

Stokes Theorem as Generalized Abstract Vapid Nonsense

“We’ll only use as much category theory as is necessary.
[*famous last words*]” –Roman Abramovich¹

Ananth Venkatesh

Massachusetts Institute of Technology (MIT)

11 December 2024

¹Attribution: (Sämann 2017) (quote is almost certainly misattributed to a Russian oligarch instead of the algebraic geometer who is likely responsible for it)

Outline

1. Stokes Theorem

2. Some Examples

3. Applications?

4. References

1.1 Manifolds

Geometry is the art of correct reasoning from incorrectly drawn figures.

— Henri Poincaré

1.1 Manifolds

Geometry is the art of correct reasoning from incorrectly drawn figures.

— Henri Poincaré

Manifolds are (very special) sets (of points)

1.1 Manifolds

Geometry is the art of correct reasoning from incorrectly drawn figures.

— Henri Poincaré

Manifolds are (very special) sets (of points)

- Manifolds are just a continuous set of points

1.1 Manifolds

Geometry is the art of correct reasoning from incorrectly drawn figures.

— Henri Poincaré

Manifolds are (very special) sets (of points)

- Manifolds are just a continuous set of points
- Moreover, at any given point, they look like a **Euclidean space**

1.2 Euclidean Spaces

- You know them as \mathbb{R}^n i.e. **vector spaces** (roughly speaking)

1.2 Euclidean Spaces

- You know them as \mathbb{R}^n i.e. **vector spaces** (roughly speaking)
- Euclidean space is the space we inhabit (not the full story; more on this later)

1.2 Euclidean Spaces

- You know them as \mathbb{R}^n i.e. **vector spaces** (roughly speaking)
- Euclidean space is the space we inhabit (not the full story; more on this later)

What actually is a manifold then?

1.2 Euclidean Spaces

- You know them as \mathbb{R}^n i.e. **vector spaces** (roughly speaking)
- Euclidean space is the space we inhabit (not the full story; more on this later)

What actually is a manifold then?

- When we say that a manifold “looks like” a Euclidean space, we’re actually talking about **homeomorphism**

1.3 Homeomorphisms

- >Topological manifold
- >Look inside
- >Euclidean space



1.4 General Topology



- Topologists do not understand shapes, so they must deform objects into meaningless blobs to count holes
- Two numbers are equal, but two topological spaces are **homeomorphic** to each other

1.4 General Topology

1.4.1 Definitions

Homeomorphism

A pair of continuous bijective (all $x \in X$ in the domain of f and all $y \in Y$ in the range of f) mappings $f : X \longrightarrow Y$ and $f^{-1} : Y \longrightarrow X$

Homeomorphic equivalence

Two topological objects X and Y are homeomorphic to each other iff there exists a **homeomorphism** between them.

1.5 Formalizing the Manifold

1.5.1 Definitions

Read: more abstract nonsense

Manifold

A k (represents intrinsic dimension) manifold is a continuous set of points X **homeomorphic** to \mathbb{R}^k

Embedding

The embedding of a manifold X is n iff $x \in \mathbb{R}^n \ \forall x \in X$

Note: If a k manifold is embedded in n space (\mathbb{R}^n), we must have $n \geq k$ (this is a trivial consequence of an advanced mathematical technique called “visualization”)

1.6 Manifolds in the Wild

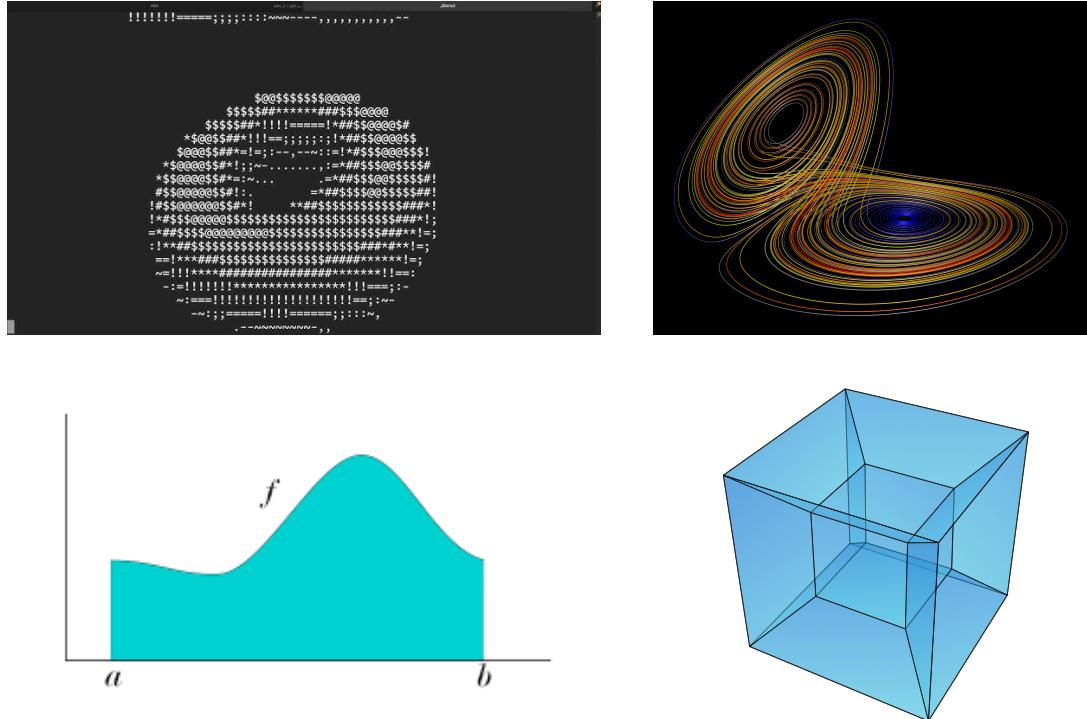
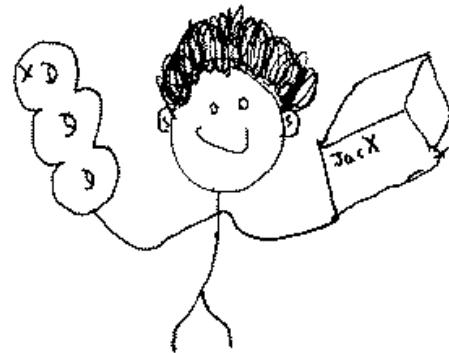


Figure 1: From left to right, top to bottom: A 2 manifold in 3 space, a 1 manifold in 3 space, a 2 manifold in 2 space, and a 2 manifold in 4 space (projected onto 3 space—the drawing is of a hypercube)

1.6 Manifolds in the Wild

1.6.1 The Prototypical Example of a Manifold



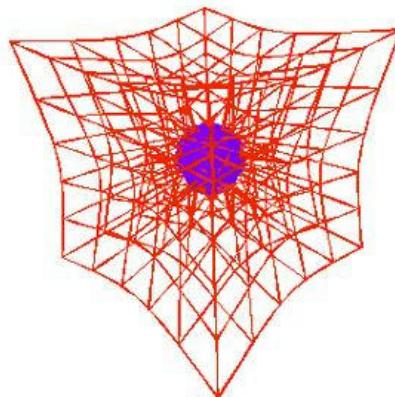
Dan Abramovic, embedding
a curve of genus 3 in
its Jacobian.

Figure 2: Don't ask me what this is because I am just as clueless as you

1.7 The Universe

We live in a **3 manifold** (possibly embedded in a *higher dimensional* space—and we have no way of knowing)!

Completely irrelevant to understanding Generalized Stokes Theorem, but very cool nevertheless.



This three-dimensional grid gives a better idea of what curved space-time might look like than the two-dimensional analogies do.

1.8 Differential Forms

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

— Emil Artin

1.8 Differential Forms

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

— Emil Artin

Remember **determinants**?

1.8 Differential Forms

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

— Emil Artin

Remember **determinants**?

Forget everything you've learned about computing the determinant algebraically.

1.8 Differential Forms

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

— Emil Artin

Remember **determinants**?

Forget everything you've learned about computing the determinant algebraically.

(These follow trivially from the general definition of the wedge product presented here.)

1.8 Differential Forms

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

— Emil Artin

Remember **determinants**?

Forget everything you've learned about computing the determinant algebraically.

(These follow trivially from the general definition of the wedge product presented here.)

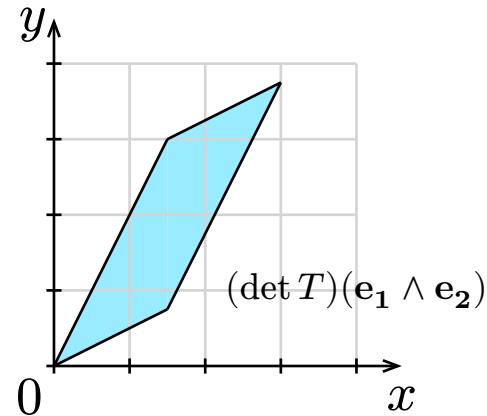
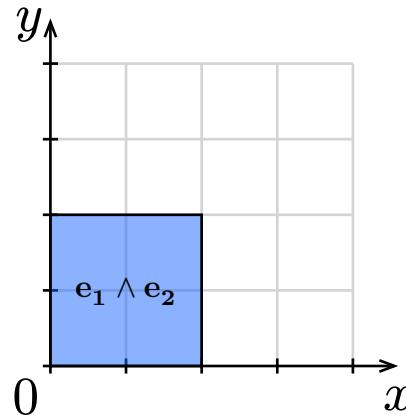
Now, consider the most abstract geometric properties the determinant should have.

1.9 Wedge Products and Bivectors

Determinants tell you how much the area of a unit square scales under some linear transformation T (**bivectors** generalize this to any coordinate system).

1.9 Wedge Products and Bivectors

Determinants tell you how much the area of a unit square scales under some linear transformation T (**bivectors** generalize this to any coordinate system).



1.10 Bivectors

- Bivectors are formed using a very scary operator called the **wedge product**.

1.10 Bivectors

- Bivectors are formed using a very scary operator called the **wedge product**.
- Wedge products between any two vectors \mathbf{a} and \mathbf{b} exist in a vector space (we are free to define the basis vectors, but they look like some wedge product, e.g. $\mathbf{e}_1 \wedge \mathbf{e}_2$)

1.11 Multivolumes

- Because the multivolume of a parallelotope (as a function of its side vectors) is multilinear*, we can (mathematicians would say *trivially* even though this is very quixotic) derive the following algebraic properties:

1.11 Multivolumes

- Because the multivolume of a parallelotope (as a function of its side vectors) is multilinear*, we can (mathematicians would say *trivially* even though this is very quixotic) derive the following algebraic properties:

*Not technically true, as we'll see later—we're actually dealing with signed multivolumes

1.11 Multivolumes

- Because the multivolume of a parallelotope (as a function of its side vectors) is multilinear*, we can (mathematicians would say *trivially* even though this is very quixotic) derive the following algebraic properties:

*Not technically true, as we'll see later—we're actually dealing with signed multivolumes

$$c(\mathbf{a} \wedge \mathbf{b}) = c\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge c\mathbf{b}$$

1.11 Multivolumes

- Because the multivolume of a parallelotope (as a function of its side vectors) is multilinear*, we can (mathematicians would say *trivially* even though this is very quixotic) derive the following algebraic properties:

*Not technically true, as we'll see later—we're actually dealing with signed multivolumes

$$c(\mathbf{a} \wedge \mathbf{b}) = c\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge c\mathbf{b}$$

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$$

1.11 Multivolumes

- Because the multivolume of a parallelotope (as a function of its side vectors) is multilinear*, we can (mathematicians would say *trivially* even though this is very quixotic) derive the following algebraic properties:

*Not technically true, as we'll see later—we're actually dealing with signed multivolumes

$$c(\mathbf{a} \wedge \mathbf{b}) = c\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge c\mathbf{b}$$

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$$

Multilinearity!

1.12 Properties of the Wedge Product

There are some others, too:

1.12 Properties of the Wedge Product

There are some others, too:

$$\mathbf{a} \wedge \mathbf{a} = 0$$

Parallelogram where opposite sides are the same is degenerate; area is zero

1.12 Properties of the Wedge Product

There are some others, too:

$$\mathbf{a} \wedge \mathbf{a} = 0$$

Parallelogram where opposite sides are the same is degenerate; area is zero

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

*This follows from the previous properties; this is why the determinant is **antisymmetric**!*

1.13 Interesting Derivation of Antisymmetry Property

By the fact that $\mathbf{u} \wedge \mathbf{u} = 0$, we have:

$$(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b}) = 0 \quad (\mathbf{u} = \mathbf{a} + \mathbf{b})$$

Distribute!

$$\mathbf{a} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} + \mathbf{b} \wedge \mathbf{b} = 0$$

Use $\mathbf{u} \wedge \mathbf{u} = 0$ again! (but for different \mathbf{u})

$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0$$

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Only the **signed multivolume** of a parallelotope as a function of its side vectors is multilinear.

1.14 Integration over 1 Manifolds

For a 1 manifold, we integrate over a 1-form that looks like dx .

1.14 Integration over 1 Manifolds

For a 1 manifold, we integrate over a 1-form that looks like dx .

- In 1 dimension, this is a standard integral $\int f(x) \, dx$

1.14 Integration over 1 Manifolds

For a 1 manifold, we integrate over a 1-form that looks like dx .

- In 1 dimension, this is a standard integral $\int f(x) dx$
- In n dimensions (where $n > 1$), you learned about this as a **line integral**:

$$\int_C \mathbf{f}(t) \cdot \frac{d\mathbf{r}(t)}{dt} dt$$

1.14 Integration over 1 Manifolds

For a 1 manifold, we integrate over a 1-form that looks like dx .

- In 1 dimension, this is a standard integral $\int f(x) dx$
- In n dimensions (where $n > 1$), you learned about this as a **line integral**:

$$\int_C \mathbf{f}(t) \cdot \frac{d\mathbf{r}(t)}{dt} dt$$

Notice the dimension of f changes with n —there are n differentials in n dimensions, and we can integrate only over 1 manifolds using a 1 form.

1.14 Integration over 1 Manifolds

For a 1 manifold, we integrate over a 1-form that looks like dx .

- In 1 dimension, this is a standard integral $\int f(x) dx$
- In n dimensions (where $n > 1$), you learned about this as a **line integral**:

$$\int_C \mathbf{f}(t) \cdot \frac{d\mathbf{r}(t)}{dt} dt$$

Notice the dimension of f changes with n —there are n differentials in n dimensions, and we can integrate only over 1 manifolds using a 1 form.

Line integrals are really just:

$$\int (f_1(t) d\mathbf{e}_1 + f_2(t) d\mathbf{e}_2 + \cdots + f_n d\mathbf{e}_n)$$

1.15 Integration over 2 Manifolds

For a 2 manifold, we integrate over a 2-form that looks like $dx \ dy$ ($dx \wedge dy$ in disguise).

1.15 Integration over 2 Manifolds

For a 2 manifold, we integrate over a 2-form that looks like $dx \ dy$ ($dx \wedge dy$ in disguise).

- In 2 dimensions, an example is a surface integral (for calculating flux) $\iint f(x, y) \cdot n \ dS$

1.15 Integration over 2 Manifolds

For a 2 manifold, we integrate over a 2-form that looks like $dx \ dy$ ($dx \wedge dy$ in disguise).

- In 2 dimensions, an example is a surface integral (for calculating flux) $\iint \mathbf{f}(x, y) \cdot \mathbf{n} \ dS$
- A general integral over a 2 manifold in 2 dimensions looks like:
 $\int \mathbf{f}(u, v) \ du \wedge dv$

1.15 Integration over 2 Manifolds

For a 2 manifold, we integrate over a 2-form that looks like $dx \ dy$ ($dx \wedge dy$ in disguise).

- In 2 dimensions, an example is a surface integral (for calculating flux) $\iint f(x, y) \cdot n \ dS$
- A general integral over a 2 manifold in 2 dimensions looks like:
 $\int f(u, v) \ du \wedge dv$

We can integrate over any surface of an n dimensional space, where our differential is a **bivector** formed by two of $d\mathbf{e}_1, \dots, d\mathbf{e}_n$. There are therefore $\binom{n}{2}$ differentials that make up the 2-form in n space, and the function f we integrate over must have this dimension.

1.15 Integration over 2 Manifolds

For a 2 manifold, we integrate over a 2-form that looks like $dx \ dy$ ($dx \wedge dy$ in disguise).

- In 2 dimensions, an example is a surface integral (for calculating flux) $\iint f(x, y) \cdot \mathbf{n} \ dS$

- A general integral over a 2 manifold in 2 dimensions looks like:

$$\int f(u, v) \ du \wedge dv$$

We can integrate over any surface of an n dimensional space, where our differential is a **bivector** formed by two of $d\mathbf{e}_1, \dots, d\mathbf{e}_n$. There are therefore $\binom{n}{2}$ differentials that make up the 2-form in n space, and the function f we integrate over must have this dimension.

$$\int \left(f_1(u, v) \ d\mathbf{e}_1 \wedge \mathbf{e}_2 + \cdots + f_n(u, v) \ d\mathbf{e}_1 \wedge \mathbf{e}_n + \cdots + f_{\binom{n}{2}}(u, v) \ d\mathbf{e}_{n-1} \wedge \mathbf{e}_n \right)$$

1.16 Flux

What are we *actually* doing when we calculate **flux**?

1.16 Flux

What are we *actually* doing when we calculate **flux**?

- Note that we always calculate the flux of a vector field \mathbf{F} (dim 3) over a surface (dim 2) embedded in xyz space (dim 3).
- Note also that $\binom{3}{2} = 3$, the dimension of \mathbf{F} , as we would expect.

1.16 Flux

What are we *actually* doing when we calculate **flux**?

- Note that we always calculate the flux of a vector field \mathbf{F} (dim 3) over a surface (dim 2) embedded in xyz space (dim 3).
- Note also that $\binom{3}{2} = 3$, the dimension of \mathbf{F} , as we would expect.

There are three differentials that make up the 2-form in xyz space: $dx \wedge dy$, $dx \wedge dz$, and $dy \wedge dz$.

1.16 Flux

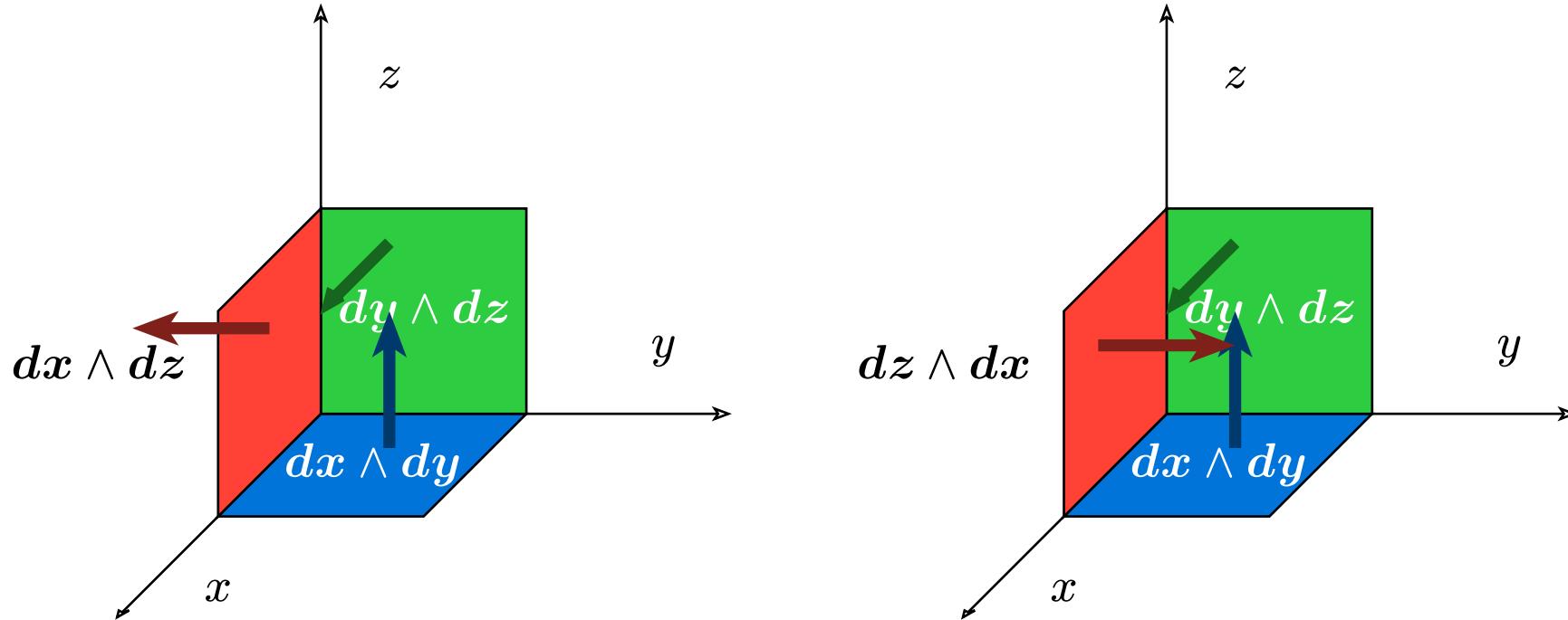
What are we *actually* doing when we calculate **flux**?

- Note that we always calculate the flux of a vector field \mathbf{F} (dim 3) over a surface (dim 2) embedded in xyz space (dim 3).
- Note also that $\binom{3}{2} = 3$, the dimension of \mathbf{F} , as we would expect.

There are three differentials that make up the 2-form in xyz space: $dx \wedge dy$, $dx \wedge dz$, and $dy \wedge dz$.

Let $\mathbf{F} = \langle A(x, y, z), B(x, y, z), C(x, y, z) \rangle$. We will calculate the flux outward through each surface in the 2-form. We will talk more about orientation later, but for the purpose of flux, we define the normal vector as being the cross product of two surface vectors (with right hand rule).

1.17 Surface Integrals



We want flux to be positive when anything “enters” this cube and negative when stuff “leaves”—but the mathematics of differential forms allow us to integrate any way we like

1.18 The Flux Integral

$$\int (A(x, y, z) \ dy \wedge dz + B(x, y, z) \ dz \wedge dx + C(x, y, z) \ dx \wedge dy)$$

1.18 The Flux Integral

$$\int (A(x, y, z) \ dy \wedge dz + B(x, y, z) \ dz \wedge dx + C(x, y, z) \ dx \wedge dy)$$

$$\int \langle A(x, y, z), B(x, y, z), C(x, y, z) \rangle \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$$

1.18 The Flux Integral

$$\int (A(x, y, z) \ dy \wedge dz + B(x, y, z) \ dz \wedge dx + C(x, y, z) \ dx \wedge dy)$$

$$\int \langle A(x, y, z), B(x, y, z), C(x, y, z) \rangle \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$$

$$\int \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$$

1.18 The Flux Integral

$$\int (A(x, y, z) \ dy \wedge dz + B(x, y, z) \ dz \wedge dx + C(x, y, z) \ dx \wedge dy)$$

$$\int \langle A(x, y, z), B(x, y, z), C(x, y, z) \rangle \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$$

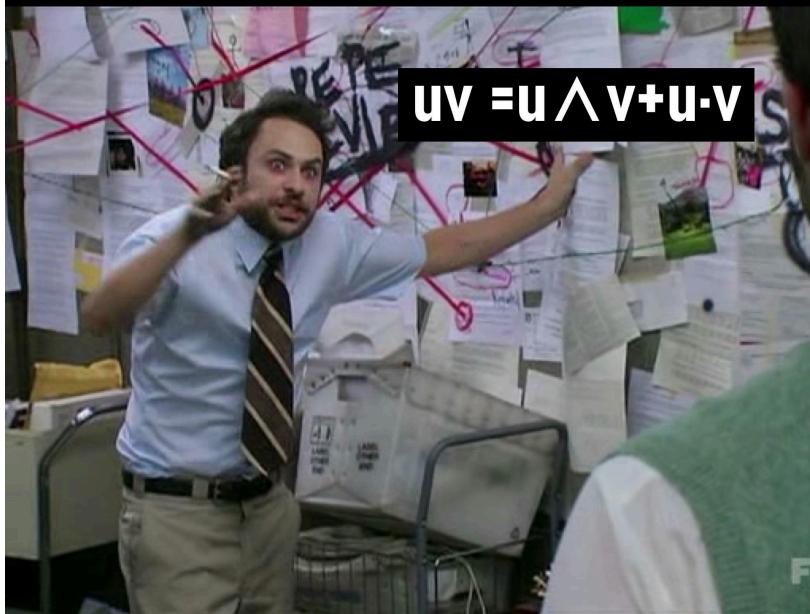
$$\int \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$$

How do we calculate such a monstrosity?

What is the meaning of this abstract nonsense?

1.19 The Pullback

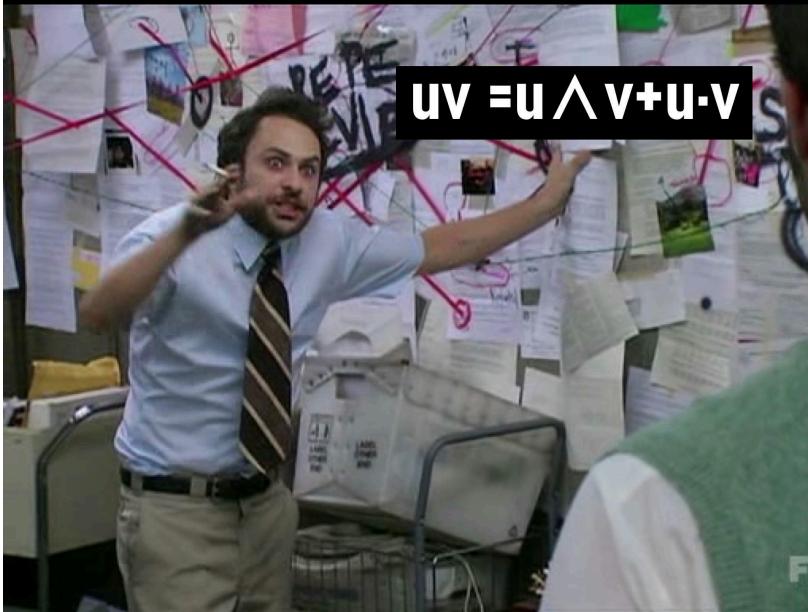
going off on an ill advised tangent about Clifford Algebra when someone calls the cross product unintuitive, be like:



- There is no going back after this (your mind may be warped by upcoming abstractions)

1.19 The Pullback

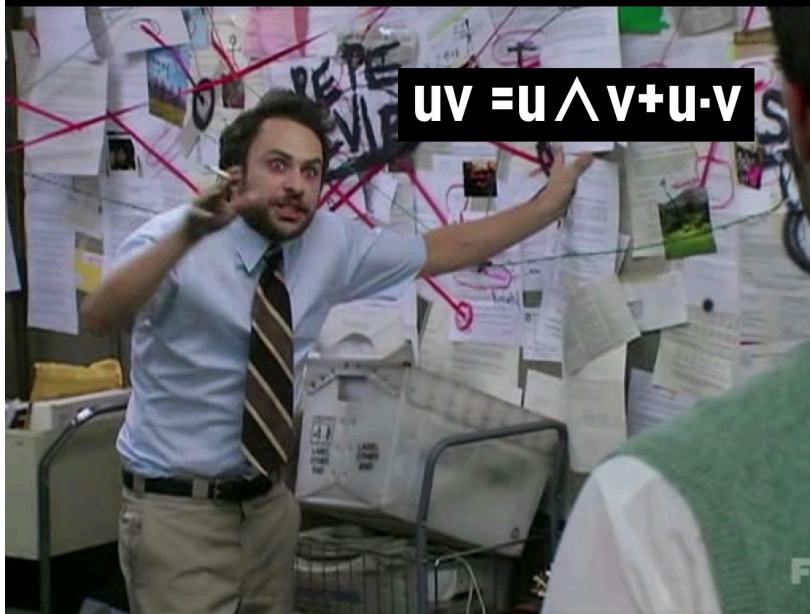
going off on an ill advised tangent about Clifford Algebra when someone calls the cross product unintuitive, be like:



- There is no going back after this (your mind may be warped by upcoming abstractions)
- We note that we can represent F , a 2 manifold, using a 2 dimensional parameterization in terms of u, v

1.19 The Pullback

going off on an ill advised tangent about Clifford Algebra when someone calls the cross product unintuitive, be like:



- There is no going back after this (your mind may be warped by upcoming abstractions)
- We note that we can represent F , a 2 manifold, using a 2 dimensional parameterization in terms of u, v
- We now need to “convert” $dy \wedge dz, dz \wedge dx, dx \wedge dy$ to $du \wedge dv$

1.20 Mathematical Maneuvering

- Remember that a manifold is **homeomorphic** to a Euclidean space (in this case, \mathbb{R}^2). This means there exists a mapping* $x, y, z \rightarrow u, v$, so we can write $u(x, y, z), v(x, y, z)$.

*where x, y, z are in some subset of \mathbb{R}^3 (occupied by the manifold)

- This mapping is called a “**chart**,” which is part of an “**atlas**” for the manifold
 - Mathematicians take this map analogy very seriously because they cannot distinguish between homeomorphic objects in real life

Topology is precisely the mathematical discipline that allows the passage from local to global...

— René Thom (itinerant theoretician)

Don’t let your mind be warped by these abstractions!

1.20 Mathematical Maneuvering

1.20.1 Computation

- Given $u(x, y, z), v(x, y, z)$ and $\mathbf{F}(u, v)$, consider the inverse map $u, v \rightarrow x, y, z$ that must also exist by the conditions of **homeomorphism** to find $x(u, v), y(u, v), z(u, v)$.
- Now, take differentials using the chain rule:

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv, dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

- We can now use the properties of the **wedge product** to evaluate any product of dx , dy , and dz in terms of either $du \wedge dv$ or $dv \wedge du$ (but remember $dv \wedge du = -du \wedge dv$).
- This gives an integral of the form:

$$\int \mathbf{G}(u, v) \, du \wedge dv = \iint \mathbf{G}(u, v) \, du \, dv$$

1.21 Integration with Differential Forms

- What we just did was construct a specific integral for a 2 manifold in 3 space, but this procedure can in general be done for a k manifold in n space, where the function we are integrating has dimension $\binom{n}{k}$.

1.21 Integration with Differential Forms

- What we just did was construct a specific integral for a 2 manifold in 3 space, but this procedure can in general be done for a k manifold in n space, where the function we are integrating has dimension $\binom{n}{k}$.
- We can also take wedge products of more than 2 vectors to get multivectors, but these have only two canonical orientations
 - More on this later, but for now just know that we can always flip terms to get them in a certain order by multiplying either by 1, positive orientation, or -1 , negative orientation

1.22 Notational Hacks

- We call the entire expression after the integral sign, when expressed with multivectors, a **differential form**, commonly denoted ω
- Note that multivectors won't always have a wedge product symbol (for one or zero (we'll encounter those later) manifolds we only need a differential or scalar, respectively)
- Common notation is to use ω for a differential form and Ω for a manifold, just to be extra confusing

1.22 Notational Hacks

- We call the entire expression after the integral sign, when expressed with multivectors, a **differential form**, commonly denoted ω
 - Note that multivectors won't always have a wedge product symbol (for one or zero (we'll encounter those later) manifolds we only need a differential or scalar, respectively)
 - Common notation is to use ω for a differential form and Ω for a manifold, just to be extra confusing
- We use a completely made-up, contrived operation called the **pullback** to turn this abstract mess of symbols into a readable integral we can evaluate
 - Basically, use an intrinsic coordinate system that represents the manifold itself and not the space it's embedded in

1.23 Mandatory Content Warning

The undergrad category theorist

- Superiority complex despite nothing to show for it
- Burned out mathematical beauty receptors.
- Not even interested in the fields that category theory was invented to deal with.
- No free will. Slave to abstraction.
- Warped sense of what mathematics is.
- Already considers "specializing" in foundations despite being unaware of centuries worth of mathematical developments.



Don't let this be you. Stop before its too late!

1.24 Orientability

Before we continue,

1.24 Orientability

Before we continue,

A mathematician who can only generalise is like a monkey who can only climb up a tree, and a mathematician who can only specialise is like a monkey who can only climb down a tree. In fact neither the up monkey nor the down monkey is a viable creature. A real monkey must find food and escape his enemies and so must be able to incessantly climb up and down. A real mathematician must be able to generalise and specialise.

— George Polya

1.25 Differential Forms and Orientability

- The **antisymmetry** property of differential forms give us a convenient way to determine the orientation of a parameterization of some manifold.

1.25 Differential Forms and Orientability

- The **antisymmetry** property of differential forms give us a convenient way to determine the orientation of a parameterization of some manifold.
- The sign of the wedge product coefficient after converting to a “**positively-oriented form**” gives the orientation of a given representation

1.25 Differential Forms and Orientability

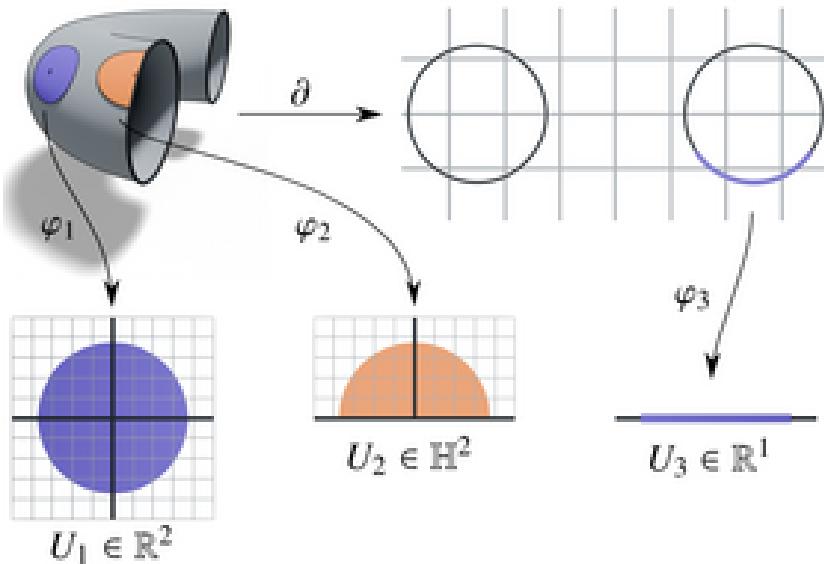
- The **antisymmetry** property of differential forms give us a convenient way to determine the orientation of a parameterization of some manifold.
- The sign of the wedge product coefficient after converting to a “**positively-oriented form**” gives the orientation of a given representation
- In 1 space, we say the canonical positively oriented form is dx
- In 2 space, it is $dx \wedge dy$, by convention
- In 3 space, $dx \wedge dy \wedge dz$

1.25 Differential Forms and Orientability

- The **antisymmetry** property of differential forms give us a convenient way to determine the orientation of a parameterization of some manifold.
- The sign of the wedge product coefficient after converting to a “**positively-oriented form**” gives the orientation of a given representation
- In 1 space, we say the canonical positively oriented form is dx
- In 2 space, it is $dx \wedge dy$, by convention
- In 3 space, $dx \wedge dy \wedge dz$

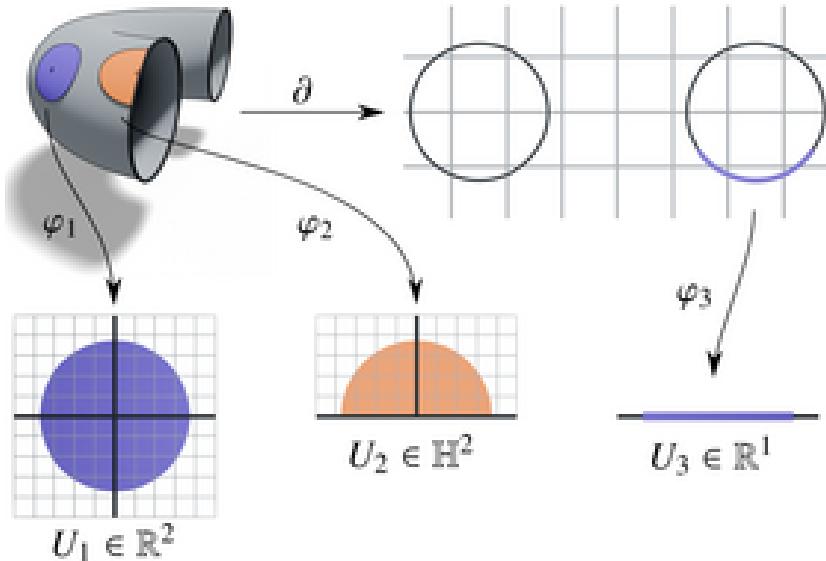
More on 0 space later ...

1.26 Boundaries



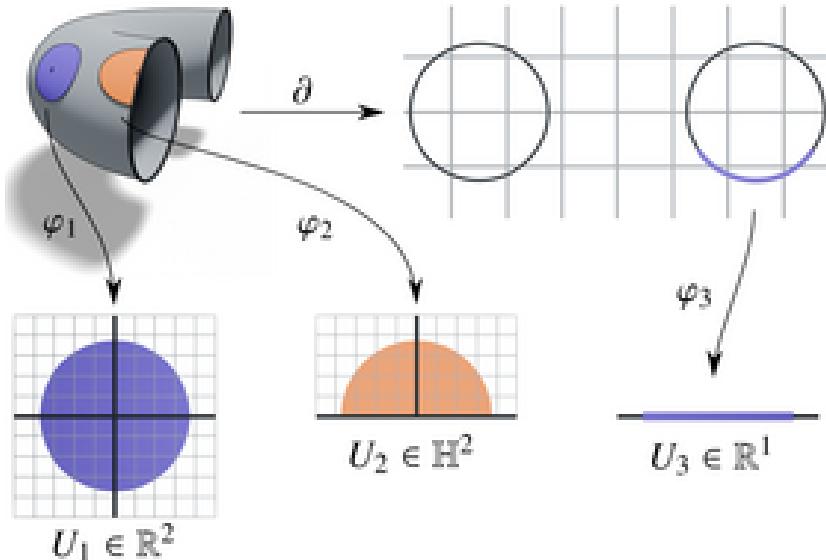
- To define the boundary, we consider **homeomorphic equivalence classes**

1.26 Boundaries



- To define the boundary, we consider **homeomorphic equivalence classes**
- Specifically, a point within a k manifold M will have a local neighborhood of points homeomorphic to a k ball

1.26 Boundaries



- To define the boundary, we consider **homeomorphic equivalence classes**
- Specifically, a point within a k manifold M will have a local neighborhood of points homeomorphic to a k ball
- At the boundary, points have a local neighborhood homeomorphic to a half k ball

1.27 Properties of Boundaries

The boundary of a k manifold is a set of closed $k - 1$ manifolds

1.27 Properties of Boundaries

The boundary of a k manifold is a set of closed $k - 1$ manifolds

“It is the snobbishness of the young to suppose that a theorem is trivial because the proof is trivial.”

— Henry Whitehead

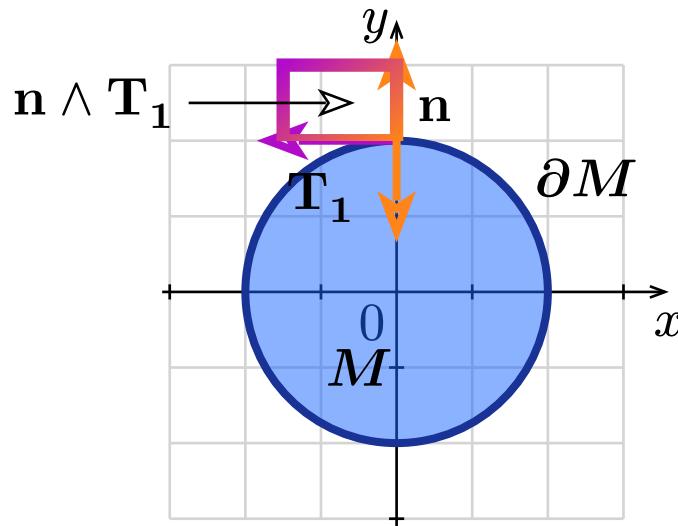
1.28 Proof of Properties of a Boundary

If you're interested:

our definition $\Rightarrow \exists \varphi : U \rightarrow \mathbb{H}^k$ where U is the neighborhood of points around some $P \in \partial M$ and \mathbb{H}^k is a half k ball, formally defined as $\mathbb{H}^k = \{x \in \mathbb{R}^k \mid x_k \geq 0\}$ for some intrinsic coordinate system where $x_k = 0$ for points along the boundary in U (otherwise a local neighborhood around these points would be homeomorphic to \mathbb{R}^k , not \mathbb{H}^k).^{*} Thus, ∂M (around P) is homeomorphic to $\{x \in \mathbb{H}^k \mid x_k = 0\}$, which is homeomorphic to \mathbb{R}^{k-1} (simply reparameterize by eliminating x_k and keeping all other parameters the same—for the inverse map add $x_k = 0$). We'll later see how to prove this boundary is closed using Generalized Stokes Theorem!

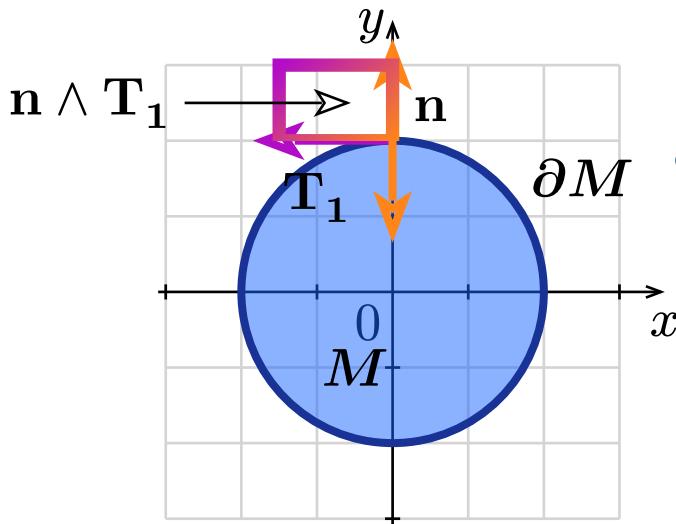
^{*}These are quite shaky foundations indeed, but a full proof would result in too much brain damage to present in its entirety.

1.29 Induced Orientations



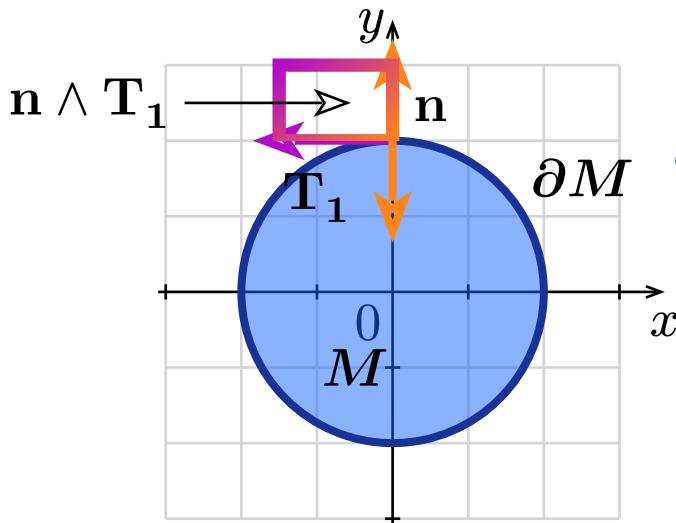
- When we have an n manifold M with a known positive orientation in n space, we can determine the orientation of its boundary ∂M (its “**induced orientation**”)

1.29 Induced Orientations



- When we have an n manifold M with a known positive orientation in n space, we can determine the orientation of its boundary ∂M (its “**induced orientation**”)
- For each parameter defining ∂M (in this case there is only one, t), draw the tangent vector T (which for this M we can write as $\frac{d\mathbf{r}}{dt} dt$ where \mathbf{r} is the mapping from 1-dimensional t space to the ∂M manifold embedded in 2 space)

1.29 Induced Orientations



- When we have an n manifold M with a known positive orientation in n space, we can determine the orientation of its boundary ∂M (its “**induced orientation**”)
- For each parameter defining ∂M (in this case there is only one, t), draw the tangent vector T (which for this M we can write as $\frac{d\mathbf{r}}{dt} dt$ where \mathbf{r} is the mapping from 1-dimensional t space to the ∂M manifold embedded in 2 space)
- Compute the normal vector n orthogonal to all tangent vectors and not pointing into M

1.30 Induced Orientations and Wedge Products

- Now, write $\mathbf{n} \wedge T_1$ as some expression of the form $A dx \wedge dy$, where $dx \wedge dy$ is the known positive orientation of M .

1.30 Induced Orientations and Wedge Products

- Now, write $\mathbf{n} \wedge T_1$ as some expression of the form $A dx \wedge dy$, where $dx \wedge dy$ is the known positive orientation of M .
- The orientation of ∂M is given by $\text{sign}(A)$
 - Note that this is either *positive* or *negative* (there are only ever 2 canonical orientations of a manifold, regardless of dimension)

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

(and casually making rigorous use of infinitesimals)

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

(and casually making rigorous use of infinitesimals)

- Since ∂M is an $n - 1$ manifold, there exists some parameterization $\varphi(x_1, \dots, x_{n-1}) :$
 $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

(and casually making rigorous use of infinitesimals)

- Since ∂M is an $n - 1$ manifold, there exists some parameterization $\varphi(x_1, \dots, x_{n-1}) :$
 $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$
- Consider all tangent vectors of the form $\mathbf{T}_{\mathbf{x}_i}$, given by $\frac{\partial \varphi}{\partial x_i} dx_i$

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

(and casually making rigorous use of infinitesimals)

- Since ∂M is an $n - 1$ manifold, there exists some parameterization $\varphi(x_1, \dots, x_{n-1}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$
- Consider all tangent vectors of the form $\mathbf{T}_{\mathbf{x}_i}$, given by $\frac{\partial \varphi}{\partial x_i} dx_i$
- Compute the normal vector \mathbf{n} satisfying $\mathbf{n} \cdot \mathbf{T}_{\mathbf{x}_i} = 0 \ \forall \mathbf{T}_{\mathbf{x}_i}$ and $\mathbf{n} \cdot d\rho < 0$ where ρ is a basis vector of $T_P M$ not along ∂M and $T_P M$ is the tangent space of M

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

(and casually making rigorous use of infinitesimals)

- Since ∂M is an $n - 1$ manifold, there exists some parameterization $\varphi(x_1, \dots, x_{n-1}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$
- Consider all tangent vectors of the form $\mathbf{T}_{\mathbf{x}_i}$, given by $\frac{\partial \varphi}{\partial x_i} dx_i$
- Compute the normal vector \mathbf{n} satisfying $\mathbf{n} \cdot \mathbf{T}_{\mathbf{x}_i} = 0 \ \forall \ \mathbf{T}_{\mathbf{x}_i}$ and $\mathbf{n} \cdot d\rho < 0$ where ρ is a basis vector of $T_P M$ not along ∂M and $T_P M$ is the tangent space of M
- Express $\mathbf{n} \wedge T_{x_1} \wedge \dots \wedge T_{x_{n-1}}$ in the form $A \ de_1 \wedge \dots \wedge de_n$ where $de_1 \wedge \dots \wedge de_n$ is the canonical positive orientation of M

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

(and casually making rigorous use of infinitesimals)

- Since ∂M is an $n - 1$ manifold, there exists some parameterization $\varphi(x_1, \dots, x_{n-1}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$
- Consider all tangent vectors of the form $\mathbf{T}_{\mathbf{x}_i}$, given by $\frac{\partial \varphi}{\partial x_i} dx_i$
- Compute the normal vector \mathbf{n} satisfying $\mathbf{n} \cdot \mathbf{T}_{\mathbf{x}_i} = 0 \ \forall \mathbf{T}_{\mathbf{x}_i}$ and $\mathbf{n} \cdot d\rho < 0$ where ρ is a basis vector of $T_P M$ not along ∂M and $T_P M$ is the tangent space of M
- Express $\mathbf{n} \wedge T_{x_1} \wedge \dots \wedge T_{x_{n-1}}$ in the form $A \ de_1 \wedge \dots \wedge de_n$ where $de_1 \wedge \dots \wedge de_n$ is the canonical positive orientation of M
- The orientation of point P is given by $\text{sign}(A)$

1.32 The Prototypical Non-Orientable Surface

When the prof says you can only use one side
of a piece of paper as a formula sheet.

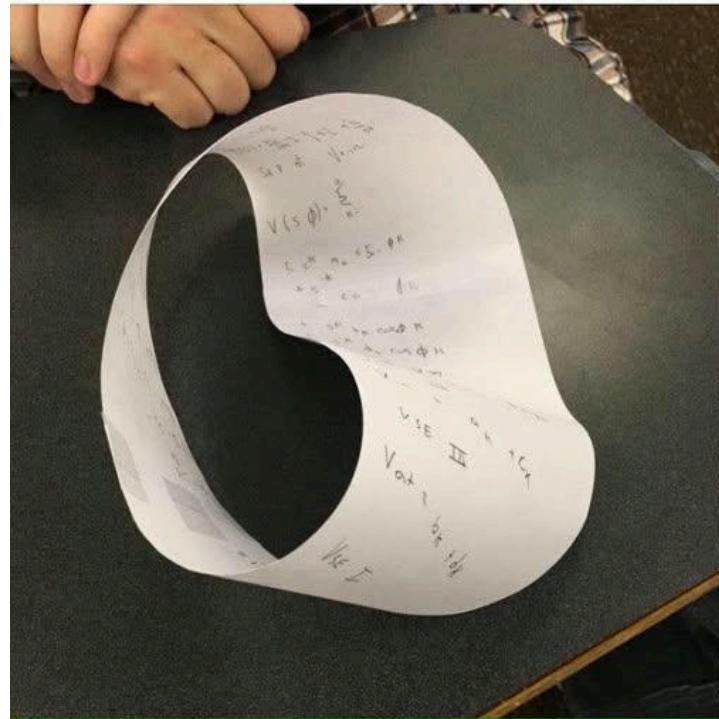


Figure 1: Not possible to define a unique \mathbf{n} for every point (any point) on the surface

1.33 Exterior Derivative

Consider a nasty differential form like:

$$\omega = M(x, y, z) \ dy \wedge dz + N(x, y, z) \ dz \wedge dx + P(x, y, z) \ dx \wedge dy$$

where $\mathbf{F} = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$ (remember flux??)

1.33 Exterior Derivative

Consider a nasty differential form like:

$$\omega = M(x, y, z) \ dy \wedge dz + N(x, y, z) \ dz \wedge dx + P(x, y, z) \ dx \wedge dy$$

where $\mathbf{F} = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$ (remember flux??)

Can we *differentiate* ω ?

1.34 Digression to Introduce the Scale of the Problem

We will need to use some very simple notions of category theory, an **esoteric subject** noted for its **difficulty** and **irrelevance**.

— G. Moore and N. Seiberg, 1989

1.34 Digression to Introduce the Scale of the Problem

We will need to use some very simple notions of category theory, an **esoteric subject** noted for its **difficulty** and **irrelevance**.

— G. Moore and N. Seiberg, 1989

What we want to do is turn an n form into an $n + 1$ form.

1.34 Digression to Introduce the Scale of the Problem

We will need to use some very simple notions of category theory, an **esoteric subject** noted for its **difficulty** and **irrelevance**.

— G. Moore and N. Seiberg, 1989

What we want to do is turn an n form into an $n + 1$ form.

1.34.3 Why?

Because we still want to integrate the resulting form (we want a differential, not a derivative)! Think of how, in 1 dimension, we take $f(x) \rightarrow f'(x) dx$ to produce a differential we can integrate over.

1.34 Digression to Introduce the Scale of the Problem

We will need to use some very simple notions of category theory, an **esoteric subject** noted for its **difficulty** and **irrelevance**.

— G. Moore and N. Seiberg, 1989

What we want to do is turn an n form into an $n + 1$ form.

1.34.4 Why?

Because we still want to integrate the resulting form (we want a differential, not a derivative)! Think of how, in 1 dimension, we take $f(x) \rightarrow f'(x) dx$ to produce a differential we can integrate over.

You should be thinking of the Fundamental Theorem of Calculus right now (we'll get back to this).

1.35 Definition of the Exterior Derivative

Exterior Derivative

Given an n form $\omega = f \ dx_1 \wedge \dots \wedge dx_n$, we define the exterior derivative $d\omega = df \wedge dx_1 \wedge \dots \wedge dx_n$, an $n + 1$ form (trivial to see given we have added another wedge product and df is a 1 form)

1.35 Definition of the Exterior Derivative

Exterior Derivative

Given an n form $\omega = f dx_1 \wedge \dots \wedge dx_n$, we define the exterior derivative $d\omega = df \wedge dx_1 \wedge \dots \wedge dx_n$, an $n + 1$ form (trivial to see given we have added another wedge product and df is a 1 form)

Back to our example (let $\mathcal{D} = \langle dx, dy, dz \rangle$),

1.35 Definition of the Exterior Derivative

Exterior Derivative

Given an n form $\omega = f \ dx_1 \wedge \dots \wedge dx_n$, we define the exterior derivative $d\omega = df \wedge dx_1 \wedge \dots \wedge dx_n$, an $n + 1$ form (trivial to see given we have added another wedge product and df is a 1 form)

Back to our example (let $\mathcal{D} = \langle dx, dy, dz \rangle$),

$$d\omega = (\nabla f \cdot \mathcal{D}) \wedge dy \wedge dz + (\nabla g \cdot \mathcal{D}) \wedge dz \wedge dx + (\nabla h \cdot \mathcal{D}) \wedge dx \wedge dy$$

1.35 Definition of the Exterior Derivative

Exterior Derivative

Given an n form $\omega = f \ dx_1 \wedge \dots \wedge dx_n$, we define the exterior derivative $d\omega = df \wedge dx_1 \wedge \dots \wedge dx_n$, an $n + 1$ form (trivial to see given we have added another wedge product and df is a 1 form)

Back to our example (let $\mathcal{D} = \langle dx, dy, dz \rangle$),

$$d\omega = (\nabla f \cdot \mathcal{D}) \wedge dy \wedge dz + (\nabla g \cdot \mathcal{D}) \wedge dz \wedge dx + (\nabla h \cdot \mathcal{D}) \wedge dx \wedge dy$$

Terms of the form $da \wedge \dots \wedge da \wedge \dots \wedge db = 0$, so we exclude them:

1.35 Definition of the Exterior Derivative

Exterior Derivative

Given an n form $\omega = f \ dx_1 \wedge \dots \wedge dx_n$, we define the exterior derivative $d\omega = df \wedge dx_1 \wedge \dots \wedge dx_n$, an $n + 1$ form (trivial to see given we have added another wedge product and df is a 1 form)

Back to our example (let $\mathcal{D} = \langle dx, dy, dz \rangle$),

$$d\omega = (\nabla f \cdot \mathcal{D}) \wedge dy \wedge dz + (\nabla g \cdot \mathcal{D}) \wedge dz \wedge dx + (\nabla h \cdot \mathcal{D}) \wedge dx \wedge dy$$

Terms of the form $da \wedge \dots \wedge da \wedge \dots \wedge db = 0$, so we exclude them:

$$d\omega = f_x \ dx \wedge dy \wedge dz + g_y \ dy \wedge dz \wedge dx + h_z \ dz \wedge dx \wedge dy$$

$$d\omega = (f_x + g_y + h_z) \ dx \wedge dy \wedge dz$$

1.36 Powers of the Exterior Derivative

That looks familiar!

1.36 Powers of the Exterior Derivative

That looks familiar!

It's $\operatorname{div} \mathbf{F} dV$ (using the fact that $dV = dx \wedge dy \wedge dz$)

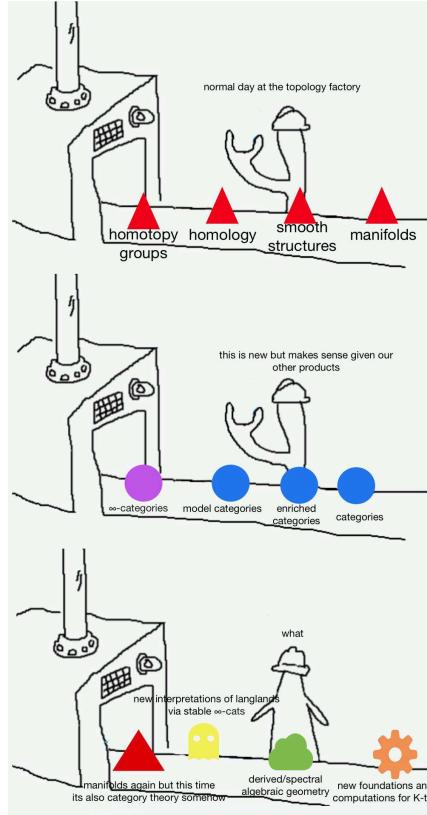
1.36 Powers of the Exterior Derivative

That looks familiar!

It's $\operatorname{div} \mathbf{F} dV$ (using the fact that $dV = dx \wedge dy \wedge dz$)

- Flux becomes divergence?
- Where have you seen this before??

1.37 Stokes Theorem



- How Generalized Stokes Theorem is actually defined using homotopy type theory
- Involves topos theory and other abstract nonsense
- We will stick to the algebraic topology definition, which is only slightly less deranged than the category theoretic one

1.38 Generalized Stokes Theorem

Topological Definition of Generalized Stokes Theorem

$$\int_{\partial M} \omega = \int_M d\omega$$

1.38 Generalized Stokes Theorem

Topological Definition of Generalized Stokes Theorem

$$\int_{\partial M} \omega = \int_M d\omega$$

Absolute perfection.

Nothing more remains to be said.

1.38 Generalized Stokes Theorem

Topological Definition of Generalized Stokes Theorem

$$\int_{\partial M} \omega = \int_M d\omega$$

Absolute perfection.

Nothing more remains to be said.

This presentation could end here and you would leave knowing a fundamental truth of the universe.

Outline

1. Stokes Theorem

2. Some Examples

3. Applications?

4. References

2.1 Deriving Stokes Theorem from Generalized Stokes Theorem

Let's consider what Stokes Theorem tells us about a 2 manifold in 3 space:

2.1 Deriving Stokes Theorem from Generalized Stokes Theorem

Let's consider what Stokes Theorem tells us about a 2 manifold in 3 space:

$$\int_{\partial M^1 \subset \mathbb{R}^3} \omega = \int_{M^2 \subset \mathbb{R}^3} d\omega$$

↑ Abuse of notation to demonstrate what we mean

2.1 Deriving Stokes Theorem from Generalized Stokes Theorem

Let's consider what Stokes Theorem tells us about a 2 manifold in 3 space:

$$\int_{\partial M^1 \subset \mathbb{R}^3} \omega = \int_{M^2 \subset \mathbb{R}^3} d\omega$$

↑ Abuse of notation to demonstrate what we mean

We know ω must be a 1 form, but this isn't helpful until we can assign a value to ω . To keep things simple, let's make the left-hand integral a line integral, so:

$$\omega = f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz$$

$$\int_{\partial M} \omega = \int_C \mathbf{F} \cdot d\mathbf{r} \text{ where } \mathbf{F} = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$$

2.2 Stokes Theorem, Revisited

We then have:

$$\begin{aligned} d\omega &= (f_y \ dy + f_z \ dz) \wedge dx + (g_x \ dx + g_z \ dz) \wedge dy + (h_x \ dx + h_y \ dy) \wedge dz \\ d\omega &= (g_x - f_y) \ dx \wedge dy + (f_z - h_x) \ dz \wedge dx + (h_y - g_z) \ dy \wedge dz \end{aligned}$$

2.2 Stokes Theorem, Revisited

We then have:

$$\begin{aligned} d\omega &= (f_y \ dy + f_z \ dz) \wedge dx + (g_x \ dx + g_z \ dz) \wedge dy + (h_x \ dx + h_y \ dy) \wedge dz \\ d\omega &= (g_x - f_y) \ dx \wedge dy + (f_z - h_x) \ dz \wedge dx + (h_y - g_z) \ dy \wedge dz \end{aligned}$$

Hey that looks familiar!

2.2 Stokes Theorem, Revisited

We then have:

$$\begin{aligned} d\omega &= (f_y \ dy + f_z \ dz) \wedge dx + (g_x \ dx + g_z \ dz) \wedge dy + (h_x \ dx + h_y \ dy) \wedge dz \\ d\omega &= (g_x - f_y) \ dx \wedge dy + (f_z - h_x) \ dz \wedge dx + (h_y - g_z) \ dy \wedge dz \end{aligned}$$

Hey that looks familiar!

$$d\omega = \text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \ ds$$

It's the **flux** of the **curl** of \mathbf{F} !!

2.2 Stokes Theorem, Revisited

We then have:

$$\begin{aligned} d\omega &= (f_y \ dy + f_z \ dz) \wedge dx + (g_x \ dx + g_z \ dz) \wedge dy + (h_x \ dx + h_y \ dy) \wedge dz \\ d\omega &= (g_x - f_y) \ dx \wedge dy + (f_z - h_x) \ dz \wedge dx + (h_y - g_z) \ dy \wedge dz \end{aligned}$$

Hey that looks familiar!

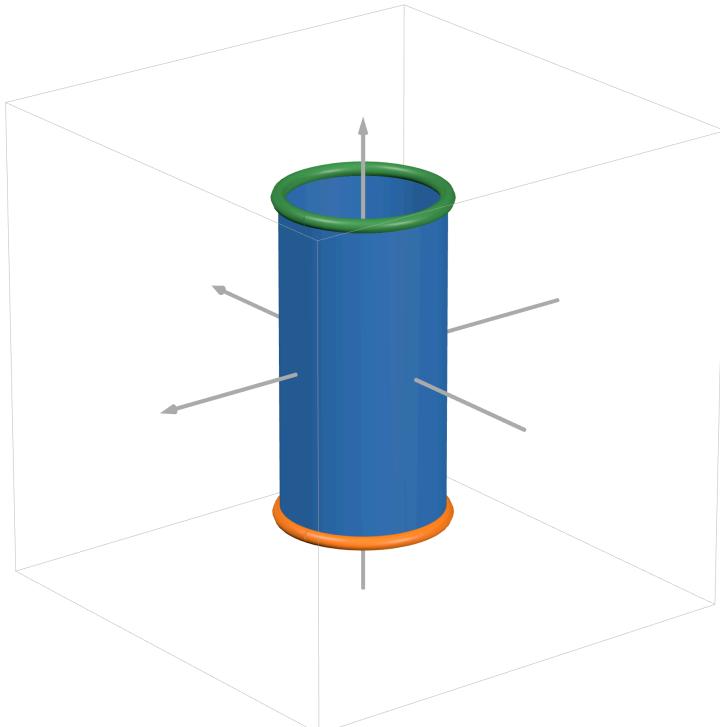
$$d\omega = \text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \ dS$$

It's the **flux** of the **curl** of \mathbf{F} !!

By Generalized Stokes,

$$\int_{\partial M} \omega = \int_M d\omega \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \ dS$$

2.3 An Interesting Example Involving Stokes Theorem



- Using a clever choice of \mathbf{F} , we will show that the orientation of the boundaries at opposite ends of a cylinder are opposite each other.
- Note that ∂M consists of two manifolds C_1 , C_2
- $\int_{\partial M} \omega = \varepsilon_1 \int_{C_1} \omega + \varepsilon_2 \int_{C_2} \omega$ where $\varepsilon_1, \varepsilon_2$ are the orientations of C_1 and C_2 , respectively

2.4 Orientation Theory

We want to show $\varepsilon_1 = -\varepsilon_2$ or $\varepsilon_1 + \varepsilon_2 = 0$.

2.4 Orientation Theory

We want to show $\varepsilon_1 = -\varepsilon_2$ or $\varepsilon_1 + \varepsilon_2 = 0$.

If we could write $\varepsilon_1 \int_{C_1} \omega + \varepsilon_2 \int_{C_2} \omega$ as $(\varepsilon_1 + \varepsilon_2) \int_C \omega$ for some C , then we derive a constraint for $\varepsilon_1 + \varepsilon_2$ using Stokes Theorem. For this to happen, we need $\int_{C_1} \omega = \int_{C_2} \omega = \int_C \omega$ (i.e. the line integral of \mathbf{F} over either C_1 or C_2 gives the same result).

2.4 Orientation Theory

We want to show $\varepsilon_1 = -\varepsilon_2$ or $\varepsilon_1 + \varepsilon_2 = 0$.

If we could write $\varepsilon_1 \int_{C_1} \omega + \varepsilon_2 \int_{C_2} \omega$ as $(\varepsilon_1 + \varepsilon_2) \int_C \omega$ for some C , then we derive a constraint for $\varepsilon_1 + \varepsilon_2$ using Stokes Theorem. For this to happen, we need $\int_{C_1} \omega = \int_{C_2} \omega = \int_C \omega$ (i.e. the line integral of \mathbf{F} over either C_1 or C_2 gives the same result).

This tells us \mathbf{F} is independent of z . Since C_1 and C_2 occupy the same region of xy space, if \mathbf{F} depends only on these two parameters, the line integrals over C_1 and C_2 are equivalent and we can factor out $\varepsilon_1 + \varepsilon_2$ from the boundary integral.

2.5 Stokes Theorem and Orientation

For some $\mathbf{F}(x, y)$ on M , Stokes Theorem tells us

$$\int_{\partial M} \omega = \varepsilon_1 \int_{\textcolor{teal}{C}_1} \omega + \varepsilon_2 \int_{\textcolor{orange}{C}_2} \omega = (\varepsilon_1 + \varepsilon_2) \int_{\textcolor{teal}{C}_1} \omega = \int_M d\omega = \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

2.5 Stokes Theorem and Orientation

For some $\mathbf{F}(x, y)$ on M , Stokes Theorem tells us

$$\int_{\partial M} \omega = \varepsilon_1 \int_{\textcolor{teal}{C}_1} \omega + \varepsilon_2 \int_{\textcolor{orange}{C}_2} \omega = (\varepsilon_1 + \varepsilon_2) \int_{\textcolor{teal}{C}_1} \omega = \int_M d\omega = \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

If $\varepsilon_1 + \varepsilon_2 = 0$ as we supposed,

$$(\varepsilon_1 + \varepsilon_2) \int_{\textcolor{teal}{C}_1} \omega = 0 = \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

2.5 Stokes Theorem and Orientation

For some $\mathbf{F}(x, y)$ on M , Stokes Theorem tells us

$$\int_{\partial M} \omega = \varepsilon_1 \int_{\mathbf{C}_1} \omega + \varepsilon_2 \int_{\mathbf{C}_2} \omega = (\varepsilon_1 + \varepsilon_2) \int_{\mathbf{C}_1} \omega = \int_M d\omega = \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

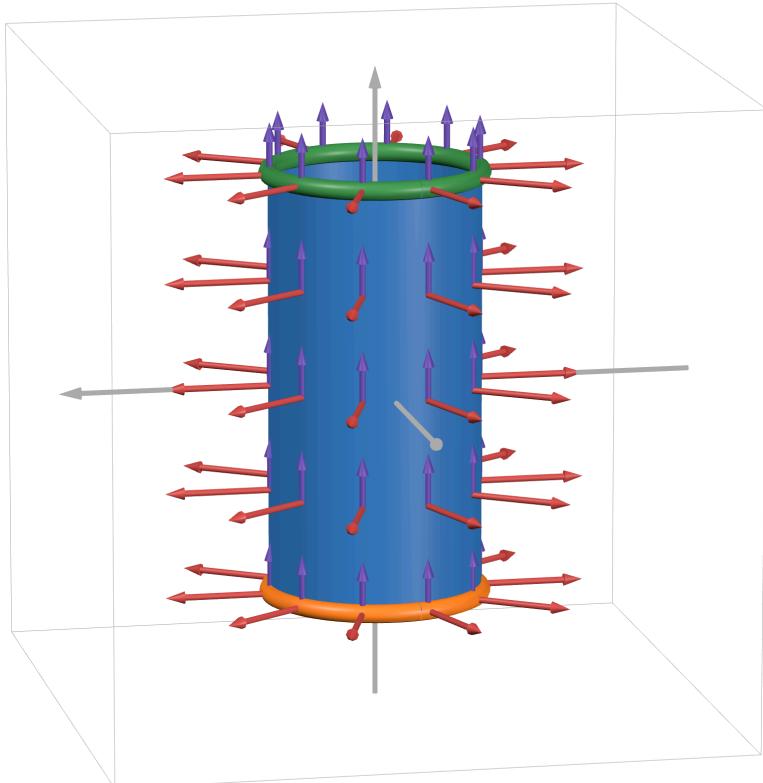
If $\varepsilon_1 + \varepsilon_2 = 0$ as we supposed,

$$(\varepsilon_1 + \varepsilon_2) \int_{\mathbf{C}_1} \omega = 0 = \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

Hence, in order to prove $\varepsilon_1 + \varepsilon_2 = 0$, we must choose $\mathbf{F}(x, y) \mid \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS = 0$.

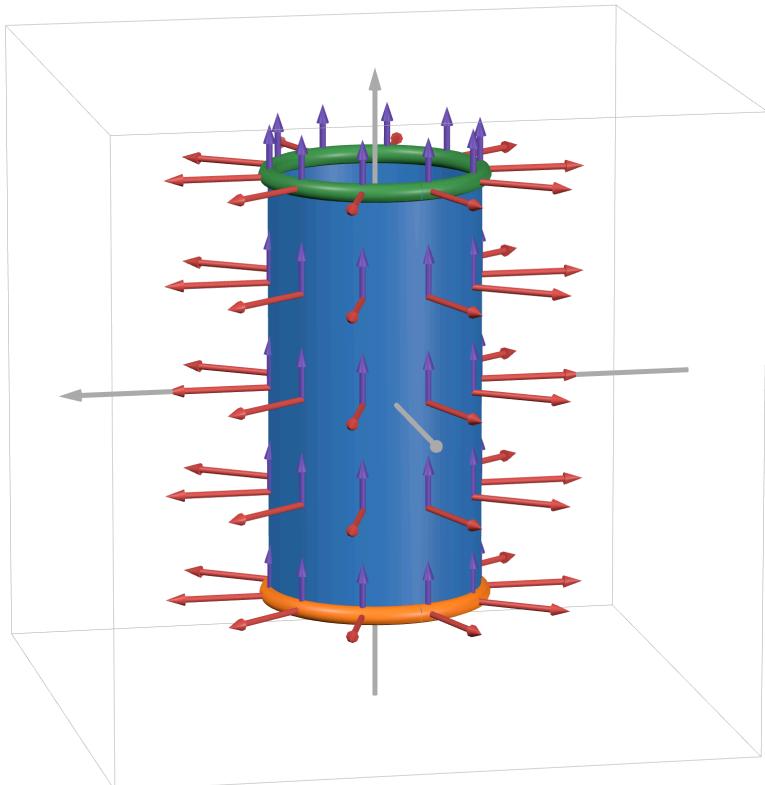
The beauty of this is that we can choose *any* differential form ω and Generalized Stokes Theorem must still hold (in the case of the 3-dimensional Stokes Theorem, it means we can choose any \mathbf{F} since the abstraction of differential forms has been thrown out).

2.6 Completing the Proof (Example)



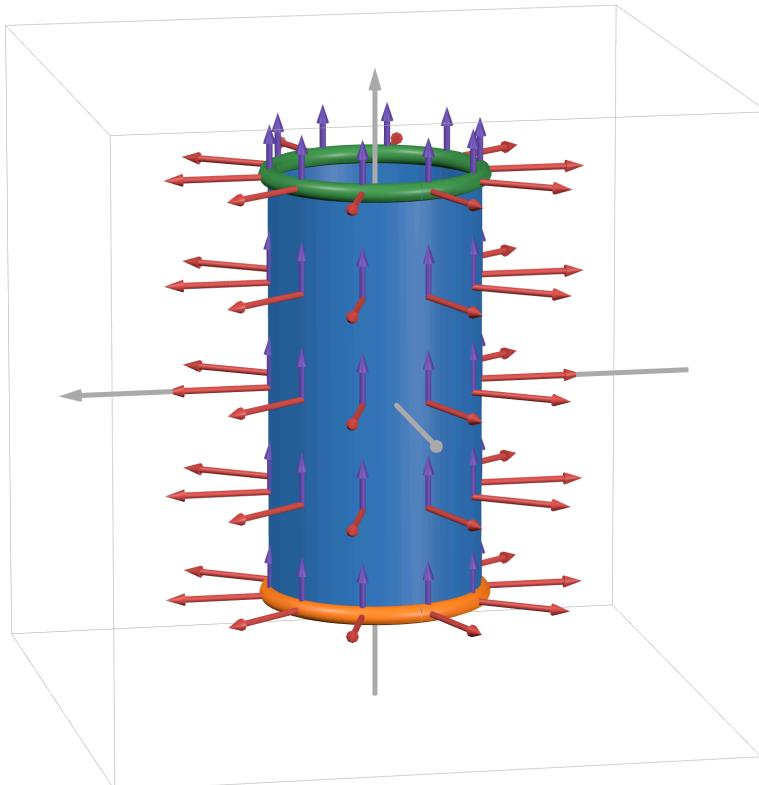
- Consider $\text{curl}(\mathbf{F})$ pointing vertically upward (or downward)

2.6 Completing the Proof (Example)



- Consider $\text{curl}(\mathbf{F})$ pointing vertically upward (or downward)
- We see that $\text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} = 0$, so we must have $\int_S \text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS = 0$

2.6 Completing the Proof (Example)



- Consider $\text{curl}(\mathbf{F})$ pointing vertically upward (or downward)
- We see that $\text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} = 0$, so we must have $\int_S \text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS = 0$
- When is this true? Whenever \mathbf{F} lies in the xy plane (i.e. $h = 0 \mid h_x, h_y, h_z = 0$ and, by the fact that \mathbf{F} is a function of x and y only, $f_z, g_z = 0$).

2.7 Recap of Stokes Theorem Example

- We found that some \mathbf{F} exists such that $\int_M d\omega = 0$, so we conclude that $\int_{\partial M} \omega = 0$, by Stokes Theorem

2.7 Recap of Stokes Theorem Example

- We found that some \mathbf{F} exists such that $\int_M d\omega = 0$, so we conclude that $\int_{\partial M} \omega = 0$, by Stokes Theorem
- We can divide ∂M into two boundaries and, since the integral of ω over both is the same, we can factor out the sum of the boundary orientations $\varepsilon_1 + \varepsilon_2$

2.7 Recap of Stokes Theorem Example

- We found that some \mathbf{F} exists such that $\int_M d\omega = 0$, so we conclude that $\int_{\partial M} \omega = 0$, by Stokes Theorem
- We can divide ∂M into two boundaries and, since the integral of ω over both is the same, we can factor out the sum of the boundary orientations $\varepsilon_1 + \varepsilon_2$
- The fact that $\int_{\partial M} \omega = 0$ implies $\varepsilon_1 + \varepsilon_2 = 0 \Rightarrow \varepsilon_1 = -\varepsilon_2$

2.7 Recap of Stokes Theorem Example

- We found that some \mathbf{F} exists such that $\int_M d\omega = 0$, so we conclude that $\int_{\partial M} \omega = 0$, by Stokes Theorem
- We can divide ∂M into two boundaries and, since the integral of ω over both is the same, we can factor out the sum of the boundary orientations $\varepsilon_1 + \varepsilon_2$
- The fact that $\int_{\partial M} \omega = 0$ implies $\varepsilon_1 + \varepsilon_2 = 0 \Rightarrow \varepsilon_1 = -\varepsilon_2$
- **Therefore, the boundaries at opposite ends of a cylinder have opposite orientations!**

2.8 The Power of Stokes

When you see a line integral that
is hard to compute



Figure 1: Literally the sole use of Stokes Theorem
(not Generalized Stokes Theorem, which is much more powerful, as we'll see)

Outline

1. Stokes Theorem

2. Some Examples

3. Applications?

4. References

3.1 Reality Check

Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now, as in the past, are inspired by the art of mathematics rather than by any prospect of ultimate usefulness.

— Eric Temple Bell

3.1 Reality Check

Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now, as in the past, are inspired by the art of mathematics rather than by any prospect of ultimate usefulness.

— Eric Temple Bell

Questions we must answer:

- Is Stokes Theorem truly useless?
- Have we as mathematicians failed in our quest to understand the universe?
- What is the nature of spacetime?

3.2 The Nature of Manifolds and their Boundaries

To appreciate the true power of Stokes Theorem, consider this trivial application:

$$\int_{\partial(\partial M)} \omega = \int_{\partial M} d\omega = \int_M d(d\omega) = \int_M 0 = 0$$

3.2 The Nature of Manifolds and their Boundaries

To appreciate the true power of Stokes Theorem, consider this trivial application:

$$\int_{\partial(\partial M)} \omega = \int_{\partial M} d\omega = \int_M d(d\omega) = \int_M 0 = 0$$

Therefore, we have concluded

$$\int_{\partial(\partial M)} \omega = 0 \quad \forall \omega$$

3.2 The Nature of Manifolds and their Boundaries

To appreciate the true power of Stokes Theorem, consider this trivial application:

$$\int_{\partial(\partial M)} \omega = \int_{\partial M} d\omega = \int_M d(d\omega) = \int_M 0 = 0$$

Therefore, we have concluded

$$\int_{\partial(\partial M)} \omega = 0 \quad \forall \omega$$

In other words, the integral of *anything* over the boundary of the boundary of *any* manifold M is zero. This is only possible if the manifold we are integrating over is the empty set \emptyset .

3.3 Closed Manifold Boundaries

Thus, the boundary of a manifold's boundary doesn't exist.

3.3 Closed Manifold Boundaries

Thus, the boundary of a manifold's boundary doesn't exist.

Phrased differently,

Proposition of Closed Manifold Boundaries

The boundary of any manifold M , ∂M , is **closed**.

where we say that a manifold is “closed” if it has no boundary.

3.3 Closed Manifold Boundaries

Thus, the boundary of a manifold's boundary doesn't exist.

Phrased differently,

Proposition of Closed Manifold Boundaries

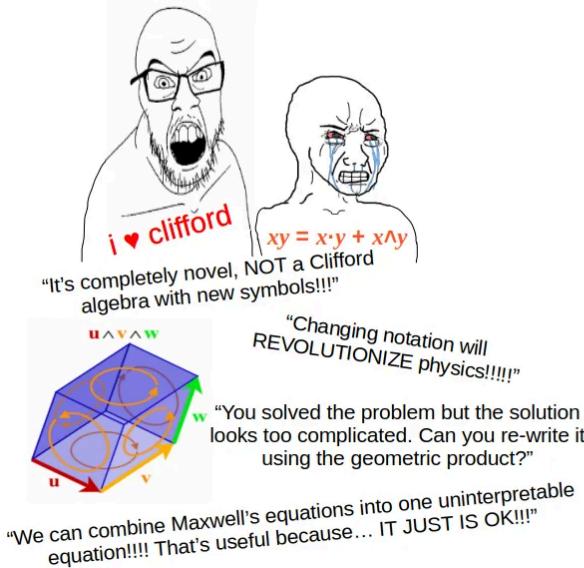
The boundary of any manifold M , ∂M , is **closed**.

where we say that a manifold is “closed” if it has no boundary.

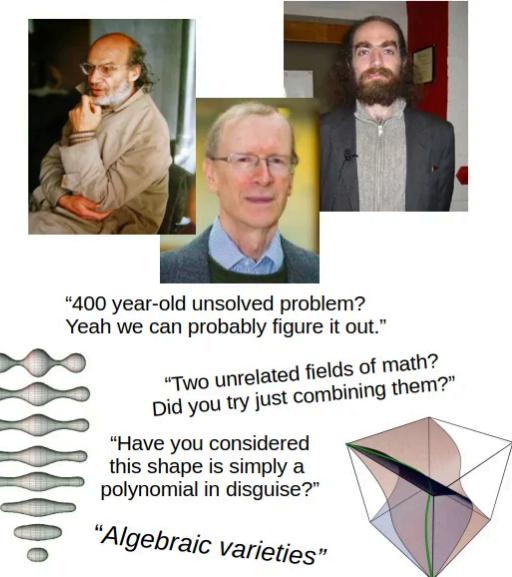
Remember when we stated this?? (and didn't prove it)

3.4 The Superiority of Generalized Stokes

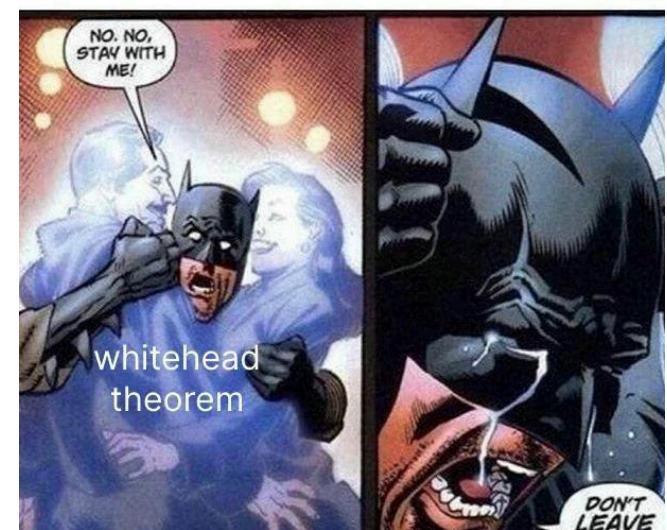
Average geometric algebra fan



Average algebraic geometry enjoyer



Me doing homotopy theory over an arbitrary ∞ -topos



3.4 The Superiority of Generalized Stokes

3.4.1 Higher Topological Powers

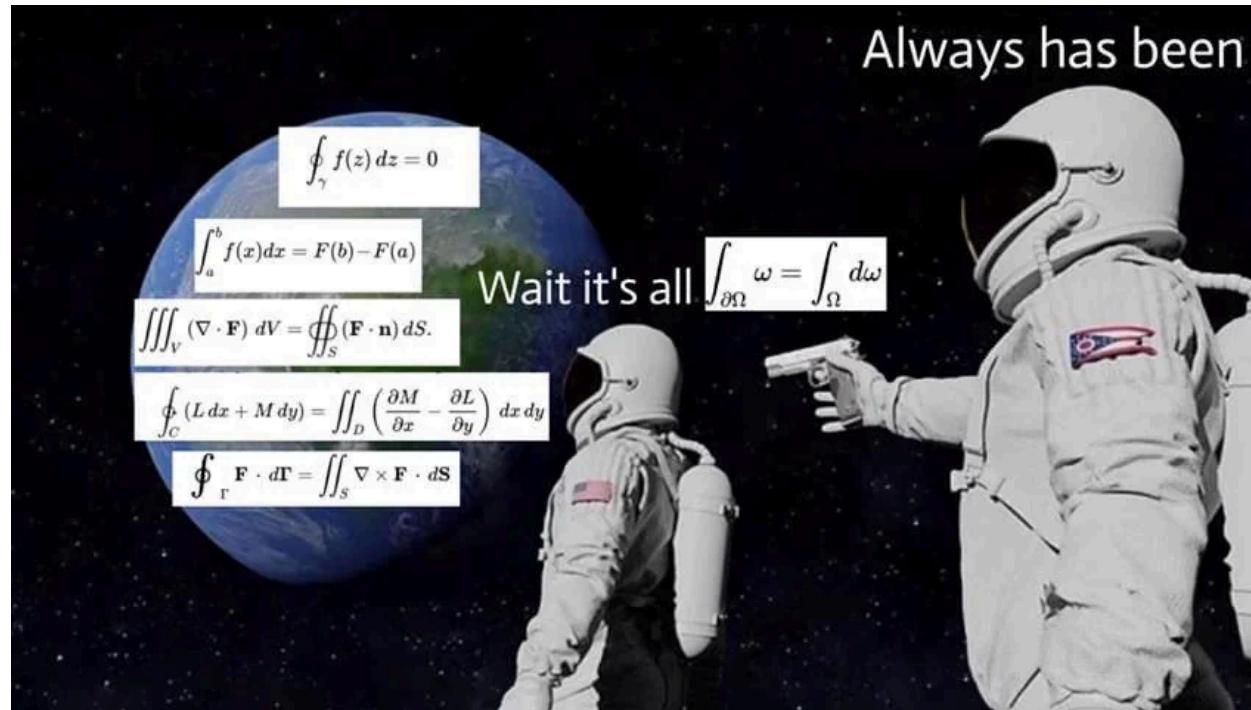


Figure 1: You have been lied to through all of calculus

3.5 Speedrunning Calculus with Generalized Stokes Theorem

“Novel” theorems (for higher-dimensional manifolds):

- Divergence Theorem
 - M is a 3 manifold in 3 space
- Stokes Theorem (already derived)
 - M is a 2 manifold in 3 space
- Green’s Theorem
 - M is a 2 manifold in 2 space

3.5 Speedrunning Calculus with Generalized Stokes Theorem

“Novel” theorems (for higher-dimensional manifolds):

- Divergence Theorem
 - M is a 3 manifold in 3 space
- Stokes Theorem (already derived)
 - M is a 2 manifold in 3 space
- Green’s Theorem
 - M is a 2 manifold in 2 space

“Degenerate” theorems (for one or zero dimensional manifolds):

- Fundamental Theorem of Calculus
 - M is a 1 manifold in 1 space

3.6 Divergence Theorem

M is a 3 manifold in 3 space

3.6 Divergence Theorem

M is a 3 manifold in 3 space

Let $\omega = \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$ (2 form in 3 space), integrated over ∂M

3.6 Divergence Theorem

M is a 3 manifold in 3 space

Let $\omega = \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$ (2 form in 3 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = \text{div}(\mathbf{F}) dx \wedge dy \wedge dz$

3.6 Divergence Theorem

M is a 3 manifold in 3 space

Let $\omega = \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$ (2 form in 3 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = \operatorname{div}(\mathbf{F}) dx \wedge dy \wedge dz$

M is a volume V , ∂M is its surface S , ω is the flux of \mathbf{F} through S : $\mathbf{F} \cdot \hat{\mathbf{n}} dS$, $d\omega = \operatorname{div}(\mathbf{F}) dV$

3.6 Divergence Theorem

M is a 3 manifold in 3 space

Let $\omega = \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$ (2 form in 3 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = \operatorname{div}(\mathbf{F}) dx \wedge dy \wedge dz$

M is a volume V , ∂M is its surface S , ω is the flux of \mathbf{F} through S : $\mathbf{F} \cdot \hat{\mathbf{n}} dS$, $d\omega = \operatorname{div}(\mathbf{F}) dV$

$$\int_{\partial M} \omega = \int_M d\omega$$

3.6 Divergence Theorem

M is a 3 manifold in 3 space

Let $\omega = \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$ (2 form in 3 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = \operatorname{div}(\mathbf{F}) dx \wedge dy \wedge dz$

M is a volume V , ∂M is its surface S , ω is the flux of \mathbf{F} through S : $\mathbf{F} \cdot \hat{\mathbf{n}} dS$, $d\omega = \operatorname{div}(\mathbf{F}) dV$

$$\int_{\partial M} \omega = \int_M d\omega$$

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iiint_V \operatorname{div}(\mathbf{F}) dV$$



3.7 Green's Theorem

M is a 2 manifold in 2 space

3.7 Green's Theorem

M is a 2 manifold in 2 space

Let $\omega = \mathbf{F} \cdot \langle dx, dy \rangle$ (1 form in 2 space), integrated over ∂M

3.7 Green's Theorem

M is a 2 manifold in 2 space

Let $\omega = \mathbf{F} \cdot \langle dx, dy \rangle$ (1 form in 2 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = \text{curl}(\mathbf{F}) dx \wedge dy$ (here we only consider the z component of curl, $N_x - M_y$)

3.7 Green's Theorem

M is a 2 manifold in 2 space

Let $\omega = \mathbf{F} \cdot \langle dx, dy \rangle$ (1 form in 2 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = \text{curl}(\mathbf{F}) dx \wedge dy$ (here we only consider the z component of curl, $N_x - M_y$)

M is an area A , ∂M is a curve C , ω is the line integral of \mathbf{F} over C : $\mathbf{F} \cdot d\mathbf{r}$, $d\omega = \text{curl}(\mathbf{F}) dA$

3.7 Green's Theorem

M is a 2 manifold in 2 space

Let $\omega = \mathbf{F} \cdot \langle dx, dy \rangle$ (1 form in 2 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = \text{curl}(\mathbf{F}) dx \wedge dy$ (here we only consider the z component of curl, $N_x - M_y$)

M is an area A , ∂M is a curve C , ω is the line integral of \mathbf{F} over C : $\mathbf{F} \cdot d\mathbf{r}$, $d\omega = \text{curl}(\mathbf{F}) dA$

$$\int_{\partial M} \omega = \int_M d\omega$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_A \text{curl}(\mathbf{F}) dA$$



3.8 The Fundamental Theorem of Calculus

M is a 1 manifold in 1 space

3.8 The Fundamental Theorem of Calculus

M is a 1 manifold in 1 space

Let $\omega = f(x)$ (0 form in 0 space), integrated over ∂M

3.8 The Fundamental Theorem of Calculus

M is a 1 manifold in 1 space

Let $\omega = f(x)$ (0 form in 0 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = f'(x) dx$

3.8 The Fundamental Theorem of Calculus

M is a 1 manifold in 1 space

Let $\omega = f(x)$ (0 form in 0 space), integrated over ∂M

Compute $d\omega$ to find $d\omega = f'(x) dx$

M is an interval $[a, b]$, ∂M is a set of points $\{a, b\}$ with orientations $\{\varepsilon_a, \varepsilon_b\}$, ω is the integral of $f(x)$ over that set of oriented points, $d\omega = f'(x) dx$

3.9 Integration over the Zero Manifold

An integral over a continuous manifold is an infinite (Riemann) sum, where at each point we multiply the integrand by dx . Essentially, we evaluate our differential form ω at each point, but make sure to choose a positive orientation for all points in the manifold. If a point is negatively oriented, we obtain the negative of the integral over the positively oriented point, so we multiply that integral by -1 to obtain the “positively oriented” integral value.

3.9 Integration over the Zero Manifold

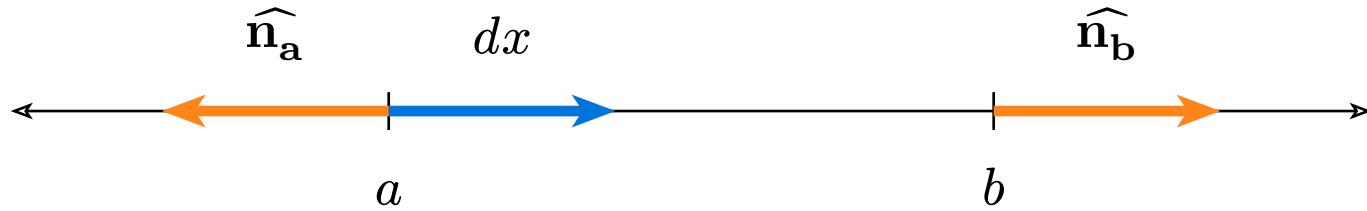
An integral over a continuous manifold is an infinite (Riemann) sum, where at each point we multiply the integrand by dx . Essentially, we evaluate our differential form ω at each point, but make sure to choose a positive orientation for all points in the manifold. If a point is negatively oriented, we obtain the negative of the integral over the positively oriented point, so we multiply that integral by -1 to obtain the “positively oriented” integral value.

3.10 Orientation of the Zero Manifold

3.10.1 What is the orientation of $\{a, b\}$ when dx points from a to b ?

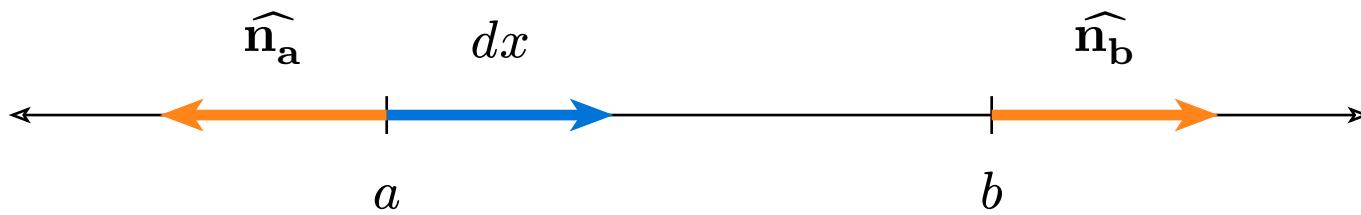
3.10 Orientation of the Zero Manifold

3.10.2 What is the orientation of $\{a, b\}$ when dx points from a to b ?



3.10 Orientation of the Zero Manifold

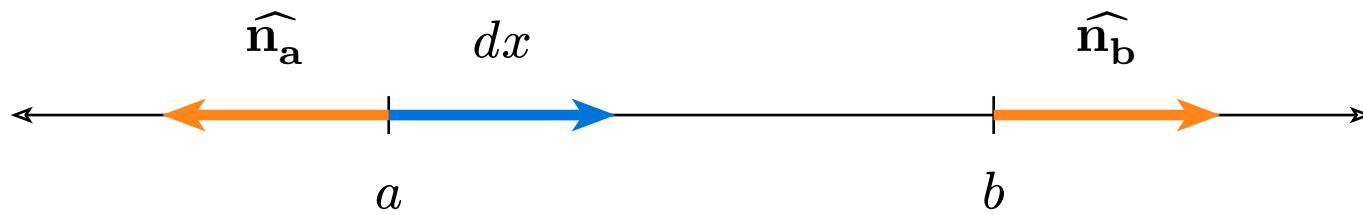
3.10.3 What is the orientation of $\{a, b\}$ when dx points from a to b ?



Expressing $\widehat{\mathbf{n}}_a$ in terms of dx leads to a negative coefficient, so the orientation of a is negative. $\widehat{\mathbf{n}}_b$ points in the same direction as dx , so it will have a positive coefficient when expressed as a multiple of dx , meaning it has positive orientation.

3.10 Orientation of the Zero Manifold

3.10.4 What is the orientation of $\{a, b\}$ when dx points from a to b ?



Expressing $\widehat{\mathbf{n}_a}$ in terms of dx leads to a negative coefficient, so the orientation of a is negative. $\widehat{\mathbf{n}_b}$ points in the same direction as dx , so it will have a positive coefficient when expressed as a multiple of dx , meaning it has positive orientation.

Notice how a^+ and b^- have opposite orientations—**just like the cylinder!**

3.11 Deriving the Fundamental Theorem of Calculus

$$\begin{aligned}\int_{\partial M} \omega &= \int_M d\omega \\ \int_{\{a,b\}} f(x) &= \int_{[a,b]} f'(x) \, dx\end{aligned}$$

3.11 Deriving the Fundamental Theorem of Calculus

$$\begin{aligned}\int_{\partial M} \omega &= \int_M d\omega \\ \int_{\{a,b\}} f(x) &= \int_{[a,b]} f'(x) \, dx\end{aligned}$$

We let $\varepsilon_a = -1$ and $\varepsilon_b = 1$ and evaluate the left integral:

$$\int_{\{a,b\}} f(x) = \varepsilon_a f(a) + \varepsilon_b f(b) = -f(a) + f(b) = f(b) - f(a)$$

3.11 Deriving the Fundamental Theorem of Calculus

$$\begin{aligned}\int_{\partial M} \omega &= \int_M d\omega \\ \int_{\{a,b\}} f(x) &= \int_{[a,b]} f'(x) \, dx\end{aligned}$$

We let $\varepsilon_a = -1$ and $\varepsilon_b = 1$ and evaluate the left integral:

$$\int_{\{a,b\}} f(x) = \varepsilon_a f(a) + \varepsilon_b f(b) = -f(a) + f(b) = f(b) - f(a)$$

Thus,

$$f(b) - f(a) = \int_a^b f'(x) \, dx$$



3.12 Final Words

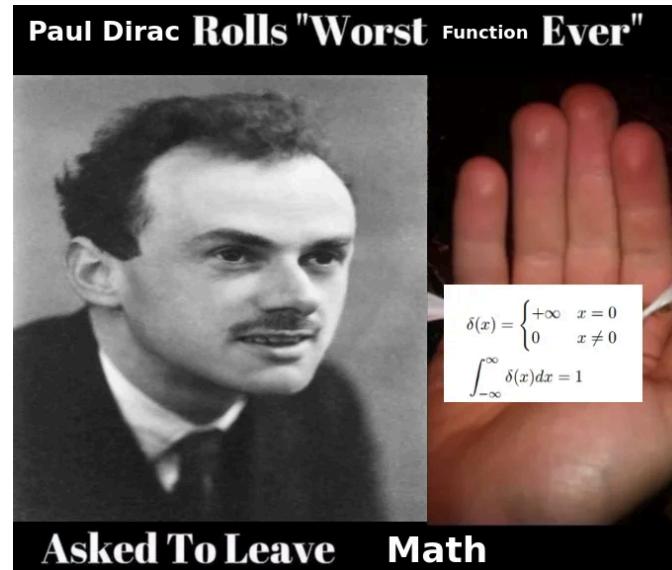
Category theory is the subject where you can leave the definitions as exercises.

— John Baez

3.12 Final Words

Category theory is the subject where you can leave the definitions as exercises.

— John Baez



3.13 Thank you!



Figure 2: I'll try my best to avoid it

4. References

4. References

Chonoles, Zev. n.d.. Accessed December 9, 2024. [https://math.uchicago.edu/~chonoles/
miscellany/quotations/](https://math.uchicago.edu/~chonoles/miscellany/quotations/)

Kulish, Dmytro, Yaroslav Ibragimov, and Dang Minh Cong. 2023. “The Generalized Stokes Theorem and Its Applications”. Boston, MA. May 2023. [https://math.mit.edu/research/
highschool/primes/YuliasDream/2023/slides/2-4-IKM.pdf](https://math.mit.edu/research/highschool/primes/YuliasDream/2023/slides/2-4-IKM.pdf)

nLab authors. 2024b. “Differential Form”

nLab authors. 2024d. “Manifold with Boundary”

nLab authors. 2024c. “Orientation”

nLab authors. 2024a. “Stokes Theorem”

4. References

- Nolte, David D. 2019. “Looking Under the Hood of the Generalized Stokes Theorem”. December 2019. <https://galileo-unbound.blog/2019/12/16/looking-under-the-hood-of-the-generalized-stokes-theorem/>
- Presman, Rick. 2012. “The Generalized Stokes Theorem”. University of Chicago Department of Mathematics. August 2012. <https://math.uchicago.edu/~may/REU2012/REUPapers/Presman.pdf>
- Sämann, Christian. 2017. “An M5-Brane Model”. Edinburgh, UK. May 2017. <https://christian-saemann.de/talksfolder/5.12.2017%20Dublin%20Institute%20for%20Advanced%20Studies.pdf>

4.1 Other References

- Separate discussions with Evan Chen PhD '25 and Angelo Farfan '28
- MIT OpenCourseWare 18.02 Lecture on Stokes Theorem
- Mathematical Folklore