Degenerate Quantum Codes Beating the Quantum Hamming Bound

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***Abstract*—In this paper we discuss degenerate codes and whether or not they can beat the Quantum Hamming Bound (QHB). In this pursuit, we cover concatenated cat codes and Toric codes that beat ”perfect” codes in terms of quantum channel capacity and error threshold respectively, and Bacon-Shor subsystem codes which beat the subsystem QHB. We review what restrictions exist to beating the QHB, such as the quantum Singleton bound, which must be considered. Finally, we outline our approach towards finding degenerate codes that beat the QHB along with a table which gives some insight. After an in-depth review of the literature and various attempts at finding violations of the QHB we have come to the conclusion that there may not be a strict violation of the QHB despite the existence of some codes surpassing ”perfect” codes in other measures.**

I. Introduction

Degenerate codes and non-degenerate codes are two distinct types of subspace codes in the world of quantum error correction. Non-degenerate codes have errors that are all distinguishable from each other. Each error affects a given code state differently. Degenerate codes, however, have errors that are not distinguishable from each other. A degenerate code may include two or more errors whose effects on a state are identical. Because of this, we have tighter bounds on non-degenerate codes than degenerate codes. Degenerate codes have the potential to be very useful; with multiple errors affecting the code in the same way, fewer correctable states may be needed to cover all errors. The Quantum Hamming Bound (QHB) is a vital proven upper limit to the number of errors a non-degenerate code can correct; however, whether this is true or not for degenerate codes remains a mystery. In this paper, we explain our approach to determining if degenerate codes can surpass the QHB and explore what might be done to pursue this endeavor.

Towards this goal, the paper has the following sections. In Section II, we cover background information relevant to the research. In Section III, we will discuss existing violations of the QHB. Section IV explores restrictions on viable codes that attempt to beat the bound. Finally in Section V, we will walk through our process, approaches, and results relating to this problem.

II. Background

The number of syndromes of the stabilizers of a code is 2^{n-k}, where n is the number of physical qubits and k is the number of logical qubits. In any $[[n,k,d]]$ code, d is the distance of the code where $d = 2t+1$ and t is the number of correctable errors. Eq. 1 refers to the QHB as it applies to non-degenerate codes.

$$\sum_{j=0}^{t} \binom{n}{j} 3^j \leq 2^{n-k} \tag{1}$$

The left hand side of (1) counts the number of correctable errors of a given code. $\binom{n}{j}$ is necessary to cover all the different choices of qubits for the errors to act on and 3^j accounts for each combination of the three Pauli errors (X, Y, Z), which provide a basis for any arbitrary single qubit error. Essentially, the QHB states that the number of correctable errors must be less than the number of different syndromes. This inequality holds true for non-degenerate codes but it has been suggested that the bound might be violated by some degenerate codes.

Non-degenerate codes that meet the QHB, that is when the left side of (1) is exactly equal to the right side, are preferred over those that do not. These codes are called ”perfect” codes because each correctable error is perfectly matched with a unique syndrome. When a non-degenerate code meets the QHB, it must have a high t value, and thus high distance, which in turn allows for better error detection and correction. If degenerate codes are able to beat the QHB, the distance of the code will be even higher than if it met the QHB. This seems as if it would come naturally to degenerate codes. With many different errors mapping to the same states, we would hope to be able to use less syndromes to correct the same amount of errors. If this is true, we should be able to have higher distances for degenerate codes and thereby beat the QHB. This means we are looking for degenerate codes that have the same number of physical and logical qubits as any given non-degenerate code but also have a higher distance than their non-degenerate counterparts.

III. Existing Violations of the QHB

There have been many attempts to find degenerate codes that beat the QHB using various methods. One route for this is to look at quantum channel capacity. The QHB describes code distance and helps us gauge how high of a physical error rate the code can handle. Quantum channel capacity tells us more directly about the rate of physical errors that the code can handle. The rate of physical errors that can be handled decreases as the fidelity decreases and codes with non-zero capacities for systems with low fidelities allow us to do error correction with less reliable systems.

Beating the QHB by having a high t value would violate the equation directly. However, attaining a lower fidelity with non-zero channel capacity than a perfect code achieves a similar effect. Although there may be low weight error types that are not correctable by the code, causing t to be too small to violate the QHB equation directly, there may be higher weight errors that are correctable, which allows for the code to correct errors better than a perfect code on average. Thus for noisy systems, achieving minimum fidelities for non-zero channel capacity lower than a ”perfect” code can beat the QHB in some regard.

DiVincenzo-Shor-Smolin looks at random coding as detailed in Bennett-DiVincenzo-Smolin-Wooters [2], which is an example of a perfect code, as a basis for a code that meets the QHB. Random coding has a minimum fidelity of 0.81071 for non-zero capacity [1], yet the paper managed to find a code that had an even lower fidelity, the concatenated cat code. The concatenated cat code [[n, k, d]] = [[25, 1, 5]] has a fidelity of 0.80944 [1]. This may seem like a minor difference but what it tells us is significant. The fact that the concatenated cat code has a fidelity that is at all smaller than random coding means that it is ”better” than a perfect code in some regimes, and thus can be said to beat the QHB. Here are more details on the concatenated cat code that might be of interest:

Inner Code Stabilizers

$$\langle X_1X_2, X_1X_3, X_1X_4, X_1X_5\rangle$$
$$X_L = X_1$$
$$Z_L = Z_1Z_2Z_3Z_4Z_5$$

Outer Code Stabilizers

$$\langle Z_1Z_2, Z_1Z_3, Z_1Z_4, Z_1Z_5\rangle$$
$$X_L = X_1X_2X_3X_4X_5$$
$$Z_L = Z_1$$

The concatenated cat code unfortunately does not beat the QHB. As shown below, the left side of (1) is well below the right side:

$$\sum_{j=0}^{2} \binom{25}{j} 3^j \leq 2^{25-1}$$
$$1 + 75 + 2700 \leq 2^{24}$$
$$2776 \leq 16777216$$

While quantum channel capacity might help guide us towards a code that can beat the QHB, it does not necessarily reveal these codes consistently.

Another major class of degenerate codes that beat the QHB in terms of channel capacity are rotated Toric codes. Toric codes are surface codes that loop around to connect an edge of a rectangular grid with its opposing side as if the qubits are laid out on the surface of a torus. Rotated Toric codes have stabilizers with two Pauli X measurements with vertically adjacent ancilla and two Pauli Z measurements horizontally rather than all X or all Z, providing traslational invariance [3]. The codes we are interested in are a family of rotated Toric codes with parameters $[[8d^2, 2, d]]$ and the two logical Pauli X and Pauli Z operations implemented as single qubit X operations along an entire row or column or Z operations along an entire row or column.

These codes have very high thresholds relative to other codes. Similar to studying minimum fidelity for non-zero channel capacity, threshold calculations allow for beating the QHB without technically violating the QHB inequality. The error correction threshold is the minimum physical error at which the error correction cannot reduce the error arbitrarily. These codes do not surpass the QHB using the value for t directly in the inequality. However, as Kay proves, this family of codes can surpass "perfect" code thresholds [4]. The QHB is surpassed by showing that the threshold is exceeded in some regimes.

Another route taken for finding degenerate codes that beat the QHB is subsystem codes. Subsystem codes remove part of an existing code so that we are left with a smaller code with similar properties. The subsystem quantum hamming bound (SQHB) is likewise different from the standard QHB. The SQHB looks at subsystem codes as $[[n, K, R, d]]_q$ where $K = 2^k$ and $R = 2^r$. Like with standard codes, k is the number of physical qubits. Meanwhile, r is the number of gauge qubits and q is the base of qudits, which in our case $q = 2$ because we are dealing with qubits.

$$\sum_{j=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{j} (q^2-1)^j \leq q^n/KR \tag{2}$$

The equivalent of this equation, using $q = 2$, $K = 2^k$, and $R = 2^r$, is as follows [10]:

$$\sum_{j=0}^{t} \binom{n}{j} 3^j \leq 2^{n-k-r} \tag{3}$$

One degenerate subsystem code that has been analyzed is the Bacon-Shor subsystem code. It has been proven that the [[9, 1, 4, 3]] and the [[16, 1, 9, 4]] Bacon-Shor codes both beat the SQHB [9]. Furthermore, Klappenecker and Sarvepalli proved that the Bacon-Shor codes of the form $[[(2t+1)^2, 1, 4t^2, 2t+1]]$ beat the SQHB, which [10]. This means they can handle higher error rates than most subsystem codes, but unfortunately it does not help us find a code that can beat the QHB.

IV. Restrictions on Violations of the QHB

While searching for new violations of the QHB it is helpful to keep in mind a number of restrictions on the values of $n, k,$ and t that a code can have. One such bound, as proven by Rains, asserts that $t \leq \lfloor \frac{n+1}{6} \rfloor$ for both degenerate and non-degenerate codes [5]. This bound, which we will refer to as the Rains bound, implies that degenerate codes tend to have low distances. Codes with higher distances have better error correction in general, so this limits the space of degenerate codes that can potentially beat the QHB.

Another general bound that applies to both degenerate and non-degenerate codes is the Quantum Singleton Bound (QSB). The QSB states that $n - k \geq 2(d-1)$ [6]. We can use the relationship that $d \geq 2t+1$ to get the bound $n - k \geq 4t$.

The Rains bound and QSB are visualized in Fig. 1 for n values up to 25. From the table, we can see that the QSB removes sections of n, k and t values from consideration in diagonal swaths starting from the top left and tapering as it continues down and to the right. However, those values that the QSB eliminates from consideration are largely restricted to small values of n if the trend shown continues. The Rains bound removes values along the left of the table and although it is vaguely periodic, removing more or less k values in each row, it seems to be growing steadily more distant from the low values of k on average.

There are a number of additional restrictions on violations of the QHB that have been proven. For instance, it has been proven by Ashikhmin and Litsyn that degenerate codes must obey the QHB asymptotically [7], which makes it seem as though large codes that violate the QHB should be harder to find than small codes. Also, Gottesman showed that single error correcting binary stabilizer codes are unable to violate the QHB [8]. In sum, this makes it hard to imagine degenerate codes that beat the QHB directly.

V. Approaches to Beating the QHB

In our approaches to finding new codes, we realized that certain criteria must first be met. For stabilizer codes, linearly independent generator set of stabilizers must be defined in addition to functional logical operators for each qubit (at least X and Z) under the condition that each logical operator commutes with all stabilizers and other logical operators with the exception of the other operator upon the same qubit. For example, given a code for three logical qubits, X_1 must commute with all stabilizers, both other logical X operators, and Z_2 and Z_3, but must anticommute with Z_1.

n\k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	2	2	1	1	1																				
6	2	2	1	1	1	1																			
7	2	2	2	1	1	1	1																		
8	2	2	2	2	1	1	1	1																	
9	3	2	2	2	2	1	1	1	1																
10	3	3	2	2	2	2	1	1	1	1															
11	3	3	3	2	2	2	1	1	1	1	1														
12	3	3	3	2	2	2	2	1	1	1	1	1													
13	3	3	3	3	2	2	2	2	1	1	1	1	1												
14	4	3	3	3	3	2	2	2	2	1	1	1	1	1											
15	4	4	3	3	3	3	2	2	2	2	1	1	1	1	1										
16	4	4	4	3	3	3	2	2	2	2	2	1	1	1	1	1									
17	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1								
18	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1							
19	5	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1						
20	5	5	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1					
21	5	5	5	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1				
22	5	5	5	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1	1			
23	5	5	5	5	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1	1		
24	6	5	5	5	5	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1	1	
25	6	6	5	5	5	4	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1	1

Fig. 1. Table of t values required to beat the QHB for each combination of n and k up to $n = 25$. Orange entries violate the QSB, yellow entries violate Rains bound, red violate both, and green are still acceptable candidate values for a degenerate code after these two restrictions.

Additionally, the stabilizer set that is chosen must be optimized for maximum correctable error weight, thus maximizing distance. Considering the QHB asserts an upper bound on the weight related to 2^{n-k}, it is clearly rather difficult to find a means to exceed or even meet the upper limit of the bound exactly; relatively few "perfect" codes have been able to succeed at this. In our attempts, we generated a chart of the minimum possible correctable error weights for a code given n and k in order to beat the QHB (Fig. 1) and attempted to create codes with those parameters, to no avail. Our general approach in selecting $[[n, k, 2t + 1]]$ combinations to work with was to select a code with the lowest k value possible given n and t. For example, experimenting with [[12,3,5]] rather than [[12,4,5]] due to a lower k value, which permits for more physical qubits per logical qubit, thus allowing more scope of error detection or correction. Beyond that, we looked for a greater number of physical qubits n for given k and t values to allow for more stabilizers and more potential error syndromes. Thus we attempted finding codes for green cells of a given t value in Fig. 1 that do not have a green cell below or to the left.

An extremely time consuming method (that we elected not to implement in the interest of time) that is guaranteed to generate results over time is a brute force search algorithm that would assess all possible stabilizer combinations and return the best set of $n - k$ stabilizers that all commute and maximize the error distance; however, even then there is no guarantee of surpassing the QHB.

A promising alternative to subspace codes for beating the QHB that we explored was subsystem codes. Because these codes had violations of the subystem QHB that we could definitively verify, testing degenerate subsystem codes seemed to be a more promising avenue for finding original codes that beat the QHB. Despite testing many codes, including those found by Aly and Klappenecker [11], and attempting to determine new codes, we were unable to discover any that surpassed this theoretical upper limit.

Further, inspecting the proof of the QHB for non-degenerate codes, we were unable to find significant parts that didn't also apply to degenerate codes; the portion of the proof relying on non-degeneracy did not allow us to make any new or interesting conclusions when removed. As we progressed, we became more convinced that few codes can surpass subsystem QHB and none truly surpass the QHB, but were unable to determine any conclusive proof.

VI. Conclusion

Degenerate codes that beat the QHB are at best, very difficult to find. Our research does not prove whether or not these degenerate codes exist, but we are inclined to think that they likely do not. The subsystem QHB is clearly possible to beat but these subsystem codes are not full degenerate codes. The fact that subsystem codes that beat the subsystem QHB are easier to find actually implies that standard degenerate codes may not actually beat the QHB.

Additionally, the QHB is a higher threshold than the subsystem QHB for any codes with the same n and k values, meaning that it is less likely for a subspace code to be more optimal for error correction than its corresponding subspace code. However, for impure codes that do surpass the subsystem QHB, it will be interesting to research and investigate whether they hold any true benefit over a subspace code of the same size. Further research on the QHB would likely be accelerated only by either proof of the bound being absolute or discovery of counterexamples, which likely would require some element of brute force search utilized in the process.

Contributions

Neema Badihian—Researched existing violations of QHB, participated in group discussions, drafted introduction, background, existing violations, abstract, and conclusion.

Saaketh Rayaprolu—Researched existing violations and degenerate subsystem codes of QHB, participated in group discussions, wrote code to generate n, k, t combinations for the table, drafted approaches section.

Cole Schroeder—Researched existing violations, restrictions, and subsystem QHB, participated in group discussions, created table of n, k, t values, created the in-depth outline for the paper, collected references, drafted restrictions section, and revised.

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