Theorem 1. *Suppose that the state $|\psi\rangle$ is real. If we substitute the column set $\mathcal{A}_n$ with $\mathcal{A}'_n$ in the problem (**??**), the optimal solution of the restricted problem is also optimal for the original problem.*

1 The proof of Theorem 1

In this section, we prove Theorem 1. The proof is based on the following lemma.

Lemma 1. *Suppose y is a real vector and satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}'_n$ such that a is a real vector. Then, y satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}_n$.*

Proof. Fix $a \in \mathcal{A}$ and suppose that a represents the state $|\phi\rangle$. Now, consider to write $|\phi\rangle$ as in the form (**??**). The case $k = 0$ is trivial since then $a \in \mathcal{A}'_n$. Suppose that $|\phi_i\rangle$ can be written as $\frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx} i^{c^\top x} |Rx + t\rangle$ with $k > 0$ and $a^\dagger y = \alpha + i\beta \quad (\alpha, \beta \in \mathbb{R})$. The following two states

$$|\phi_+\rangle \coloneqq \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx} |Rx + t\rangle, \quad |\phi_-\rangle \coloneqq \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx + c^\top x} |Rx + t\rangle$$

belongs to $\mathcal{A}'_n$, and denote the column vectors of $|\phi_+\rangle$ and $|\phi_-\rangle$ as a_+ and a_-, respectively. Then, we have $a_+^\dagger y = \alpha + \beta, a_-^\dagger y = \alpha - \beta$ from the assumption, and

$$\left|a^\dagger y\right| = \sqrt{\alpha^2 + \beta^2} \leq |\alpha| + |\beta| = \max\{|\alpha + \beta|, |\alpha - \beta|\} \leq 1,$$

which completes the proof. □

Now, we are ready to prove Theorem 1.

Theorem 1. *Suppose that the state $|\psi\rangle$ is real. If we substitute the column set $\mathcal{A}_n$ with $\mathcal{A}'_n$ in the problem (**??**), the optimal solution of the restricted problem is also optimal for the original problem.*

Proof. Let x^* and y^* be the optimal solutions of the restricted primal and dual problems, namely, the problem (**??**) and the problem (**??**) with the column set $\mathcal{A}'_n$ instead of $\mathcal{A}_n$. We can assure such solutions always exists. Now, we show that the x^*, y^* are optimal not only for the restricted problems but also for the original problems.

Let OPT be the optimal value for the original problem. Since x^* can be a feasible solution for the original primal problem, it is clear that $\mathrm{OPT} \leq \|x^*\|_1$. By the strong duality theorem, OPT is also the optimal value for the original dual problem. From the lemma 1, we can see that y^* is a feasible solution for the original dual problem and $\mathrm{OPT} \geq \mathrm{Re}\left\{b^\dagger y^*\right\}$. Again, by applying the strong duality theorem to the restricted problems, we have $\|x^*\|_1 = \mathrm{Re}\left\{b^\dagger y^*\right\}$, which means that $\mathrm{OPT} = \|x^*\|_1 = \mathrm{Re}\left\{b^\dagger y^*\right\}$. Therefore, x^* and y^* are also optimal solutions for the original problems. □