Stabilizer Extent Calculation by Column Generation

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1 Introduction

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2 Preliminaries

We denote the entire set of n-qubit stabilizer states as $\mathcal{S}_n := \{|\phi_\alpha\rangle\}$. We also define the density matrices as $\sigma_\alpha := |\phi_\alpha\rangle\langle\phi_\alpha|$. The size of this $\mathcal{S}_n$ scales superexponentially as $|\mathcal{S}_n| = 2^n \prod_{k=0}^{n-1}(2^{n-k}+1) = 2^{\mathcal{O}(n^2)}$ [1, Proposition 1]. See also Table 1 for the size of $\mathcal{S}_n$.

Table 1: The size of $\mathcal{S}_n$, the data size of A_n in sparse matrix format, and the data size of a matrix we used in the first step of Algorithm 1 for each n.

n	5	6	7	8	9	10
$\lvert\mathcal{S}_n\rvert$	2.42e+06	3.15e+08	8.13e+10	4.18e+13	4.29e+16	8.79e+19
size of A_n	2.3 KiB	25.9 KiB	367.2 KiB	6.2 MiB	128.0 MiB	3.1 GiB
proposed	2.3 KiB	25.9 KiB	367.2 KiB	6.2 MiB	128.0 MiB	3.1 GiB

The *Robustness of Magic* is introduced in [?] to quantify a n-qubit state ρ, expressed as density matrix, and defined as follows:

$$\mathcal{R}(\rho) := \min_{c\in\mathbb{R}^{|\mathcal{S}_n|}} \left\{ \|c\|_1 \;\middle|\; \rho = \sum_{\alpha=1}^{|\mathcal{S}_n|} c_\alpha \sigma_\alpha \right\}$$

The *stabilizer extent* is introduced in [2, Definition 3] to quantify a normalized n-qubit state ψ, expressed as state vector, and is defined as follows:

$$\xi(\psi) := \min_{c\in\mathbb{C}^{|\mathcal{S}_n|}} \left\{ \|c\|_1^2 \;\middle|\; |\psi\rangle = \sum_{\alpha=1}^{|\mathcal{S}_n|} c_\alpha |\phi_\alpha\rangle \right\} \tag{1}$$

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In this paper, we focus on the calculation of the stabilizer extent. This definition of the stabilizer extent can be simplified as complex L^1-norm minimization problem:

$$\sqrt{\xi(\psi)} = \min_{x \in \mathbb{C}^{|\mathcal{S}_n|}} \{\|x\|_1 \mid A_n x = b\} \tag{2}$$

Here, we define $A_n \in \mathbb{C}^{2^n \times |\mathcal{S}_n|}$ as $(A_n)_{ij} \coloneqq \langle i|\phi_j\rangle$ and $b \in \mathbb{C}^{2^n}$ as $b_i \coloneqq \langle i|\psi\rangle$ using the computational basis $\{|i\rangle\}_{i=0}^{2^n-1}$. As in [3], the problem (2) is a second order cone program (SOCP). Thus, by defining $\mathcal{A}_n$ as the columns set $\{a_j\}$ of A_n, its dual problem can be derived as [3, Appendix A][4, Section 5.1.6]

$$\sqrt{\xi(\psi)} = \max_{y \in \mathbb{C}^{2^n}} \left\{\mathrm{Re}(b^\dagger y) \;\middle|\; \left|a_j^\dagger y\right| \le 1 \text{ for all } a_j \in \mathcal{A}_n\right\} \tag{3}$$

where $\dagger$ denotes the conjugate transpose. However, the actual objective function of Lagrange dual problem of (2) is not $\mathrm{Re}(b^\dagger y)$ but $-\mathrm{Re}(b^\dagger y)$. We flipped the sign for simplicity, since this operation does not affect the optimal solution due to $\left|a_j^\dagger y\right| = \left|a_j^\dagger(-y)\right|$.

Further, in order to describe our algorithm in later sections, we denote a function $\mathrm{SolveSOCP}(\mathcal{C}, b)$ which takes a columns set $\mathcal{C} \subseteq \mathcal{A}$ and a vector b, and returns the optimal primal solution x dual optimal solution $\boldsymbol{y}$ of the SOCP problem (3). In actual numerical computation, this function can be realized by just solving the corresponding primal problem (2).

3 Scaling up The Exact Stabilizer Extent Calculation

In the preceding sections, we introduced two quantum resource measures: the Robustness of Magic and the stabilizer extent. Despite both being efficiently quantifiable through convex optimization problems, solving them directly for $n > 5$ qubit systems becomes impractical due to the superexponential growth of stabilizer states, $|\mathcal{S}_n|$. To address this challenge, we proposed employing the classical optimization technique known as column generation (CG) method for Robustness of Magic calculation [5] However, it remained unclear whether the same approach could be applied to calculate the stabilizer extent. Here, we demonstrate that leveraging the specific structure of stabilizer states enables a similar method to work effectively for calculating the stabilizer extent as well.

3.1 Core Subroutine: Calculating Overlap

Firstly, for given $b \in \mathbb{C}^{2^n}$ corresponding to the state $|\psi\rangle$, we consider the following problem:

$$\max_{a_j \in \mathcal{A}_n} \left|a_j^\dagger b\right| \tag{4}$$

This problem plays a crucial role in the CG method, thus we need to solve this problem efficiently. To this end, we introduce the following proposition. As a well-known fact, the stabilizer states have a simple form as shown in the following proposition.

Proposition 1 ([6, Theorem 2], [7, Section 5], [8, Theorem 5.(ii)]). *All stabilizer states can be written in the following form:*

$$\begin{cases} |\phi\rangle \coloneqq |t\rangle & \text{if } k = 0 \\ |\phi\rangle \coloneqq \dfrac{1}{2^{k/2}} \displaystyle\sum_{x=0}^{2^k-1} (-1)^{x^\top Q x} i^{c^\top x} |Rx + t\rangle & \text{if } k > 0 \end{cases} \tag{5}$$

where $Q \in \mathbb{F}_2^{k\times k}$, $c \in \mathbb{F}_2^k$, $R \in \mathbb{F}_2^{k\times(n-k)}$, $t \in \mathbb{F}_2^{n-k}$ *and* $\mathrm{rank}(R) = k$. *Also, any state that can be written in this form is a stabilizer state.*

By modifying the form slightly, we can obtain the following more convenient form. The proof is given in Appendix A.1.

Theorem 1. *The form* (5) *with the following conditions enumerates all the stabilizer states without any duplication or omission:*

- Q *is a upper triangular* $\mathbb{F}_2^{k\times k}$ *matrix.*
- R *is a* $\mathrm{rank}\, k$ $\mathbb{F}_2^{k\times(n-k)}$ *rref (reduced row echelon form) matrix.*
- t *belongs to the complement of the row space of* R.

Let ϕ be a one of stabilizer state in the standard form (5) with $k > 0$, which means $|\psi\rangle = \sum_{x=0}^{2^n-1}(-1)^{x^\top Qx} i^{c^\top x} |Rx + t\rangle$. Then, by denoting a as the corresponding column in A_n of ϕ, the overlap between $|\phi\rangle$ and $|\psi\rangle$ is

$$a^\dagger b = \langle\psi|\phi\rangle = \sum_{x=0}^{2^n-1}(-1)^{x^\top Qx} i^{c^\top x} \langle Rx + t|\phi\rangle = \sum_{x=0}^{2^n-1}(-1)^{x^\top Qx} i^{c^\top x} b^\dagger_{Rx+t}.$$

In the following, we define $P_x := b_x^\dagger$, and for the simplicity, we assume that $k = n, R = I_n, t = 0$. This assumption is not restrictive since the other cases can be easily reduced to this case. Now, what we want to solve is basically equivalent to the following problem:

$$\max_{c\in\mathbb{F}_2^n, Q\in\mathbb{F}_2^{n\times n}} \left\{ \left| \sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x \right| \right\}.$$

However, if we resort to a naive approach, the time complexity of this problem is $\mathcal{O}\left(2^{n+n(n+1)/2} 2^n n^2\right)$, where $2^{n+n(n+1)/2}$ is the number of the possible c, Q, 2^n is the number of term in the sum, and n^2 is the computational cost per each term. We can reduce the time complexity by using the following theorem. The detailed proof is given in Appendix A.2.

Theorem 2. *We can solve the problem* (3.1) *in* $\mathcal{O}\left(2^{n+n(n+1)/2}\right)$ *time complexity and* $\mathcal{O}(2^n)$ *space complexity.*

The basic algorithm to solve this problem is the depth-first search (DFS) algorithm. Consequently, we can solve the problem (3.1).

Theorem 3. *Complexity of computing all stabilizer overlaps Computation of* $A_n^\dagger y$ *can be done in time complexity of* $\mathcal{O}(n|S_n|)$ *and space complexity of* $\mathcal{O}(2^n)$.

3.2 CG method for the stabilizer extent calculation

Next, we introduce the CG method outlined in Algorithm 1. This iterative algorithm solves a subproblem restricted to $\mathcal{C} \subseteq \mathcal{A}_n$ per each iterations. It begins with a small subset $\mathcal{C}_0$ and progressively adds columns $\mathcal{C}'_k$ that violate the constraints of the dual problem (3), and terminate if there are no more violated columns. For further implementation details, we direct the reader to [5], There are two key aspects of this algorithm: the initialization process and the optimality of the solution. We will discuss these in subsequent sections.

Algorithm 1: Exact stabilizer extent calculation by Column Generation

Input: vector b corresponding to the state ψ

Output: Exact stabilizer extent $\xi(\psi)$

$\mathcal{C}_0 \leftarrow$ Partial set of $\mathcal{A}_n$ `/* Initialize using top overlaps */`
for $k = 0, 1, 2, \dots$ **do**
  $\boldsymbol{x}_k, \boldsymbol{y}_k \leftarrow$ `SolveSOCP`$(\mathcal{C}_k, \boldsymbol{b})$
  $\mathcal{C}'_k \leftarrow \left\{ \boldsymbol{a} \in \mathcal{A}_n \;\middle|\; \left|\boldsymbol{a}^\dagger \boldsymbol{y_k}\right| > 1 \right\}$ `/* Use of subroutine in Section 3.1 */`
  if $\mathcal{C}'_k = \emptyset$ **then**
    return $\xi(\psi) = \|\boldsymbol{x}_k\|_1$
  $\mathcal{C}_{k+1} \leftarrow \mathcal{C}_{k+1} \cup \mathcal{C}'_k$

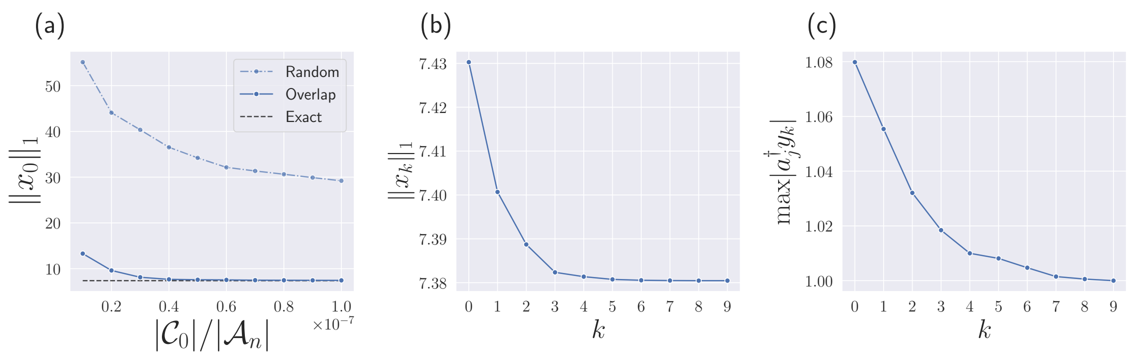

Figure 1: (a) $\|x_0\|_1$ in the Algorithm 1, which can be obtained from `SolveSOCP`$(\mathcal{C}_0, b)$, for a random 8-qubit state. The ratio $|\mathcal{C}_0|/|\mathcal{A}_n|$ varies from 10^{-8} to 10^{-7}. We can get much better results with the top overlap heuristics compared to the random selection of $\mathcal{C}_0$. (b) The convergence of the CG method for the same state. The max violation becomes 1.00 after 10 iterations, which means the optimal solution is found.

3.2.1 Initialization

In the initial step of Algorithm 1, we select a subset $\{a_j\} = \mathcal{C}_0 \subseteq \mathcal{A}_n$ in descending order of $\left|a_j^\dagger b\right|$, which can be computed efficiently as stated in Theorem 3. This heuristic can be justified by the interpretation of $\left|a_j^\dagger b\right| = |\langle\phi_j|\psi\rangle|$ as the "closeness" between the states $|\psi\rangle$ and $|\phi_j\rangle$. Hence, choosing the states based on their overlaps is a reasonable choice. The numerical experiments result in [5] also support the effectiveness of this heuristic. In the case of a random pure 8-qubit state, even if we use as small subset as $|\mathcal{C}_0| = 10^{-7}|\mathcal{A}_n|$, the $\|x_0\|_1$ obtained closely approximates the exact value and outperforms randomly selected $\mathcal{C}_0$.

3.2.2 Optimality of the solution

The terminate criterion for Algorithm 1 is the absence of columns that violate the dual constraints $\left|a_j^\dagger y_k\right| \leq 1$, which is checked by the subroutine in Section 3.1. This means the optimal solution for the dual problem (3) is found, and the primal solution x_k is also optimal thank to the strong duality of the SOCP problem. Consequently, we can affirm that Algorithm 1 is certain to find the exact stabilizer extent for any n-qubit state $|\psi\rangle$ once it terminates. The convergence of the CG method is also confirmed in the numerical experiments. For the same 8-qubit state, $\max_{a_j \in \mathcal{A}} \left| a_j^\dagger y_k \right|$ reaches 1.00 after 10 iterations, indicating the discovery of the optimal solution.

3.3 For the case $|\psi\rangle$ is Real

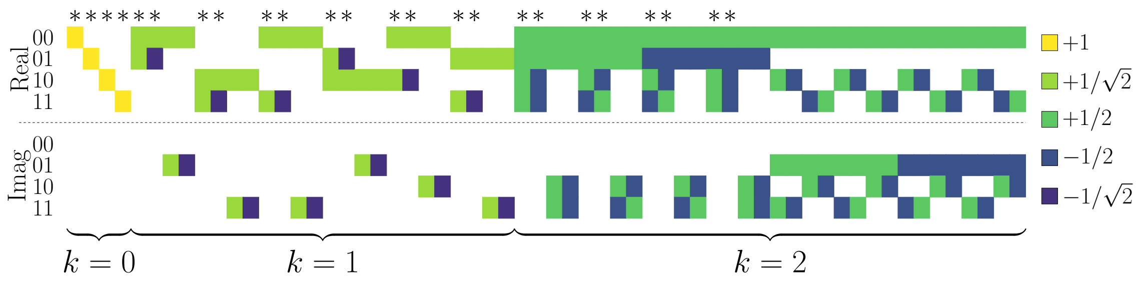

Figure 2: Visualization of the matrix A_n with $n = 2$. The upper half corresponds to the real part, and the lower half corresponds to the imaginary part. The j-th column of this represents the column a_j and its state $|\phi_j\rangle$. The k below the matrix corresponds to the standard form (5). By restricting the matrix A_n to the starred columns which are real vectors, we can obtain the matrix A'_n.

In some cases, the state $|\psi\rangle$ could be real. For example, ... We show that in such cases the problem can be further simplified. We define the subset of the states in $\mathcal{S}_n$ with real coefficients as $\mathcal{T}_n$, and the corresponding subset of the columns in $\mathcal{A}_n$ as $\mathcal{A}'_n$. Then, the next theorem holds.

Theorem 4. *Suppose that the state $|\psi\rangle$ is real. If we substitute the column set $\mathcal{A}_n$ with $\mathcal{A}'_n$ in the problem (2), the optimal solution of the restricted problem is also optimal for the original problem.*

Thanks to theorem 4, we can reduce the size of the column set size by a factor of 2^n.

4 Approx Solutions

5 Discussion

In this paper, we have shown that todo.

There is still room for improvement in some specific cases. As for Robustness of Magic, there is a marvelous algorithm proposed in [**?**] which focuses on copies of symmetric pure magic states, and we enhanced this result in [5]. Applying such techniques to the stabilizer extent calculation would be promising and is left for future work.

Furthermore, there is some more future direction. For example, todo.

Acknowledgements.—

We would like to thank N. Marumo for valuable comments on the manuscript. N.Y. wishes to thank JST PRESTO No. JPMJPR2119 and the support from IBM Quantum. This work was supported by JST Grant Number JPMJPF2221. This work was supported by JST ERATO Grant Number JPMJER2302 and JST CREST Grant Number JPMJCR23I4, Japan.

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A Fast Algorithm for Overlap

In this section, we will explain the algorithm in detail and introduce some heuristics to improve the efficiency.

A.1 Efficient Enumeration of Stabilizer States

Proof. Main Ideas come from [6]. Firstly, we show that the mapping $\{(Q, c, R, t)\} \to \mathcal{S}_n$ is injective. We can say that

$$\begin{aligned}
&\left\{R_1 x + t_1 \,\middle|\, x \in \mathbb{F}_2^{n-k}\right\} = \left\{R_2 x + t_2 \,\middle|\, x \in \mathbb{F}_2^{n-k}\right\} \\
\iff &\operatorname{Im}(R_1) = \operatorname{Im}(R_2) \wedge (t_2 - t_1) \in \operatorname{Im}(R_1) \\
\iff &R_1 = R_2 \wedge t_1 = t_2.
\end{aligned}$$

The last equivalence is due to the property of the rref matrix and the complement condition. Since Q is a upper triangular matrix, we can uniquely determine Q and c for given state $|\phi\rangle$. Thus, if two states are the same, then the corresponding (Q, c, R, t) are also the same, which means that the mapping is injective.

Next, we show that the mapping is surjective. Since the mapping is injective, we only have to show that the cardinality of the domain is equal to that of the codomain, i.e., $|\mathcal{S}_n|$. It is known that the number of rank k $\mathbb{F}_2^{k\times(n-k)}$ rref matrices is $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_2$, which is a q-binomial coefficient with $q = 2$. Therefore, the number of Q, c, R, t is $2^{k(k+1)/2}, 2^k, \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_2, 2^{n-k}$, respectively, and the total number of states is

$$2^n + \sum_{k=1}^{n} 2^{k(k+1)/2} 2^k \begin{bmatrix} n \\ k \end{bmatrix}_2 2^{n-k} = 2^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_2 2^{k(k+1)/2} = 2^n \prod_{k=1}^{n} (2^k + 1) = |\mathcal{S}_n|.$$

In the second last equation, we used the q-binomial theorem. Therefore, the mapping is surjective, which concludes the proof. □

In theorem **??**, we used $\mathbb{F}_2$. However, in the viewpoint of the dfs algorithm, it is more practical to use $\{0,1\} \subset \mathbb{Z}$ and allow the term $c^\top x$ to be any integer. Therefore, the following corollary is useful.

Corollary 1. *In theorem* **??**, *We can substitute* $\mathbb{F}_2$ *with* $\{0,1\} \subset \mathbb{Z}$.

Proof. By changing $\mathbb{F}_2$ to $\{0,1\} \subset \mathbb{Z}$, the term $(-1)^{x^\top Qx}$ is invariant, and the term $i^{c^\top x}$ is multiplied by -1 iff $p \equiv 2,3 \pmod 4$, where p is the number of i such that $c_i = 1$ and $x_i = 1$. Now, we consider the following form:

$$\begin{cases} |\phi\rangle := |t\rangle & \text{if } k = 0, \\ |\phi\rangle := \dfrac{1}{2^{k/2}} \displaystyle\sum_{x=0}^{2^k-1} (-1)^{x^\top (Q+Q')x} i^{c^\top x} |Rx+t\rangle & \text{if } k > 0, \end{cases} \tag{6}$$

where $Q \in \{0,1\}^{k\times k}, c \in \{0,1\}^k, R \in \{0,1\}^{k\times(n-k)}, t \in \{0,1\}^{n-k}, \operatorname{rank}(R) = k$ and $Q'_{ij} = 1$ iff $(i<j) \wedge (c_i = c_j = 1)$. Now, if the pair (Q,c,R,t) in (6) is the same as that of the original form (5), then the two states are representing the exactly same state since

$$(-1)^{x^\top Q'x} = (-1)^{\binom{p}{2}} = \begin{cases} 1 & \text{if } p \equiv 0,1 \pmod 4, \\ -1 & \text{if } p \equiv 2,3 \pmod 4. \end{cases}$$

Therefore, by identifying the $Q+Q'$ in $\mathbb{Z}$ with new Q'' in $\mathbb{F}_2$, we can conclude the proof. □

A.2 Calculating the Overlap

Proof. Define $x := \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix}$, $c := \begin{bmatrix} c_0 \\ \overline{c} \end{bmatrix}$, and $Q := \begin{bmatrix} Q_{00} & Q_0^\top \\ 0 & \overline{Q} \end{bmatrix}$ (x_0, c_0 and Q_{00} are all in $\{0,1\}$). Since $x^\top Qx = x_0(Q_{00} + Q_0^\top \overline{x}) + \overline{x}^\top \overline{Q}\overline{x}$ and $c^\top x = c_0 x_0 + \overline{c}^\top \overline{x}$, we can rewrite the state as

$$\begin{aligned} |\phi\rangle &= \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx} i^{c^\top x} |x\rangle \\ &= \sum_{\overline{x}=0}^{2^{k-1}-1} (-1)^{\overline{x}^\top \overline{Q}\overline{x}} i^{\overline{c}^\top \overline{x}} \Big(|2\overline{x}\rangle + (-1)^{Q_{00}+Q_0^\top \overline{x}} i^{c_0} |2\overline{x}+1\rangle \Big) \\ &= \sum_{\overline{x}=0}^{2^{k-1}-1} (-1)^{\overline{x}^\top \overline{Q}\overline{x}} i^{\overline{c}^\top \overline{x}} |\overline{x}'\rangle \end{aligned}$$

by defining $|\overline{x}'\rangle := |2\overline{x}\rangle + (-1)^{Q_{00}+Q_0^\top \overline{x}} i^{c_0} |2\overline{x}+1\rangle$.

Thus, we can compute the overlap recursively with very small computational cost per each step. This leads to the efficient calculation of the overlaps, which concludes the proof. □

A.3 Branch Cut For The DFS

In the previous section, we explained the efficient algorithm for the overlap calculation. However, this algorithm can be much more faster by using the branch cut heuristics we will introduce in this section.

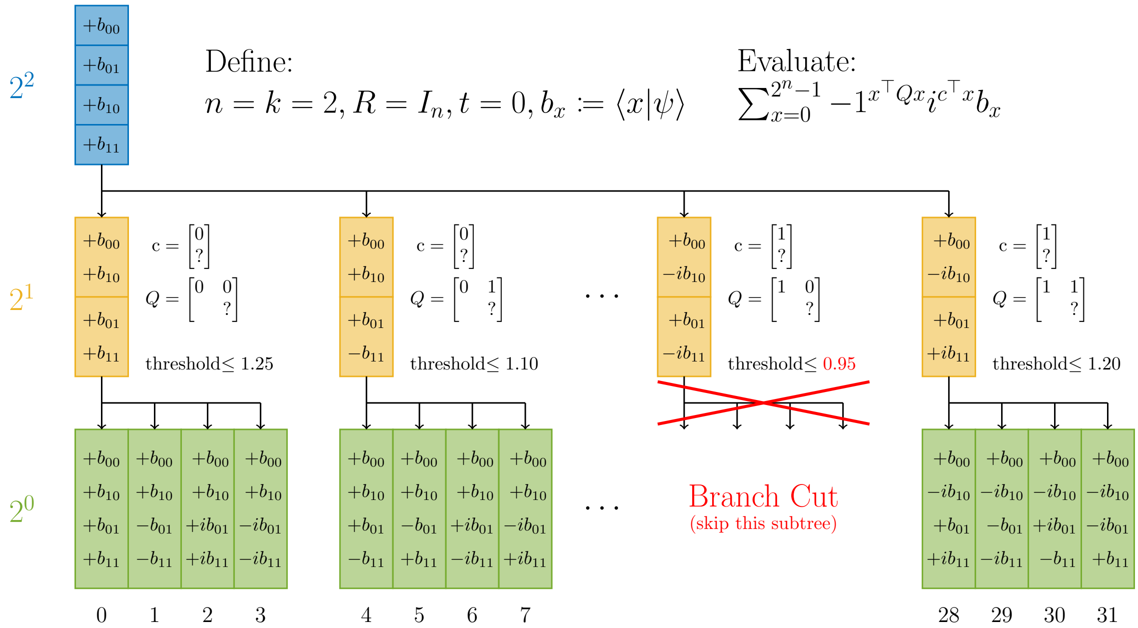

Figure 3: The visualization of the DFS algorithm. The DFS algorithm is a recursive algorithm that calculates the overlap of the stabilizer states by recursively calculating the overlap of the substates. The algorithm is efficient since the overlap of the substates can be calculated with very small computational cost.

Firstly, please recall that we are maximizing the following:

$$\max_{c,Q}\left\{\left|\sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x\right|\right\}$$

This value can easily evaluate as the following:

$$\max_{c,Q}\left\{\left|\sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x\right|\right\} \leq \max_{c,Q}\left\{\sum_{x=0}^{2^n-1} \left|-1^{x^\top Qx} i^{c^\top x} P_x\right|\right\} = \sum_{x=0}^{2^n-1} |P_x|$$

However, since each coefficient takes only $1, -1, i$ or $-i$, we can obtain more tight bound by

$$\max_{c,Q}\left\{\left|\sum_{x=0}^{2^n-1} -1^{x^\top Qx} i^{c^\top x} P_x\right|\right\} \leq \max_{c,Q}\left\{\left|\sum_{x=0}^{2^n-1} i^{c_x} P_x\right|\right\} \tag{7}$$

where $c_x \in \{0, 1, 2, 3\}$ is the independent variable for each x. Let $\mathcal{P} \coloneqq \sum_{x=0}^{2^n-1} i^{c_x^*} P_x$ be the one of optimal solutions for the problem (7). Then, without loss of generality, we can assume that $\frac{\pi}{2} \leq \arg \mathcal{P} < \frac{3\pi}{2}$, and by sorting and multiplying $i, -1$ or $-i$ to P_x appropriately, we can also assume that

$$0 \leq \arg(P_0) \leq \arg(P_1) \leq \cdots \leq \arg(P_{2^n-1}) < \pi/2. \tag{8}$$

If all c_x satisfies $\arg(\mathcal{P}) - \pi/4 \leq \arg(i^{c_x} P_x) < \arg(\mathcal{P}) + \pi/4$, then c_x is optimal for $\mathcal{P}$. Therefore, we can justify the following Algorithm 2 by moving $\arg(\mathcal{P})$ in the range of $[\pi/2, 3\pi/2)$. Also refer to the figure 4 for the visualization of this algorithm. The time complexity of this algorithm is $O(n2^n)$ due to the sorting of 2^n elements.

Algorithm 2: Branch Cut Algorithm

Input: Coefficients P_x for $x = 0, 1, \ldots, 2^n - 1$

Output: The answer for the problem (7)

Sort and modify the coefficients P_x so that the condition (8) is satisfied

ans $\leftarrow 0, \quad c_x \leftarrow 0 \quad$ for all x

for $x \leftarrow 0$ **to** $2^n - 1$ **do**

  ans $\leftarrow \max\left(\text{ans}, \left|\sum_{x=0}^{2^n-1} i^{c_x} P_x\right|\right)$

  $c_x \leftarrow c_x + 1$

return ans

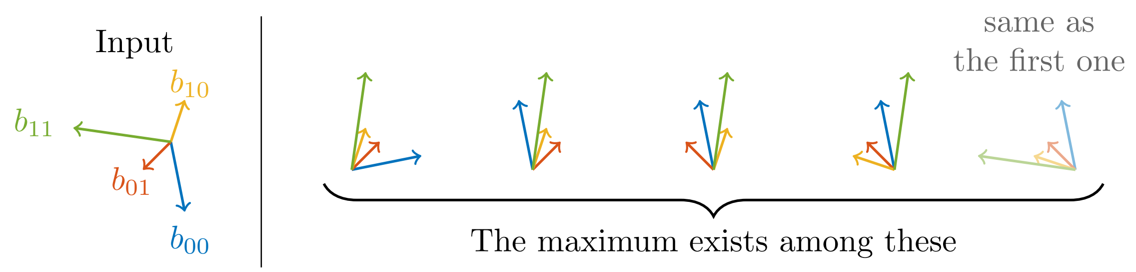

Figure 4: The visualization of Algorithm 2. Suppose that $n = 2$ and P_x are represented as the vectors in the complex plane (for example, $P_{00} = 1 - 5i$) by the left figure. Then, by sorting and iterating the loop in Algorithm 2, we can obtain 2^n patterns of the coefficients c_x as the right figure. The maximum of for the problem (7) exists among these 2^n patterns.

B The proof of Theorem 4

In this section, we prove Theorem 4. The proof is based on the following lemma.

Lemma 1. *Suppose y is a real vector and satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}'_n$ such that a is a real vector. Then, y satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}_n$.*

Proof. Fix $a \in \mathcal{A}$ and suppose that a represents the state $|\phi\rangle$. Now, consider to write $|\phi\rangle$ as in the form (5). The case $k = 0$ is trivial since then $a \in \mathcal{A}'_n$. Suppose that $|\phi_i\rangle$ can be written as $\frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Q x} i^{c^\top x} |Rx + t\rangle$ with $k > 0$ and $a^\dagger y = \alpha + i\beta \quad (\alpha, \beta \in \mathbb{R})$. The following two states

$$|\phi_+\rangle := \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Q x} |Rx + t\rangle, \quad |\phi_-\rangle := \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Q x + c^\top x} |Rx + t\rangle$$

belongs to $\mathcal{A}'_n$, and denote the column vectors of $|\phi_+\rangle$ and $|\phi_-\rangle$ as a_+ and a_-, respectively. Then, we have $a_+^\dagger y = \alpha + \beta, a_-^\dagger y = \alpha - \beta$ from the assumption, and

$$\left|a^\dagger y\right| = \sqrt{\alpha^2 + \beta^2} \leq |\alpha| + |\beta| = \max\{|\alpha + \beta|, |\alpha - \beta|\} \leq 1,$$

which completes the proof. □

Now, we are ready to prove Theorem 4.

Theorem 4. *Suppose that the state $|\psi\rangle$ is real. If we substitute the column set $\mathcal{A}_n$ with $\mathcal{A}'_n$ in the problem (2), the optimal solution of the restricted problem is also optimal for the original problem.*

Proof. Let x^* and y^* be the optimal solutions of the restricted primal and dual problems, namely, the problem (2) and the problem (3) with the column set $\mathcal{A}'_n$ instead of $\mathcal{A}_n$. We can assure such solutions always exists. Now, we show that the x^*, y^* are optimal not only for the restricted problems but also for the original problems.

Let OPT be the optimal value for the original problem. Since x^* can be a feasible solution for the original primal problem, it is clear that $\mathrm{OPT} \leq \|x^*\|_1$. By the strong duality theorem, OPT is also the optimal value for the original dual problem. From the lemma 1, we can see that y^* is a feasible solution for the original dual problem and $\mathrm{OPT} \geq \mathrm{Re}\left\{b^\dagger y^*\right\}$. Again, by applying the strong duality theorem to the restricted problems, we have $\|x^*\|_1 = \mathrm{Re}\left\{b^\dagger y^*\right\}$, which means that $\mathrm{OPT} = \|x^*\|_1 = \mathrm{Re}\left\{b^\dagger y^*\right\}$. Therefore, x^* and y^* are also optimal solutions for the original problems. $\square$