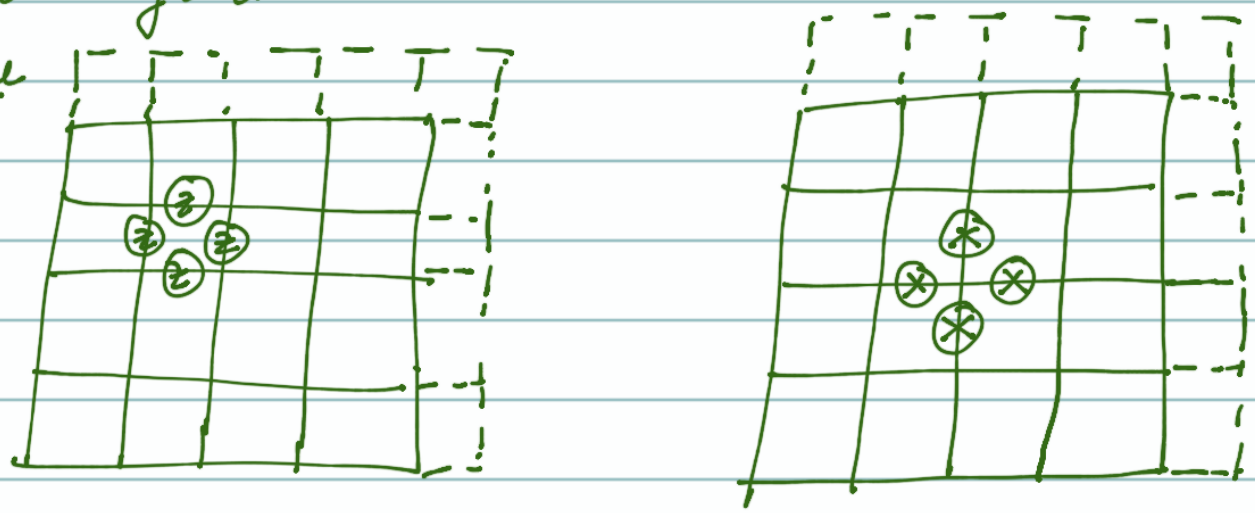
- A qubit is placed on every edge
- No. of qubits $= 2L^2$

Stabilizers of Toric Code

- A Z-type stabilizer on every face of the grid

- An X-type stabilizer at every vertex of the grid

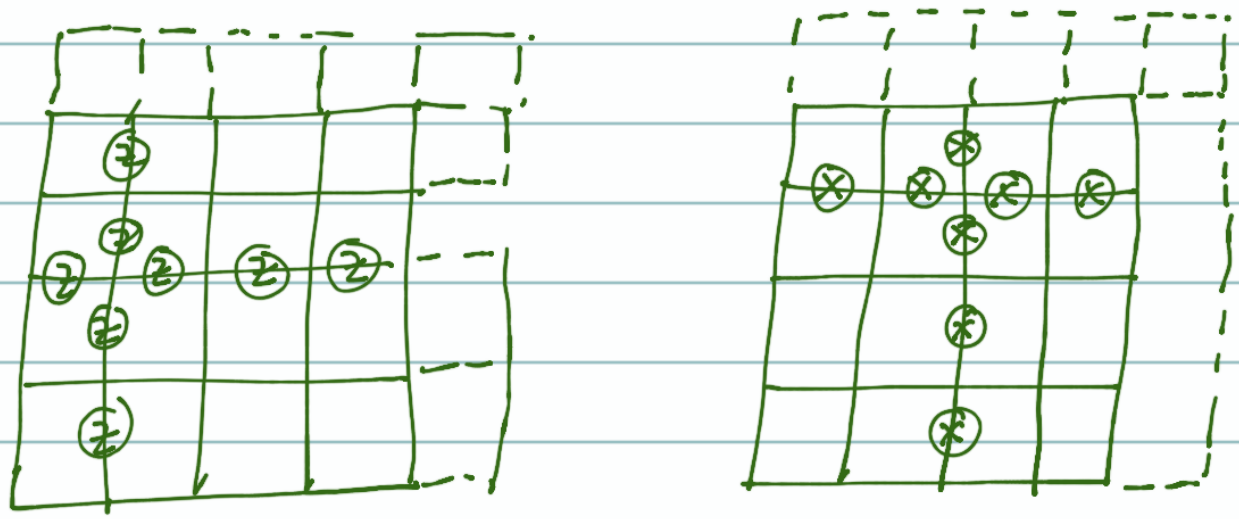
Example



- Note that all Z-type and X-type stabilizers commute.

- All "boundaries" are generated by the stabilizers and hence part of stabilizer group

Logical operators of the Toric Code



- The logical operators are cycles in the grid which are not boundaries.
- "Cycle" property allows to commute with all the stabilizer.

- Minimum weight of a logical operator $= L =$ minimum distance of the toric code.

- Toric code is a CSS code.

$$\boxed{\text{Toric Code is a } [[2L^2, 1, L]] \text{ CSS Code}}$$

- Dimension of toric code

$$= n - \left((n-k_1) + (n-k_2)\right)$$

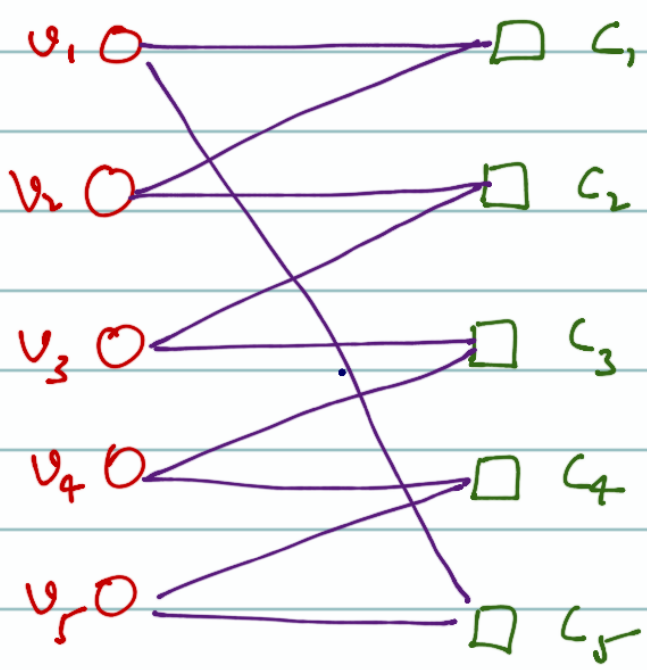
$$= 2L^2 - \left((L^2-1) + (L^2-1)\right) = 2$$

Tanner Graph of a classical Error-Correcting Code

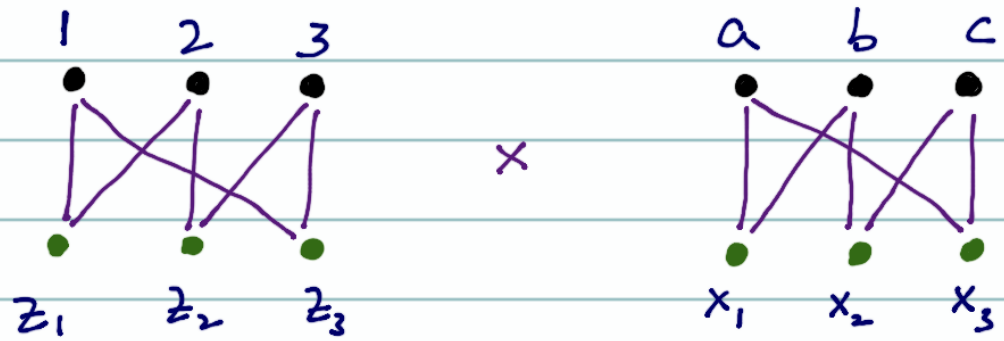
- Bipartite graph $T(V \cup C, E)$

  - V: Variable nodes (Columns of H)
  - C: check nodes (rows of H)
  - Edge $(i,j) \in E$ if $h_{ij} = 1$

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

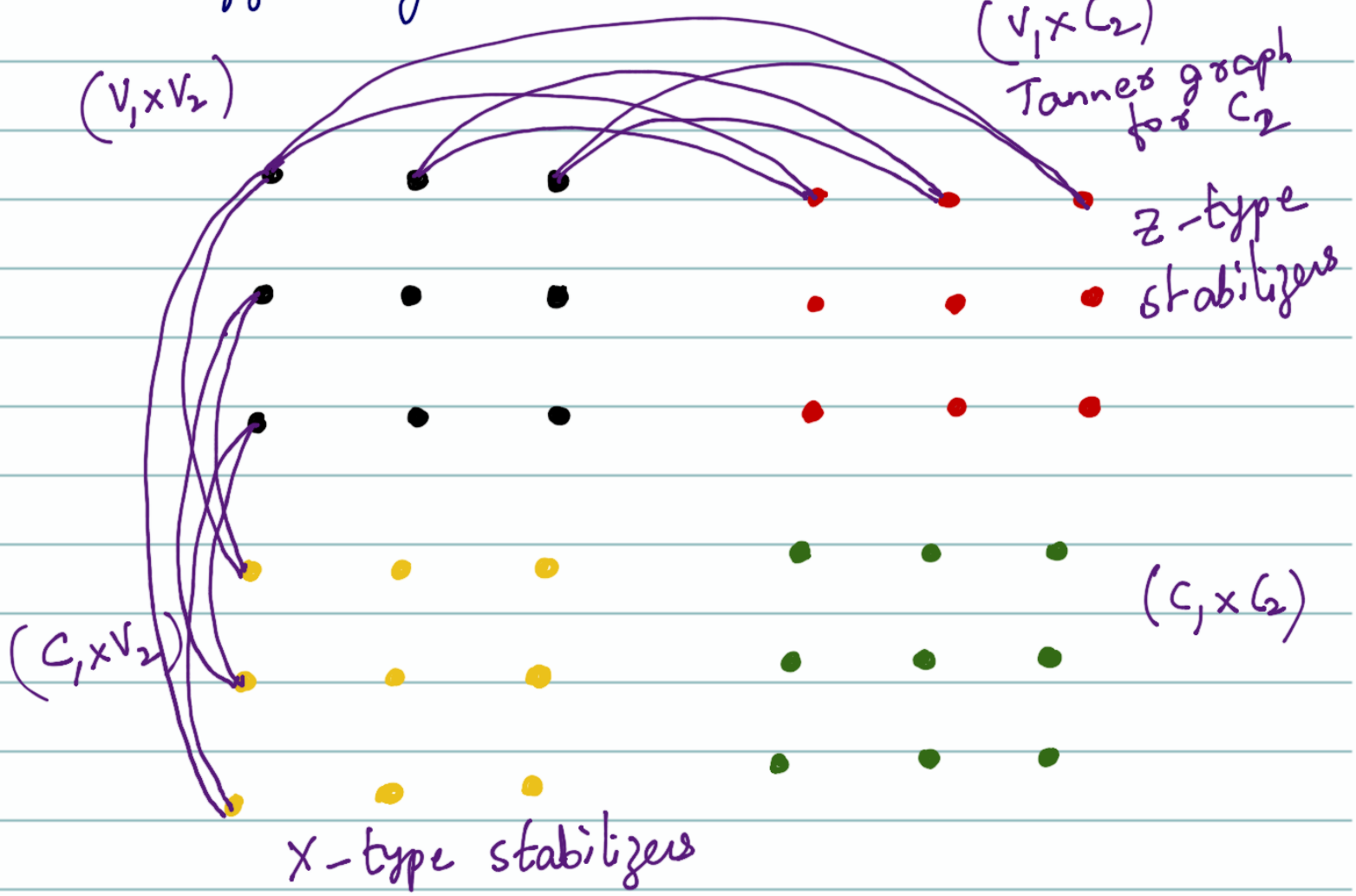


Interpreting Toric Code as a graph product code



Two Tanner graphs of 3-bit repetition code

Toric Code represented differently



Hypergraph Product Codes

- Take arbitrary classical codes C_1, C_2 with parity check matrices H_1, H_2
- $T(H_1)$ & $T(H_2)$ be their Tanner graphs
- Do the graph product of the Tanner graphs exactly like in the case of Toric Code
- The resulting quantum code is called "Hypergraph Product Code"

Parameters of Hypergraph Product Codes

- Say the component codes have parameters
$$(n_1, k_1, d_1) \text{ and } (n_2, k_2, d_2)$$

- Length of the code
$$= n_1 n_2 + (n_1 - k_1)(n_2 - k_2).$$

- Hypergraph product has the CSS structure for stabilizers

- stabilizers themselves are given by
$$H_X = \begin{pmatrix} I \otimes H_2 & H_1^T \otimes I \end{pmatrix}$$
$$H_Z = \begin{pmatrix} H_1 \otimes I & I \otimes H_2^T \end{pmatrix}$$

Check:- $\boxed{H_X H_Z^T = 0}$ & hence dual-containing property is satisfied.

Dimension of HGP Codes

Transpose Code $C^T \rightarrow$ obtained by interchanging the roles of variable & check nodes

Say $\mathcal{T}(H) = \mathcal{T}(V, C, E) \rightarrow$ Tanner graph of Code C

then $\boxed{\dim(C) = |V| - |C| + \dim(C^T)}$

Recall that

$$H_X = \begin{bmatrix} I \otimes H_2 & H_1^T \otimes I \end{bmatrix} \rightarrow$$ Transpose of the product Code $C_2 \otimes C_1^T$

$$H_Z = \begin{bmatrix} H_1 \otimes I & I \otimes H_2^T \end{bmatrix} \rightarrow$$ Transpose of the product Code $C_1 \otimes C_2^T$

Dimension of HGP Code will turn out to be $K_1 K_2 + K_1^T K_2^T$

Minimum Distance of HGP Codes

$$d_{min}(Q_s) = \min\{d_x, d_z\}$$

$$d_x \geq \min\{d_1, d_2^T\}$$

$$d_z \geq \min\{d_1^T, d_2\}$$

$$d_{min}(Q_s) \geq \min\{d_1, d_2, d_1^T, d_2^T\}$$

Some conditions for equality can be derived but this is by far the best expression.

Quantum LDPC Codes

→ Low weight stabilizers

→ Motivated by measurements which have to be spatially local

→ HGP Codes are QLDPC if the Component Codes are picked as classical LDPC Codes.

→ If classical LDPC codes are picked with parameters $[n, \Theta(n), \Theta(n)]$,

Quantum HGP Code will have parameters $[[n, \Theta(n), O(\sqrt{n})]]$

History of QLDPC Codes

(Courtesy Error Correction Zoo)

k	d	Code.
2	$\sqrt{n}$	Toric Code
2	$\sqrt{n\sqrt{\log n}}$	Freedman-Meyer-Luo
$\Theta(n)$	$\sqrt{n}$	Hypergraph Product
$\sqrt{n}/\log n$	$\sqrt{n}\log n$	High Dimensional Expander (HDX)
$\sqrt{n}$	$\sqrt{n}\log^c n$	tensor product HDX
$n^{3/5}/\text{polylog}\, n$	$n^{3/5}/\text{polylog}\, n$	fiber-bundle
$\log n$	$n/\log n$	Lifted Product (LP)
$\Theta(n)$	$\Theta(n)$	Expander LP.
$\Theta(n)$	$\Theta(n)$	Quantum Tanner
$\Theta(n)$	$\Theta(n)$	Dinur-Hsieh-Lin-Vidick

Quick Idea of Lifted Product Codes

For Hypergraph product Codes

$$H_X = \begin{bmatrix} I \otimes H_2 & H_1^T \otimes I \end{bmatrix}$$

$$H_Z = \begin{bmatrix} H_1 \otimes I & I \otimes H_2^T \end{bmatrix}$$

H_1, H_2 are binary matrices

In lifted product Codes, H_1, H_2 take elements from $\mathbb{F}_2$ algebra (group algebra)

$$\mathbb{F}_2[x]/(x^l - 1)$$

- To get the final Code, each element is replaced by an $l \times l$ circulant matrix & hence the name "lifted product Code"

- One key ingredient of the Construction is expansion property for both the Code & its transpose"

Quantum Tanner Codes

- Lifted product Codes used lifting based on Abelian groups

- To get the "Constant rate" and "Constant distance", lifts based on "Non-Abelian groups" are required.

- Also in lifted product Codes, "Expansion + good local Code" are used

- In Quantum Tanner Code, "Expansion + "Kronecker product of local Codes which are robustly testable" used.

- Lifted product Codes are "implemented". Tanner Codes are not yet for multiple reasons.

Quantum Computing Technologies

Superconducting qubits - IBM, Google

- Circuit loop with current in loops (Cooper pairs)
- Superconducting metal at very low temp.
- Follow quantum mechanics rules
- Measurement applying microwave or magnetic flux pulses.

Ion Traps - IonQ, Quantinuum

- Ions trapped in electromagnetic field
- Quantum info. stored in electronic states of ions
- Lasers induce quantum gates

Quantum Computing Technologies

Photonic / Bosonic Computing – Xanadu, PsiQuantum

- Quantum Continuous Variable Systems
- Quantization of modes of an electromagnetic field
- Quantum Harmonic Oscillator
- Qubits are subspaces of infinite dimensional Hilbert Space

Neutral Atoms – Pascal, QuEra

- Rubidium atoms / Rydberg atoms. Each atom has wide range of potential excited states each at different energy levels.
- Lasers act as optical tweezers, trap individual atoms
- Cools atoms to almost zero. Very good coherence times
- Multi-qubit gates

Constant Overhead Fault-Tolerant quantum Computation with reconfigurable Atom Arrays. (Nature Physics 2024)

- QLDPC Codes have good rate & distance

- Implementing QLDPC Codes involves non local operations + long range Connectivity between qubits

- Product structure inherent in QLDPC Codes to implement nonlocal Syndrome extraction through (neutral) atom rearrangement

- Perhaps for the first time,

  QLDPC Codes based architectures outperform surface Codes with as few as several hundred physical qubits

Several Open Questions

1. Decoding of QLDPC Codes efficiently
   - BP doesn't work because of short cycles
   - BP + OSD algorithm is by far the best known algorithm now

1+. Good finite length Codes with spatial locality + some long range connections.

2. Other Error Models - In Some Systems, most failures are erasures and Codes / decoding need to be designed for erasures

3. Biased Error Models - X-type errors much more probable than Z-type errors say.

4. Performing logical operations with these QLDPC Codes - (requires a certain closure under multiplication)

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