Theorem 1. *Suppose that the state $|\psi\rangle$ is real. If we substitute the matrix A_n with the matrix A'_n in the problem (**??**), the optimal solution of the restricted problem is also optimal for the original problem.*

1 The proof of Theorem 1

In this section, we prove Theorem 1. As we have stated in section **??**, by substituting the matrix A_n with the matrix A'_n, we consider the restricted problems of primal problem (**??**) and dual problem (**??**). Let x^* and y^* be the optimal solutions of the restricted primal and dual problems, respectively. We can assure such solutions always exists. Now, we show that the x^*, y^* are optimal not only for the restricted problems but also for the original problems.

Lemma 1. *Suppose y is a real vector and satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}_n$ such that a is a real vector. Then, y satisfies $\left|a^\dagger y\right| \leq 1$ for all $a \in \mathcal{A}_n$.*

Proof. We check the all states $|\phi_i\rangle \in \mathcal{S}_n$ respectively. It is trivial for the case $k = 0$ since the corresponding columns both exist in $\mathcal{S}_n$ and $\mathcal{T}_n$. We set $|\phi_i\rangle = \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Q x} i^{c^\top x} |Rx + t\rangle$ with $k > 0$ and $\langle \phi_i | y \rangle = \alpha + i\beta (\alpha, \beta \in \mathbb{R})$. The following two states

$$|\phi_+\rangle \coloneqq \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Q x} |Rx + t\rangle, \quad |\phi_-\rangle \coloneqq \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Q x + c^\top x} |Rx + t\rangle$$

are in $\mathcal{T}_n$, and satisfy $\langle \phi_+ | y \rangle = \alpha + \beta, \langle \phi_- | y \rangle = \alpha - \beta$. From the assumption, we have

$$|\langle \phi_i | y \rangle| = \sqrt{\alpha^2 + \beta^2} \leq |\alpha| + |\beta| = \max\{|\alpha + \beta|, |\alpha - \beta|\} \leq 1,$$

which completes the proof. □

Theorem 2. *The optimal solutions for the restricted problems x^* and y^* are also optimal for the original problems.*

Proof. Let OPT be the optimal value for the original primal problem. Since x^* can be a feasible solution for the original primal problem, it is clear that $\mathrm{OPT} \leq \|x^*\|_1$. By the strong duality theorem, OPT is also the optimal value for the original dual problem. From the lemma 1, we can see that y^* is a feasible solution for the original dual problem and $\mathrm{OPT} \geq \langle \psi | y^* \rangle$. Again, by applying the strong duality theorem to the restricted problems, we have $\|x^*\|_1 = \langle \psi | y^* \rangle$, which means that $\mathrm{OPT} = \|x^*\|_1 = \langle \psi | y^* \rangle$. Therefore, x^* and y^* are also optimal for the original problems. □