Stokes Theorem as Generalized Abstract Vapid Nonsense

“We’ll only use as much category theory as is necessary. [*famous last words*]” –Roman Abramovich[1]

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11 December 2024

[1]Attribution: “Higher Gauge Theory: The Fundamentals” (quote is almost certainly misattributed to a Russian oligarch instead of the algebraic geometrist who is likely responsible for it)

Outline

1.1 Manifolds

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- Manifolds are just a continuous set of points
- Moreover, at any given point, they look like a **Euclidean space**

1.2 Euclidean Spaces

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- Euclidean space is the space we inhabit (not the full story; more on this later)

What actually is a manifold then?

- When we say that a manifold "looks like" a Euclidean space, we're actually talking about **homeomorphism**

1.3 Homeomorphisms

1.4 General Topology

- Topologists do not understand shapes, so they must deform objects into meaningless blobs to count holes
- Two numbers are equal, but two topological spaces are **homeomorphic** to each other

1.4 General Topology

1.4.1 Definitions

Homeomorphism

A pair of continuous bijective (all $x \in X$ in the domain of f and all $y \in Y$ in the range of f) mappings $f : X \longrightarrow Y$ and $f^{-1} : Y \longrightarrow X$

Homeomorphic equivalence

Two topological objects X and Y are homeomorphic to each other iff there exists a **homeomorphism** between them.

1.5 Formalizing the Manifold

1.5.1 Definitions

Read: more abstract nonsense

Manifold

A k (represents intrinsic dimension) manifold is a continuous set of points X **homeomorphic** to $\mathbb{R}^k$

Embedding

The embedding of a manifold X is n iff $x \in \mathbb{R}^n \ \forall x \in X$

Note: If a k manifold is embedded in n space ($\mathbb{R}^n$), we must have $n \geq k$ (this is a trivial consequence of an advanced mathematical technique called "visualization")

1.6 Manifolds in the Wild

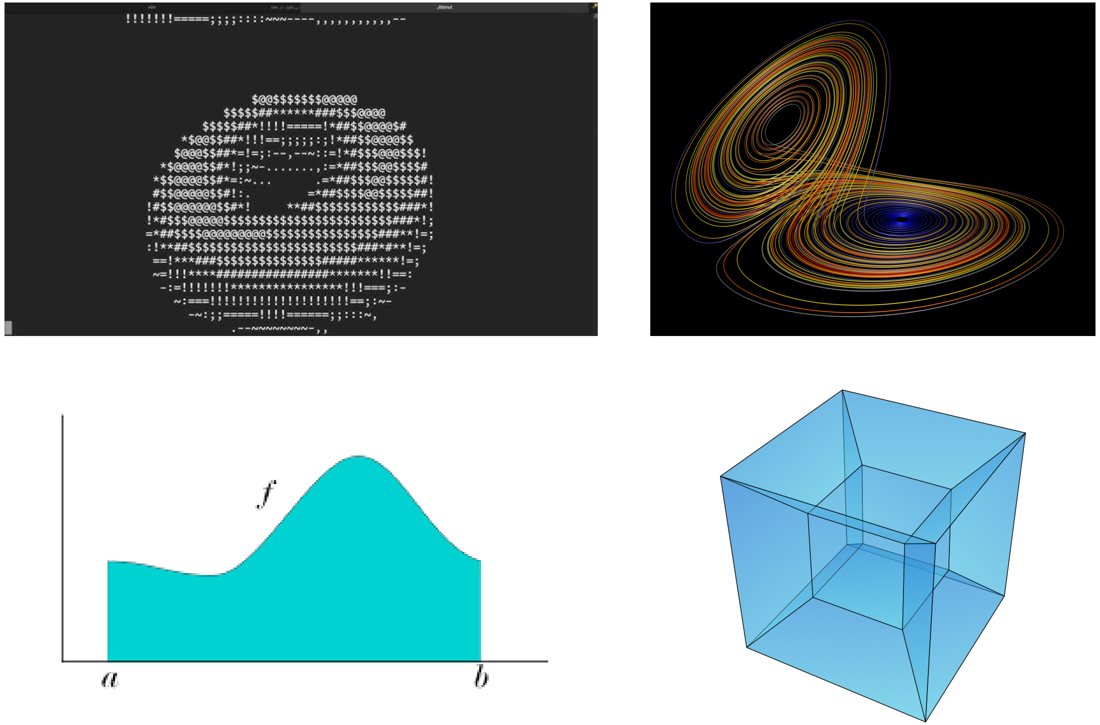


Figure 1: From left to right, top to bottom*: A 2 manifold in 3 space, a 1 manifold in 3 space, a 2 manifold in 2 space, and a 2 manifold in 4 space (projected onto 3 space–the drawing is of a hypercube)*

1.6 Manifolds in the Wild

1.6.1 The Prototypical Example of a Manifold

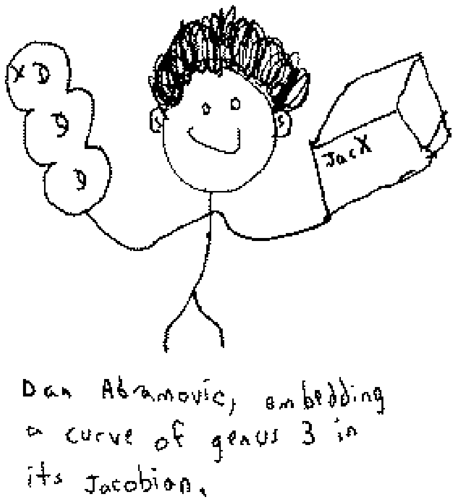
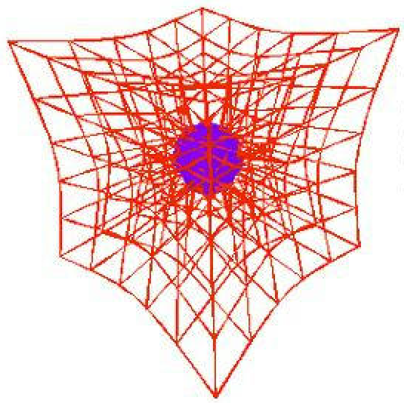


Figure 2: Don't ask me what this is because I am just as clueless as you

1.7 The Universe

We live in a **3 manifold** (possibly embedded in a *higher dimensional* space–and we have no way of knowing)!

Completely irrelevant to understanding Generalized Stokes Theorem, but very cool nevertheless.



This three-dimensional grid gives a better idea of what curved space-time might look like than the two-dimensional analogies do.

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Now, consider the most abstract geometric properties the determinant should have.

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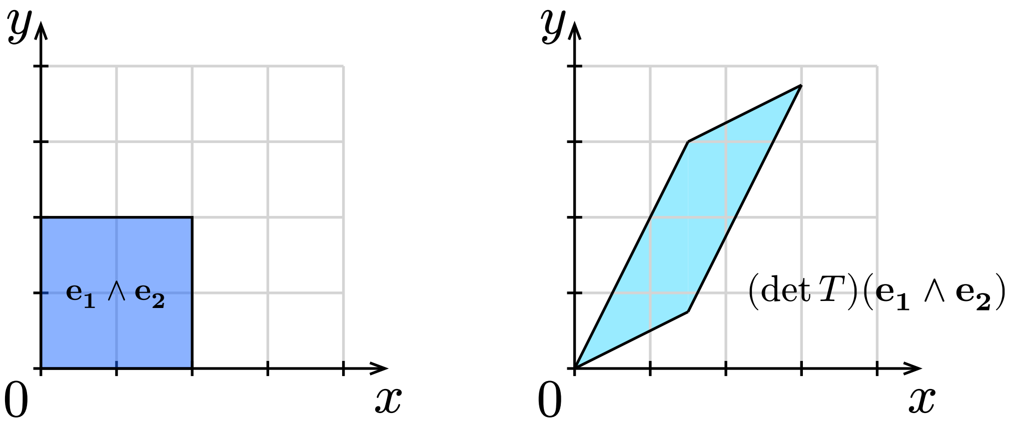

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- Bivectors are formed using a very scary operator called the **wedge product**.
- Wedge products between any two vectors $\mathbf{a}$ and $\mathbf{b}$ exist in a vector space (we are free to define the basis vectors, but they look like some wedge product, e.g. $\mathbf{e_1} \wedge \mathbf{e_2}$

1.11 Multivolumes

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Multilinearity!

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$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$

This follows from the previous properties; this is why the determinant is ***antisymmetric****!*

1.13 Interesting Derivation of Antisymmetry Property

By the fact that $\mathbf{u} \wedge \mathbf{u} = 0$, we have:

$$(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b}) = 0 \quad (\mathbf{u} = \mathbf{a} + \mathbf{b})$$

Distribute!

$$\mathbf{a} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} + \mathbf{b} \wedge \mathbf{b} = 0$$

Use $\mathbf{u} \wedge \mathbf{u} = 0$ again! (but for different $\mathbf{u}$)

$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0$$
$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Only the **signed multivolume** of a parallelotope as a function of its side vectors is multilinear.

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Line integrals are really just:

$$\int\ (f_1(t)\ d\mathbf{e_1} + f_2(t)\ d\mathbf{e_2} + \cdots + f_n\ d\mathbf{e_n})$$

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We can integrate over any surface of an n dimensional space, where our differential is a **bivector** formed by two of $d\mathbf{e_1}$, ..., $d\mathbf{e_n}$. There are therefore $\binom{n}{2}$ differentials that make up the 2-form in n space, and the function $\mathbf{f}$ we integrate over must have this dimension.

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$$\int\ (f_1(u, v)\ d\mathbf{e_1} \wedge \mathbf{e_2} + \cdots + f_2(u, v)\ d\mathbf{e_1} \wedge \mathbf{e_n} + \cdots + f_n(u, v)\ d\mathbf{e_{n-1}} \wedge \mathbf{e_n})$$

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- Note that we always calculate the flux of a vector field $\mathbf{F}$ (dim 3) over a surface (dim 2) embedded in xyz space (dim 3).
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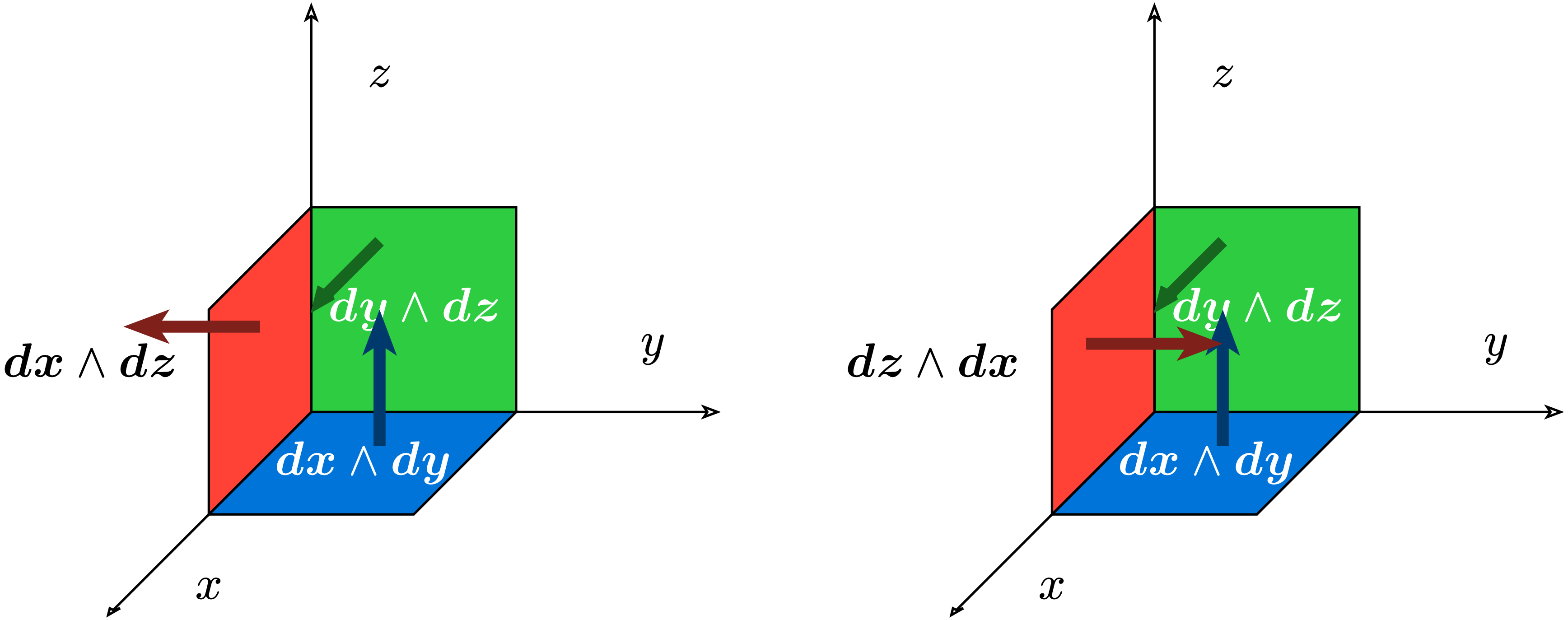
Let $\mathbf{F} = \langle A(x,y,z), B(x,y,z), C(x,y,z)\rangle$. We will calculate the flux outward through each surface in the 2-form. We will talk more about orientation later, but for the purpose of flux, we define the normal vector as being the cross product of two surface vectors (with right hand rule).

1.17 Surface Integrals



We want flux to be positive when anything “enters” this cube and negative when stuff “leaves”–but the mathematics of differential forms allow us to integrate any way we like

1.18 The Flux Integral

$$\int (A(x,y,z)\ dy \wedge dz + B(x,y,z)\ dz \wedge dx + C(x,y,z)\ dx \wedge dy)$$

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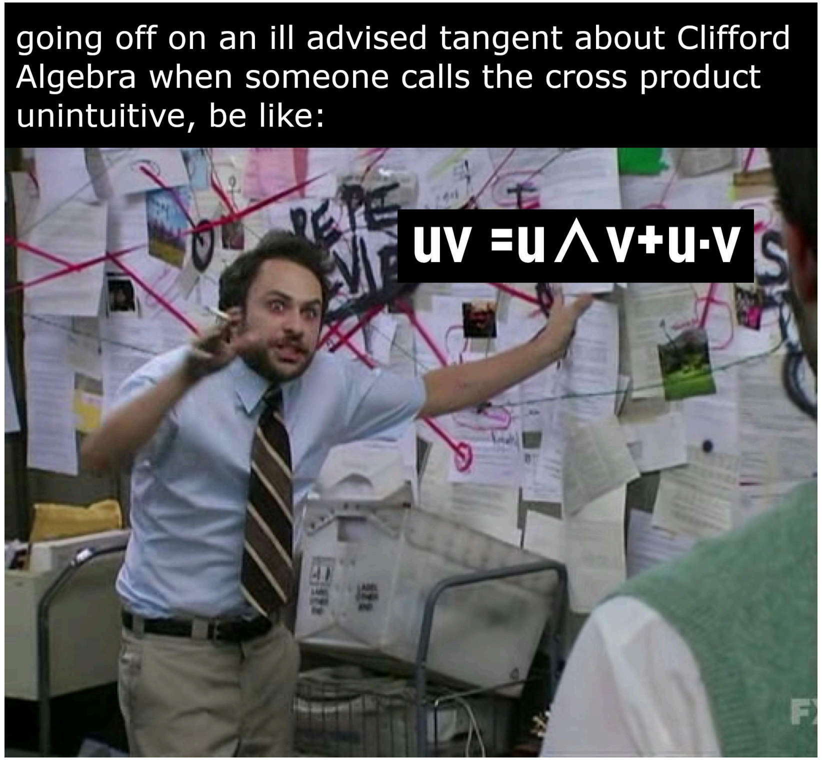
$$\int \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$$

How do we calculate such a monstrosity?
What is the meaning of this abstract nonsense?

1.19 The Pullback



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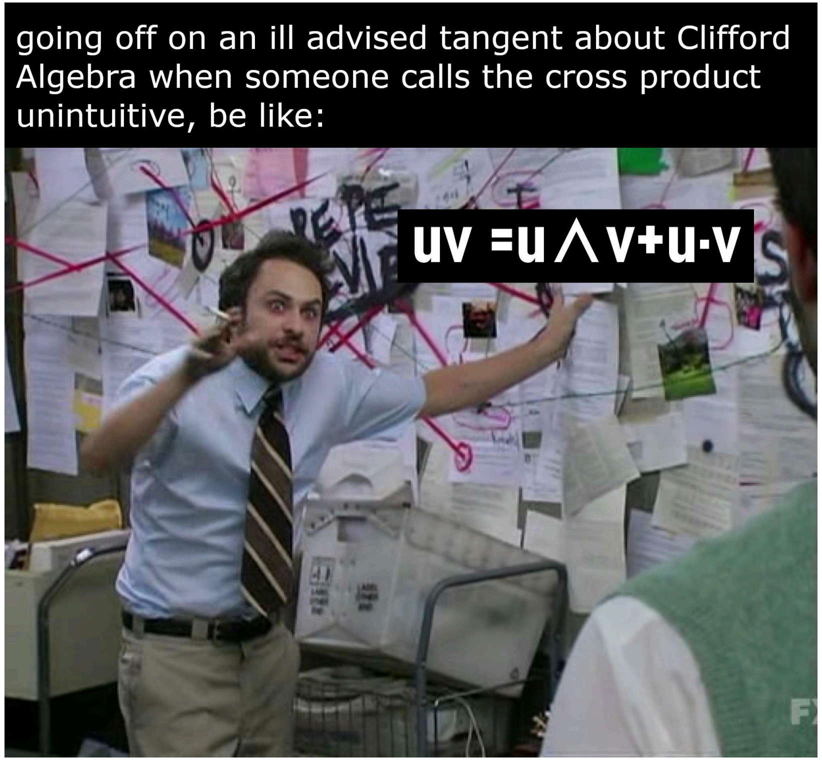

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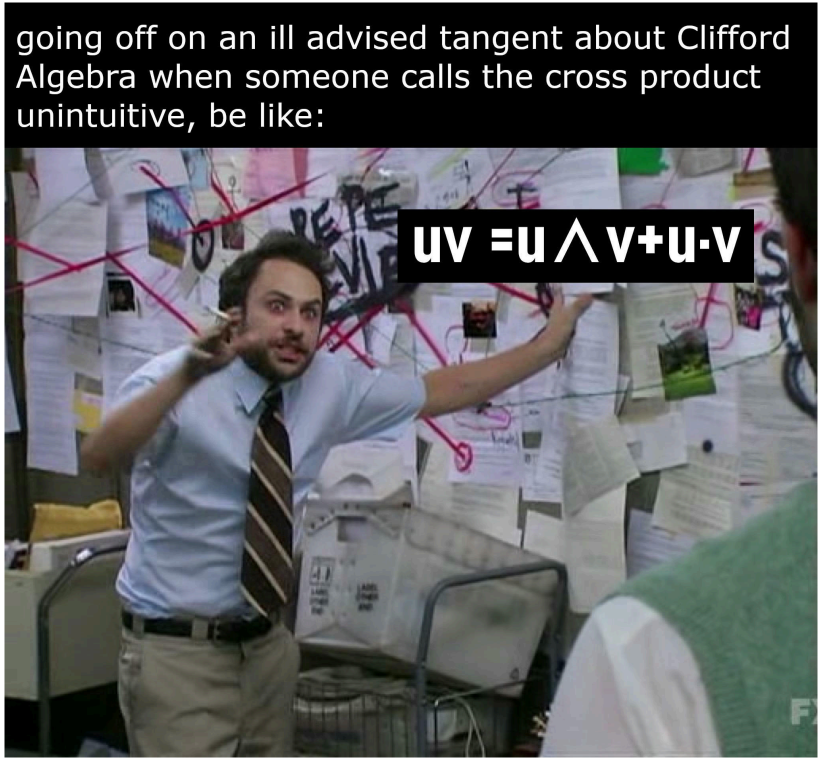

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- We note that we can represent $\boldsymbol{F}$, a 2 manifold, using a 2 dimensional parameterization in terms of u, v
- We now need to "convert" $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$ to $du \wedge dv$

1.20 Mathematical Maneuvering

- Remember that a manifold is **homeomorphic** to a Euclidean space (in this case, $\mathbb{R}^2$). This means there exists a mapping* $x, y, z \to u, v$, so we can write $u(x, y, z), v(x, y, z)$.

*where x, y, z are in some subset of $\mathbb{R}^3$ (occupied by the manifold)

- This mapping is called a "**chart**," which is part of an "**atlas**" for the manifold
  - Mathematicians take this map analogy very seriously because they cannot distinguish between homeomorphic objects in real life

> Topology is precisely the mathematical discipline that allows the passage from local to global...
>
> — René Thom (itinerant theoretician)

Don't let your mind be warped by these abstractions!

1.20 Mathematical Maneuvering

1.20.1 Computation

- Given $u(x, y, z), v(x, y, z)$ and $\boldsymbol{F}(u, v)$, consider the inverse map $u, v \to x, y, z$ that must also exist by the conditions of **homeomorphism** to find $x(u, v), y(u, v), z(u, v)$.
- Now, take differentials using the chain rule:

$$dx = \frac{\partial x}{\partial u}\ du + \frac{\partial x}{\partial v}\ dv,\ dy = \frac{\partial y}{\partial u}\ du + \frac{\partial y}{\partial v}\ dv,\ dz = \frac{\partial z}{\partial u}\ du + \frac{\partial z}{\partial v}\ dv$$

- We can now use the properties of the **wedge product** to evaluate any product of dx, dy, and dz in terms of either $du \wedge dv$ or $dv \wedge du$ (but remember $dv \wedge du = -du \wedge dv$).
- This gives an integral of the form:

$$\int \boldsymbol{G}(u, v)\ du \wedge dv = \iint \boldsymbol{G}(u, v)\ du\ dv$$

1.21 Integration with Differential Forms

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- We can also take wedge products of more than 2 vectors to get multivectors, but these have only two canonical orientations
  - More on this later, but for now just know that we can always flip terms to get them in a certain order by multiplying either by 1, positive orientation, or -1, negative orientation

1.22 Notational Hacks

- We call the entire expression after the integral sign, when expressed with multivectors, a **differential form**, commonly denoted ω
  - Note that multivectors won't always have a wedge product symbol (for one or zero (we'll encounter those later) manifolds we only need a differential or scalar, respectively)
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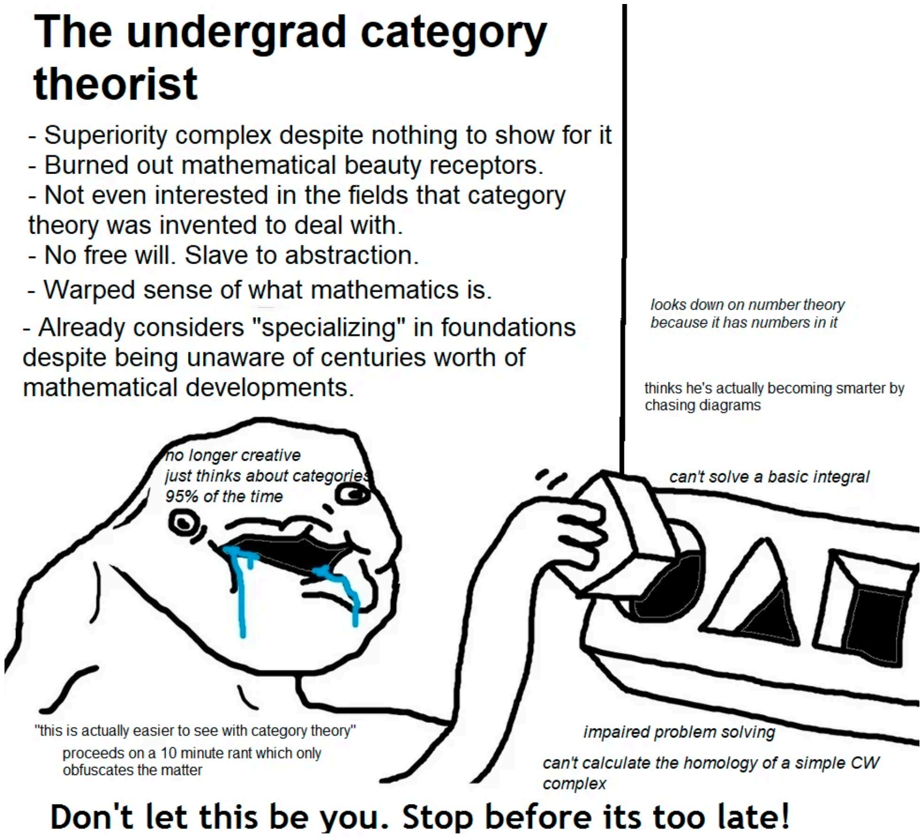
  - Common notation is to use ω for a differential form and Ω for a manifold, just to be extra confusing
- We use a completely made-up, contrived operation called the **pullback** to turn this abstract mess of symbols into a readable integral we can evaluate
  - Basically, use an intrinsic coordinate system that represents the manifold itself and not the space it's embedded in

1.23 Mandatory Content Warning



1.24 Orientability

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> A mathematician who can only generalise is like a monkey who can only climb up a tree, and a mathematician who can only specialise is like a monkey who can only climb down a tree. In fact neither the up monkey nor the down monkey is a viable creature. A real monkey must find food and escape his enemies and so must be able to incessantly climb up and down. A real mathematician must be able to generalise and specialise.
>
> — George Polya

1.25 Differential Forms and Orientability

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- In 2 space, it is $dx \wedge dy$, by convention
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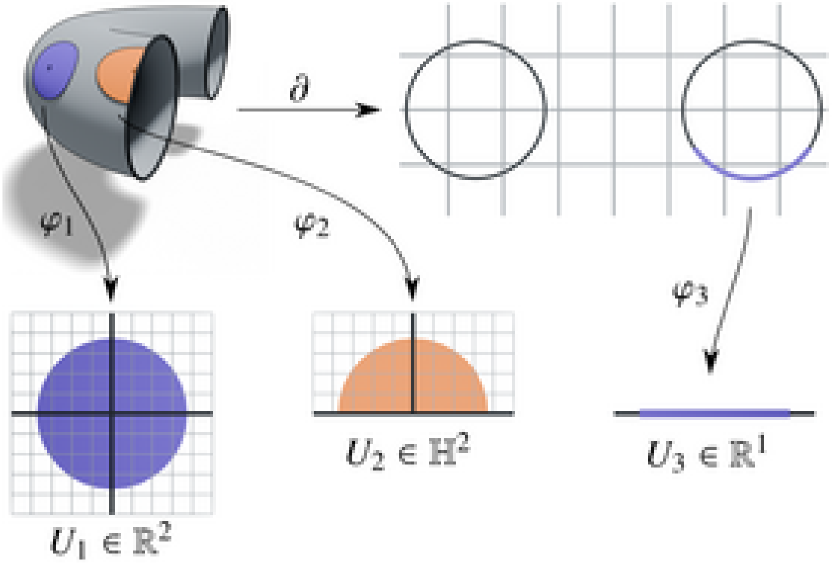
More on 0 space later ...

1.26 Boundaries



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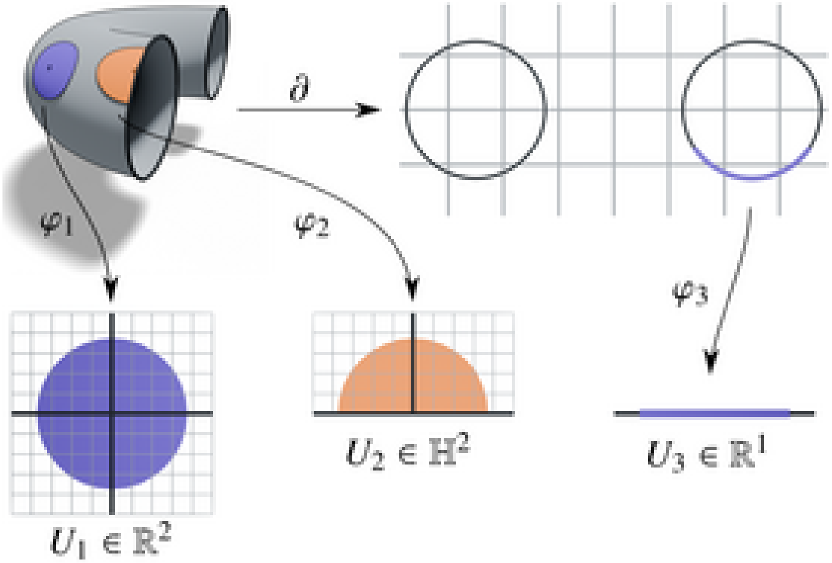

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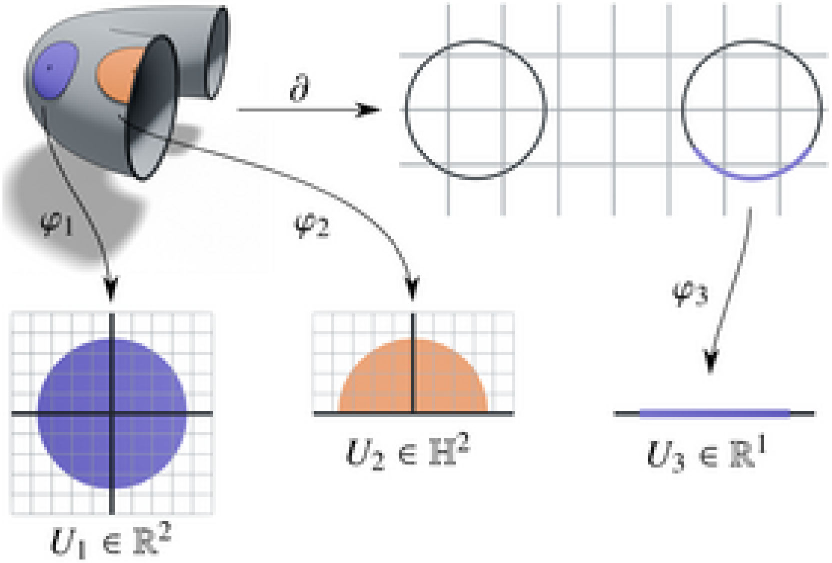

- To define the boundary, we consider **homeomorphic equivalence classes**
- Specifically, a point within a k manifold M will have a local neighborhood of points homeomorphic to a k ball
- At the boundary, points have a local neighborhood homeomorphic to a half k ball

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> "It is the snobbishness of the young to suppose that a theorem is trivial because the proof is trivial."
>
> — Henry Whitehead

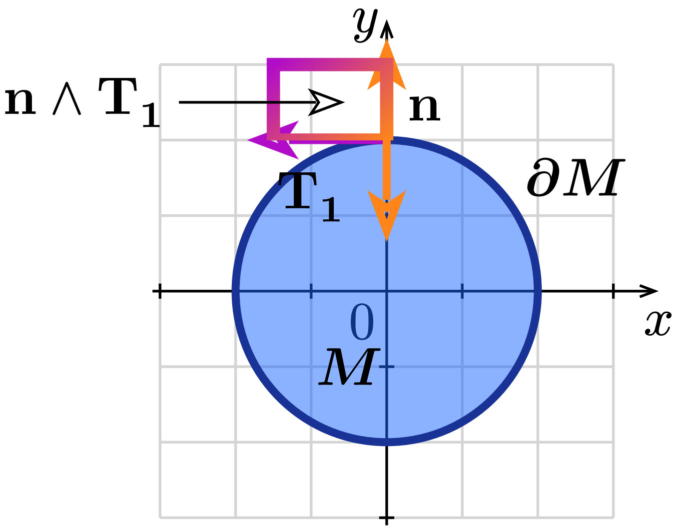
1.28 Proof of Properties of a Boundary

If you're interested:

our definition $\Rightarrow \exists\ \varphi : U \to \mathbb{H}^k$ where U is the neighborhood of points around some $P \in \partial M$ and $\mathbb{H}^k$ is a half k ball, formally defined as $\mathbb{H}^k = \{x \in \mathbb{R}^k \mid x_k \geq 0\}$ for some intrinsic coordinate system where $x_k = 0$ for points along the boundary in U (otherwise a local neighborhood around these points would be homeomorphic to $\mathbb{R}^k$, not $\mathbb{H}^k$).* Thus, ∂M (around P) is homeomorphic to $\{x \in \mathbb{H}^k \mid x_k = 0\}$, which is homeomorphic to $\mathbb{R}^{k-1}$ (simply reparameterize by eliminating x_k and keeping all other parameters the same–for the inverse map add $x_k = 0$). We'll later see how to prove this boundary is closed using Generalized Stokes Theorem!

*These are quite shaky foundations indeed, but a full proof would result in too much brain damage to present in its entirety.

1.29 Induced Orientations



- When we have an n manifold M with a known positive orientation in n space, we can determine the orientation of its boundary ∂M (its "**induced orientation**")

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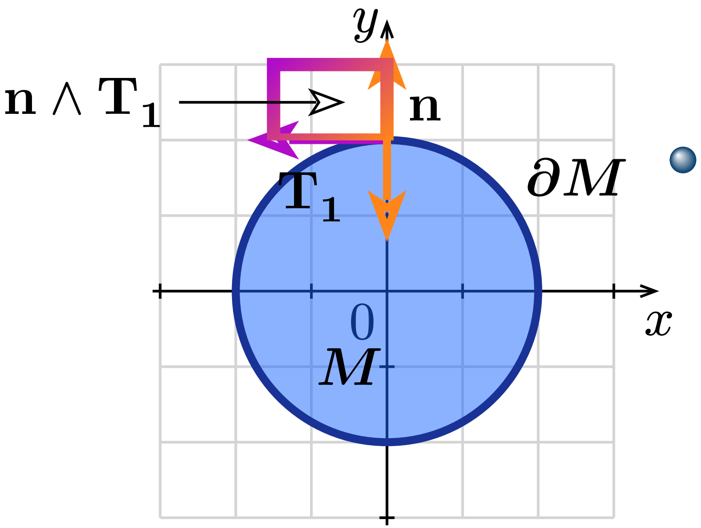

- When we have an n manifold M with a known positive orientation in n space, we can determine the orientation of its boundary ∂M (its "**induced orientation**")
- For each parameter defining ∂M (in this case there is only one, t), draw the tangent vector T (which for this M we can write as $\frac{d\mathbf{r}}{dt}dt$ where $\mathbf{r}$ is the mapping from 1-dimensional t space to the ∂M manifold embedded in 2 space

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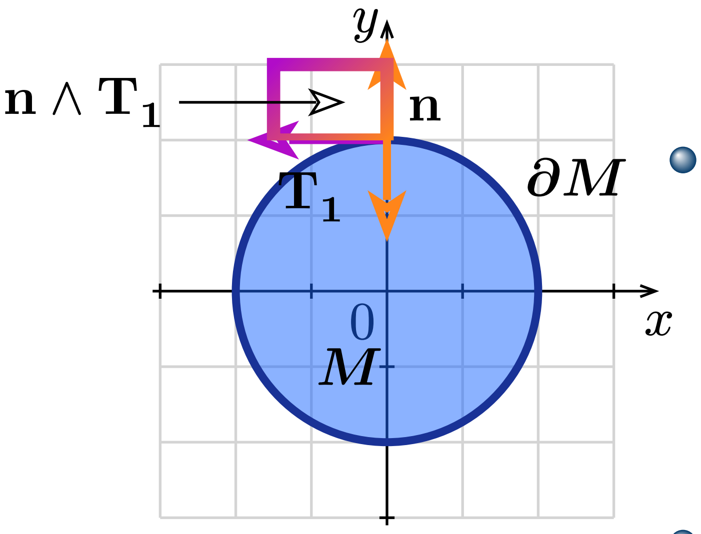

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- Compute the normal vector $\mathbf{n}$ orthogonal to all tangent vectors and not pointing into M

1.30 Induced Orientations and Wedge Products

- Now, write $\mathbf{n} \wedge \mathbf{T_1}$ as some expression of the form $A\ dx \wedge dy$, where $dx \wedge dy$ is the known positive orientation of M.

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- Now, write $\mathbf{n} \wedge \mathbf{T_1}$ as some expression of the form $A\ dx \wedge dy$, where $dx \wedge dy$ is the known positive orientation of M.
- The orientation of ∂M is given by $\operatorname{sign}(A)$
  - Note that this is either *positive* or *negative* (there are only ever 2 canonical orientations of a manifold, regardless of dimension)

1.31 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

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1.32 The Prototypical Non-Orientable Surface

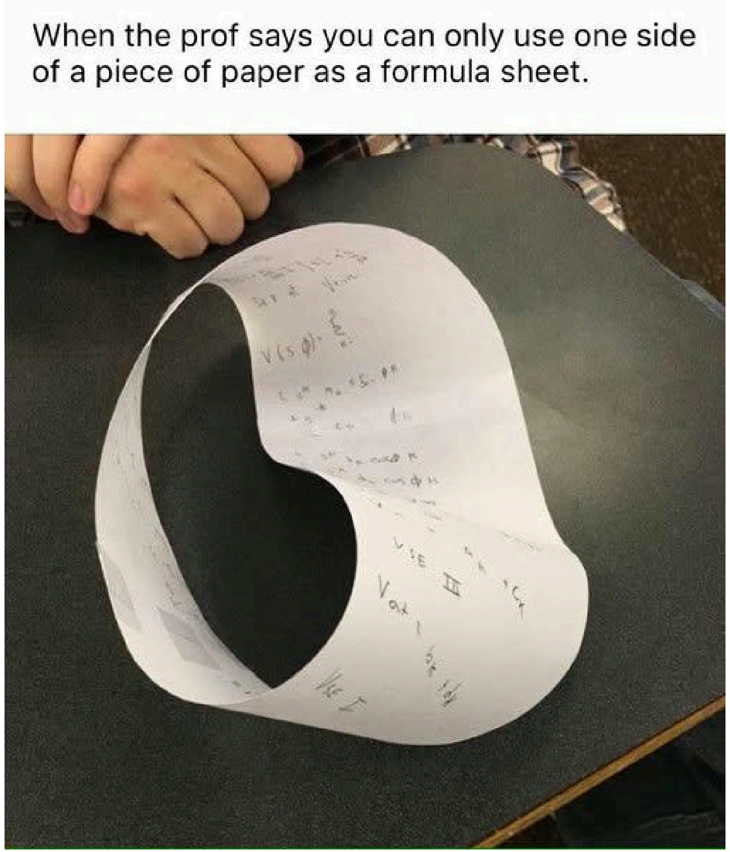

Figure 1: Not possible to define a unique $\mathbf{n}$ *for every point (any point) on the surface*

1.33 Exterior Derivative

Consider a nasty differential form like:

$\omega = M(x, y, z)\ dy \wedge dz + N(x, y, z)\ dz \wedge dx + P(x, y, z)\ dx \wedge dy$

where $\mathbf{F} = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$ (remember flux??)

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Can we *differentiate* ω?

1.34 Digression to Introduce the Scale of the Problem

We will need to use some very simple notions of category theory, an **esoteric subject** noted for its **difficulty** and **irrelevance**.

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You should be thinking of the Fundamental Theorem of Calculus right now (we'll get back to this).

1.35 Definition of the Exterior Derivative

Exterior Derivative

Given an n form $\omega = f\ dx_1 \wedge \ldots \wedge dx_n$, we define the exterior derivative $d\omega = df \wedge dx_1 \wedge \ldots \wedge dx_n$, an $n + 1$ form (trivial to see given we have added another wedge product and df is a 1 form)

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$$d\omega = f_x\ dx \wedge dy \wedge dz + g_y\ dy \wedge dz \wedge dx + h_z\ dz \wedge dx \wedge dy$$
$$d\omega = \left(f_x + g_y + h_z\right)\ dx \wedge dy \wedge dz$$

1.36 Powers of the Exterior Derivative

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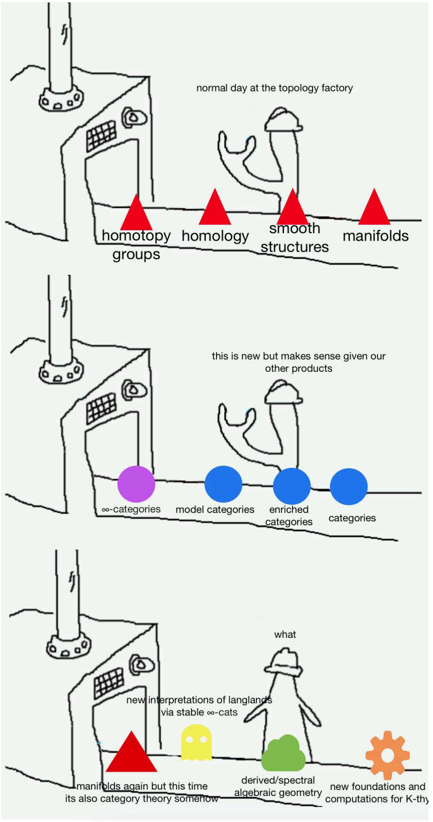
- Flux becomes divergence?
- Where have you seen this before??



- How Generalized Stokes Theorem is actually defined using homotopy type theory
- Involves topos theory and other abstract nonsense
- We will stick to the algebraic topology definition, which is only slightly less deranged than the category theoretic one

1.38 Generalized Stokes Theorem

Topological Definition of Generalized Stokes Theorem

$$\int_{\partial M} \omega = \int_M d\omega$$

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This presentation could end here and you would leave knowing a fundamental truth of the universe.

Outline

2.1 Deriving Stokes Theorem from Generalized Stokes Theorem

Let’s consider what Stokes Theorem tells us about a 2 manifold in 3 space:

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↑ Abuse of notation to demonstrate what we mean

We know ω must be a 1 form, but this isn't helpful until we can assign a value to ω. To keep things simple, let's make the left-hand integral and line integral, so:

$\omega = f(x, y, z)\ dx + g(x, y, z)\ dy + h(x, y, z)\ dz$

$\int_{\partial M} \omega = \int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$

2.2 Stokes Theorem, Revisited

We then have:

$$d\omega = \left(f_y\ dy + f_z\ dz\right) \wedge dx + \left(g_x\ dx + g_z\ dz\right) \wedge dy + \left(h_x\ dx + h_y\ dy\right) \wedge dz$$
$$d\omega = \left(g_x - f_y\right)\ dx \wedge dy + \left(f_z - h_x\right)\ dz \wedge dx + \left(h_y - g_z\right)\ dy \wedge dz$$

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$$d\omega = \mathrm{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}}\ dS$$

It’s the **flux** of the **curl** of **F**!!

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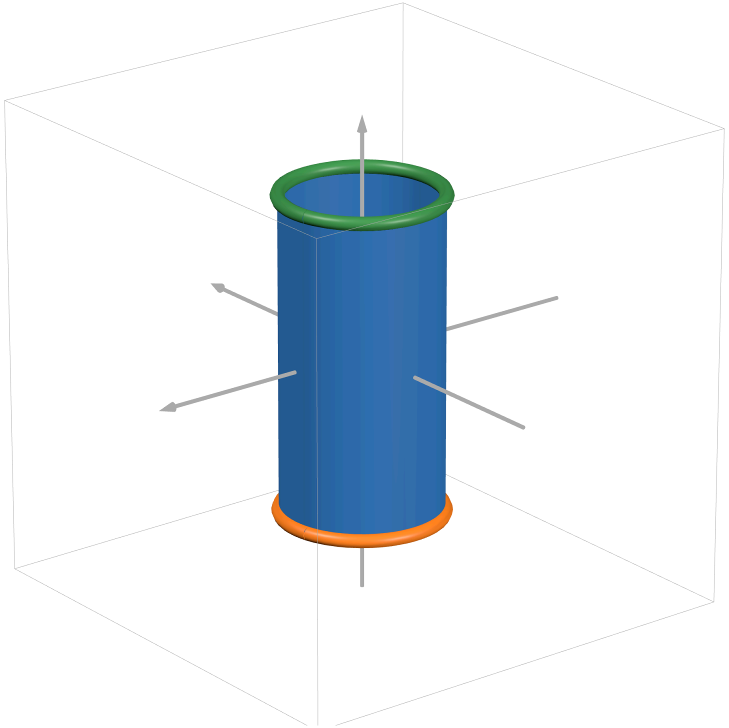
$$d\omega = \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}}\ dS$$

It's the **flux** of the **curl** of $\mathbf{F}$!!

By Generalized Stokes,

$$\int_{\partial M} \omega = \int_M d\omega \Rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}}\ dS$$

2.3 An Interesting Example Involving Stokes Theorem



- Using a clever choice of $\mathbf{F}$, we will show that the orientation of the boundaries at opposite ends of a cylinder are opposite each other.
- Note that ∂M consists of two manifolds C_1, C_2
- $\int_{\partial M} \omega = \varepsilon_1 \int_{C_1} \omega + \varepsilon_2 \int_{C_2} \omega$ where $\varepsilon_1, \varepsilon_2$ are the orientations of C_1 and C_2, respectively

2.4 Orientation Theory

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This tells us $\mathbf{F}$ is independent of z. Since C_1 and C_2 occupy the same region of xy space, if $\mathbf{F}$ depends only on these two parameters, the line integrals over C_1 and C_2 are equivalent and we can factor our $\varepsilon_1 + \varepsilon_2$ from the boundary integral.

2.5 Stokes Theorem and Orientation

For some $\mathbf{F}(x, y)$ on M, Stokes Theorem tells us

$$\int_{\partial M} \omega = \varepsilon_1 \int_{\boldsymbol{C_1}} \omega + \varepsilon_2 \int_{\boldsymbol{C_2}} \omega = (\varepsilon_1 + \varepsilon_2) \int_{\boldsymbol{C_1}} \omega = \int_M d\omega = \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$$

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If $\varepsilon_1 + \varepsilon_2 = 0$ as we supposed,

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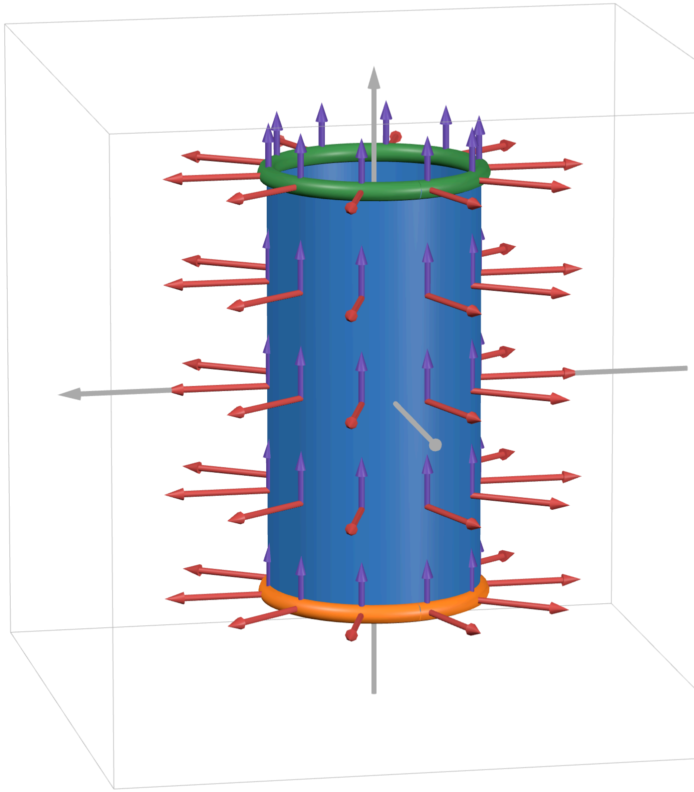
Hence, in order to prove $\varepsilon_1 + \varepsilon_2 = 0$, we must choose $\mathbf{F}(x, y) \mid \int_S \operatorname{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}}\ dS = 0$. The beauty of this is that we can choose *any* differential form ω and Generalized Stokes Theorem must still hold (in the case of the 3-dimensional Stokes Theorem, it means we can choose any $\mathbf{F}$ since the abstraction of differential forms has been thrown out).

2.6 Completing the Proof (Example)



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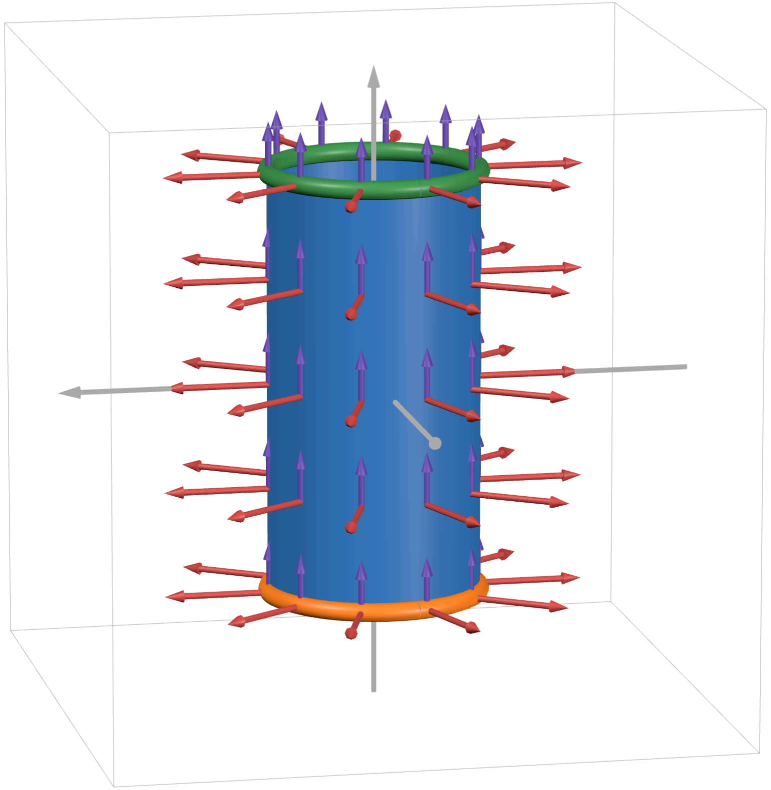

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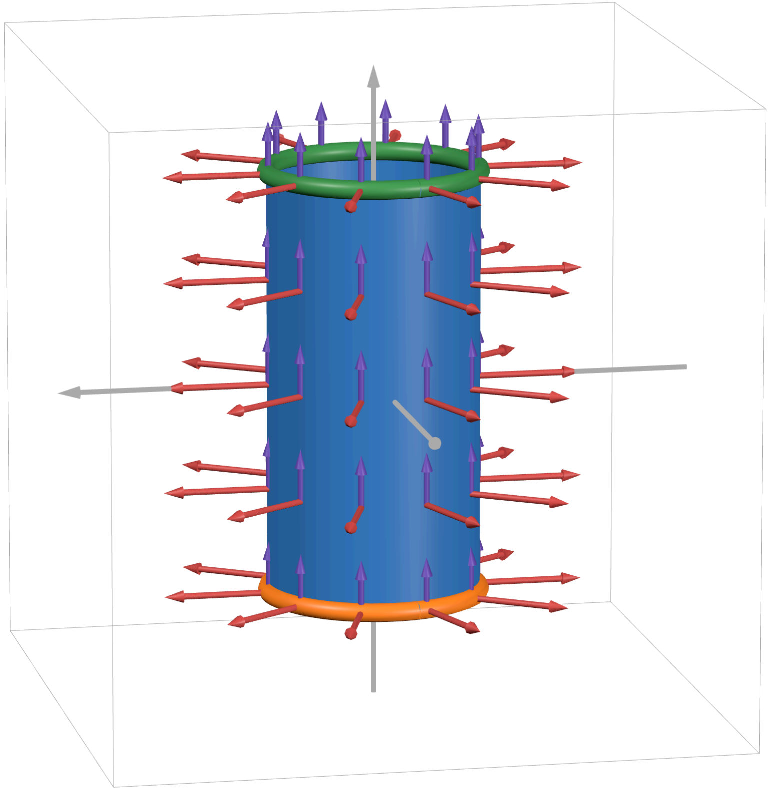

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- We see that $\mathrm{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}} = 0$, so we must have $\int_S \mathrm{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = 0$
- When is this true? Whenever $\mathbf{F}$ lies in the xy plane (i.e. $h = 0 \mid h_x, h_y, h_z = 0$ and, by the fact that $\mathbf{F}$ is a function of x and y only, $f_z, g_z = 0$).

2.7 Recap of Stokes Theorem Example

- We found that some $\mathbf{F}$ exists such that $\int_M d\omega = 0$, so we conclude that $\int_{\partial M} \omega = 0$, by Stokes Theorem

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- We can divide ∂M into two boundaries and, since the integral of ω over both is the same, we can factor out the sum of the boundary orientations $\varepsilon_1 + \varepsilon_2$
- The fact that $\int_{\partial M} = 0$ implies $\varepsilon_1 + \varepsilon_2 = 0 \Rightarrow \varepsilon_1 = -\varepsilon_2$
- **Therefore, the boundaries at opposite ends of a cylinder have opposite orientations**!

2.8 The Power of Stokes

Figure 1: Literally the sole use of Stokes Theorem
(not Generalized Stokes Theorem, *which is much more powerful, as we'll see)*

Outline

3.1 Reality Check

Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now, as in the past, are inspired by the art of mathematics rather than by any prospect of ultimate usefulness.

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Questions we must answer:

- Is Stokes Theorem truly useless?
- Have we as mathematicians failed in our quest to understand the universe?
- What is the nature of spacetime?

3.2 The Nature of Manifolds and their Boundaries

To appreciate the true power of Stokes Theorem, consider this trivial application:

$$\int_{\partial(\partial M)} \omega = \int_{\partial M} d\omega = \int_M d(d\omega) = \int_M 0 = 0$$

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In other words, the integral of *anything* over the boundary of the boundary of *any* manifold M is zero. This is only possible if the manifold we are integrating over is the empty set $\emptyset$.

3.3 Closed Manifold Boundaries

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Proposition of Closed Manifold Boundaries

The boundary of any manifold M, ∂M, is **closed**.

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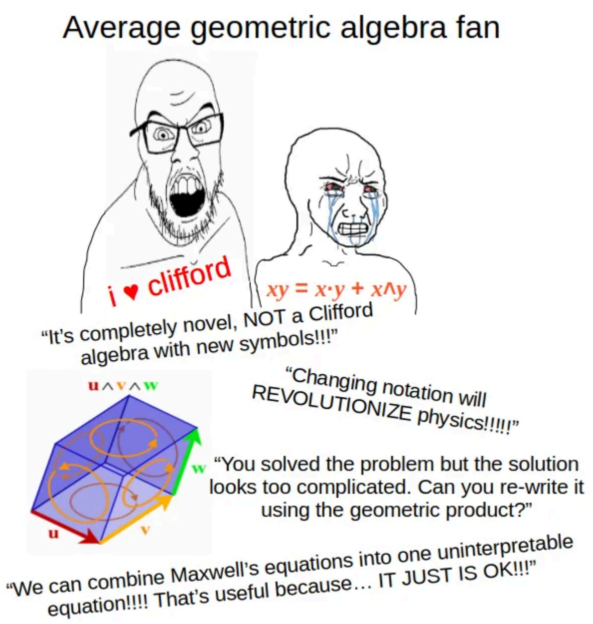
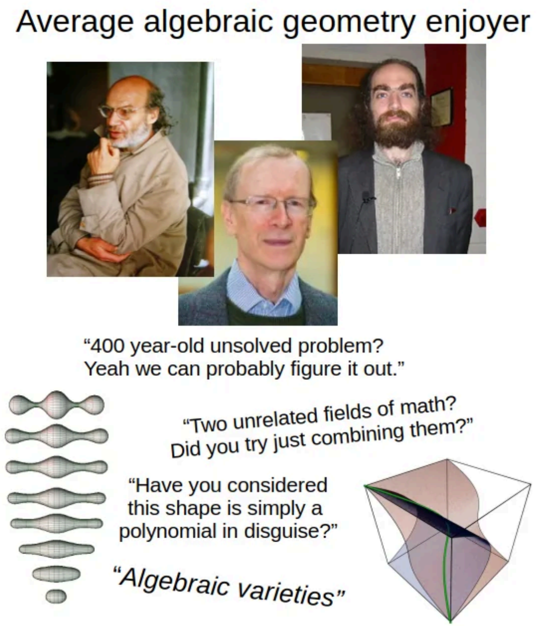
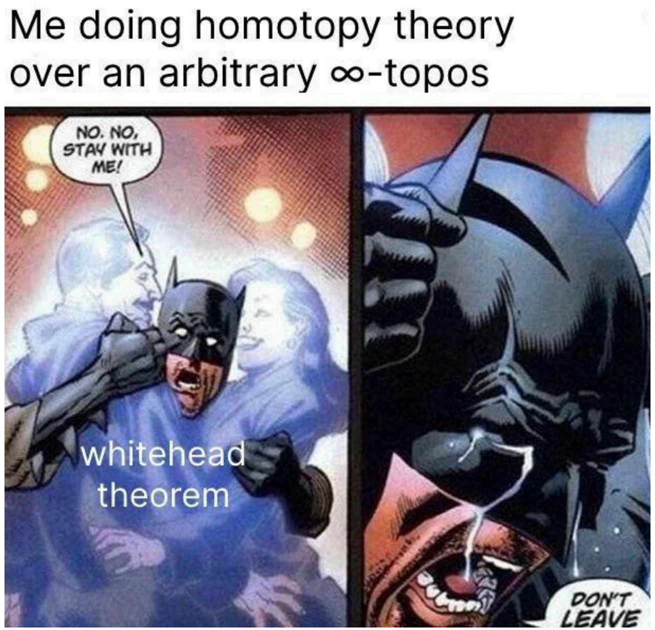
Remember when we stated this?? (and didn't prove it)

3.4 The Superiority of Generalized Stokes







3.4 The Superiority of Generalized Stokes

3.4.1 Higher Topological Powers

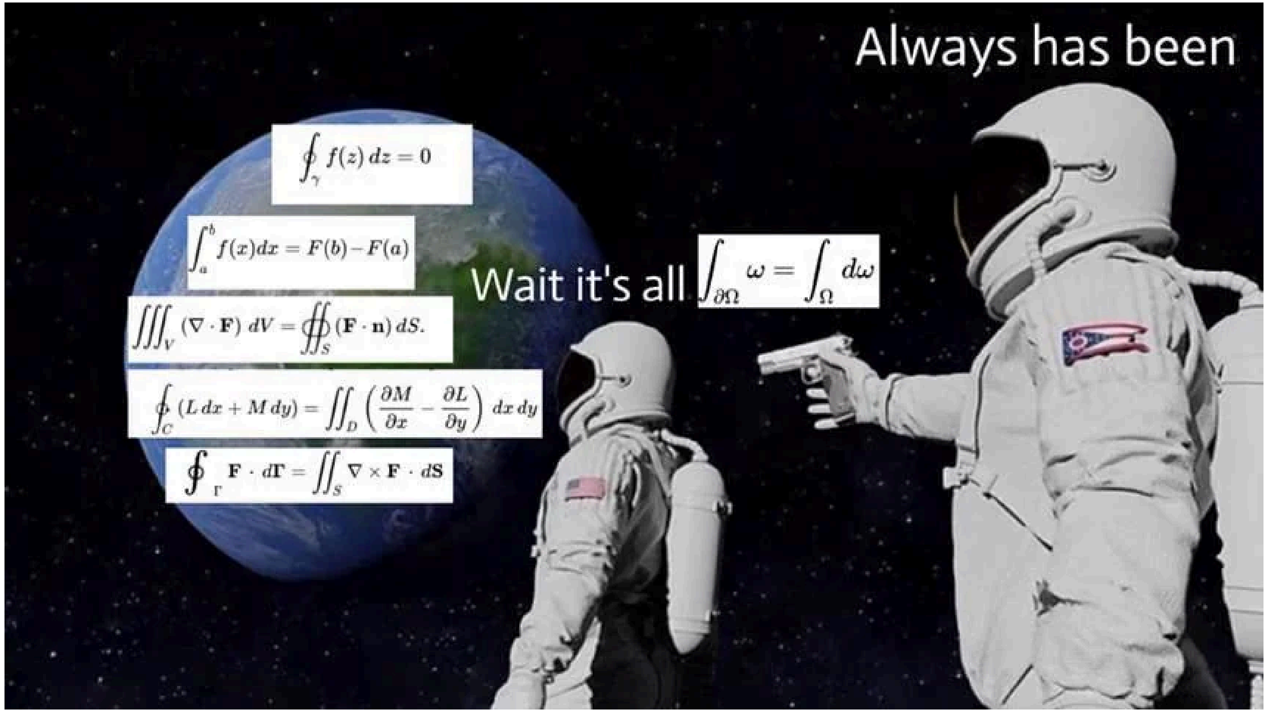


Figure 1: You have been lied to through all of calculus

3.5 Speedrunning Calculus with Generalized Stokes Theorem

"Novel" theorems (for higher-dimensional manifolds):

- Divergence Theorem
  - M is a 3 manifold in 3 space
- Stokes Theorem (already derived)
  - M is a 2 manifold in 3 space
- Green's Theorem
  - M is a 2 manifold in 2 space

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"Degenerate" theorems (for one or zero dimensional manifolds):

- Fundamental Theorem of Calculus
  - M is a 1 manifold in 1 space

3.6 Divergence Theorem

M is a 3 manifold in 3 space

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M is a volume V, ∂M is its surface S, ω is the flux of $\mathbf{F}$ through S: $\mathbf{F} \cdot \hat{\mathbf{n}}\ dS$, $d\omega = \operatorname{div}(\mathbf{F})\ dV$

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$$\int_{\partial M} \omega = \int_M d\omega$$

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\ dS = \iiint_V \operatorname{div}(\mathbf{F})\ dV$$

■

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$$\int_{\partial M} \omega = \int_M d\omega$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_A \text{curl}(\mathbf{F})\ dA$$

$\blacksquare$

3.8 The Fundamental Theorem of Calculus

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Compute $d\omega$ to find $d\omega = f'(x)\ dx$

M is an interval $[a, b]$, ∂M is a set of points $\{a, b\}$ with orientations $\{\varepsilon_a, \varepsilon_b\}$, ω is the integral of $f(x)$ over that set of oriented points, $d\omega = f'(x)\ dx$

3.9 Integration over the Zero Manifold

An integral over a continuous manifold is an infinite (Riemann) sum, where at each point we multiply the integrand by dx. Essentially, we evaluate our differential form ω at each point, but make sure to choose a positive orientation for all points in the manifold. If a point is negatively oriented, we obtain the negative of the integral over the positively oriented point, so we multiply that integral by -1 to obtain the "positively oriented" integral value.

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3.10 Orientation of the Zero Manifold

3.10.1 What is the orientation of $\{a, b\}$ when dx points from a to b?

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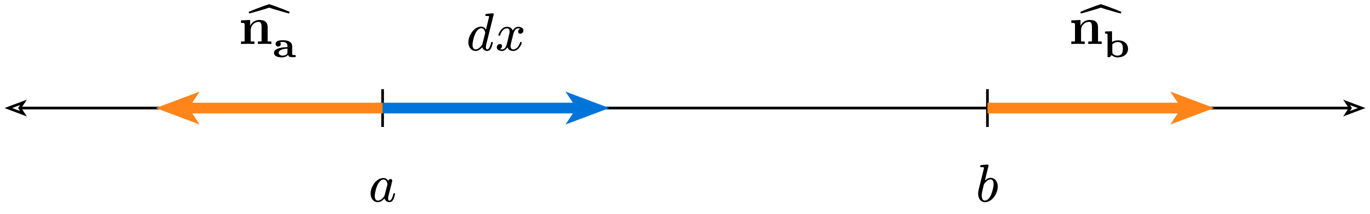

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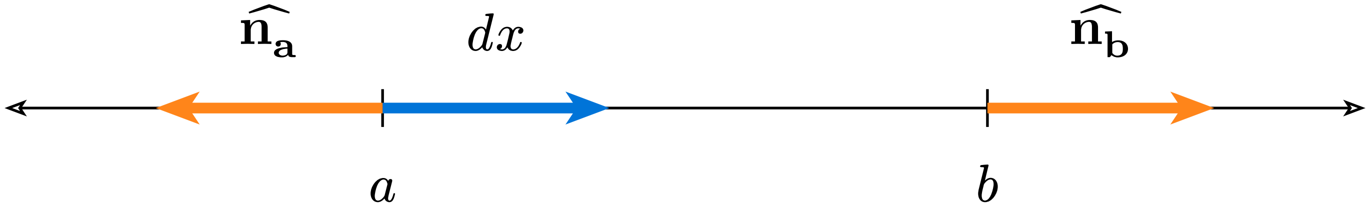

Expressing $\widehat{\mathbf{n_a}}$ in terms of dx leads to a negative coefficient, so the orientation of a is negative. $\widehat{\mathbf{n_b}}$ points in the same direction as dx, so it will have a positive coefficient when expressed as a multiple of dx, meaning it has positive orientation.

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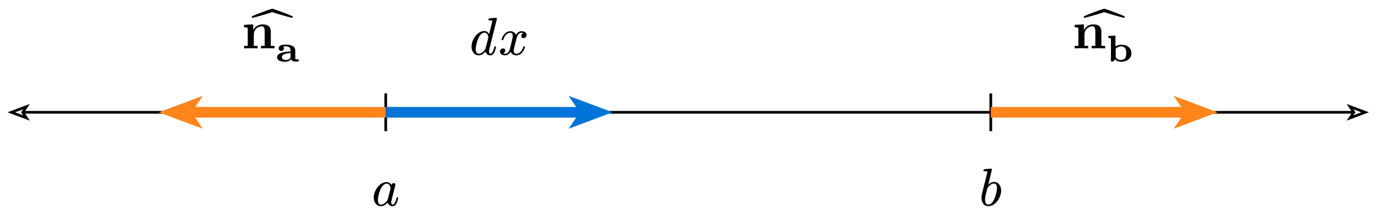

Expressing $\widehat{\mathbf{n_a}}$ in terms of dx leads to a negative coefficient, so the orientation of a is negative. $\widehat{\mathbf{n_b}}$ points in the same direction as dx, so it will have a positive coefficient when expressed as a multiple of dx, meaning it has positive orientation.

Notice how a^+ and b^- have opposite orientations–**just like the cylinder!**

3.11 Deriving the Fundamental Theorem of Calculus

$$\int_{\partial M} \omega = \int_M d\omega$$
$$\int_{\{a,b\}} f(x) = \int_{[a,b]} f'(x)\ dx$$

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We let $\varepsilon_a = -1$ and $\varepsilon_b = 1$ and evaluate the left integral:

$$\int_{\{a,b\}} f(x) = \varepsilon_a f(a) + \varepsilon_b f(b) = -f(a) + f(b) = f(b) - f(a)$$

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Thus,

$$f(b) - f(a) = \int_a^b f'(x)\ dx$$

■

3.12 Final Words

Category theory is the subject where you can leave the definitions as exercises.

— John Baez

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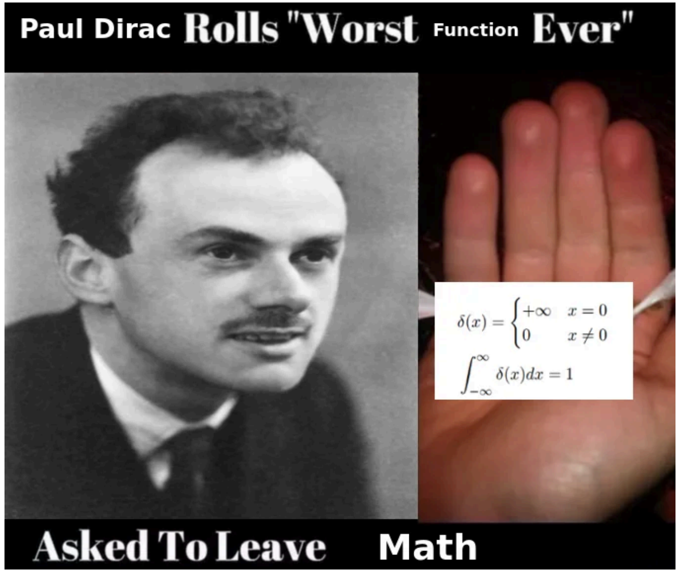

3.13 Thank you!

Figure 2: I'll try my best to avoid it

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4.1 Other References

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- Mathematical Folklore