CS 754 Project : Reweighted ℓ_1 Minimisation

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***Abstract*—We tested out and analysed the results of some of the reweighted minimisation algorithms from the paper ”Enhancing Sparsity by Reweighted ℓ_1 Minimization”. Apart from the main algorithm, we implemented the reweighted Gauss Dantzig selector, error correction using reweighted ℓ_1 minimization, applied an ISTA based version of the algorithm for Hitomi Snapshot reconstruction, and also carried out reconstruction of medical images.**

I. Testing Reweighted ℓ_1 Minimisation and Some Variations

First we test out the reweighted ℓ_1 minimization algorithm for 4 reweighted iterations on a sparse signal x of length $n = 256$, and $m = 100$ measurements while varying ϵ and the sparsity k. The k nonzero values and the forward matrix A are chosen from a standard normal distribution and the columns of A are normalised. We also test the unweighted version on the same setup. We plot the reconstruction probabilities after 100 trials on each value of ϵ and the 0th iteration result of every epsilon trial is used to find the probability for the unweighted case. (Here success is declared if $||x - x_{recon}||_{\ell_\infty} < 1e-3$).

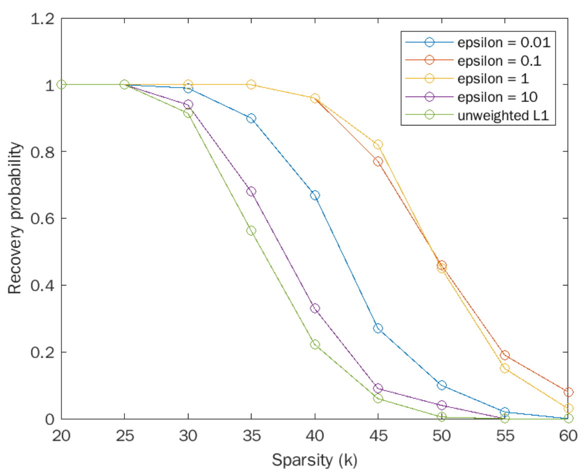

Fig. 1. Reconstruction probabilities for different ϵ and unweighted case

Our result is similar to the papers result. The unweighted version works well upto k = 25, while the best case is when ϵ is around 0.1-1, and it gives perfect reconstruction upto k = 35, which shows the improvement over unweighted case. The performance of weighted algorithm decays as we increase ϵ to 10 or decrease it to 0.01.

Adaptive ϵ:

We test the performance of the algorithm with an adaptive choice of ϵ as given in the paper by plotting reconstruction probabilities for 100 trials on the same previous setup. We

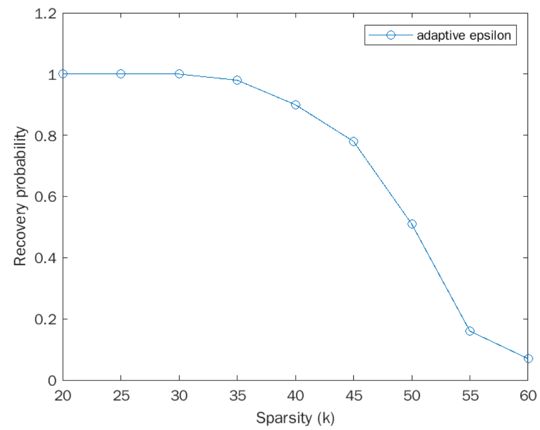

Fig. 2. Reconstruction probability with adaptive epsilon

can see that the performance is close (just slightly worse) to the optimal performance of $\epsilon = 1$ in the previous case. The probability decays rather slowly with k and it works perfectly till k = 30.

atan penalty based reweighing:

As mentioned in the paper we can use different type of penalties to approximate the ℓ_0 norm. One example given was the penalty: $g(x) = \sum_{i=1}^{n} atan(|xi|/\epsilon)$. This leads to weight updates as $w_i = 1/(x_i^2 + \epsilon^2)$. We tested this version of reweighted ℓ_1 minimization on the same setup as before.

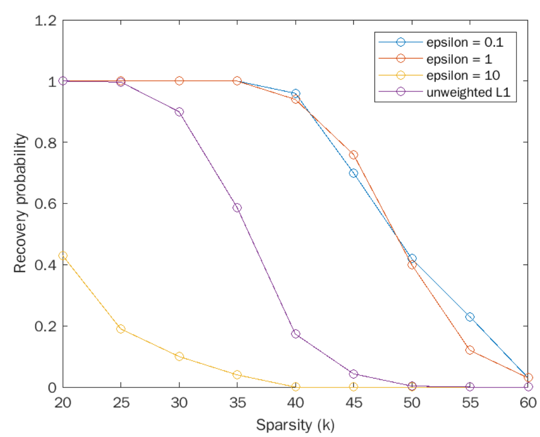

Fig. 3. Reconstruction probability with atan based reweighing

In this case, we can see that the performance for ϵ around 0.1 to 1 is optimal and performance degrades very sharply as we increase the value of ϵ to 10. The peak performance is similar to that of the log penalty based reweighing, with perfect reconstruction till k = 35.

II. NOISE AWARE REWEIGHTED ℓ_1 MINIMISATION

We test the noise aware algorithm in the paper with x and A sampled as earlier and mean zero gaussian noise added to Ax with std. deviation $\sigma = 0.04$ and δ selected as $\delta^2 = \sigma^2(m + 2\sqrt{2m})$. We set $\epsilon = 1$ and sparsity $k = 10$ for this experiment and look at the histograms of the relative ℓ_2 errors for reweighted and unweighted reconstruction over 100 trials. (Here rel. ℓ_2 err. = $||x - x_{recon}||_{\ell_2}/||x||_{\ell_2}$)

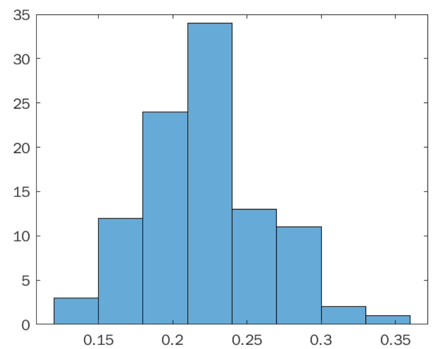

Fig. 4. Relative l_2 error histogram (unweighted reconstruction)

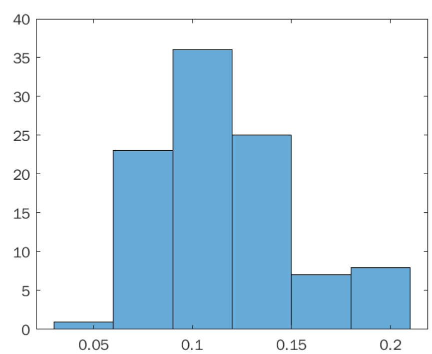

Fig. 5. Relative l_2 error histogram (reweighted reconstruction)

We can see a clear improvement of reweighted version over unweighted version. The error in reweighted case is centered around 0.1-0.15, while for unweighted case it is around 0.2-0.25.

III. REWEIGHTED GAUSS-DANTZIG

We implement the reweighted gauss-dantzig algorithm which involves a reweighted dantzig selector step followed by a refined gauss-dantzig estimate in each iteration. We choose x to have size n = 256 and is created by sampling a_i from standard normal distribution and setting $x_i = a_i + sign(a_i)$ for all non-zero locations i of x (This ensures that magnitude of non zero elements is > 1). The sparsity k = 10 and $\epsilon = 0.1$ for this experiment and matrix A is sampled as before (m = 100). A noise of $\sigma = 0.1$ is added to Ax and we set $\delta = 1$. Parameter $\alpha = 0.25$ in the gauss-dantzig refinement step.
We run the reweighted version for 9 iterations and compare with unweighted version. We look at the histograms of relative ℓ_2 error and of the number of correctly identified indices in the estimated support of x (out of 10) for the two reconstructions over 100 trials.

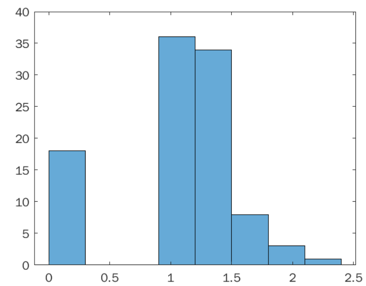

Fig. 6. Relative l_2 error histogram (unweighted reconstruction)

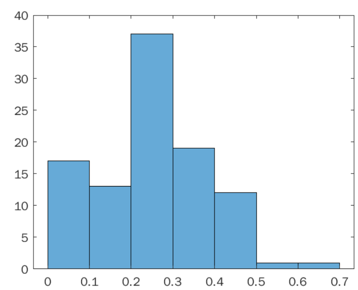

Fig. 7. Relative l_2 error histogram (reweighted reconstruction)

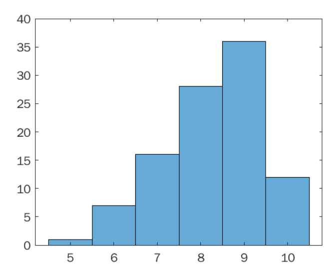

Fig. 8. no. of correctly identified indices (unweighted reconstruction)

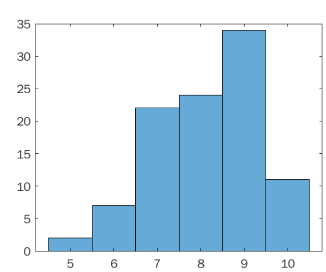

Fig. 9. no. of correctly identified indices (reweighted reconstruction)

We can see that the reweighted version produces smaller reconstruction error more reliably (most of the time rel err < 0.5), while unweighted version sometimes produces small error but often produces very large errors (greater than 1).

IV. ERROR CORRECTION

We implemented the reweighted error correction algorithm in the paper. The vector x and corrupted y (corruption is k sign flips) are generated exactly as in the paper. Matrix A is generated as before and we keep n = 128, m = 4n = 512. We set $\epsilon = \beta \times std(y)$. We observe the reconstruction probabilities (100 trials) vs k/m curves for different β and for unweighted case. We used 4 reweighted iterations. This result doesn't

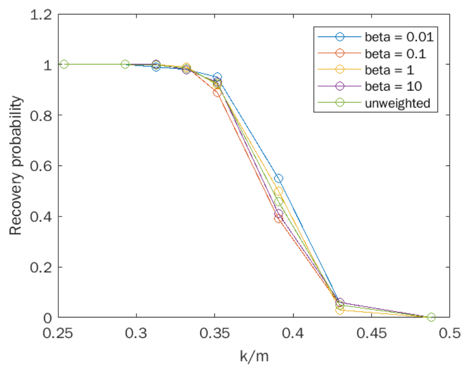

Fig. 10. Reconstruction probabilities with different β

match the result of the paper (only slight difference in our experiments is normalisation of columns of A in our case). This result doesn't suggest much improvement of reweighted approach over unweighted approach. We also tried the same approach except for using a β parameter, we directly varied ϵ (no dependance on std(y)).

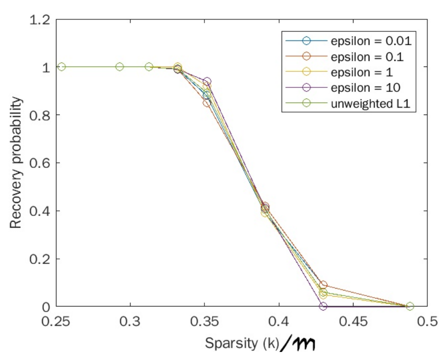

Fig. 11. Reconstruction probabilities with different ϵ

A notable difference from the papers results is that in our case, the unweighted version is performing much better, which is likely due to normalisation of the A matrix columns, since we did everything else the same. Thus we do not see much improvement over unweighted ℓ_1 minimisation in this experiment.

V. HITOMI RECONSTRUCTION

We applied the reweighted ℓ_1 reconstruction to the Hitomi Camera. For this we replaced the Basis Pursuit Problem with a LASSO problem, that is:

$$BP: \ min_x \ ||x||_{\ell_1} \ s.t. \ ||y - Ax||_{\ell_2} \leq \delta \tag{1}$$

was replaced with:

$$LASSO: \ min_x \ \lambda||x||_{\ell_1} + ||y - Ax||_{\ell_2}^2 \tag{2}$$

And the LASSO problem was solved using ISTA.

The video cars.avi was used and T = 3 consecutive frames we included in the snapshot. We work with 120×240 patch of the total frame. Noise added to the snapshot is mean 0 gaussian with $\sigma = 2$. We do patchwise reconstruction with 8×8 patch and average over overlapping patches. The 2D DCT basis is used as the sparsifying basis. So we can say $x = \Psi\theta$ and we want to reconstruct a sparse θ given $y = A\theta + \eta$, where η is the noise and $A = \Phi\Psi$ where Φ is the matrix corresponding to the snapshot measurement. We used reweighted reconstruction here as well as unweighted reconstruction for different values of λ and compare them. We keep $\epsilon = 1$ and look at the results after r reweighing iterations.

Fig. 12. Original frames

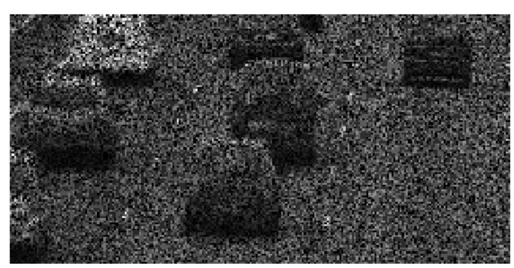

Fig. 13. Snapshot generated

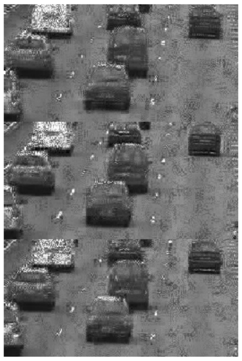

Fig. 14. Reconstruction, r = 0, $\lambda = 10$, PSNR = 19.7189

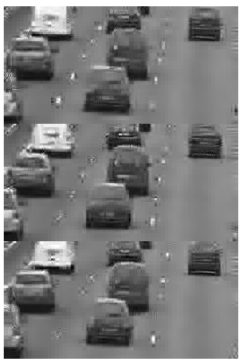

Fig. 15. Reconstruction, r = 4, $\lambda = 10$, PSNR = 23.3354

This is a low value of λ so the LASSO focusses more on the distance term than the L1 term, which causes the unweighted reconstruction to be more like the snapshot. On reweighing the sparsity of reconstructed θ is improved, so the result is more like the original. However as we will notice, this enhanced sparsity can sometimes be a bit much and blur the finer details. In the final case λ is too large (50), so the unweighted reconstruction (r = 0) is quite sparse itself, and sparseness only increases each iteration. In this case PSNR slightly decreases on increasing iterations.

Fig. 16. Reconstruction, r = 0, $\lambda = 20$, PSNR = 22.9976

In this case (λ = 20) we can see some finer details are blurred

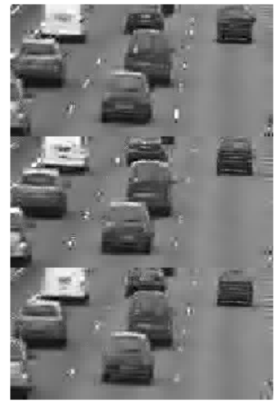

Fig. 17. Reconstruction, r = 4, $\lambda = 20$, PSNR = 23.4330

on increasing the iterations as reconstruction becomes sparse, but the PSNR still improves with iterations.

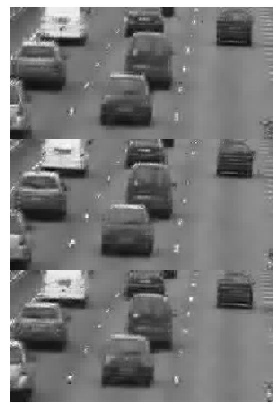

Fig. 18. Reconstruction, r = 0, $\lambda = 50$, PSNR = 23.7879

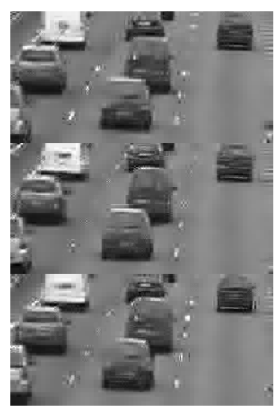

Fig. 19. Reconstruction, r = 4, $\lambda = 50$, PSNR = 23.5287

CS 754 Project Report

Project Report for IITB CS754 Advanced Image Processing 2022

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I. Reconstruction using reweighted l_1 analysis

A. Introduction

In Compressed Sensing, there are two major types of reconstruction problems. These are synthesis problems and analysis problems. Synthesis problems involve the "synthesis" of the original unknown image or vector by minimizing a norm or similar property with given constraints. Analysis problems involve analyzing the norm or a similar property of a function of the unknown vector given some constraints. Example of a synthesis problems can be,

$$min_x ||x||_1 \tag{1}$$

such that

$$y = \Phi \Psi x \tag{2}$$

while the Analysis counterpart of this problem would be,

$$min_x ||\Psi^* x||_1 \tag{3}$$

such that

$$y = \Phi x \tag{4}$$

In many cases, the above two problems would be mathematically equivalent as the basis matrix Ψ is generally orthonormal. But, they still would yield different solutions when actually implemented.

In case of reweighted l_1 analysis problems, we have the following Algorithm :

Initializing weights :

$$w_i = 1 \forall i = 1, 2, ..., n \tag{5}$$

Analysis Problem :

$$min_x ||diag(w) \Psi^* x|| \tag{6}$$

such that

$$y = \Phi x \tag{7}$$

Updating weights :

$$w_i = (\Psi^* x)_i \tag{8}$$

B. Experiments Performed

We performed experiments involving the usual analysis problem on medical images as well as the reweighted analysis problem on the same images. We used the basis matrix Ψ as the DCT matrix (we are assuming sparsity of images in DCT basis). For reweighted analysis, we need to pre-multiply Ψ by the weight matrix $W = diag(w)$ where w is the vector of weights.

For the measurement matrix Φ, we have used two different schemes. First, we tried to directly randomize Φ as an $m \times n$ matrix. Next, we tried out random sampling in the Fourier domain. This means that we would take the 2D Fourier transform of the image and randomly select any m coefficients.

We have used NESTA for the implementation of these Algorithms.

Here's one of our test cases in the experiment. This was the original image that we used for getting the measurements through the two different schemes mentioned above:

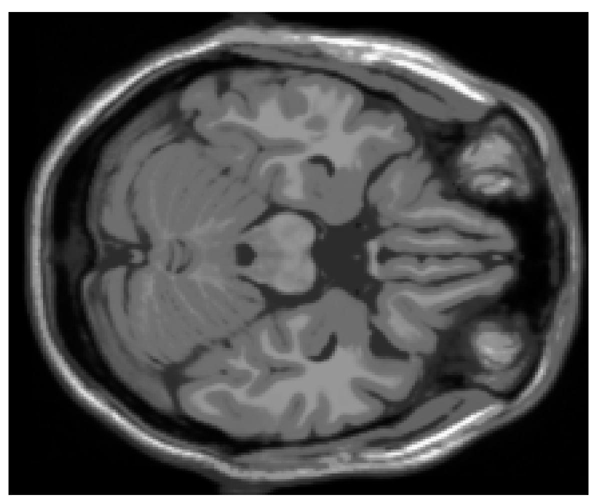

Fig. 1. Original Image from a brain MRI scan

Here's the image reconstructed by taking measurements using the first scheme (random $m \times n$ Φ) and solving usual analysis problem:

Fig. 2. Image reconstructed using first scheme and analysis problem

Now, here's the image reconstructed by taking measurements using the first scheme and solving reweighted analysis problem for 5 iterations:

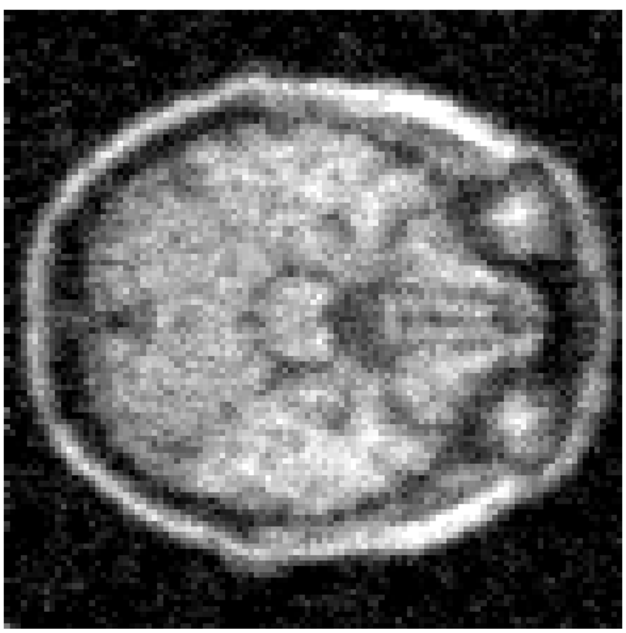

Fig. 3. Image reconstructed using first scheme and reweighted analysis problem

Now, we used the second scheme. Here's the image reconstructed by taking measurements using the second scheme (random sampling in Fourier domain) and solving usual analysis problem:

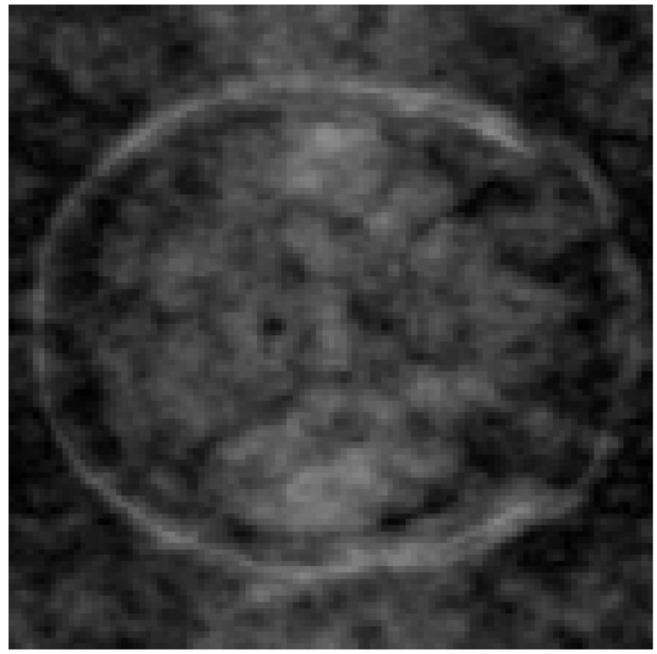

Fig. 4. Image reconstructed using second scheme and analysis problem

Now, here's the image reconstructed by taking measurements using the second scheme and solving reweighted analysis problem for 5 iterations:

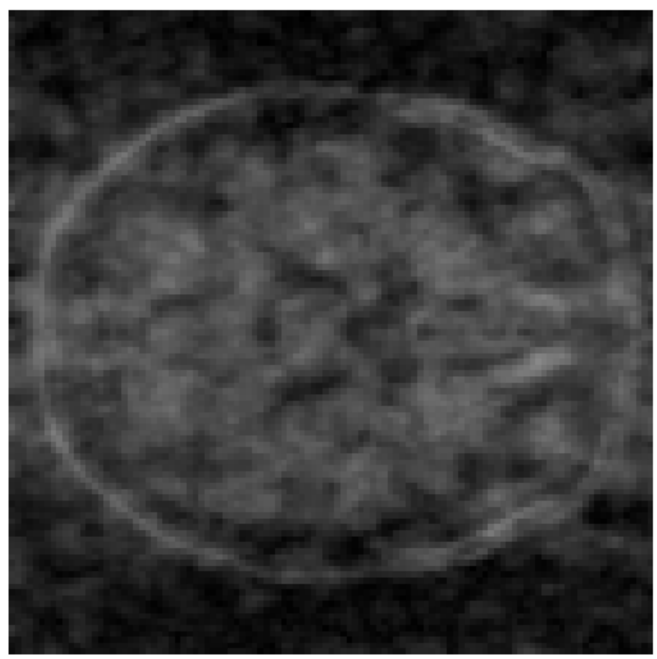

Fig. 5. Image reconstructed using second scheme and reweighted analysis problem

C. Observations

As can be seen from the figures, the first scheme works pretty well under the condition that number of measurements are the same. We had kept number of measurements same ($m = 3270$) for the analysis problem based experiments just to know which method works better. We can clearly see much of the finer details in the image reconstructed using reweighted l_1 analysis with first measurement scheme. Also, the contrast of this image is enhanced as compared to the usual analysis problems. In the second scheme, we can see that there is an artifact in the form of a faint horizontal or vertical line in the images. Here, again the contrast is enhanced if we use reweighted l_1 analysis.

References

[1] Enhancing Sparsity by Reweighted $l1$ Minimization Link

[2] NESTA: A fast and accurate first-order method for sparse recovery, Stephen Becker, Jerome Bobin and Emmanuel J. Candes Link