Problem 2 :

Given compressive measurements $y = \Phi x + \eta$, $\|\eta\|_2 \le \varepsilon$. The set of indices of non-zero elements of x is S, which is known for obtaining the oracular solution. And we are given that $\Phi_S^T \Phi_S$ is invertible.

(a) To obtain the oracular solution $\tilde{x}$, we need to only solve for the elements of $\tilde{x}$ at indices in S. let us denote this vector as x_S. We have:

$$y = \Phi x + \eta = \Phi_S x_S + \eta$$

since elements on indices outside S are 0.
Since we don't know η, we can use the following optimisation problem to obtain $\tilde{x}_S$:

$\rightarrow$ minimise $L(\theta) = \|y - \Phi_S \theta\|_2^2$, $\theta \in \mathbb{R}^{|S|}$

The minimiser is $\tilde{x}_S$.

The above is a good way of obtaining $\tilde{x}_S$ if η has elements which are mean 0 gaussian, i.e., $\eta_i \sim \mathcal{N}(0, \sigma^2)$. It is a generally used procedure in most cases anyways. Now let us solve the problem, (note it is convex)

$$L(\theta) = \|y - \Phi_S \theta\|_2^2 \Rightarrow \frac{\partial L}{\partial \theta} = -2 \Phi_S^T (y - \Phi_S \theta)$$

At minimiser $\tilde{x}_S$, $\frac{\partial L}{\partial \theta} = 0$

Date:

$$\frac{\partial L}{\partial \theta}(\tilde{x}_S) = 0 \Rightarrow -2\Phi_S^T(y - \Phi_S \tilde{x}_S) = 0$$

$$\Rightarrow \Phi_S^T \Phi_S \tilde{x}_S = \Phi_S^T y$$

$$\Rightarrow \tilde{x}_S = (\Phi_S^T \Phi_S)^{-1} \Phi_S^T y \quad (\because \Phi_S^T \Phi_S \text{ is invertible})$$

$$\Rightarrow \boxed{\tilde{x}_S = \Phi_S^{+} y}, \text{ where } \Phi_S^{+} = (\Phi_S^T \Phi_S)^{-1} \Phi_S^T$$

is pseudo-inverse of Φ_S

We can obtain $\tilde{x}$ from $\tilde{x}_S$ by padding zeroes at all indices in $\tilde{x}_{S^c}$.

i.e. $\tilde{x}_{S^c} = 0$, $\tilde{x}_S = \Phi_S^+ y$, completely gives us $\tilde{x}$.

Since $L(\theta)$ is ~~con~~ strictly convex, this $\tilde{x}_S$ is the unique minimiser.

(b) In this part, let us first prove a result:

Result 1: Let $\theta \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, A has a SVD, $A = UDV^T$, ($U - m \times m$, $V - n \times n$) ~~Di...~~ and let ~~...~~ maximum singular value be D_{max} and minimum singular value be D_{min}. Then

$$D_{min} \|\theta\|_2 \leq \|A\theta\|_2 \leq D_{max} \|\theta\|_2$$

Proof: note that $\|A\theta\|_2^2 = \|UDV^T\theta\|_2^2$

$= \|DV^T\theta\|_2^2$ ($\because U$ is unitary / orthogonal)

let $\beta = V^T\theta \in \mathbb{R}^n$

$$\Rightarrow \|A\theta\|_2^2 = \|D\beta\|_2^2$$

let the singular values in the diagonal of D be $D_1, D_2, D_3, \ldots, D_r$. ($r = \min(m, n)$)

Date:

$$\Rightarrow \|D\beta\|_2^2 = \sum_{i=1}^{r} |D_i \beta_i|^2$$

$D_{max} \geq D_i \geq D_{min} \geq 0 \quad \forall i$, so

$$\sum_{i=1}^{r} D_{min}^2 |\beta_i|^2 \leq \|D\beta\|_2^2 \leq \sum_{i=1}^{r} D_{max}^2 |\beta_i|^2.$$
$$= D_{min}^2 \|\beta\|_2^2 \qquad\qquad = D_{max}^2 \|\beta\|_2^2$$

$$\Rightarrow D_{min} \|\beta\|_2 \leq \|D\beta\|_2 \leq D_{max} \|\beta\|_2$$

and

$$\|\beta\|_2 = \|V^T\theta\|_2 = \|\theta\|_2 \quad (V \text{ is orthogonal})$$
$$\|D\beta\|_2 = \|A\theta\|_2$$

$$\Rightarrow D_{min} \|\theta\|_2 \leq \|A\theta\|_2 \leq D_{max} \|\theta\|_2 \qquad \blacksquare$$

Also note that if we picked a suitable β the equality holds for that β (upper or lower). That is, there exist $\theta_1, \theta_2 \in \mathbb{R}^n$, such that $\|A\theta_1\|_2 = D_{min}\|\theta_1\|_2$ and $\|A\theta_2\|_2 = D_{max}\|\theta_2\|_2$. Keep this in mind as well. (here $\theta_1, \theta_2 \neq 0$)

Now, with this result, we can proceed:

$$\|\tilde{x} - x\|_2 = \|\tilde{x}_s - x_s\|_2 = \|\Phi_s^+ y - x_s\|_2$$

and
$$\Phi_s^+ y = \Phi_s^+ (\Phi_s x_s + \eta) = \Phi_s^+ \Phi_s x_s + \Phi_s^+ \eta$$
$$= x_s + \Phi_s^+ \eta$$

here,
$$\Phi_s^+ \Phi_s = (\Phi_s^T \Phi_s)^{-1} \Phi_s^T \Phi_s$$
$$= I$$

$$\Rightarrow \|\tilde{x} - x\|_2 = \|\Phi_s^+ y - x_s\|_2 = \|\Phi_s^+ \eta\|_2$$

Date:

And from result 1, if $\|\Phi_S^{+}\|_{min}$ and $\|\Phi_S^{+}\|_{max} = \|\Phi_S^{+}\|_2$ denote the minimum and maximum singular values of Φ_S^{+}, we have:

$$\|\Phi_S^{+}\|_{min}\|\eta\|_2 \le \|\Phi_S^{+}\eta\|_2 \le \|\Phi_S^{+}\|_{max}\|\eta\|_2$$

$$\Rightarrow \|\Phi_S^{+}\|_{min}\|\eta\|_2 \le \|\tilde{x} - x\|_2 \le \|\Phi_S^{+}\|_{max}\|\eta\|_2$$
$$= \|\Phi_S^{+}\|_2\|\eta\|_2$$

(C) Given $k = |S|$, lets ~~xxxx~~ consider a vector, θ, which is non-zero only on indices in S. $\therefore$ θ is k-sparse. But this also ~~s~~ means that θ is atmost $2k$ sparse. So, we can say:

$$(1-\delta_{2k})\|\theta\|_2^2 \le \|\Phi\theta\|_2^2 \le (1+\delta_{2k})\|\theta\|_2^2$$

we know $\|\theta\|_2 = \|\theta_S\|_2$ and $\Phi\theta = \Phi_S\theta_S$. So, we can say that $\forall\ z \in \mathbb{R}^k$.

$$(1-\delta_{2k})\|z\|_2^2 \le \|\Phi_S z\|_2^2 \le (1+\delta_{2k})\|z\|_2^2$$

$$\Rightarrow \sqrt{1-\delta_{2k}}\,\|z\|_2 \le \|\Phi_S z\|_2 \le \sqrt{1+\delta_{2k}}\,\|z\|_2$$

$\|\Phi_S\|_{min}$, $\|\Phi_S\|_{max}$ are min and max, singular values of Φ_S. from what we showed earlier, there exist $z_1, z_2 \in \mathbb{R}^k$, such that ($z_1, z_2 \neq 0$)

$$\|\Phi_S z_1\|_2 = \|\Phi_S\|_{min}\|z_1\|_2, \quad \|\Phi_S z_2\|_2 = \|\Phi_S\|_{max}\|z_2\|_2$$

$$\Rightarrow \|\Phi_S\|_{min}\|z_1\|_2 \ge \sqrt{1-\delta_{2k}}\,\|z_1\|_2$$
$$\Rightarrow \|\Phi_S\|_{min} \ge \sqrt{1-\delta_{2k}}$$

and $\|\Phi_S\|_{max}\|z_2\|_2 \le \sqrt{1+\delta_{2k}}\,\|z_2\|_2 \Rightarrow \|\Phi_S\|_{max} \le \sqrt{1+\delta_{2k}}$

Date:

Now with this, let us use the following:
Let Φ_S have a SVD

$$\Phi_S = U D V^T.$$

Then Φ_S^+ has a SVD

$$\Phi_S^+ = (\Phi_S^T \Phi_S)^{-1} \Phi_S^T = V (D^T D)^{-1} D^T U^T$$
$$= V D^+ U^T$$

where D^+ has diagonal values which are reciprocal of diagonal values of D, that is:

$$D^+_{ii} = \frac{1}{D_{ii}} \qquad (\text{here } D_{ii} \neq 0)$$

and rest of its elements are 0. So, singular values of Φ_S^+ are <u>reciprocal</u> of singular values of Φ_S. In particular:

$$\|\Phi_S^+\|_{max} = \frac{1}{\|\Phi_S\|_{min}} \quad \text{and} \quad \|\Phi_S^+\|_{min} = \frac{1}{\|\Phi_S\|_{max}}$$

and using

$$\sqrt{1-\delta_{2k}} \leq \|\Phi_S\|_{min} \leq \|\Phi_S\|_{max} \leq \sqrt{1+\delta_{2k}}$$

we get

$$\Rightarrow \frac{1}{\sqrt{1+\delta_{2k}}} \leq \|\Phi_S^+\|_{min} \leq \|\Phi_S^+\|_{max} \leq \frac{1}{\sqrt{1-\delta_{2k}}}$$

which proves the result that

$$\frac{1}{\sqrt{1+\delta_{2k}}} \leq \|\Phi_S^+\|_2 = \|\Phi_S^+\|_{max} \leq \frac{1}{\sqrt{1-\delta_{2k}}}$$

Date:

(d) From results of (b) and (c) part, we have:

$$\frac{\|\eta\|_2}{\sqrt{1+\delta_{2k}}} \le \|\Phi_S^+\|_{min}\|\eta\|_2 \le \|x-\tilde{x}\|_2 \le \|\Phi_S^+\|_{max}\|\eta\|_2 \le \frac{\|\eta\|_2}{\sqrt{1-\delta_{2k}}}$$

The bound of error $\|\eta\|_2$ is ε, so in the <u>worst case</u> ($\|\eta\|_2 = \varepsilon$), we have:

$$\Rightarrow \quad \frac{\varepsilon}{\sqrt{1+\delta_{2k}}} \le \|x-\tilde{x}\|_2 \le \frac{\varepsilon}{\sqrt{1-\delta_{2k}}}$$

On the other hand, theorem 3 for ~~an~~ a k-sparse x gives us an error bound:
(here x^* is the reconstruction using BP)

$$\|x-x^*\|_2 \le \frac{C_0}{\sqrt{k}}\|x-x_S\|_1 + C_1\varepsilon = C_1\varepsilon$$

So, $\|x-x^*\|_2 \le C_1\varepsilon$, ($C_1$ depends only on δ_{2k}, which is fixed for given Φ)
which means, given Φ,
$\|x-x^*\|_2$ is $O(\varepsilon)$ [big O notation]
and $\|x-\tilde{x}\|_2$
is $O(\varepsilon)$ and also $\Omega(\varepsilon)$. $\therefore$ $\|x-\tilde{x}\|_2$ is $\Theta(\varepsilon)$
Thus $\|x-\tilde{x}\|_2$ is only a constant factor ~~worse~~ better than $\|x-x^*\|_2$, ~~as~~ and we have:

$$\|x-\tilde{x}\|_2 \ge \frac{\varepsilon}{\sqrt{1+\delta_{2k}}} \ge \left(\frac{1}{C_1\sqrt{1+\delta_{2k}}}\right)\|x-x^*\|_2$$

This shows that, $\|x-x^*\|_2 \le C_1\sqrt{1+\delta_{2k}}\,\|x-\tilde{x}\|_2$
giving us the required justification.
That is, x^* is different from x not more than a constant factor times $\|\tilde{x}-x\|_2$.

Problem 5:

$P1: \min_x \|x\|_1 \quad \text{s.t.} \quad \|y - \Phi x\|_2 \leq \varepsilon$

LASSO: $\min_x J(x)$, $J(x) = \|y - \Phi x\|_2^2 + \lambda \|x\|_1$

Given x is a minimizer of LASSO, define $\varepsilon' = \|y - \Phi x\|_2$. (for a given $\lambda > 0$)

For all vectors θ, (of same dim as x) $J(\theta) \geq J(x)$

$\Rightarrow \|y - \Phi \theta\|_2^2 + \lambda \|\theta\|_1 \geq \|y - \Phi x\|_2^2 + \lambda \|x\|_1$

$\Rightarrow \|y - \Phi \theta\|_2^2 + \lambda \|\theta\|_1 \geq (\varepsilon')^2 + \lambda \|x\|_1$

$\Rightarrow \lambda \|\theta\|_1 \geq ((\varepsilon')^2 - \|y - \Phi \theta\|_2^2) + \lambda \|x\|_1 \quad - (1)$

now consider all θ satisfying $\|y - \Phi\theta\|_2 \leq \varepsilon'$.

Thus, $\Rightarrow (\varepsilon')^2 - \|y - \Phi \theta\|_2^2 \geq 0$

From (1), this gives us:

$\Rightarrow \lambda \|\theta\|_1 \geq ((\varepsilon')^2 - \|y - \Phi \theta\|_2^2) + \lambda \|x\|_1 \geq \lambda \|x\|_1$

$\Rightarrow \|\theta\|_1 \geq \|x\|_1 \quad (\because \lambda > 0)$

Thus, $\forall$ θ with $\|y - \Phi \theta\|_2 \leq \varepsilon'$, we have $\|\theta\|_1 \geq \|x\|_1$.

Thus x is a minimiser of the problem P_1.

(Note x also satisfies the required constraint, i.e., $\|y - \Phi x\|_2 \leq \varepsilon'$)

Date:

Problem 6:

n subjects, $k << n$ are infected. Dorfman pooling with group size g. [n/g is integer]

So, number of groups, $m = \frac{n}{g}$

All m pools are tested in 1st stage. If R of the m groups test +ve, all members in the R groups are tested in the second stage. So, number of total tests, $T = m + Rg$, as Rg members are tested in 2nd stage.

Now, assuming the k cases to be uniformly randomly distributed, Let $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$ be random variables representing the number of cases in the groups 1 to m.

We know $\sum_{i=1}^{m} \beta_i = k$, total +ve cases.

So, the random variable R, which is the number of non-zero β_i (i.e., rows with non 0 cases) is given by:

$$R = \sum_{i=1}^{m} \mathbb{I}[\beta_i > 0].$$ (here $\mathbb{I}[X]$ is indicator R.V. of event X)

So the random variable T is:

$$T = m + g \sum_{i=1}^{m} \mathbb{I}[\beta_i > 0]$$

So, the average number of tests $\langle T \rangle$, is the same as $\mathbb{E}[T]$ (expectation of T)

Date:

$$\langle T\rangle = \mathbb{E}\left[m + g\sum_{i=1}^{m} \mathbb{I}[\beta_i > 0]\right]$$

$$= m + g\sum_{i=1}^{m} \mathbb{E}[\mathbb{I}[\beta_i > 0]]$$

$$= m + g\sum_{i=1}^{m} \Pr[\beta_i > 0] \qquad [\because \mathbb{E}[\mathbb{I}(X)] = \Pr[X] \rightarrow \text{probability of event } X]$$

Note that due to uniformity and symmetry $\Pr[\beta_i > 0] = p \;\; \forall 1 \le i \le m$.
~~[illegible]~~ That is all groups ~~[illegible]~~ i have the same value of $\Pr[\beta_i > 0] = p$. $\Rightarrow \langle T\rangle = m + mgp$.
Note

$$p = \Pr[\beta_i > 0] = 1 - \Pr[\beta_i = 0] \qquad (\because \beta_i \text{ takes values in } \mathbb{Z}^+ \cup \{0\})$$

$\Pr[\beta_i = 0]$ can be found by taking number of ways to sample g ~~[illegible]~~ subjects ~~[illegible]~~ from the $(n-k)$ non-infected subjects: $n_1 = \binom{n-k}{g}$, and dividing it by the total

number of ways to pick g out of n subjects: $n_2 = \binom{n}{g}$. $\Pr[\beta_i = 0] = \frac{n_1}{n_2}$

$$\Rightarrow \quad p = 1 - \Pr[\beta_i = 0] = 1 - \frac{n_1}{n_2} = 1 - \frac{\binom{n-k}{g}}{\binom{n}{g}}$$

$$\Rightarrow \langle T\rangle = m + gmp = m\left(1 + g\left(1 - \frac{\binom{n-k}{g}}{\binom{n}{g}}\right)\right)$$

$$\langle T\rangle = \frac{n}{g} + n\left(1 - \frac{\binom{n-k}{g}}{\binom{n}{g}}\right)$$

$$= \frac{n}{g} + n\left(1 - \frac{(n-k)!}{n!} \cdot \frac{(n-g)!}{(n-g-k)!}\right)$$

Date:

Now consider the worst case, i.e., when T takes its maximum value. Note that since there are k total cases, there can be atmost k groups which give positive result in 1st stage, and this is when k groups have 1 case each, and rest of the groups have 0 cases. That is, the maximum value of the R.V., R is:

$$R_{max} = k \geq R.$$

so

$$T_{max} = m + g R_{max} = m + kg \geq T.$$

T_{max} is the worst case number of tests. If n, k are fixed and g is the variable which we want to optimise, then we can write: $T_{max}(g) = \frac{n}{g} + kg$

$$T_{max}(g) = \frac{n}{g} + kg, \Rightarrow \frac{dT_{max}}{dg} = k - \frac{n}{g^2} = 0$$

$$\Rightarrow \boxed{g^* = \sqrt{\frac{n}{k}}}$$ is the group size which minimises $T_{max}(g)$.

So, for worst case:

→ Optimal group size, $g^* = (n/k)^{1/2}$

→ optimal number of groups, $m^* = \frac{n}{g^*} = (nk)^{1/2}$

Problem 1:

(1) For Φ, if $\delta_{2s} = 1$. Then $\forall$ 2s sparse θ, we have

$$0 \leq \|\Phi\theta\|_2^2 \leq 2\|\theta\|_2^2$$ → not necessary however

and thus it may be possible that $\|\Phi\theta\|_2 = 0$ for some non-zero 2s-sparse θ. If S_1 is the set of indices of non-zero elements ~~indices~~ of θ, then $\|\Phi_{S_1}\theta_{S_1}\|_2 = 0$ $[|S_1| \leq 2s]$

$$\Rightarrow \Phi_{S_1}\theta_{S_1} = 0$$

$$\Rightarrow \sum_{i\in S_1} \Phi_i \theta_i = 0, \text{ and } \theta_{S_1} \neq 0$$

This means that columns of Φ_{S_1} are linearly ~~independent~~ dependent, i.e., ~~we~~ Φ may have 2s linearly dependent columns.

(2) $\|\Phi(x^* - x)\|_{\ell_2} = \|(\Phi(x^*) - y) + (y - \Phi x)\|_{\ell_2}$

by triangular inequality, we have:

$$\|(\Phi x^* - y) + (y - \Phi x)\|_{\ell_2} \leq \|\Phi x^* - y\|_{\ell_2} + \|y - \Phi x\|_{\ell_2}$$

We also know: $y = \Phi x + \eta$ where $\varepsilon \geq \|\eta\|_{\ell_2}$

so $\|y - \Phi x\|_{\ell_2} = \|\eta\|_{\ell_2} \leq \varepsilon$

Also to find x^*, a constraint is that

$\|y - \Phi x^*\|_{\ell_2} \leq \varepsilon$ [constraint in BP]

So: $\|y - \Phi x^*\|_{\ell_2} + \|y - \Phi x\|_{\ell_2} \leq \varepsilon + \varepsilon = 2\varepsilon$

$$\Rightarrow \|\Phi(x^* - x)\|_{\ell_2} \leq \|\Phi x^* - y\|_{\ell_2} + \|y - \Phi x\|_{\ell_2} \leq 2\varepsilon$$

(3) First let us show $\|x\|_{\ell_2} \le \|x\|_{\ell_\infty} s^{1/2}$ $\forall$ s-sparse $x \in \mathbb{R}^n$ for some given n.
for this let x_i be the ith element of x, and S_0 be the set of indices with non-zero elements of x: (here $|S_0| = s$)

$$\|x\|_{\ell_2}^2 = \sum_{i=1}^{n} |x_i|^2 = \sum_{i \in S_0} |x_i|^2$$

(here $\|x\|_{\ell_\infty} = \max\limits_{1 \le i \le n} |x_i|$)

$$\le \sum_{i \in S_0} \|x\|_{\ell_\infty}^2 = |S_0| \, \|x\|_{\ell_\infty}^2 \le s \|x\|_{\ell_\infty}^2$$

So $\|x\|_{\ell_2}^2 \le s \|x\|_{\ell_\infty}^2 \Rightarrow \|x\|_{\ell_2} \le s^{\frac{1}{2}} \|x\|_{\ell_\infty}$

Note that h_{T_j} is also s-sparse, so
$\|h_{T_j}\|_{\ell_2} \le s^{1/2} \|h_{T_j}\|_{\ell_\infty}$

now we know that all non-zero coefficients of $h_{T_{j-1}}$ are larger (in magnitude) than all coefficients of h_{T_j}, by the way all h_{T_j}'s are chosen. If $h_{T_{j-1}} = z$

$$\Rightarrow \|z\|_{\ell_1} = \sum_{i \in S} |z_i| \ge \sum_{i \in S} \|h_{T_j}\|_{\ell_\infty}$$

where S is indices of z with non-zero coefficients. and $|z_i| \ge \|h_{T_j}\|_{\ell_\infty}$ follows from the earlier argument. So: $\|z\|_{\ell_1} \ge \sum_{i \in S} \|h_{T_j}\|_{\ell_\infty} = s \|h_{T_j}\|_{\ell_\infty}$

$$\Rightarrow \|z\|_{\ell_1} \ge s \|h_{T_j}\|_{\ell_\infty} \ge s^{\frac{1}{2}} \|h_{T_j}\|_{\ell_2}$$

$$\Rightarrow \|h_{T_j}\|_{\ell_2} \le s^{\frac{1}{2}} \|h_{T_j}\|_{\ell_\infty} \le \frac{\|h_{T_{j-1}}\|_{\ell_1}}{s^{1/2}}$$

$\therefore$ proved.

(4) From last part $\|h_{T_j}\|_{\ell_2} \leq s^{-\frac{1}{2}} \|h_{T_{j-1}}\|_{\ell_1}$ $\forall j \geq 2$

So $\sum_{j\geq 2} \|h_{T_j}\|_{\ell_2}$

$$\leq \sum_{j\geq 2} s^{-\frac{1}{2}} \|h_{T_{j-1}}\|_{\ell_1}$$

$$= s^{-\frac{1}{2}} (\|h_{T_1}\|_{\ell_1} + \|h_{T_2}\|_{\ell_1} + \dots)$$

now note that $T_0^c = T_1 \cup T_2 \cup T_3 \dots$

$$= \bigcup_{j\geq 1} T_j$$

so

$$\|h_{T_0^c}\|_{\ell_1} = \sum_{i\in T_0^c} |h_i| = \sum_{i\in T_1} |h_i| + \sum_{i\in T_2} |h_i| + \sum_{i\in T_3} |h_i| + \dots$$

$$= \sum_{j\geq 1} \|h_{T_j}\|_{\ell_1} = \|h_{T_1}\|_{\ell_1} + \|h_{T_2}\|_{\ell_1} + \dots \boxed{\geq \sum_{j\geq 2} \|h_{T_{j-1}}\|_{\ell_1}}$$

$$\therefore \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \leq s^{-\frac{1}{2}} (\|h_{T_1}\|_{\ell_1} + \|h_{T_2}\|_{\ell_1} + \dots) = s^{-\frac{1}{2}} \sum_{j\geq 2} \|h_{T_{j-1}}\|_{\ell_1}$$

$$\leq s^{-\frac{1}{2}} \|h_{T_0^c}\|_{\ell_1}$$

(5) note that $(T_0 \cup T_1)^c = \bigcup_{j\geq 2} T_j = T_2 \cup T_3 \cup T_4 \cup \dots$

So $h_{(T_0\cup T_1)^c} = \sum_{j\geq 2} h_{T_j} \Rightarrow \|h_{(T_0\cup T_1)^c}\|_{\ell_2} = \|\sum_{j\geq 2} h_{T_j}\|_{\ell_2}$

$\|\sum_{j\geq 2} h_{T_j}\|_{\ell_2} \leq \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2}$ (by triangular inequality)

from last part $\sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \leq s^{-\frac{1}{2}} \|h_{T_0^c}\|_{\ell_1}$

$$\therefore \|h_{(T_0\cup T_1)^c}\|_{\ell_2} \leq \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \leq s^{-\frac{1}{2}} \|h_{T_0^c}\|_{\ell_1}$$

(6) $\|x^*\|_{\ell_1} = \|x+h\|_{\ell_1}$ is a minimum for the BP optimisation problem

so $\forall\ z$ with $\|y - \Phi z\|_{\ell_2} \leq \varepsilon \Rightarrow \|z\|_{\ell_1} \geq \|x^*\|_{\ell_1}$

since $\|y - \Phi x\|_{\ell_2} = \|\eta\|_{\ell_2} \leq \varepsilon$

$\Rightarrow \|x\|_{\ell_1} \geq \|x^*\|_{\ell_1}$

so

$$\|x\|_{\ell_1} \geq \|x+h\|_{\ell_1} = \sum_{i\in(T_0 \cup T_0^c)} |x_i + h_i|$$

$$= \sum_{i\in T_0} |x_i + h_i| + \sum_{i \in T_0^c} |x_i + h_i|$$

now $|x_i + h_i| \geq |x_i| - |h_i| \Rightarrow \sum_{i\in T_0} |x_i + h_i| \geq \sum_{i\in T_0} |x_i| - \sum_{i\in T_0^c} |h_i|$

(triangle inequality)

$$\Rightarrow \sum_{i\in T_0} |x_i + h_i| \geq \|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1}$$

similarly, $|x_i + h_i| \geq |h_i| - |x_i|$

$$\Rightarrow \sum_{i\in T_0^c} |x_i + h_i| \geq \cancel{\|h_{T_0}\|} \ \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1}$$

$$\Rightarrow \|x\|_{\ell_1} \geq \|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1}$$

(7) From last part:

$$\|x\|_{\ell_1} \geq \|x_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1}$$

$$\Rightarrow \|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1} + \|x\|_{\ell_1} + \|x_{T_0^c}\|_{\ell_1} - \|x_{T_0}\|_{\ell_1}$$
$$= \|h_{T_0}\|_{\ell_1} + \|x_{T_0^c}\|_{\ell_1} + (\|x\|_{\ell_1} - \|x_{T_0}\|_{\ell_1})$$

now we know $x = x_{T_0} + x_{T_0^c}$.

$$\Rightarrow \|x\|_{\ell_1} = \|x_{T_0} + x_{T_0^c}\|_{\ell_1} \leq \|x_{T_0}\|_{\ell_1} + \|x_{T_0^c}\|_{\ell_1}$$
(by ℓ_1 norm triangular inequality)

$$\Rightarrow (\|x\|_{\ell_1} - \|x_{T_0}\|_{\ell_1}) \leq \|x_{T_0^c}\|_{\ell_1}$$
(Infact an equality holds)

So from this

$$\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1} + \|x_{T_0^c}\|_{\ell_1} + (\|x\|_{\ell_1} - \|x_{T_0}\|_{\ell_1})$$
$$\leq \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$$

$\therefore$ proved.

(8) Results from previous parts:

(i) $\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \leq s^{-\frac{1}{2}} \|h_{T_0^c}\|_{\ell_1}$

(ii) $\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$

From (i) and (ii), clearly, $\|h_{(T_0\cup T_1)^c}\|_{\ell_2} \leq s^{-\frac{1}{2}}\|h_{T_0}\|_{\ell_1} + 2 s^{-\frac{1}{2}}\|x_{T_0^c}\|_{\ell_1}$

now let $e_0 \equiv s^{-\frac{1}{2}} \|x_{T_0^c}\|_{\ell_1}$
$$= s^{-\frac{1}{2}}\|x - x_{T_0}\|_{\ell_1} = s^{-\frac{1}{2}}\|x - x_s\|_{\ell_1}$$

and note that

$$\|h_{T_0}\|_{\ell_1} = \sum_{i \in T_0} |h_i| \leq |T_0| \sqrt{\left(\sum_{i\in T_0} |h_i|^2\right) \times \frac{1}{|T_0|}}$$

(by RMS $\geq$ AM inequality)

$$\Rightarrow \|h_{T_0}\|_{\ell_1} \leq \sqrt{|T_0|}\sqrt{\sum_{i\in T_0}|h_i|^2} = \sqrt{s}\,\|h_{T_0}\|_{\ell_2}$$

$$\Rightarrow s^{-\frac{1}{2}}\|h_{T_0}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_2}$$

This gives us $\|h_{(T_0\cup T_1)^c}\|_{\ell_2} \leq \|h_{T_0}\|_{\ell_2} + 2e_0$

Date:

(9) Firstly, by cauchy schwarz inequality, ($|\langle x, y\rangle| \leq \|x\|_{\ell_2}\|y\|_{\ell_2}$)

$$|\langle \Phi h_{T_0 \cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0\cup T_1}\|_{\ell_2} \|\Phi h\|_{\ell_2}$$

now, from an earlier part, we have:

$$\|\Phi h\|_{\ell_2} = \|\Phi(x^* - x)\|_{\ell_2} \leq 2\varepsilon \quad - (1)$$

Moreover, by the **RIP of Φ** and since $h_{T_0\cup T_1}$ is 2s sparse, we have:

$$\|\Phi h_{T_0\cup T_1}\|_{\ell_2}^2 \leq (1+\delta_{2s})\|h_{T_0\cup T_1}\|_{\ell_2}^2$$

$$\Rightarrow \|\Phi h_{T_0\cup T_1}\|_{\ell_2} \leq \sqrt{1+\delta_{2s}}\,\|h_{T_0\cup T_1}\|_{\ell_2} \quad (2)$$

From (1) and (2), we can conclude:

$$|\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle| \leq 2\varepsilon\sqrt{1+\delta_{2s}}\,\|h_{T_0\cup T_1}\|_{\ell_2}$$

(10) In the paper Lemma 2.1 states and proves that

$$|\langle \Phi x, \Phi x'\rangle| \leq \delta_{s+s'}\|x\|_{\ell_2}\|x'\|_{\ell_2}$$

where x is s-sparse and x' is s'-sparse

In our case h_{T_0} and h_{T_j} are both s-sparse $\forall j$. So directly from Lemma 2.1:

$$|\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle| \leq \delta_{s+s}\|h_{T_0}\|_{\ell_2}\|h_{T_j}\|_{\ell_2}$$
$$= \delta_{2s}\|h_{T_0}\|_{\ell_2}\|h_{T_j}\|_{\ell_2}$$

The same holds for T_1 in place of T_0 as T_1 is also s-sparse.

(11) note that T_0 and T_1 are disjoint. So

$$\|h_{T_0 \cup T_1}\|_{\ell_2}^2 = \sum_{i \in T_0 \cup T_1} |h_i|^2 = \sum_{i \in T_0} |h_i|^2 + \sum_{i \in T_1} |h_i|^2$$

$$= \|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2$$

so, by RMS $\geq$ AM inequality, we get:

$$\sqrt{\left(\frac{\|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2}{2}\right)} \geq \frac{1}{2}\left(\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}\sqrt{\left(\|h_{T_0 \cup T_1}\|_{\ell_2}^2\right)} \geq \frac{1}{2}\left(\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2}\right)$$

$$\Rightarrow \sqrt{2}\,\|h_{T_0 \cup T_1}\|_{\ell_2} \geq \|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2} \quad \blacksquare$$

(12) From the RIP of Φ and using the fact that $h_{T_0 \cup T_1}$ is $2s$-sparse, we get:

$$(1 - \delta_{2s}) \|h_{T_0 \cup T_1}\|_{\ell_2}^2 \leq \|\Phi h_{T_0 \cup T_1}\|_{\ell_2}^2$$

and $$\|\Phi h_{T_0 \cup T_1}\|_{\ell_2}^2 = \left(\langle \Phi h_{T_0 \cup T_1}, \Phi h_{T_0 \cup T_1} \rangle\right)$$

$$= \left(\langle \Phi h_{T_0 \cup T_1}, \left(\Phi h - \sum_{j \geq 2} \Phi h_{T_j}\right) \rangle\right) \qquad \left[\because h = h_{T_0 \cup T_1} + \sum_{j \geq 2} h_{T_j}\right]$$

$$= \langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle - \sum_{j \geq 2} \langle \Phi h_{T_0 \cup T_1}, \Phi h_{T_j} \rangle$$

$$= \langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle - \sum_{j \geq 2} \langle \Phi h_{T_0}, \Phi h_{T_j} \rangle - \sum_{j \geq 2} \langle \Phi h_{T_1}, \Phi h_{T_j} \rangle$$

$$\leq |\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle| + \sum_{j \geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + \sum_{j \geq 2} |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \qquad (\because h_{T_0 \cup T_1} = h_{T_0} + h_{T_1})$$

Date:

From result in previous parts, we have:

$$\|\Phi h_{T_0 \cup T_1}\|_{\ell_2}^2 \le |\langle \Phi h_{T_0 \cup T_1}, \Phi h\rangle| + \sum_{j\geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle| + \sum_{j\geq 2} |\langle \Phi h_{T_1}, \Phi h_{T_j}\rangle|$$

$$\le 2\varepsilon \sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_{\ell_2} + \sum_{j\geq 2} \left(\delta_{2s}(\|h_{T_0}\|_{\ell_2} + \|h_{T_1}\|_{\ell_2})\, \|h_{T_j}\|_{\ell_2}\right)$$

$$\le 2\varepsilon \sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_{\ell_2} + \sqrt{2}\,\delta_{2s}\left(\sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \|h_{T_0 \cup T_1}\|_{\ell_2}\right)$$

$$= \|h_{T_0 \cup T_1}\|_{\ell_2} \left(2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2}\right)$$

Thus, we have proved that:

$$(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_{\ell_2}^2 \le \|\Phi h_{T_0 \cup T_1}\|_{\ell_2}^2 \le \|h_{T_0 \cup T_1}\|_{\ell_2} \left(2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2}\right)$$

(13) from result of last part: (cancelling $0 \neq \|h_{T_0 \cup T_1}\|_{\ell_2}$ on both sides)

$$(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_{\ell_2} \le (2\sqrt{1+\delta_{2s}})\,\varepsilon + \sqrt{2}\,\delta_{2s} \sum_{j\geq 2} \|h_{T_j}\|_{\ell_2}$$

(note that if $\|h_{T_0 \cup T_1}\|_{\ell_2} = 0$ then the inequality holds anyways) $(\varepsilon \geq 0)$

now, from result of earlier parts: $\sum_{j\geq 2} \|h_{T_j}\|_{\ell_2} \le s^{-\frac{1}{2}} \|h_{T_0^c}\|_{\ell_1}$

So, $(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_{\ell_2} \le (2\sqrt{1+\delta_{2s}})\,\varepsilon + \sqrt{2}\,\delta_{2s}\, s^{-\frac{1}{2}} \|h_{T_0^c}\|_{\ell_1}$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_{\ell_2} \le \alpha\varepsilon + \rho\, s^{-\frac{1}{2}} \|h_{T_0^c}\|_{\ell_2}$$

where $\alpha \equiv \dfrac{2\sqrt{1+\delta_{2s}}}{(1-\delta_{2s})}$, $\rho \equiv \dfrac{\sqrt{2}\,\delta_{2s}}{(1-\delta_{2s})}$

(14) From last part and part 7-8, we have:

$$(\|h_{T_0}\|_{\ell_1} \le s^{1/2}\|h_{T_0}\|_{\ell_2})$$

$$\|h_{T_0^c}\|_{\ell_1} \le s^{1/2}\|h_{T_0}\|_{\ell_2} + 2\|x_{T_0^c}\|_{\ell_1}$$
$$= s^{1/2}\|h_{T_0}\|_{\ell_2} + 2\, s^{1/2}(\|x - x_s\|_{\ell_1} s^{-1/2})$$
$$= s^{1/2}(\|h_{T_0}\|_{\ell_2} + 2e_0)$$
$$\Rightarrow \quad s^{-1/2}\|h_{T_0^c}\|_{\ell_1} \le \|h_{T_0}\|_{\ell_2} + 2e_0$$

So from last part:

$$\|h_{T_0 \cup T_1}\|_{\ell_2} \le \alpha\varepsilon + \rho(s^{-1/2}\|h_{T_0^c}\|_{\ell_1})$$
$$\le \alpha\varepsilon + \rho\|h_{T_0}\|_{\ell_2} + 2\rho e_0$$

now

$$\|h_{T_0}\|_{\ell_2}^2 + \|h_{T_1}\|_{\ell_2}^2 = \|h_{T_0 \cup T_1}\|_{\ell_2}^2 \ge \|h_{T_0}\|_{\ell_2}^2$$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_{\ell_2} \ge \|h_{T_0}\|_{\ell_2}$$

So,

$$\Rightarrow \|h_{T_0 \cup T_1}\|_{\ell_2} \le \alpha\varepsilon + \rho\|h_{T_0}\|_{\ell_2} + 2\rho e_0$$
$$\le \alpha\varepsilon + \rho\|h_{T_0 \cup T_1}\|_{\ell_2} + 2\rho e_0 \quad \blacksquare$$

(15) $$\|h\|_{\ell_2} = \|h_{(T_0 \cup T_1)} + h_{(T_0 \cup T_1)^c}\|_{\ell_2} \qquad [\because h = h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}]$$
$$\le \|h_{T_0 \cup T_1}\|_{\ell_2} + \|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \qquad \text{(triangle ineq.)}$$

and we proved in part 8 that:

$$\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \le \|h_{T_0}\|_{\ell_2} + 2e_0 \qquad (\because \|h_{T_0}\|_{\ell_2} \le \|h_{T_0 \cup T_1}\|_{\ell_2})$$
$$\le \|h_{T_0 \cup T_1}\|_{\ell_2} + 2e_0$$

So

$$\|h\|_{\ell_2} \le \|h_{T_0 \cup T_1}\|_{\ell_2} + (\|h_{T_0 \cup T_1}\|_{\ell_2} + 2e_0)$$
$$= 2\|h_{(T_0 \cup T_1)}\|_{\ell_2} + 2e_0$$

From last part, we have:

$$\|h_{T_0 \cup T_1}\|_{\ell_2} \leq (1-\rho)^{-1} (\alpha \varepsilon + 2\rho e_0)$$

So, putting it together:

$$\|h\|_{\ell_2} \leq 2\|h_{T_0 \cup T_1}\|_{\ell_2} + 2e_0$$

$$\leq 2\left[(1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0) + e_0\right]$$

$$= 2(1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0 + (1-\rho)e_0)$$

$$= 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0) \quad \blacksquare$$

(16) $\|h\|_{\ell_1} = \|h_{T_0} + h_{T_0^c}\|_{\ell_1} = \|h_{T_0 \cup T_0^c}\|_{\ell_1}$ (T_0, T_0^c are disjoint)

$$= \sum_{i \in T_0 \cup T_0^c} |h_i| = \sum_{i \in T_0} |h_i| + \sum_{i \in T_0^c} |h_i|$$

$$\cancel{= \|h_{T_0}\|} = \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1}$$

In lemma 2.2 we proved:

$$\|h_{T_0}\|_{\ell_1} \leq s^{1/2}\|h_{T_0}\|_{\ell_2} \leq s^{1/2}\|h_{T_0 \cup T_1}\|_{\ell_2}$$

$$\leq s^{1/2}(\alpha \times 0 + \rho s^{-1/2}\|h_{T_0^c}\|_{\ell_1})$$

$$= \rho\|h_{T_0^c}\|_{\ell_1} \quad (\varepsilon = 0)$$

$\Rightarrow \|h_{T_0}\|_{\ell_1} \leq \rho\|h_{T_0^c}\|_{\ell_1}$

So, $\|h\|_{\ell_1} \leq \|h_{T_0^c}\|_{\ell_1}(1+\rho)$

and from prev. parts $\underbrace{\|h_{T_0^c}\|_{\ell_1} \leq 2(1-\rho)^{-1}\|x_{T_0^c}\|_{\ell_1}}$ explanation

$\Rightarrow \|h\|_{\ell_1} \leq 2(1+\rho)(1-\rho)^{-1}\|x_{T_0^c}\|_{\ell_1}$

$\therefore$ proved.

(we proved earlier

$$\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$$

$$\leq \rho\|h_{T_0^c}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$$

$$\Rightarrow (1-\rho)\|h_{T_0^c}\|_{\ell_1} \leq 2\|x_{T_0^c}\|_{\ell_1})$$

Assignment - 2 (Problem - 3)

Aditya Kudre

February 2022

1 Problem - 3

The definition of the Restricted Isometry Constant (RIC) is as follows: the RIC δ_s of a matrix A of size $m \times n$ is the smallest value x such that:

$$(1-x)||\theta||^2 \leq ||A\theta||^2 \leq (1+x)||\theta||^2 \tag{1}$$

where θ is an s-sparse $n \times 1$ vector.

Thus, if there exists a y such that $(1-y)||\theta||^2 \leq ||A\theta||^2 \leq (1+y)||\theta||^2$ then $\delta_s \leq y$. Now suppose we have two positive integers s and t with corresponding RICs δ_s and δ_t respectively and $s \leq t$, then it is clear that for any t-sparse vector θ_t,

$$(1-\delta_t)||\theta_t||^2 \leq ||A\theta_t||^2 \leq (1+\delta_t)||\theta||^2 \tag{2}$$

Any s-sparse vector θ_s will also be a t-sparse vector as $s \leq t$. Thus, for any s-sparse vector θ_s,

$$(1-\delta_t)||\theta_s||^2 \leq ||A\theta_s||^2 \leq (1+\delta_t)||\theta||^2 \tag{3}$$

From our previous conclusion, we can conclude that for such a number δ_t then $\delta_s \leq \delta_t$.

Q.E.D.

Assignment - 2 (Problem - 4)

Aditya Kudre

February 2022

1 Details of the Paper

- **Title :** Measurement Matrix Design for Compressive Sensing–Based MIMO Radar
- **Journal :** IEEE Transactions on Signal Processing
- **Authors :** Yao Yu, Athina P. Petropulu, Fellow, IEEE, and H. Vincent Poor, Fellow, IEEE
- **Publication Month and Year :** November 2011
- **Link to the Paper :** https://ieeexplore.ieee.org/document/5955141

2 Imaging System

In this system, we have a set of M_t transmitters and N_r receivers. Each transmitter transmits periodic narrow-band pulses of duration T_p and pulse repetition interval T. Let L denote the number of $T - s$-spaced samples of the transmitted waveforms within one pulse. T_s is also the time duration of one sub-pulse, $T_s = \frac{T_p}{L}$. The compressive receiver is pre-multiplies by matrix Φ_l a T_s-sampled version of the received pulse. The size of Φ_l is $M \times (L + L\ )$, where $L\ = \frac{2R_u}{c}$ where R_u is the maximum unambiguous range. A suitable basis matrix Ψ_{lm} is also chosen and a new matrix $S_{lm} = \Phi_l \Psi_{lm}$ is constructed.

3 Sensing Matrix Construction

In this paper, we are trying to optimize the sum of coherence of two columns of the matrix S which is $\Sigma_{k \neq k'} \mu_{k,k'}$ along with a new property called SIR (Signal to Interference Ratio). This interference occurring in the radar measurement is assumed to be i.i.d. Gaussian with zero mean i.e. $n_{lm}(t)\ N(0, \sigma^2)$. Specifically, we have the following optimization problem:

$$min_{\Phi}(\Sigma_{k \neq k'} \mu_{k,k'} + \frac{\lambda}{SIR}) \tag{1}$$

Where, λ is a tradeoff coefficient.

Here, we need to enforce the constraint on the sensing matrix such that the columns of S should have unity norm. Applying this condition, we get that,

$$SIR = \frac{1}{Tr(\Phi^H \Phi)} \tag{2}$$

This is a convex optimization problem and can be solved using the CVX functionality in Matlab.

4 Simulation Results

This design improves the Signal to Interference Ratio in signals thereby reducing the effects of interference caused in Radar Signals as can be seen from the graph below:

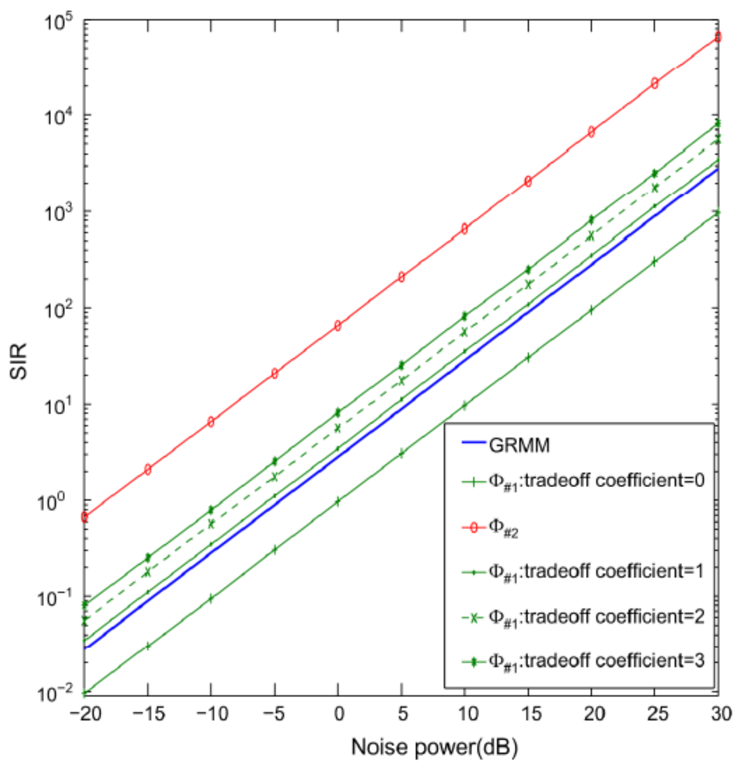

Figure 1: SIR values as the tradeoff coefficient is varied