Stabilizer Extent Calculation by Column Generation

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1 Introduction

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2 Preliminaries

Let $\mathcal{S}_n \coloneqq \{|\phi_\alpha\rangle\}$ be the entire set of n-qubit stabilizer states. We also define the density matrix for $|\phi_\alpha\rangle$ as $\sigma_\alpha \coloneqq |\phi_\alpha\rangle\langle\phi_\alpha|$. The size of $\mathcal{S}_n$ scales superexponentially as $|\mathcal{S}_n| = 2^n \prod_{k=0}^{n-1}(2^{n-k}+1) = 2^{\mathcal{O}(n^2)}$ [1, Proposition 1]. See also Table 1 for the size of $\mathcal{S}_n$.

Table 1: The size of $\mathcal{S}_n$, the data size of A_n in sparse matrix format [2], and the average time to quantify 10 Haar random pure state with our proposed algorithm.

n	5	6	7	8	9	10
$\lvert\mathcal{S}_n\rvert$	2.42e+06	3.15e+08	8.13e+10	4.18e+13	4.29e+16	8.79e+19
size of A_n	2.3 KiB	25.9 KiB	367.2 KiB	6.2 MiB	128.0 MiB	3.1 GiB
proposed	1 s	10 s	1 min	10 min	(Approx)	(Approx)

The *Robustness of Magic* (RoM) is introduced in [3] to quantify an n-qubit state ρ, represented by density matrix, and defined as follows:

$$\mathcal{R}(\rho) \coloneqq \min_{c \in \mathbb{R}^{|\mathcal{S}_n|}} \left\{ \|c\|_1 \;\middle|\; \rho = \sum_{j=1}^{|\mathcal{S}_n|} c_j \sigma_j \right\}$$

On the other hand, the *stabilizer extent* is introduced in [4, Definition 3] to quantify an normalized n-qubit state ψ, represented by state vector, and is defined as follows:

$$\xi(\psi) \coloneqq \min_{c \in \mathbb{C}^{|\mathcal{S}_n|}} \left\{ \|c\|_1^2 \;\middle|\; |\psi\rangle = \sum_{j=1}^{|\mathcal{S}_n|} c_j |\phi_j\rangle \right\} \tag{1}$$

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In this paper, our focus lies on computing the stabilizer extent. This definition of the stabilizer extent can be simplified as complex L^1-norm minimization problem:

$$\sqrt{\xi(\psi)} = \min_{x \in \mathbb{C}^{|\mathcal{S}_n|}} \{\|x\|_1 \mid A_n x = b\} \tag{2}$$

Here, we define $A_n \in \mathbb{C}^{2^n \times |\mathcal{S}_n|}$ as $(A_n)_{ij} := \langle i|\phi_j\rangle$ and $b \in \mathbb{C}^{2^n}$ as $b_i := \langle i|\psi\rangle$ using the computational basis $\{|i\rangle\}_{i=0}^{2^n-1}$. As in [5], the problem (2) is a second order cone program (SOCP). Thus, by defining $\mathcal{A}_n$ as the columns set $\{a_j\}$ of A_n, its dual problem can be derived as [5, Appendix A][6, Section 5.1.6]

$$\sqrt{\xi(\psi)} = \max_{y \in \mathbb{C}^{2^n}} \left\{ \mathrm{Re}(b^\dagger y) \;\middle|\; \left|a_j^\dagger y\right| \le 1 \text{ for all } a_j \in \mathcal{A}_n \right\} \tag{3}$$

where $\dagger$ denotes the conjugate transpose. While the true objective function of the Lagrange dual problem corresponding to (2) is not $\mathrm{Re}(b^\dagger y)$ but $-\mathrm{Re}(b^\dagger y)$, we flipped the sign for simplicity. This is valid as it does not alter the optimal solution, owing to $\left|a_j^\dagger y\right| = \left|a_j^\dagger(-y)\right|$.

Further, in order to describe our algorithm in later sections, we denote a function $\mathrm{SolveSOCP}(\mathcal{C}, b)$ which takes a columns set $\mathcal{C} \subseteq \mathcal{A}$ and a vector b, and returns the optimal primal solution x dual optimal solution y of the SOCP problem (3). In actual numerical computation, this function can be realized by just solving the corresponding primal problem (2).

3 Scaling up The Exact Stabilizer Extent Calculation

In the preceding sections, we introduced two quantum resource measures: the Robustness of Magic and the stabilizer extent. Despite both being efficiently quantifiable through convex optimization problems, solving them directly for $n > 5$ qubit systems becomes impractical due to the superexponential growth of the number of stabilizer states $|\mathcal{S}_n|$. To address this challenge, we proposed employing the classical optimization technique known as column generation (CG) method [**?**] for Robustness of Magic calculation [7]. However, it remained unclear whether the same approach could be applied to calculate the stabilizer extent. Here, we demonstrate that leveraging the specific structure of stabilizer states enables a similar method to work effectively for calculating the stabilizer extent as well.

3.1 Core Subroutine: Calculating Overlap

Before we start to consider about the stabilizer extent, we define *fidelity* of $b \in \mathbb{C}^{2^n}$ as

$$\sqrt{F(b)} := \max_{a_j \in \mathcal{A}_n} \left|a_j^\dagger b\right| = \max_{\phi \in \mathcal{S}_n} |\langle\phi|\psi\rangle|,$$

which is the maximal overlap between y and the stabilizer states [4, Definition 4][5]. It is known that fidelity is deeply related to stabilizer extend [4, Theorem 4] [5, Theorem 4], and fidelity for mixed state was important when computing Robustness of Magic [7]. Further, we will show fidelity plays a crucial role in our proposed algorithm in later sections. In this section, we show that how to compute the fidelity efficiently up to 8-qubit systems.

To this end, we introduce the following proposition. As a well-known fact, the stabilizer states have a simple form as shown in the following proposition.

Proposition 1 ([8, Theorem 2], [9, Section 5], [10, Theorem 5.(ii)]). *All stabilizer states can be written in the following form:*

$$\begin{cases} |\phi\rangle := |t\rangle & \text{if } k = 0 \\ |\phi\rangle := \dfrac{1}{2^{k/2}} \displaystyle\sum_{x=0}^{2^k-1} (-1)^{x^\top Q x} i^{c^\top x} |Rx + t\rangle & \text{if } k > 0 \end{cases} \tag{4}$$

where $Q \in \mathbb{F}_2^{k\times k}$, $c \in \mathbb{F}_2^k$, $R \in \mathbb{F}_2^{n\times k}$, $t \in \mathbb{F}_2^n$ *and* $\operatorname{rank}(R) = k$. *Also, any state that can be written in this form is a stabilizer state.*

By modifying the form slightly, we can obtain the following more convenient form. The proof is given in Appendix A.1.

Theorem 1. *The form* (4) *under the following conditions enumerates all stabilizer states without any duplication or omission.*

- Q *is a upper triangular* $\mathbb{F}_2^{k\times k}$ *matrix.*
- R *is a* $\mathbb{F}_2^{n\times k}$ *rref (reduced row echelon form) matrix satisfies* $\operatorname{rank}(R) = k$.
- t *belongs to the complement of the row space of* R.

Let ϕ be a one of stabilizer state in the standard form (4) with $k > 0$, which means $|\phi\rangle = \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Q x} i^{c^\top x} |Rx + t\rangle$. Then, by denoting $a \in \mathcal{A}$ as the corresponding vector of ϕ, the overlap between a and b is

$$a^\dagger b = \langle\phi|\psi\rangle = \frac{1}{2^{k/2}} \sum_{x=0}^{2^n-1} (-1)^{x^\top Q x} i^{c^\top x} \langle Rx + t|\phi\rangle = \sum_{x=0}^{2^n-1} (-1)^{x^\top Q x} i^{c^\top x} \left(\frac{1}{2^{k/2}} b_{Rx+t}^\dagger \right).$$

In the following, we define $P_x := \frac{1}{2^{k/2}} b_x^\dagger$, and for the simplicity, we fix $k = n, R = I_n, t = 0$. This assumption is not restrictive since the other cases can be easily reduced to this case. Recalling that what we want to solve is $\max_{a_j \in \mathcal{A}_n} \left| a_j^\dagger b \right|$, this is basically equivalent to the following problem:

$$\max_{c \in \mathbb{F}_2^n, Q \in \mathbb{F}_2^{n\times n}} \left\{ \left| \sum_{x=0}^{2^n-1} -1^{x^\top Q x} i^{c^\top x} P_x \right| \right\}. \tag{5}$$

If we solve (5) naively, the time complexity is $\mathcal{O}\left(2^{n+n(n+1)/2} 2^n n^2\right)$, where $2^{n+n(n+1)/2}$ is the number of the possible c, Q, 2^n is the number of term in the sum, and n^2 is the computational cost per each term. However, we can reduce this time complexity by using the following theorem.

Theorem 2. *The problem* (5) *can be solved in* $\mathcal{O}\left(2^{n+n(n+1)/2}\right)$ *time complexity and* $\mathcal{O}(2^n)$ *space complexity.*

The basic algorithm to solve this problem is the depth-first search (DFS) algorithm, which is describe in Figure 1. The detailed proof is given in Appendix A.2. Consequently, the next theorem is obtained.

Theorem 3. *Computation of* $A_n^\dagger b$ *can be done in time complexity of* $\mathcal{O}(|S_n|)$ *and space complexity of* $\mathcal{O}(2^n)$. *Thus, we can compute the fidelity in the same complexity.*

As a remark, the fidelity a random 8-qubit pure state can be computed in ?? minutes with a *laptop* powered by Intel(R) Core(TM) i7-10510U CPU with 16 GB RAM. It's worth noting that the algorithm's speed is enhanced by branch cutting within the DFS algorithm, as detailed in Appendix A.3. We will use this algorithm as a subroutine in the later sections.

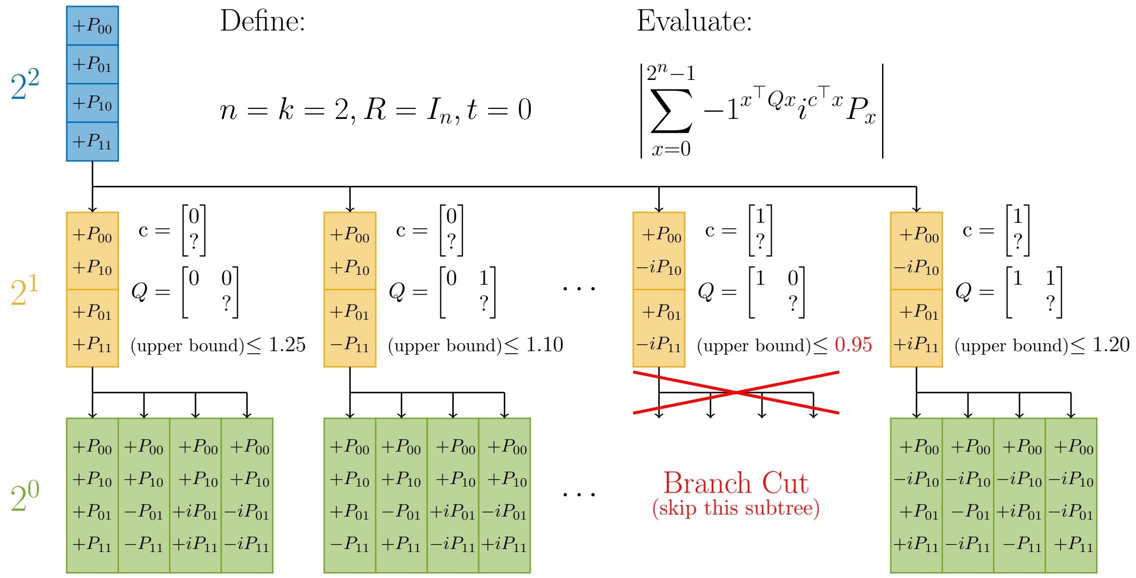

Figure 1: Visualization of the DFS algorithm for Theorem 2. The DFS algorithm is a recursive procedure to calculates the overlap. Memory usage is limited to only $\sum_{i=0}^{n} 2^n$. During computation, the maximal solution is either lower bounded by the current best solution or, in certain cases, 1. Thus, we can terminate branches if the upper bound of the current branch is inferior to the current best solution. Also refer to Appendix A.3.

3.2 CG method for the stabilizer extent calculation

Next, we introduce the CG method outlined in Algorithm 1. This is exactly the algorithm to compute the stabilizer extent $\xi(\psi)$, and is a iterative algorithm that solves a subproblem restricted to $\mathcal{C} \subseteq \mathcal{A}_n$ per each iterations. It begins with a small subset $\mathcal{C}_0$ and progressively adds columns $\mathcal{C}'_k$ that violate the constraints of the dual problem (3), and terminate if there are no more violated columns. For further implementation details, we direct the reader to [7], There are two key aspects of this algorithm: the initialization process and the optimality of the solution. We will discuss these in subsequent sections.

3.2.1 Initialization

In the initial step of Algorithm 1, we select a subset $\{a_j\} = \mathcal{C}_0 \subseteq \mathcal{A}_n$ in descending order of $\left|a_j^\dagger b\right|$, which can be computed efficiently as stated in Theorem 3. This choice can be justified with various interpretations. One of them is to consider $\left|a_j^\dagger b\right| = |\langle\phi_j|\psi\rangle|$ as the "closeness" between the states $|\psi\rangle$ and $|\phi_j\rangle$. Hence, choosing the states based on their overlaps is a reasonable choice. The numerical experiments result in [7] also support the effectiveness of this heuristic. In the case of a random pure 8-qubit state, even if we use as small subset as $|\mathcal{C}_0| = 10^{-7}|\mathcal{A}_n|$, the $\|x_0\|_1$ obtained closely approximates the exact value and outperforms randomly selected $\mathcal{C}_0$.

3.2.2 Optimality of the solution

The terminate criterion for Algorithm 1 is the absence of columns that violate the dual constraints $\left|a_j^\dagger y_k\right| \leq 1$, which can be checked efficiently by Theorem 3 as well. This means the optimal solution for the dual problem (3) is found, and the primal solution x_k is also optimal thank to the strong duality of the SOCP problem. Consequently, we can affirm

Algorithm 1: Exact stabilizer extent calculation by Column Generation

Input: vector b corresponding to the state ψ

Output: Exact stabilizer extent $\xi(\psi)$

$\mathcal{C}_0 \leftarrow$ Partial set of $\mathcal{A}_n$ `/* Initialize using top overlap` $\left|a_j^\dagger b\right|$ `*/`

for $k = 0, 1, 2, \ldots$ **do**

  $x_k, y_k \leftarrow$ `SolveSOCP`$(\mathcal{C}_k, \boldsymbol{b})$

  $\mathcal{C}' \leftarrow \left\{a \in \mathcal{A}_n \;\middle|\; \left|a^\dagger y_k\right| > 1\right\}$ `/* Use of subroutine in Section 3.1 */`

  if $\mathcal{C}' = \emptyset$ **then**

    return $\xi(\psi) = \|x_k\|_1$

  $\mathcal{C}_{k+1} \leftarrow \mathcal{C}_k \cup \mathcal{C}'$

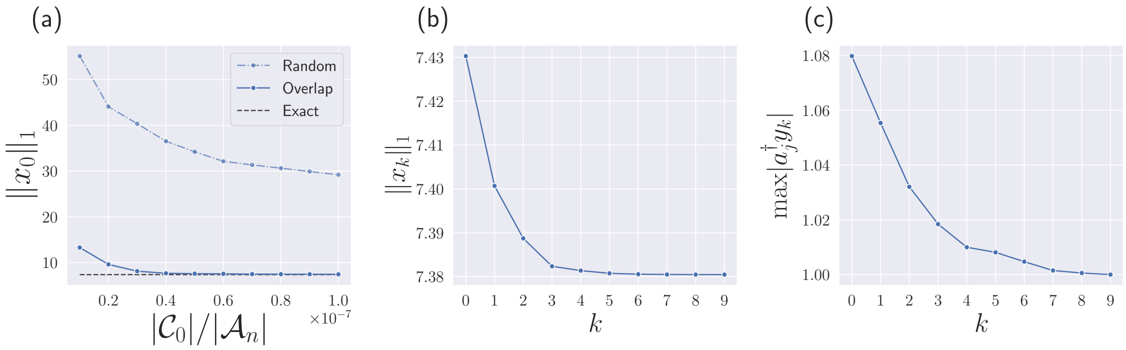

Figure 2: (a) $\|x_0\|_1$ in the Algorithm 1, which can be obtained from `SolveSOCP`$(\mathcal{C}_0, b)$, for a random 8-qubit state. The ratio $|\mathcal{C}_0|/|\mathcal{A}_n|$ varies from 10^{-8} to 10^{-7}. We can get much better results with the top overlap heuristics compared to the random selection of $\mathcal{C}_0$. (b) The convergence of the CG method for the same state. The max violation becomes 1.00 after 10 iterations, which means the optimal solution is found.

that Algorithm 1 is certain to find the exact stabilizer extent for any n-qubit state $|\psi\rangle$ once it terminates. The convergence of the CG method is also confirmed in numerical experiments. For the same 8-qubit state as in the initialization, $\max_{a_j \in \mathcal{A}} \left|a_j^\dagger y_k\right|$ reaches 1.00 after 10 iterations, indicating the discovery of the optimal solution.

3.3 For the case $|\psi\rangle$ is Real

In some cases, the state $|\psi\rangle$ could be real. For example, todo . We show that in such cases the problem can be further simplified. We define the subset of the states in $\mathcal{S}_n$ with real coefficients as $\mathcal{T}_n$, and the corresponding subset of the columns in $\mathcal{A}_n$ as $\mathcal{A}_n'$. Then, the next theorem holds.

Theorem 4. *Suppose that the state $|\psi\rangle$ is real. If we substitute the column set $\mathcal{A}_n$ with $\mathcal{A}_n'$ in the problem (2), the optimal solution of the restricted problem is also optimal for the original problem.*

Thanks to theorem 4, we can reduce the size of the column set size by a factor of 2^n. numerical experiments result?

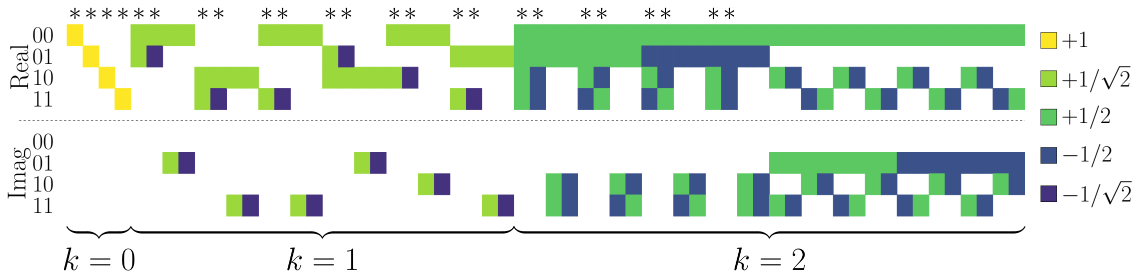

Figure 3: Visualization of the matrix A_n with $n = 2$. The upper half corresponds to the real part, and the lower half corresponds to the imaginary part. The j-th column of this represents the column a_j and its state $|\phi_j\rangle$. The k below the matrix corresponds to the standard form (4). By restricting the matrix A_n to the starred columns which are real vectors, we can obtain the matrix A'_n.

4 Approx Solutions

TBD

How should we explain the approx solutions?

5 Discussion

In this paper, we have shown that todo.

There is still room for improvement in some specific cases. As for Robustness of Magic, there is a marvelous algorithm proposed in [11] which focuses on copies of symmetric pure magic states, and we enhanced this result in [7]. Applying such techniques to the stabilizer extent calculation would be promising and is left for future work.

Furthermore, there is some more future direction. For example, todo.

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A Fast Algorithm for Overlap

In this section, we will explain the detail of dfs algorithm in Section 3.1 and introduce some heuristics to improve the efficiency.

A.1 Efficient Enumeration of Stabilizer States

In this section, we prove the Theorem 1.

Theorem 1. *The form (4) under the following conditions enumerates all stabilizer states without any duplication or omission.*

- *Q is a upper triangular $\mathbb{F}_2^{k\times k}$ matrix.*
- *R is a $\mathbb{F}_2^{n\times k}$ rref (reduced row echelon form) matrix satisfies $\operatorname{rank}(R) = k$.*
- *t belongs to the complement of the row space of R.*

Proof. Main Ideas come from [8]. The assertion is trivial for $k = 0$. We will only consider the case $k > 0$. Define $f : (Q, c, R, t) \mapsto |\phi\rangle$ as the mapping from (Q, c, R, t) to the corresponding state $|\phi\rangle$. Firstly, we show that f is injective. We can say that

$$\begin{aligned} &\left\{R_1 x + t_1 \,\middle|\, x \in \mathbb{F}_2^{n-k}\right\} = \left\{R_2 x + t_2 \,\middle|\, x \in \mathbb{F}_2^{n-k}\right\} \\ \iff &\operatorname{Im}(R_1) = \operatorname{Im}(R_2) \wedge (t_2 - t_1) \in \operatorname{Im}(R_1) \\ \iff &R_1 = R_2 \wedge t_1 = t_2. \end{aligned}$$

The last equivalence is due to the property of the rref matrix and the complement condition. Given that Q is an upper triangular matrix, both Q and c can be uniquely reconstructed from the coefficients of the state $|\phi\rangle$. Consequently, for any $|\phi\rangle$, the values of (Q, c, R, t) can be uniquely determined, establishing injectivity in the mapping f.

Next, we show that f is surjective. Since f is injective, we only have to show that the cardinality of the domain is equal to that of the codomain, i.e., $-2^n + |\mathcal{S}_n|$. It is known that the number of $\mathbb{F}_2^{n\times k}$ rref matrices with $\mathrm{rank}(R)=k$ is $\begin{bmatrix} n \\ k \end{bmatrix}_2$, which is a q-binomial coefficient with $q=2$. Therefore, the number of Q, c, R, t is $2^{k(k+1)/2}$, 2^k, $\begin{bmatrix} n \\ k \end{bmatrix}_2$, 2^{n-k}, respectively, and the total number of states is

$$\sum_{k=1}^{n} 2^{k(k+1)/2} 2^k \begin{bmatrix} n \\ k \end{bmatrix}_2 2^{n-k} = -2^n + 2^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_2 2^{k(k+1)/2} = -2^n + 2^n \prod_{k=1}^{n} (2^k+1) = -2^n + |\mathcal{S}_n|.$$

In the second last equation, we used the q-binomial theorem. Therefore, the mapping is surjective, which concludes the proof. □

In theorem 1, we used $\mathbb{F}_2$. However, from the perspective of the dfs algorithm, it is more practical to use $\{0,1\} \subset \mathbb{Z}$ and permit the term $c^\top x$ to be any integer value. Hence, the subsequent corollary is valuable.

Corollary 1. *In theorem 1, We can substitute $\mathbb{F}_2$ with $\{0,1\} \subset \mathbb{Z}$.*

Proof. By changing $\mathbb{F}_2$ to $\{0,1\} \subset \mathbb{Z}$, the term $(-1)^{x^\top Qx}$ is invariant, and the term $i^{c^\top x}$ is multiplied by -1 iff $p \equiv 2, 3 \pmod 4$, where p is the number of i such that $c_i = 1$ and $x_i = 1$. Now, we consider the following form:

$$\begin{cases} |\phi\rangle := |t\rangle & \text{if } k = 0, \\ |\phi\rangle := \dfrac{1}{2^{k/2}} \displaystyle\sum_{x=0}^{2^k-1} (-1)^{x^\top (Q+Q')x} i^{c^\top x} |Rx+t\rangle & \text{if } k > 0, \end{cases} \tag{6}$$

where $Q \in \{0,1\}^{k\times k}, c \in \{0,1\}^k, R \in \{0,1\}^{n\times k}, t \in \{0,1\}^n, \mathrm{rank}(R) = k$ and $Q'_{ij} = 1$ iff $(i<j) \wedge (c_i = c_j = 1)$. Now, if the pair (Q, c, R, t) in (6) is the same as that of the original form (4), then the two states are representing the exactly same state since

$$(-1)^{x^\top Q'x} = (-1)^{\binom{p}{2}} = \begin{cases} 1 & \text{if } p \equiv 0, 1 \pmod 4, \\ -1 & \text{if } p \equiv 2, 3 \pmod 4. \end{cases}$$

Therefore, by identifying the $Q+Q'$ in $\mathbb{Z}$ with new Q'' in $\mathbb{F}_2$, we can conclude the proof. □

A.2 Calculating the Overlap

In this section, we prove the Theorem 2. Be aware that the problem (5) is equivalent to the following problem thank to the corollary 1:

$$\max_{c\in\{0,1\}^n, Q\in\{0,1\}^{n\times n}} \left\{ \left| \sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x \right| \right\}.$$

Theorem 2. *The problem (5) can be solved in $\mathcal{O}\left(2^{n+n(n+1)/2}\right)$ time complexity and $\mathcal{O}(2^n)$ space complexity.*

Proof. Define $x := \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix}$ $(x_0 \in \{0,1\}, \overline{x} \in \{0,1\}^{n-1})$, $c := \begin{bmatrix} c_0 \\ \overline{c} \end{bmatrix}$ $(c_0 \in \{0,1\}, \overline{c} \in \{0,1\}^{n-1})$, and $Q := \begin{bmatrix} Q_{00} & Q_0^\top \\ 0 & \overline{Q} \end{bmatrix}$ $(Q_{00} \in \{0,1\}, Q_0 \in \{0,1\}^{n-1}, \overline{Q} \in \{0,1\}^{(n-1)\times(n-1)})$. Since $x^\top Qx = x_0(Q_{00} + Q_0^\top \overline{x}) + \overline{x}^\top \overline{Q}\overline{x}$ and $c^\top x = c_0 x_0 + \overline{c}^\top \overline{x}$, we can derive that

$$\sum_{x=0}^{2^n-1} (-1)^{x^\top Qx} i^{c^\top x} P_x = \sum_{\overline{x}=0}^{2^{n-1}-1} (-1)^{\overline{x}^\top \overline{Q}\overline{x}} i^{\overline{c}^\top \overline{x}} \left(P_{2\overline{x}} + (-1)^{Q_{00}+Q_0^\top \overline{x}} i^{c_0} P_{2\overline{x}+1} \right)$$

$$= \sum_{\overline{x}=0}^{2^{n-1}-1} (-1)^{\overline{x}^\top \overline{Q}\overline{x}} i^{\overline{c}^\top \overline{x}} P'_x \tag{7}$$

where we identify a vector $\begin{bmatrix} x_0 & x_1 & \cdots & x_{n-1} \end{bmatrix}^\top$ as a integer $\sum_{i=0}^{n-1} x_i 2^i$, and define $P'_x := P_{2\overline{x}} + (-1)^{Q_{00}+Q_0^\top \overline{x}} i^{c_0} P_{2\overline{x}+1}$. Since (7) is the same form as the original problem, this problem can be solved recursively by fixing the value c_0, Q_{00} and Q_0.

We now analyze the time complexity of this recursive algorithm. Considering each possible combination of c_0, Q_{00}, and Q_0, there are 2^{n+1} options. For each such combination, P'_x can be computed in $O(n2^{n-1})$ time. Hence, we establish the following recurrence relation:

$$T(n) = 2^{n+1}(T(n-1) + n2^{n-1}), \quad T(1) = 4.$$

Solving this recurrence relation yields

$$T(n) = 2^{n+\frac{n(n+1)}{2}} + \sum_{d=2}^{n} 2^{n+\frac{n(n+1)}{2}-\frac{d(d-1)}{2}} d$$

$$\frac{T(n)}{2^{n+\frac{n(n+1)}{2}}} = 1 + \sum_{d=2}^{n} 2^{-\frac{d(d-1)}{2}} d \le 1 + \sum_{d=2}^{n} 2^{-d+1} d \le 4 - (n+2)2^{-n+1} \to 4.$$

Hence, the time complexity is $O(2^{n+n(n+1)/2})$. □

While the algorithm and proof presented above may seem somewhat rough, our actual implementation is significantly more precise and efficient. You can access it at GitHub [12]. Moreover, we can enhance efficiency further by employing branch cut heuristics, as we will explain in the next section.

A.3 Branch Cut For The DFS

In the previous section, we explained the efficient algorithm for the overlap calculation. However, this algorithm can be much more faster by using the branch cut heuristics we will introduce in this section.

Firstly, please recall that we are maximizing the following:

$$\max_{c,Q} \left\{ \left| \sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x \right| \right\}$$

This can be easily bounded by

$$\max_{c,Q} \left\{ \left| \sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x \right| \right\} \le \max_{c,Q} \left\{ \sum_{x=0}^{2^n-1} \left| -1^{x^\top Qx} i^{c^\top x} P_x \right| \right\} = \sum_{x=0}^{2^n-1} |P_x|$$

However, since each coefficient takes only $1, -1, i$ or $-i$, we can obtain

$$\max_{c,Q} \left\{ \left| \sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x \right| \right\} \le \max_{c,Q} \left\{ \left| \sum_{x=0}^{2^n-1} i^{c_x} P_x \right| \right\} \tag{8}$$

where $c_x \in \{0, 1, 2, 3\}$ is the independent variable for each x. Let $P^* = \sum_{x=0}^{2^n-1} i^{c^*_x} P_x$ be one of the optimal solutions of (8), i.e., $|P^*| = \max_{c,Q} \left\{ \left| \sum_{x=0}^{2^n-1} i^{c_x} P_x \right| \right\}$. Then, without loss of generality, we can assume that $\frac{\pi}{2} \leq \arg P^* < \frac{3\pi}{2}$, and by sorting and multiplying $i, -1$ or $-i$ to P_x appropriately, we can also assume that

$$0 \leq \arg(P_0) \leq \arg(P_1) \leq \cdots \leq \arg(P_{2^n-1}) < \pi/2. \tag{9}$$

If all c_x satisfies $\arg(P^*) - \pi/4 \leq \arg\left(i^{c_x} P_x\right) < \arg(P^*) + \pi/4$, then c_x is optimal for P^*. Therefore, we can justify the following Algorithm 2 by moving $\arg(P^*)$ in the range of $[\pi/2, 3\pi/2)$. Also refer to the figure 4 for the visualization of this algorithm. The time complexity of this algorithm is $O(n2^n)$ due to the sorting of 2^n elements.

Algorithm 2: Branch Cut Algorithm

Input: Coefficients P_x for $x = 0, 1, \ldots, 2^n - 1$
Output: The answer for the problem (8)

```
Sort and modify the coefficients P_x so that the condition (9) is satisfied
ans ← 0,   c_x ← 0   for all x
for x ← 0 to 2^n − 1 do
    ans ← max(ans, |Σ_{x=0}^{2^n−1} i^{c_x} P_x|)
    c_x ← c_x + 1
return ans
```

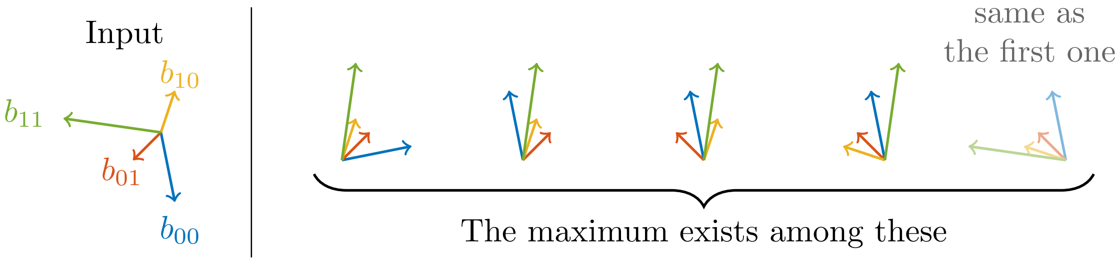

Figure 4: The visualization of Algorithm 2. Suppose that $n = 2$ and P_x are represented as the vectors in the complex plane (for example, $P_{00} = 1 - 5i$) by the left figure. Then, by sorting and iterating the loop in Algorithm 2, we can obtain 2^n patterns of the coefficients c_x as the right figure. The maximum of for the problem (8) exists among these 2^n patterns.

B The proof of Theorem 4

In this section, we prove the Theorem 4. The proof is based on the following lemma.

Lemma 1. *Suppose y is a real vector and satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}'_n$ such that a is a real vector. Then, y satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}_n$.*

Proof. Fix $a \in \mathcal{A}$ and suppose that a represents the state $|\phi\rangle$. Now, consider to write $|\phi\rangle$ as in the form (4). The case $k = 0$ is trivial since then $a \in \mathcal{A}'_n$. Suppose that $|\phi_i\rangle$ can be written as $\frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx} i^{c^\top x} |Rx + t\rangle$ with $k > 0$ and $a^\dagger y = \alpha + i\beta \quad (\alpha, \beta \in \mathbb{R})$. The following two states

$$|\phi_+\rangle \coloneqq \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx} |Rx + t\rangle, \quad |\phi_-\rangle \coloneqq \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx + c^\top x} |Rx + t\rangle$$

belongs to $\mathcal{A}'_n$, and denote the column vectors of $|\phi_+\rangle$ and $|\phi_-\rangle$ as a_+ and a_-, respectively. Then, we have $a_+^\dagger y = \alpha + \beta, a_-^\dagger y = \alpha - \beta$ from the assumption, and

$$\left|a^\dagger y\right| = \sqrt{\alpha^2 + \beta^2} \leq |\alpha| + |\beta| = \max\{|\alpha + \beta|, |\alpha - \beta|\} \leq 1,$$

which completes the proof. □

Now, we are ready to prove the Theorem 4.

Theorem 4. *Suppose that the state $|\psi\rangle$ is real. If we substitute the column set $\mathcal{A}_n$ with $\mathcal{A}'_n$ in the problem (2), the optimal solution of the restricted problem is also optimal for the original problem.*

Proof. Let x^* and y^* be the optimal solutions of the restricted primal and dual problems, namely, the problem (2) and the problem (3) with the column set $\mathcal{A}'_n$ instead of $\mathcal{A}_n$. We can assure such solutions always exists. Now, we show that the x^*, y^* are optimal not only for the restricted problems but also for the original problems.

Let OPT be the optimal value for the original problem. Since x^* can be a feasible solution for the original primal problem, it is clear that OPT $\leq \|x^*\|_1$. By the strong duality theorem, OPT is also the optimal value for the original dual problem. From the lemma 1, we can see that y^* is a feasible solution for the original dual problem and OPT $\geq \mathrm{Re}(b^\dagger y^*)$. Again, by applying the strong duality theorem to the restricted problems, we have $\|x^*\|_1 = \mathrm{Re}(b^\dagger y^*)$, which means that OPT $= \|x^*\|_1 = \mathrm{Re}(b^\dagger y^*)$. Therefore, x^* and y^* are also optimal solutions for the original problems. □