

Stokes Theorem as Generalized Abstract Vapid Nonsense

“We’ll only use as much category theory as is necessary.
[*famous last words*]” –Roman Abramovich¹

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¹Attribution: “Higher Gauge Theory: The Fundamentals” (quote is almost certainly misattributed to a Russian oligarch instead of the algebraic geometrist who is likely responsible for it)

Outline

1. Stokes Theorem

2. Some Examples

3. Applications?

1.1 Manifolds

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- Manifolds are just a continuous set of points
- Moreover, at any given point, they look like a **Euclidean space**

1.2 Euclidean Spaces

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- Euclidean space is the space we inhabit (not the full story; more on this later)

What actually is a manifold then?

- When we say that a manifold “looks like” a Euclidean space, we’re actually talking about **homeomorphism**

1.3 General Topology



- Topologists do not understand shapes, so they must deform objects into meaningless blobs to count holes
- Two numbers are equal, but two topological spaces are **homeomorphic** to each other

1.3 General Topology

1.3.1 Definitions

Homeomorphism

A pair of continuous bijective (all $x \in X$ in the domain of f and all $y \in Y$ in the range of f) mappings $f : X \longrightarrow Y$ and $f^{-1} : Y \longrightarrow X$

Homeomorphic equivalence

Two topological objects X and Y are homeomorphic to each other iff there exists a **homeomorphism** between them.

1.4 Formalizing the Manifold

1.4.1 Definitions

Read: more abstract nonsense

Manifold

A k (represents intrinsic dimension) manifold is a continuous set of points X **homeomorphic** to \mathbb{R}^k

Embedding

The embedding of a manifold X is n iff $x \in \mathbb{R}^n \ \forall x \in X$

Note: If a k manifold is embedded in n space (\mathbb{R}^n), we must have $n \geq k$ (this is a trivial consequence of an advanced mathematical technique called “visualization”)

1.5 Manifolds in the Wild

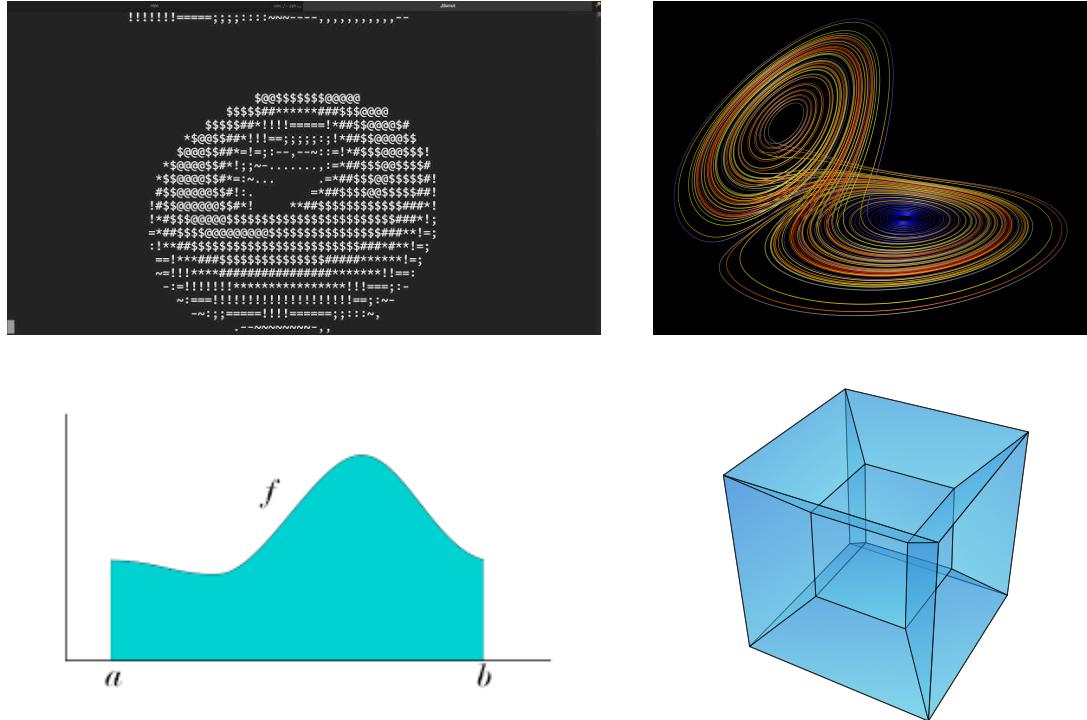
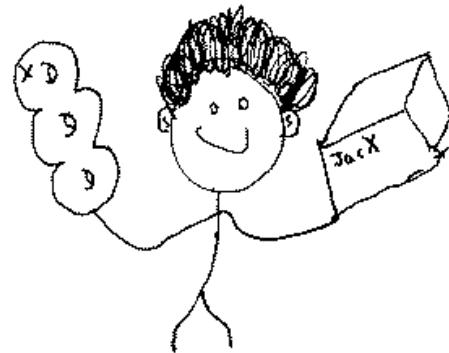


Figure 1: From left to right, top to bottom: A 2 manifold in 3 space, a 1 manifold in 3 space, a 2 manifold in 2 space, and a 2 manifold in 4 space (projected onto 3 space—the drawing is of a hypercube)

1.5 Manifolds in the Wild

1.5.1 The Prototypical Example of a Manifold



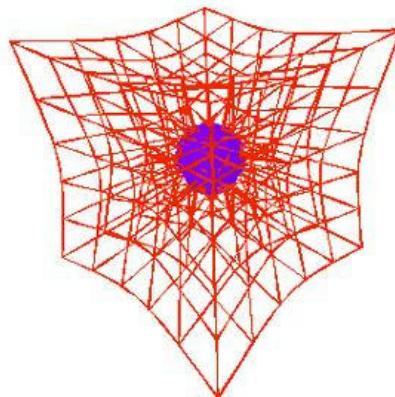
Dan Abramovic, embedding
a curve of genus 3 in
its Jacobian.

Figure 2: Don't ask me what this is because I am just as clueless as you

1.6 The Universe

We live in a **3 manifold** (possibly embedded in a *higher dimensional* space—and we have no way of knowing)!

Completely irrelevant to understanding Generalized Stokes Theorem, but very cool nevertheless.



This three-dimensional grid gives a better idea of what curved space-time might look like than the two-dimensional analogies do.

1.7 Differential Forms

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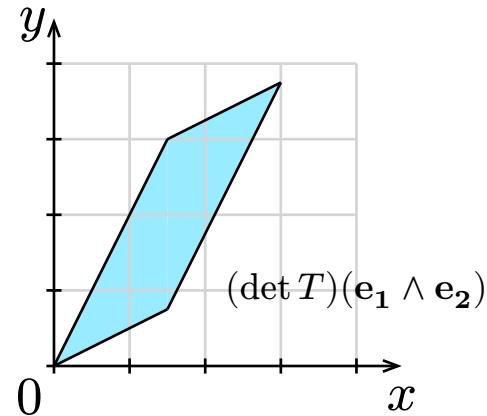
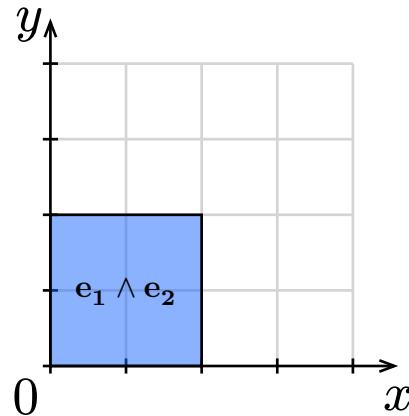
Now, consider the most abstract geometric properties the determinant should have.

1.8 Wedge Products and Bivectors

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1.9 Bivectors

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- Wedge products between any two vectors \mathbf{a} and \mathbf{b} exist in a vector space (we are free to define the basis vectors, but they look like some wedge product, e.g. $\mathbf{e}_1 \wedge \mathbf{e}_2$)

1.10 Multivolumes

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$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

*This follows from the previous properties; this is why the determinant is **antisymmetric**!*

1.12 Interesting Derivation of Antisymmetry Property

By the fact that $\mathbf{u} \wedge \mathbf{u} = 0$, we have:

$$(\mathbf{a} + \mathbf{b}) \wedge (\mathbf{a} + \mathbf{b}) = 0 \quad (\mathbf{u} = \mathbf{a} + \mathbf{b})$$

Distribute!

$$\mathbf{a} \wedge \mathbf{a} + \mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} + \mathbf{b} \wedge \mathbf{b} = 0$$

Use $\mathbf{u} \wedge \mathbf{u} = 0$ again! (but for different \mathbf{u})

$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0$$

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Only the **signed multivolume** of a parallelotope as a function of its side vectors is multilinear.

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Line integrals are really just:

$$\int (f_1(t) d\mathbf{e}_1 + f_2(t) d\mathbf{e}_2 + \cdots + f_n d\mathbf{e}_n)$$

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$$\int (f_1(u, v) \ d\mathbf{e}_1 \wedge \mathbf{e}_2 + \dots + f_2(u, v) \ d\mathbf{e}_1 \wedge \mathbf{e}_n + \dots + f_n(u, v) \ d\mathbf{e}_{n-1} \wedge \mathbf{e}_n)$$

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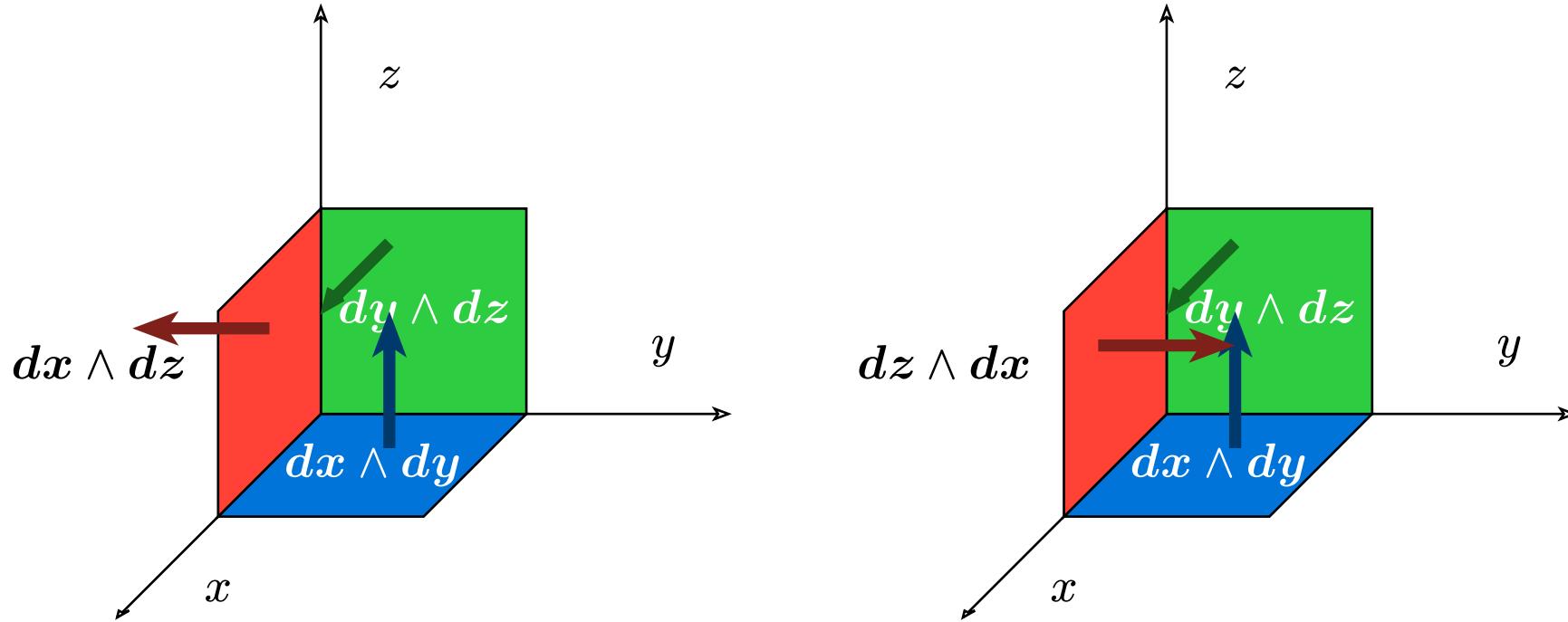
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Let $\mathbf{F} = \langle A(x, y, z), B(x, y, z), C(x, y, z) \rangle$. We will calculate the flux outward through each surface in the 2-form. We will talk more about orientation later, but for the purpose of flux, we define the normal vector as being the cross product of two surface vectors (with right hand rule).

1.16 Surface Integrals



We want flux to be positive when anything “enters” this cube and negative when stuff “leaves”—but the mathematics of differential forms allow us to integrate any way we like

1.17 The Flux Integral

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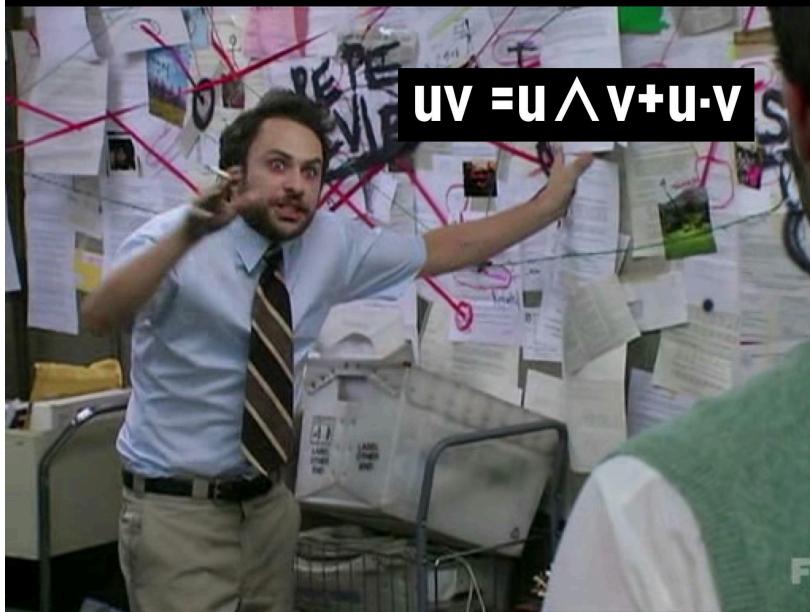
$$\int \mathbf{F} \cdot \langle dy \wedge dz, dz \wedge dx, dx \wedge dy \rangle$$

How do we calculate such a monstrosity?

What is the meaning of this abstract nonsense?

1.18 The Pullback

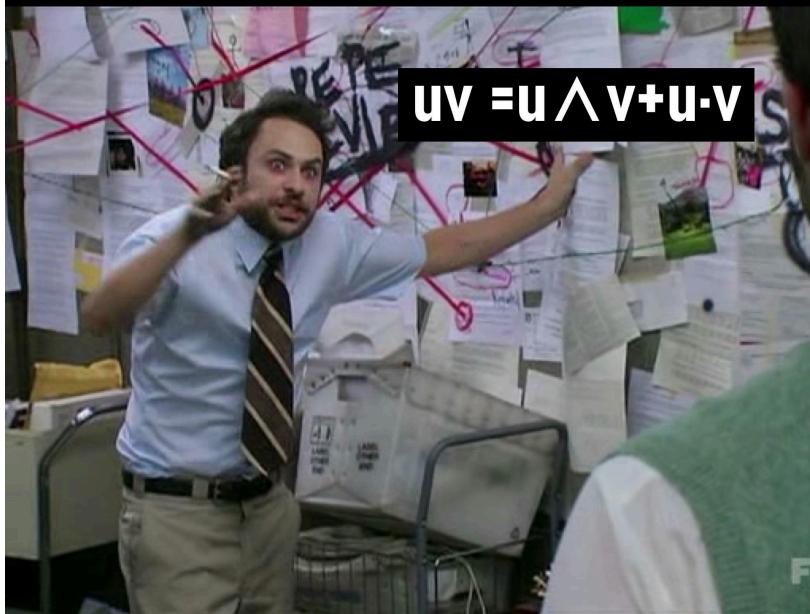
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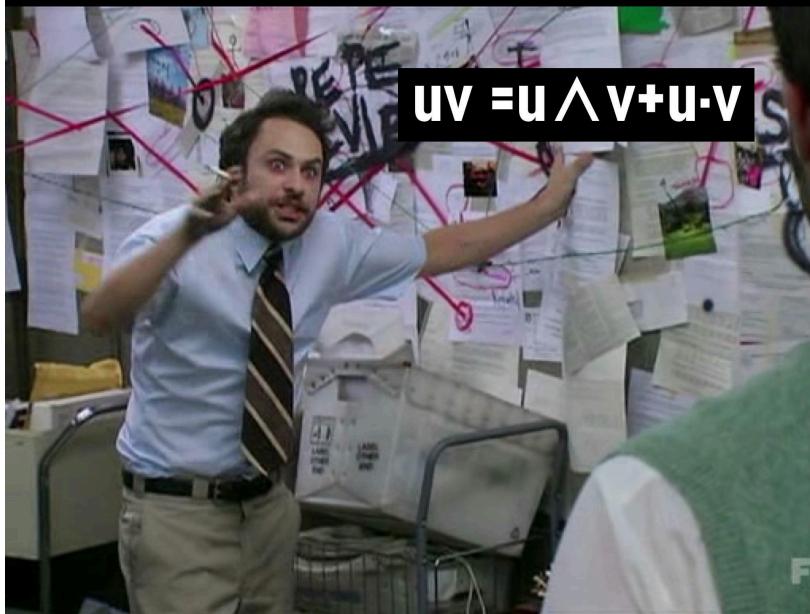
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- There is no going back after this (your mind may be warped by upcoming abstractions)
- We note that we can represent F , a 2 manifold, using a 2 dimensional parameterization in terms of u, v
- We now need to “convert” $dy \wedge dz, dz \wedge dx, dx \wedge dy$ to $du \wedge dv$

1.19 Mathematical Maneuvering

- Remember that a manifold is **homeomorphic** to a Euclidean space (in this case, \mathbb{R}^2). This means there exists a mapping* $x, y, z \rightarrow u, v$, so we can write $u(x, y, z), v(x, y, z)$.

*where x, y, z are in some subset of \mathbb{R}^3 (occupied by the manifold)

- This mapping is called a “**chart**,” which is part of an “**atlas**” for the manifold
 - Mathematicians take this map analogy very seriously because they cannot distinguish between homeomorphic objects in real life

Topology is precisely the mathematical discipline that allows the passage from local to global...

— René Thom (itinerant theoretician)

Don’t let your mind be warped by these abstractions!

1.19 Mathematical Maneuvering

1.19.1 Computation

- Given $u(x, y, z), v(x, y, z)$ and $\mathbf{F}(u, v)$, consider the inverse map $u, v \rightarrow x, y, z$ that must also exist by the conditions of **homeomorphism** to find $x(u, v), y(u, v), z(u, v)$.
- Now, take differentials using the chain rule:

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv, dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

- We can now use the properties of the **wedge product** to evaluate any product of dx , dy , and dz in terms of either $du \wedge dv$ or $dv \wedge du$ (but remember $dv \wedge du = -du \wedge dv$).
- This gives an integral of the form:

$$\int \mathbf{G}(u, v) \, du \wedge dv = \iint \mathbf{G}(u, v) \, du \, dv$$

1.20 Integration with Differential Forms

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- We can also take wedge products of more than 2 vectors to get multivectors, but these have only two canonical orientations
 - More on this later, but for now just know that we can always flip terms to get them in a certain order by multiplying either by 1, positive orientation, or -1 , negative orientation

1.21 Notational Hacks

- We call the entire expression after the integral sign, when expressed with multivectors, a **differential form**, commonly denoted ω
- Note that multivectors won't always have a wedge product symbol (for one or zero (we'll encounter those later) manifolds we only need a differential or scalar, respectively)
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 - Common notation is to use ω for a differential form and Ω for a manifold, just to be extra confusing
- We use a completely made-up, contrived operation called the **pullback** to turn this abstract mess of symbols into a readable integral we can evaluate
 - Basically, use an intrinsic coordinate system that represents the manifold itself and not the space it's embedded in

1.22 Mandatory Content Warning

The undergrad category theorist

- Superiority complex despite nothing to show for it
- Burned out mathematical beauty receptors.
- Not even interested in the fields that category theory was invented to deal with.
- No free will. Slave to abstraction.
- Warped sense of what mathematics is.
- Already considers "specializing" in foundations despite being unaware of centuries worth of mathematical developments.



Don't let this be you. Stop before its too late!

1.23 Orientability

Before we continue,

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A mathematician who can only generalise is like a monkey who can only climb up a tree, and a mathematician who can only specialise is like a monkey who can only climb down a tree. In fact neither the up monkey nor the down monkey is a viable creature. A real monkey must find food and escape his enemies and so must be able to incessantly climb up and down. A real mathematician must be able to generalise and specialise.

— George Polya

1.24 Differential Forms and Orientability

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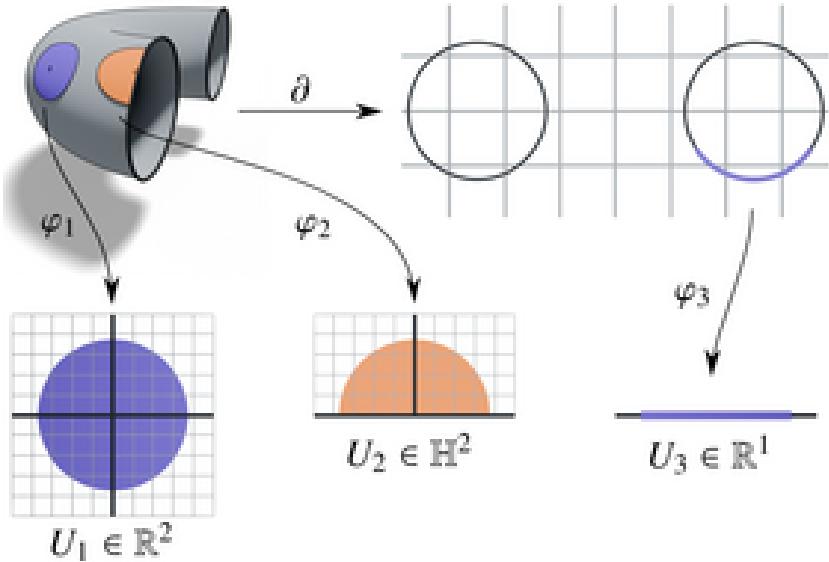
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- In 2 space, it is $dx \wedge dy$, by convention
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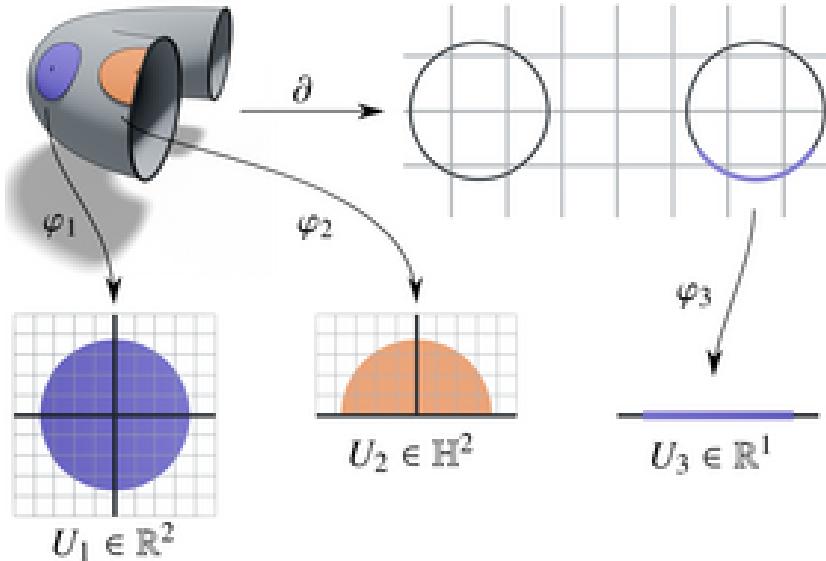
More on 0 space later ...

1.25 Boundaries



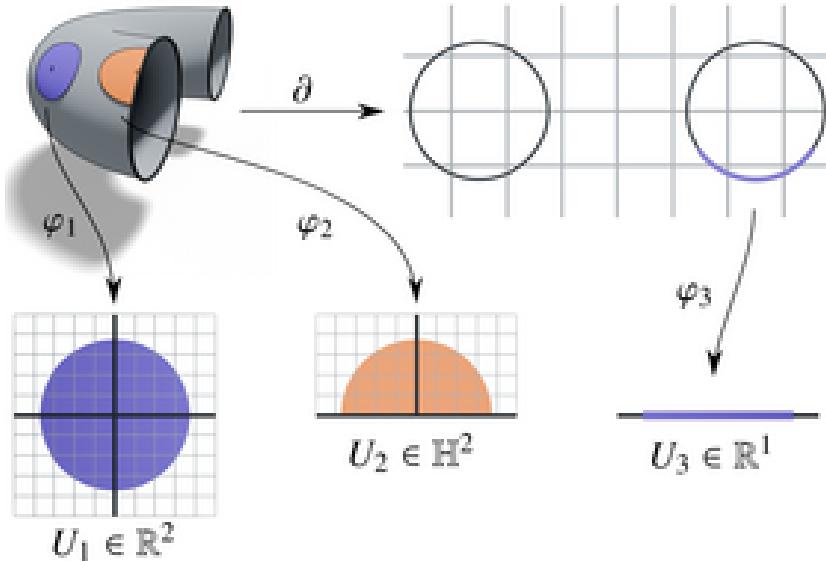
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- Specifically, a point within a k manifold M will have a local neighborhood of points homeomorphic to a k ball
- At the boundary, points have a local neighborhood homeomorphic to a half k ball

1.26 Properties of Boundaries

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“It is the snobbishness of the young to suppose that a theorem is trivial because the proof is trivial.”

— Henry Whitehead

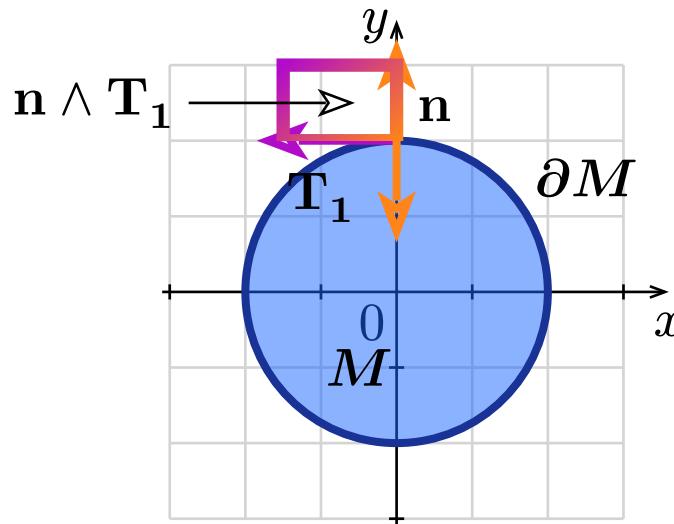
1.27 Proof of Properties of a Boundary

If you're interested:

our definition $\Rightarrow \exists \varphi : U \rightarrow \mathbb{H}^k$ where U is the neighborhood of points around some $P \in \partial M$ and \mathbb{H}^k is a half k ball, formally defined as $\mathbb{H}^k = \{x \in \mathbb{R}^k \mid x_k \geq 0\}$ for some intrinsic coordinate system where $x_k = 0$ for points along the boundary in U (otherwise a local neighborhood around these points would be homeomorphic to \mathbb{R}^k , not \mathbb{H}^k).^{*} Thus, ∂M is homeomorphic to $\{x \in \mathbb{H}^k \mid x_k = 0\}$, which is homeomorphic to \mathbb{R}^{k-1} (simply reparameterize by eliminating x_k and keeping all other parameters the same—for the inverse map add $x_k = 0$). We'll later see how to prove this boundary is closed using Generalized Stokes Theorem!

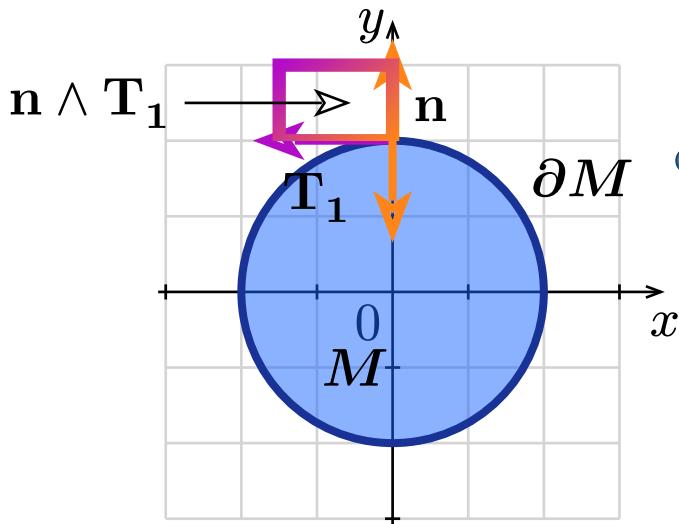
^{*}These are quite shaky foundations indeed, but a full proof would result in too much brain damage to present in its entirety.

1.28 Induced Orientations



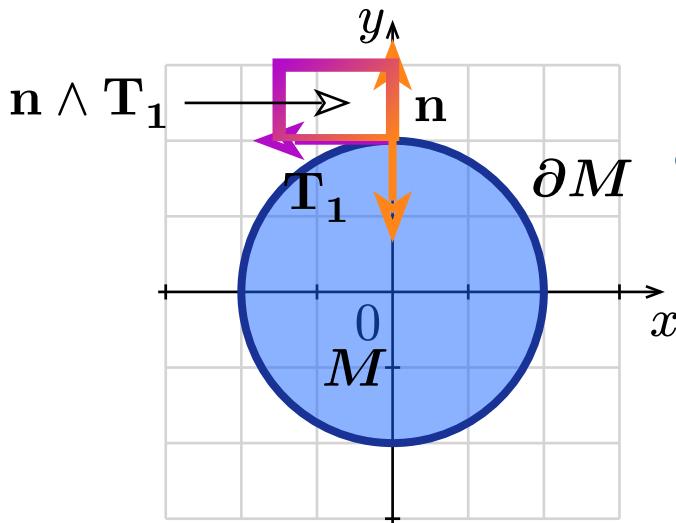
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- Compute the normal vector \mathbf{n} orthogonal to all tangent vectors and not pointing into M

1.29 Induced Orientations and Wedge Products

- Now, write $\mathbf{n} \wedge T_1$ as some expression of the form $A dx \wedge dy$, where $dx \wedge dy$ is the known positive orientation of M .

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- Now, write $\mathbf{n} \wedge T_1$ as some expression of the form $A dx \wedge dy$, where $dx \wedge dy$ is the known positive orientation of M .
- The orientation of ∂M is given by $\text{sign}(A)$
 - Note that this is either *positive* or *negative* (there are only ever 2 canonical orientations of a manifold, regardless of dimension)

1.30 Induced Orientation in Arbitrary Dimensions

Summarizing what we did, for an n manifold in n space:

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1.31 The Prototypical Non-Orientable Surface

When the prof says you can only use one side
of a piece of paper as a formula sheet.

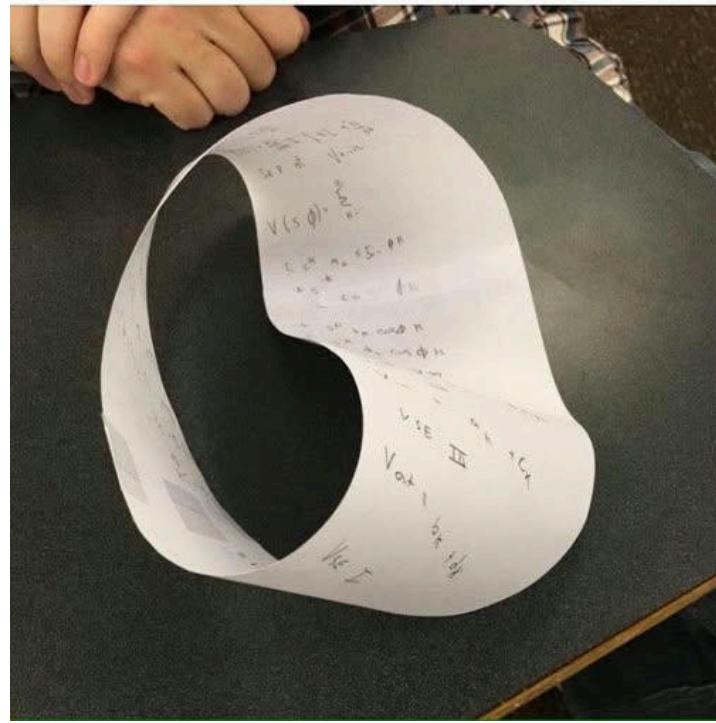


Figure 1: Not possible to define a unique \mathbf{n} for every point (any point) on the surface

1.32 Exterior Derivative

Consider a nasty differential form like:

$$\omega = M(x, y, z) \ dy \wedge dz + N(x, y, z) \ dz \wedge dx + P(x, y, z) \ dx \wedge dy$$

where $\mathbf{F} = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$ (remember flux??)

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Can we *differentiate* ω ?

1.33 Digression to Introduce the Scale of the Problem

We will need to use some very simple notions of category theory, an **esoteric subject** noted for its **difficulty** and **irrelevance**.

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You should be thinking of the Fundamental Theorem of Calculus right now (we'll get back to this).

1.34 Definition of the Exterior Derivative

Exterior Derivative

Given an n form $\omega = f \ dx_1 \wedge \dots \wedge dx_n$, we define the exterior derivative $d\omega = df \wedge dx_1 \wedge \dots \wedge dx_n$, an $n + 1$ form (trivial to see given we have added another wedge product and df is a 1 form)

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$$d\omega = f_x \, dx \wedge dy \wedge dz + g_y \, dy \wedge dz \wedge dx + h_z \, dz \wedge dx \wedge dy$$

$$d\omega = (f_x + g_y + h_z) \, dx \wedge dy \wedge dz$$

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It's $\operatorname{div} \mathbf{F} dV$ (using the fact that $dV = dx \wedge dy \wedge dz$)

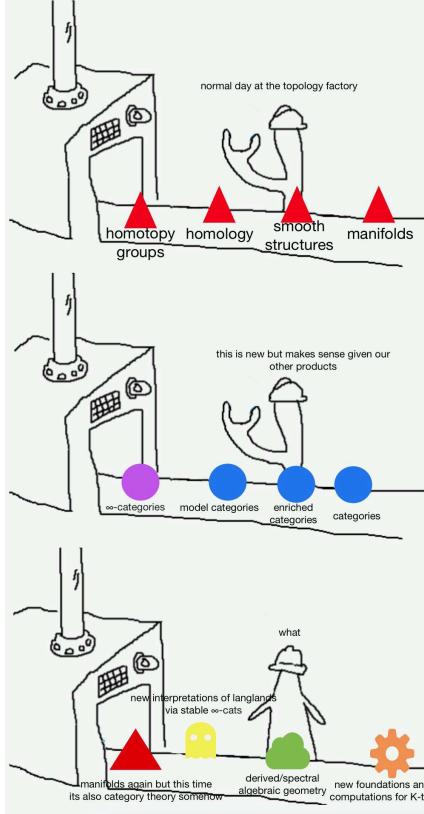
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- Flux becomes divergence?
- Where have you seen this before??

1.36 Stokes Theorem



- How Generalized Stokes Theorem is actually defined using homotopy type theory
- Involves topos theory and other abstract nonsense
- We will stick to the algebraic topology definition, which is only slightly less deranged than the category theoretic one

1.37 Generalized Stokes Theorem

Topological Definition of Generalized Stokes Theorem

$$\int_{\partial M} \omega = \int_M d\omega$$

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Nothing more remains to be said.

This presentation could end here and you would leave knowing a fundamental truth of the universe.

Outline

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2. Some Examples

3. Applications?

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