1 For the Case $|\psi\rangle$ is Real

Assume that the state $|\psi\rangle$ is real. Now, we consider the following optimization problem:

$$\begin{aligned} \underset{x \in \mathbb{C}^{|\mathcal{S}_n|}}{\text{minimize}} \quad & \|x\|_1 \\ \text{subject to} \quad & \sum_{\phi_i \in \mathcal{S}_n} x_i |\phi_i\rangle = |\psi\rangle \end{aligned}$$

$$\begin{aligned} \underset{|y\rangle \in \mathbb{C}^{2^n}}{\text{maximize}} \quad & \mathrm{Re}(\langle\psi|y\rangle) \\ \text{subject to} \quad & |\langle\phi_i|y\rangle| \leq 1, \quad \forall \phi_i \in \mathcal{S}_n \end{aligned}$$

Let $\mathcal{T}_n$ be the collection of the states in $\mathcal{S}_n$ with real coefficients. This means $\mathcal{T}_n$ is the collection of the states with $\boldsymbol{c} = 0$ and $|\mathcal{T}_n| = |\mathcal{S}_n|/(2^n)$. Now, we consider the following restricted optimization problem:

$$\begin{aligned} \underset{x \in \mathbb{R}^{|\mathcal{T}_n|}}{\text{minimize}} \quad & \|x\|_1 \\ \text{subject to} \quad & \sum_{\phi_j \in \mathcal{T}_n} x_j |\phi_j\rangle = |\psi\rangle \end{aligned}$$

$$\begin{aligned} \underset{|y\rangle \in \mathbb{R}^{2^n}}{\text{maximize}} \quad & \langle\psi|y\rangle \\ \text{subject to} \quad & |\langle\phi_j|y\rangle| \leq 1, \quad \forall \phi_j \in \mathcal{T}_n \end{aligned}$$

Let x^* and y^* be the optimal solutions of the restricted primal and dual problems, respectively. These vectors always exist since $\mathcal{T}_n$ forms a over complete basis. We now show that the x^*, y^* are optimal not only for the restricted problems but also for the original problems.

Lemma 1. *Suppose $|y\rangle$ is real and satisfies $|\langle\phi_j|y\rangle| \leq 1$ for all $\phi_j \in \mathcal{T}_n$. Then, $|y\rangle$ satisfies $|\langle\phi_i|y\rangle| \leq 1$ for all $\phi_i \in \mathcal{S}_n$.*

証明 We check the all states $|\phi_i\rangle \in \mathcal{S}_n$ respectively. It is trivial for the case $k = 0$ since the corresponding columns both exist in $\mathcal{S}_n$ and $\mathcal{T}_n$. We set $|\phi_i\rangle = \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx} i^{c^\top x} |Rx + t\rangle$ with $k > 0$ and $\langle\phi_i|y\rangle = \alpha + i\beta (\alpha, \beta \in \mathbb{R})$. The following two states

$$|\phi_+\rangle := \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx} |Rx + t\rangle, \quad |\phi_-\rangle := \frac{1}{2^{k/2}} \sum_{x=0}^{2^k-1} (-1)^{x^\top Qx + c^\top x} |Rx + t\rangle$$

are in $\mathcal{T}_n$, and satisfy $\langle\phi_+|y\rangle = \alpha + \beta, \langle\phi_-|y\rangle = \alpha - \beta$. From the assumption, we have

$$|\langle\phi_i|y\rangle| = \sqrt{\alpha^2 + \beta^2} \leq |\alpha| + |\beta| = \max\{|\alpha + \beta|, |\alpha - \beta|\} \leq 1,$$

which completes the proof. $\square$

Theorem 1 The optimal solutions for the restricted problems x^* and y^* are also optimal for the original problems.

証明 Let OPT be the optimal value for the original primal problem. Since x^* can be a feasible solution for the original primal problem, it is clear that OPT $\leq \|x^*\|_1$. By the strong duality theorem, OPT is also the optimal value for the original dual problem. From the lemma 1, we can see that y^* is a feasible solution for the original dual problem and OPT $\geq \langle\psi|y^*\rangle$. Again, by applying the strong duality theorem to the restricted problems, we have $\|x^*\|_1 = \langle\psi|y^*\rangle$, which means that OPT $= \|x^*\|_1 = \langle\psi|y^*\rangle$. Therefore, x^* and y^* are also optimal for the original problems. $\square$