1 Derivation of the Dual Problem by CVXPY formulation

In the cvxpy official userguide, the following problem is considered. We change the index from k to i for consistency with our notation.

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & G_i \boldsymbol{x} + \boldsymbol{s}_i = \boldsymbol{h}_i, \qquad i = 0, \dots, M, \\
& A\boldsymbol{x} = \boldsymbol{b}, \\
& \boldsymbol{s}_0 \succeq \boldsymbol{0}, \\
& s_{i0} \geq \|\boldsymbol{s}_{i1}\|_2, \quad i = 1, \dots, M
\end{aligned}
$$

$$
\begin{aligned}
\underset{\boldsymbol{y}}{\text{maximize}} \quad & -\sum_{i=0}^{M} \boldsymbol{h}_i^\top \boldsymbol{z}_i - \boldsymbol{b}^\top \boldsymbol{y} \\
\text{subject to} \quad & \sum_{i=0}^{M} G_i^\top \boldsymbol{z}_i + A^\top \boldsymbol{y} + \boldsymbol{c} = \boldsymbol{0}, \\
& \boldsymbol{z}_0 \succeq \boldsymbol{0}, \\
& \boldsymbol{z}_{i0} \geq \|\boldsymbol{z}_{i1}\|_2, \quad i = 1, \dots, M
\end{aligned}
$$

where $\succeq$ denotes element-wise inequality, and

$$
\begin{aligned}
\boldsymbol{s}_i &= (s_{i0}, \boldsymbol{s}_{i1}) \in \mathbb{R} \times \mathbb{R}^{r_i - 1}, \\
\boldsymbol{z}_i &= (z_{i0}, \boldsymbol{z}_{i1}) \in \mathbb{R} \times \mathbb{R}^{r_i - 1}.
\end{aligned}
$$

We adapt this problem to our problem, minimizing the complex L^1 norm.

$$
\begin{aligned}
\underset{\boldsymbol{x} \in \mathbb{C}^{|\mathcal{S}_n|}}{\text{minimize}} \quad & \|\boldsymbol{x}\|_1 \\
\text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b}
\end{aligned}
$$

Let e_i be a vector with the i-th component 1 and the others 0. Then, the problem can be written as follows.

$$
\underset{\boldsymbol{t} \in \mathbb{R}^{|\mathcal{S}_n|}, \boldsymbol{x} \in \mathbb{C}^{|\mathcal{S}_n|}}{\text{minimize}} \quad \begin{bmatrix} \boldsymbol{1}^\top & \boldsymbol{0}^\top & \boldsymbol{0}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix}
$$

$$
\text{subject to} \quad -\begin{bmatrix} e_i^\top & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & e_i^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & \mathbf{0}^\top & e_i^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix} + \boldsymbol{s}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad i = 1, \ldots, |\mathcal{S}_n|,
$$

$$
\begin{bmatrix} 0_{2^n \times |\mathcal{S}_n|} & \mathrm{Re}(A) & -\mathrm{Im}(A) \\ 0_{2^n \times |\mathcal{S}_n|} & \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(\boldsymbol{b}) \\ \mathrm{Im}(\boldsymbol{b}) \end{bmatrix},
$$

$$
s_{i0} \geq \|\boldsymbol{s}_{i1}\|_2, \quad i = 1, \ldots, |\mathcal{S}_n|
$$

$$
\begin{aligned}
&\underset{\boldsymbol{y} \in \mathbb{C}^{2^n}}{\text{maximize}} \quad -\begin{bmatrix} \mathrm{Re}(\boldsymbol{b})^\top & \mathrm{Im}(\boldsymbol{b})^\top \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\boldsymbol{y}) \\ \mathrm{Im}(\boldsymbol{y}) \end{bmatrix} \\
&\text{subject to} \quad -\sum_{i=1}^{M} \begin{bmatrix} e_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & e_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & e_i \end{bmatrix} \begin{bmatrix} z_{i0} \\ \mathrm{Re}(z_{i1}) \\ \mathrm{Im}(z_{i1}) \end{bmatrix} + \begin{bmatrix} 0_{|\mathcal{S}_n| \times 2^n} & 0_{|\mathcal{S}_n| \times 2^n} \\ \mathrm{Re}(A)^\top & \mathrm{Im}(A)^\top \\ -\mathrm{Im}(A)^\top & \mathrm{Re}(A)^\top \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\boldsymbol{y}) \\ \mathrm{Im}(\boldsymbol{y}) \end{bmatrix} + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \\
&\qquad z_{i0} \geq |z_{i1}|, \quad i = 1, \ldots, |\mathcal{S}_n|
\end{aligned}
$$

Furthermore, let the i-th column of the matrix A be $\boldsymbol{a}_i$. Then, the problem can be written as follows.

$$
\begin{aligned}
\underset{\boldsymbol{y} \in \mathbb{C}^{2^n}}{\text{maximize}} \quad & -\mathrm{Re}(\boldsymbol{b}^\dagger \boldsymbol{y}) \\
\text{subject to} \quad & z_{i0} = 1, \quad i = 1, \ldots, |\mathcal{S}_n|, \\
& z_{i1} = \boldsymbol{a}_i^\dagger \boldsymbol{y}, \quad i = 1, \ldots, |\mathcal{S}_n|, \\
& z_{i0} \geq |z_{i1}|, \quad i = 1, \ldots, |\mathcal{S}_n|
\end{aligned}
$$

$$
\begin{aligned}
\underset{\boldsymbol{y} \in \mathbb{C}^{2^n}}{\text{maximize}} \quad & -\mathrm{Re}(\boldsymbol{b}^\dagger \boldsymbol{y}) \\
\text{subject to} \quad & \left| \boldsymbol{a}_i^\dagger \boldsymbol{y} \right| \leq 1, \quad i = 1, \ldots, |\mathcal{S}_n|
\end{aligned}
$$

Even if we flip the sign of the objective function, the optimal solution does not change thanks to the absolute value in the constraints. Thus, the dual problem of the original problem is

$$
\begin{aligned}
\underset{\boldsymbol{y} \in \mathbb{C}^{2^n}}{\text{maximize}} \quad & \mathrm{Re}(\boldsymbol{b}^\dagger \boldsymbol{y}) \\
\text{subject to} \quad & \left| \boldsymbol{a}_i^\dagger \boldsymbol{y} \right| \leq 1, \quad i = 1, \ldots, |\mathcal{S}_n|.
\end{aligned} \tag{1}
$$

2 Formulation by Mosek

In the mosek official tutorial, the following problem is considered with $\mathcal{D}$ is a cone.

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{c}^f \\ \text{subject to} \quad & l^c \leq A\boldsymbol{x} \leq u^c, \\ & l^x \leq \quad \boldsymbol{x} \leq u^x, \\ & F\boldsymbol{x} + g \in \mathcal{D} \end{aligned}$$

For our problem, the problem can be written as follows.

$$\begin{aligned} \underset{\boldsymbol{t} \in \mathbb{R}^{|\mathcal{S}_n|}, \boldsymbol{x} \in \mathbb{C}^{|\mathcal{S}_n|}}{\text{minimize}} \quad & \begin{bmatrix} \boldsymbol{1}^\top & \boldsymbol{0}^\top & \boldsymbol{0}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix} + 0 \\ \text{subject to} \quad & \begin{bmatrix} \mathrm{Re}(\boldsymbol{b}) \\ \mathrm{Im}(\boldsymbol{b}) \end{bmatrix} \leq \begin{bmatrix} 0_{2^n \times |\mathcal{S}_n|} & \mathrm{Re}(A) & -\mathrm{Im}(A) \\ 0_{2^n \times |\mathcal{S}_n|} & \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix} \leq \begin{bmatrix} \mathrm{Re}(\boldsymbol{b}) \\ \mathrm{Im}(\boldsymbol{b}) \end{bmatrix}, \\ & -\infty \leq \begin{bmatrix} \boldsymbol{t} \\ \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix} \leq \infty, \\ & \begin{bmatrix} I_{3|\mathcal{S}_n|} \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix} + \boldsymbol{0} \in \left(\mathcal{Q}^3\right)^{|\mathcal{S}_n|} \quad \text{(with reordering)} \end{aligned}$$

3 Derivation of the Dual Problem by Lagrangian

We can derive the dual problem more directly using the Lagrangian. The original problem was formulated as follows.

$$\begin{aligned} \underset{\boldsymbol{x} \in \mathbb{C}^{|\mathcal{S}_n|}, \boldsymbol{t} \in \mathbb{R}^{|\mathcal{S}_n|}}{\text{minimize}} \quad & \sum_{i=1}^{|\mathcal{S}_n|} t_i \\ \text{subject to} \quad & \begin{bmatrix} \mathrm{Re}(A) & -\mathrm{Im}(A) \\ \mathrm{Im}(A) & \mathrm{Re}(A) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\boldsymbol{x}) \\ \mathrm{Im}(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(\boldsymbol{b}) \\ \mathrm{Im}(\boldsymbol{b}) \end{bmatrix}, \\ & |x_i| \leq t_i, \quad i = 1, \ldots, |\mathcal{S}_n| \end{aligned} \tag{2}$$

The Lagrangian of the problem (2) can be defined by

$$
\begin{aligned}
L(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}, \boldsymbol{s}) = & \sum_{i=1}^{|\mathcal{S}_n|} t_i \\
& + \operatorname{Re}(\boldsymbol{y}^\top)(\operatorname{Re}(A)\operatorname{Re}(\boldsymbol{x}) - \operatorname{Im}(A)\operatorname{Im}(\boldsymbol{x}) - \operatorname{Re}(\boldsymbol{b})) \\
& + \operatorname{Im}(\boldsymbol{y}^\top)(\operatorname{Im}(A)\operatorname{Re}(\boldsymbol{x}) + \operatorname{Re}(A)\operatorname{Im}(\boldsymbol{x}) - \operatorname{Im}(\boldsymbol{b})) \\
& + \sum_{i=1}^{|\mathcal{S}_n|} s_i(|x_i| - t_i) \\
= & - \operatorname{Re}(\boldsymbol{y}^\dagger \boldsymbol{b}) \\
& + \sum_{i=1}^{|\mathcal{S}_n|}(1 - s_i)t_i + \sum_{i=1}^{|\mathcal{S}_n|}\left(s_i|x_i| + \begin{bmatrix} \operatorname{Re}(\boldsymbol{a}_i^\dagger \boldsymbol{y}) \\ \operatorname{Im}(\boldsymbol{a}_i^\dagger \boldsymbol{y}) \end{bmatrix}^\top \begin{bmatrix} \operatorname{Re}(\boldsymbol{x}) \\ \operatorname{Im}(\boldsymbol{x}) \end{bmatrix} \right).
\end{aligned}
$$

By using this relation, we can obtain the Lagrange dual function as follows.

$$
\begin{aligned}
g(\boldsymbol{y}, \boldsymbol{s}) = & \inf_{\boldsymbol{x}, \boldsymbol{t}} L(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}, \boldsymbol{s}) \\
= & - \operatorname{Re}(\boldsymbol{b}^\dagger \boldsymbol{y}) \\
& + \begin{cases} 0 & \text{if } s_i \neq 1 \\ -\infty & \text{otherwise} \end{cases} \\
& + \begin{cases} 0 & \text{if } \left|\boldsymbol{a}_i^\dagger \boldsymbol{y}\right| \leq 1 \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Thus, the dual problem of the original problem is

$$
\begin{array}{ll}
\underset{\boldsymbol{y} \in \mathbb{C}^{2^n}}{\text{maximize}} & - \operatorname{Re}(\boldsymbol{b}^\dagger \boldsymbol{y}) \\
\text{subject to} & \left|\boldsymbol{a}_i^\dagger \boldsymbol{y}\right| \leq 1, \quad i = 1, \ldots, |\mathcal{S}_n|,
\end{array}
$$

and even if we flip the sign of the objective function, the optimal solution does not change thanks to the absolute value in the constraints. Thus, the dual problem of the original problem is

$$
\begin{array}{ll}
\underset{\boldsymbol{y} \in \mathbb{C}^{2^n}}{\text{maximize}} & \operatorname{Re}(\boldsymbol{b}^\dagger \boldsymbol{y}) \\
\text{subject to} & \left|\boldsymbol{a}_i^\dagger \boldsymbol{y}\right| \leq 1, \quad i = 1, \ldots, |\mathcal{S}_n|,
\end{array}
\tag{3}
$$

which is consistent with the dual problem (1).