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Elastic consequences of a single plastic event: A step towards the microscopic modeling of the flow of yield stress fluids

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Abstract. With the eventual aim of describing flowing elasto-plastic materials, we focus here on the elementary process of such a flow, a plastic event, and compute the long-range perturbation it elastically induces in a medium submitted to a global shear strain. We characterize the effect of a nearby wall on this perturbation, and quantify the importance of finite-size effects. Although most of our explicit formulae refer to 2D situations, our statements hold for 3D situations as well.

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1 Introduction

An increasing number of experiments indicate that, for many complex systems, flow at a macroscopic scale actually relies on a spatially heterogeneous behaviour at smaller scales. We focus here on a (large) sub-class, that of the systems that display a macroscopic yield stress, the so-called "yield stress fluids" (among which foams, suspensions, emulsions, colloidal glasses, ...). Typically, these systems flow homogeneously at large stress/shear rate, whereas at low shear rates they may exhibit spatial coexistence between a flowing and a frozen region [1–5] or intermittent heterogeneous flow [2]. To this point, there is little insight as to whether the mechanisms leading to such macroscopic behaviours are generic or dependent on the specific microscopic structure of the fluid and the corresponding interactions.

Various "elasto-plastic" models have been put forward to apprehend such macroscopic behaviour, to which one can add attempts aiming at the description of earthquakes In these models, the medium first responds elastically to a global forcing (either stress or strain). The deformation or stress can then locally induce a rearrangement or *plastic event*, if a local threshold is reached. Such a plastic event locally relaxes a stress that is elastically redistributed in the medium, and can trigger other local events. In this picture, the macroscopic flow is the outcome of the collectively organized sequence of local rearrangements. Although this mesoscopic description seems very reasonable, many questions remain to be answered for this scenario to be operational. First, what is (are) the basic plastic event(s), and how can it (they) be identified in a given flowing complex material? Second, what is the constitutive (dynamic) equation that describes such a single plastic event under a local forcing? Third, how does such an "event", locally relaxing stress, perturb the surrounding medium? The answer to this last question is obviously linked to the nature of the plastic event.

As to the first two questions, *i.e.* the nature and the description of the plastic event, various convincing pictures have been proposed in the literature. In a pioneering work, Bulatov and Argon introduced a phenomenological description of a single plastic event, which allowed them to describe many properties of macroscopic plastic flows [7–9]. In their simulation, the unit cell can undergo several fixed plastic deformations specific to their hexagonal geometry. Later, on the basis of molecular simulations of a Lennard-Jones glass under imposed shear stress [10], and building on earlier works by Spaepen and Argon, Falk and Langer introduced the notion of shear transformation zone (STZ), which described a local limited zone where rearrangements occur. The occurrence of very localized plastic events is most easily evidenced in foams [5], where they take the form of T1 rearrangements. Langer [11] then constructed an analytical "mean field" elasto-plastic model, introducing STZ as zones with a plastic tensorial deformation. More recently, Baret *et al.* [12], and Braun [13] performed numerical simulations on lattices for a scalar model, in which a plastic event occurs when the local stress

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reaches a yield stress. Thus, within a very general class of elasto-plastic models, the notion and the description of a plastic event is now well documented and clarified. However, a clear description of the consequences of a localized plastic event on the stress distribution in the material (the third question above) still needs to be constructed.

This is the purpose of the present paper: using a rather general description for a localized plastic event, we compute the long-range *elastic* perturbation that such an event induces in an elastic material. We characterize its symmetry and amplitude, as well as the way it is modified if the event occurs close to a solid boundary. An *a priori* counter-intuitive result which emerges from our calculations, is the crucial role played by finite-size effects in the modeling of flowing elasto-plastic materials. We limit ourselves here to the study of the elastic effects of a single event, and leave for a later report the analysis of the collective organization of the plastic events when the material is flowing.

The paper is organized as follows. In Section 2, we specify our general "elasto-plastic" model corresponding to a medium taken homogeneous and isotropic, as well as incompressible for simplicity. In Section 3, we consider an infinite geometry, and compute the elastic distortion induced by a localized shear plastic deformation, making connections to earlier studies. In Section 4, focusing on the 2D case, we consider finite-size geometries, where the system is bounded by solid walls. First, we describe how a wall affects the perturbation induced by an event occurring in its vicinity. Second, turning to a medium confined between two walls, we calculate the average stress relaxation induced by a localized shear event, and give explicit formulae to compute the whole stress field. In Section 5, we conclude and briefly highlight important consequences for the modeling of flowing systems.

2 Elasto-plastic model

We assume, following many of the previously quoted studies, that the displacements and strains are given by the simple superimposition of a plastic flow (the localized plastic events) and an elastic distortion of the medium. We further assume that the medium is homogeneous, isotropic, and linearly elastic. In addition, we focus for simplicity on the incompressible case (the compressible case can be studied following the same lines), so that the elastic properties of the medium are fully described by the shear modulus μ.

Denoting $\mathbf{u}(\mathbf{r})$ the total displacement vector at position $\mathbf{r}$, the strain tensor is given by $\boldsymbol{\epsilon} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^t)$. From our hypotheses, this total strain is the sum of an elastic strain and a plastic strain (non-zero only at the locus of plastic events):

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{el}} + \boldsymbol{\epsilon}^{\mathrm{pl}} \ . \tag{1}$$

Incompressibility corresponds to

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \ . \tag{2}$$

With the hypotheses of linear elasticity and incompressibility, the total stress tensor is $\mathbf{s} = -p\mathbf{1} + \boldsymbol{\sigma}$, where p is the pressure and $\boldsymbol{\sigma}$ verifies

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon}^{\mathrm{el}} = 2\mu\boldsymbol{\epsilon} - 2\mu\boldsymbol{\epsilon}^{\mathrm{pl}} \ . \tag{3}$$

As we have in mind the slow flow of pasty materials, we neglect inertial effects so that mechanical equilibrium simply requires

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} - p\mathbf{I}) = 0 \ . \tag{4}$$

We consider the classical situation where an applied shear (either imposed deformation or stress, usually enforced through action on the boundaries of the system) induces elastic loading of the material, up to the point where it triggers a single localized plastic event. The consequent state of the medium in response to the applied forcing is here the sum of a purely elastic response to the forcing (denoted with superscripts 0) and of the perturbation induced by the occurrence of the plastic event (denoted with superscripts 1). With these notations:

$$\begin{aligned} &\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^0 + \boldsymbol{\epsilon}^1 \ , \\ &\boldsymbol{\epsilon}^{\mathrm{el}} = \boldsymbol{\epsilon}^{\mathrm{el0}} + \boldsymbol{\epsilon}^{\mathrm{el1}} \ ; \quad \boldsymbol{\epsilon}^{\mathrm{pl}} = \boldsymbol{\epsilon}^{\mathrm{pl1}} \ , \\ &\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \boldsymbol{\sigma}^1 \ , \end{aligned} \tag{5}$$

where $\boldsymbol{\epsilon}^{\mathrm{pl1}}$ describes the localized plastic event.

We are seeking for the consequences of this event, namely the displacement and elastic stress fields $\mathbf{u}^1$ and $\boldsymbol{\sigma}^1$. Equations (2), (3) and (4) can be reformulated in a well-defined problem for the perturbation field:

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \mathbf{u}^1 = 0 \ , \\ &\boldsymbol{\nabla} \cdot (2\mu\boldsymbol{\epsilon}^1 - p^1\mathbf{I}) = 2\mu\boldsymbol{\nabla} \cdot \boldsymbol{\epsilon}^{\mathrm{pl}} \ , \end{aligned} \tag{6}$$

which must be solved with the appropriate boundary conditions. The solution can formally be written:

$$\mathbf{u}^1 = \int \mathrm{d}\mathbf{r}\ \mathbf{Q}(\mathbf{r}, \mathbf{r}')[-2\mu\boldsymbol{\nabla} \cdot \boldsymbol{\epsilon}^{\mathrm{pl}}(\mathbf{r}')] \ , \tag{7}$$

where $\mathbf{Q}$ is the Green's function for the response of the same elastic medium (with the same boundary conditions) to a point force $\mathbf{f}$ applied at $\mathbf{r}'$ [14], so that $\mathbf{u}(\mathbf{r}) = \mathbf{Q}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{f}$ is the solution of

$$\begin{aligned} &\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \ , \\ &\boldsymbol{\nabla} \cdot (2\mu\boldsymbol{\epsilon} - p\mathbf{I}) + \mathbf{f}\delta(\mathbf{r} - \mathbf{r}') = 0 \ . \end{aligned} \tag{8}$$

In an infinite medium, $\mathbf{Q}$ is the usual Oseen tensor $\mathbf{O}$. The induced stress $\boldsymbol{\sigma}^1$ is derived from (7) using (3).

The above formula (7) clearly shows that the elastic distortion in the medium originates in *variations* of the plastic strain. This directly echoes the classical study of Eshelby [15], who considered a "plastic" deformation, of constant value inside an "inclusion" and zero outside, in an infinite elastic matrix. He showed that the overall displacement in the system can be considered as generated by induced forces at the inclusion boundary (*i.e.* where the plastic strain suddenly drops to zero). Equation (7) is an integral form generalizing this result to arbitrary deformation field, geometry (boundary conditions are here implicit), and spatial dimension. In his remarkable paper, Eshelby actually computed the whole deformation field for ellipsoidal inclusions, and the far-field deformation for inclusions of arbitrary shape, for a 3D infinite system.

As we wish to deal with systems of different geometries (infinite, semi-infinite and confined by walls) and dimensions (in particular 2D), we will describe the plastic events by "point-like" plastic strains using Dirac delta-functions. This will allow us to get explicit solutions for the far-field behaviour in all these situations, at the expense, of course, of a precise description of the intimate structure and geometry of the plastic event and of its surroundings.

We first start by applying this strategy to the simple infinite geometry, that will allow us to check our procedure against Eshelby's results for the 3D case.

3 Infinite medium

We first point out that for an infinite medium, there is no distinction between a stress-driven system for which the appropriate boundary condition is

$$\boldsymbol{\sigma}^1(\infty) \to \mathbf{0}, \tag{9}$$

and a strain-controlled system for which

$$\mathbf{u}^1(\infty) \to \mathbf{0} \; . \tag{10}$$

In both situations, one can use the formalism above with $\mathbf{Q}(\mathbf{r}, \mathbf{r}')$ replaced by the Oseen tensor $\mathbf{O}(\mathbf{r}-\mathbf{r}')$, thanks to the translational invariance. The result is most simply displayed in terms of Fourier transforms (denoted by "hats"):

$$\hat{\mathbf{u}}^1(\mathbf{q}) = \hat{\mathbf{O}}(\mathbf{q}) \cdot (2\mu i \mathbf{q} \cdot \hat{\boldsymbol{\epsilon}}^{\mathrm{pl}}) \tag{11}$$

with [14]

$$\hat{\mathbf{O}}(\mathbf{q}) = \frac{1}{\mu q^2} \left(\mathbf{I} - \frac{\mathbf{q}\mathbf{q}}{q^2} \right) \; . \tag{12}$$

At this point, no assumption has been made on the nature of the plastic event, nor on the space dimension D, and the formula (11) holds quite generally (for example, one can recover some of Eshelby's results by plugging in an $\boldsymbol{\epsilon}^{\mathrm{pl}}$ constant in a sphere and zero outside).

3.1 Localized plastic shear events

To pursue analytically without dealing with too general tensorial formulae, *we now focus on a plastic event with the symmetry of the global forcing which we choose to be that of simple shear.* We will also mostly focus on the two-dimensional case $D = 2$: the $H \to \infty$ limit of Figure 1. However, the formalism of this subsection holds irrespective of the dimension, and is thus also valid for the 3D situation (see comments further).

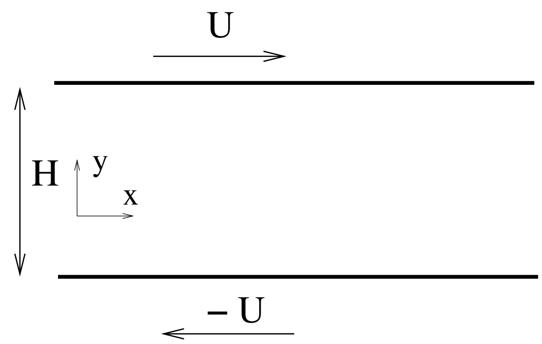

Fig. 1. The medium is sheared at an imposed strain $\gamma = 2U/H$. Section 3 deals with the limit corresponding to $H \to \infty$, $U \to \infty$ with $\frac{U}{H} = \gamma/2$ kept constant.

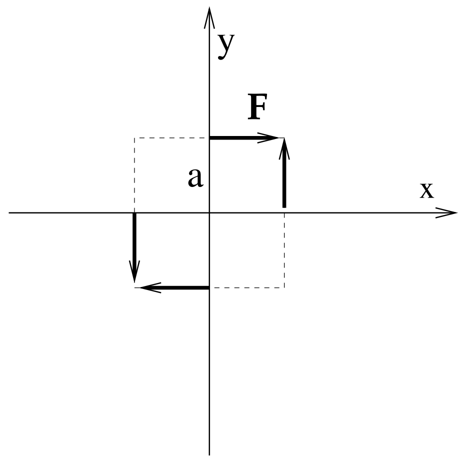

Fig. 2. The perturbation due to a localized plastic shear is equivalent to the perturbation due to a set of two dipoles of forces with $F = a^{D-1}\mu\epsilon_0$.

With the hypothesis above, the plastic deformation tensor corresponds to simple shear $\epsilon^{\mathrm{pl}}_{xy} = \epsilon^{\mathrm{pl}}_{yx}$, and no $\epsilon^{\mathrm{pl}}_{xx}, \epsilon^{\mathrm{pl}}_{yy}$. Expression (11) becomes

$$\begin{aligned} \hat{u}^1_x(\mathbf{q}) &= -2\mu(\hat{O}_{xx} \cdot iq_y + \hat{O}_{xy} \cdot iq_x)\hat{\epsilon}^{\mathrm{pl}}_{xy} \; , \\ \hat{u}^1_y(\mathbf{q}) &= -2\mu(\hat{O}_{xy} \cdot iq_y + \hat{O}_{yy} \cdot iq_x)\hat{\epsilon}^{\mathrm{pl}}_{xy} \; . \end{aligned} \tag{13}$$

As motivated at the end of Section 2, we now focus on "point-like" localized plastic events $\epsilon^{\mathrm{pl}}_{xy} = \epsilon_0 a^D \delta(\mathbf{r})$. From dimensional analysis, it is clear that the strength of an event is the product of a typical amplitude ϵ_0 and a microscopic volume a^D. Then $\hat{\epsilon}^{\mathrm{pl}}_{xy} = \epsilon_0 a^D$, and remembering that the displacement field of a dipole of forces $\mathbf{f}$ a distance $\mathbf{h}$ apart is $-i(\mathbf{h}\cdot\mathbf{q})\hat{\mathbf{O}}\cdot\mathbf{f}$, the perturbation displacement described by (13) is the one induced by the two force dipoles $(F, 2a)$ represented in Figure 2, with $F = a^{(D-1)}\mu\epsilon_0$ (or, more precisely, its limit for $a \to 0$ with the dipole strength $aF = \mu\epsilon_0 a^D$ kept constant).

Although transparent in the present Fourier representation, this equivalence between a localized plastic strain and force dipoles can be generally obtained in arbitrary geometry and dimension from (7).

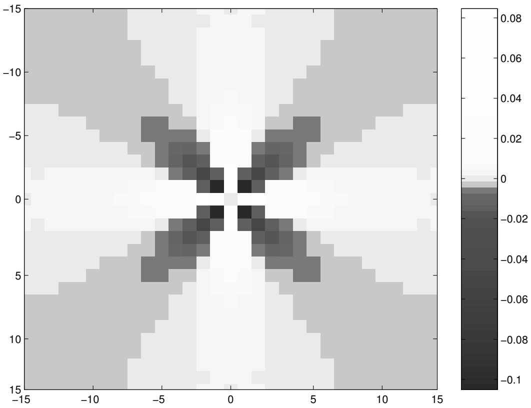

Fig. 3. Perturbation of the shear stress field for a plastic event occurring at the origin in a medium submitted to a shear strain or stress.

The shear stress perturbation corresponding to (13) can be obtained using (3), and reads:

$$\hat{\sigma}^1_{xy} = 2\mu^2(q_y^2\hat{O}_{xx} + q_x^2\hat{O}_{yy} + 2q_xq_y\hat{O}_{xy} - 1/\mu)\hat{\epsilon}^{\mathrm{pl}}_{xy} \ . \tag{14}$$

Similar formulae can be obtained for the other components of the stress (σ_{xx} and σ_{yy} in 2D), but we will mostly focus on the shear stress.

From the above formulae we can compute the *propagators* $\mathbf{P}^\infty$, G^∞ that describe the consequences in terms of displacement and elastic shear stress of a single plastic event in an infinite medium by

$$\mathbf{u}^1(\mathbf{r}) = \int \mathrm{d}\mathbf{r}'\mathbf{P}^\infty(\mathbf{r}-\mathbf{r}')\epsilon^{\mathrm{pl}}_{xy}(\mathbf{r}') \ , \tag{15}$$

$$\sigma^1_{xy}(\mathbf{r}) = 2\mu\int \mathrm{d}\mathbf{r}'G^\infty(\mathbf{r}-\mathbf{r}')\epsilon^{\mathrm{pl}}_{xy}(\mathbf{r}') \ . \tag{16}$$

In the two-dimensional case $D = 2$, we find

$$\hat{G}^\infty = -4\frac{q_x^2q_y^2}{q^4} \ , \tag{17}$$

$$G^\infty(r,\theta) = \frac{1}{\pi r^2}\cos(4\theta) \ . \tag{18}$$

Hence, in a system forced with a symmetry of simple shear, the perturbation of the shear stress due to a localized plastic event has eight lobes, four where the stress increases and four where it decreases (see Fig. 3). The amplitude of this perturbation decays away from the plastic event as a power law, $\sim 1/r^2$ in two dimensions. In three-dimensional systems, $\hat{G}^\infty = -(4q_x^2q_y^2 + q_z^2q^2)/q^4$ which yields $G^\infty = (3/4\pi r^7)(r^2(x^2+y^2) - 10x^2y^2)$, and thus again a $\cos(4\theta)$ angular dependence in the xy plane and a $\sim 1/r^3$ power law decrease.

3.2 Global effects of a plastic event

We now consider global effects of a localized plastic event in 2D $\epsilon^{\mathrm{pl}}_{xy}(\boldsymbol{r}) = a^2\epsilon_0\delta(\mathbf{r})$, namely quantities integrated over a whole layer (of constant y). The following are easily derived from our previous calculations:

$$\int_{-\infty}^{+\infty}\sigma^1_{xy}(x,y)\mathrm{d}x = 0 \ ,$$
$$\int_{-\infty}^{+\infty}u^1_x(x,y)\mathrm{d}x = \mathrm{Sign}(y)a^2\epsilon_0 \ ,$$
$$\int_{-\infty}^{+\infty}u^1_y(x,y)\mathrm{d}x = 0 \ . \tag{19}$$

The first equation states that the shear stress resulting from the plastic event is redistributed in such a way that the integrated stress on every layer is unchanged, *i.e.* there is no net release of stress over a layer, and consequently no change in the net force applied from above on the system! The second relation indicates that the average horizontal displacement in a (horizontal) layer depends only on whether it is above or below the event (but not on its distance to the event), while the third equation expresses that the average vertical displacement over a layer is zero. For simple xy shear in three dimensions, similar equations hold for quantities integrated over xz planes.

3.3 Relation to former studies

Let us now relate to former studies the results we have obtained at the end of Section 3.1, for the displacement and stress fields induced by a single localized plastic event of simple shear symmetry.

First, our results are in line with the full analytical description of Eshelby for the 3D case [15]. He computed the long-range part of the elastic perturbation (displacement and stress) for an arbitrary inclusion shape. Our formalism using a Dirac distribution to describe a plastic deformation actually gives back Eshelby's result in 3D (for the shear case considered here). We thus anticipate that (16, 18) provide the 2D formula for the long-range elastic decay around a localized plastic shear strain deformation of arbitrary shape.

To study numerically collective effects, Baret *et al.* [12] simulated elements with local yield stresses on a 2D lattice (semi-periodic boundary conditions). In contrast with the present study, they modeled plastic events by simple scalar displacements, which numerically led to a dipolar propagator to describe their consequences. As a formal check, we used our general expression (11) to compute the shear stress perturbation corresponding to their choice of plastic events, and indeed found a dipolar symmetry, consistent with the propagator they had numerically obtained on their lattice. This provides a validation of our procedure, but mostly underlines that the nature and symmetry of the elementary plastic event seriously affect the propagator describing its consequences, and therefore potentially the collective interplay of such events and the resulting macroscopic flow behaviour. Supported by the comparison with Eshelby's exact calculation, we believe that the form of plastic event used in the present study is better suited for the actual description of the flow of elastoplastic materials.

Kabla and Debrégeas [16] recently performed an explicit numerical simulation of a two-dimensional foam under shear strain. In their quasi-static procedure, at each step the length of the film is minimized at constant bubble volume, and they mimicked plastic "T1" events by reorganizing sets of four bubbles when the length of the film separating two bubbles decreases below a critical value. Among other interesting observations, they focused on the stress rearrangements following such T1 events in their simulation. Averaging over many such events, they found that statistically these stress perturbations have an eight-lobe pattern (with a slight tilt of the axes with respect to those of the macroscopic shear (*x,y*), probably due to a structuration of the foam by the flow into a slightly non-isotropic medium). They observed that this stress field coincides with that generated in an elastic medium by a set of dipoles equivalent to Figure 2. Thus, quite remarkably, their cellular simulation of a rather disordered medium yields *upon averaging* a picture consistent with the simple elastic continuum approach followed here.

Finally, in a series of papers on the deformation of plastic amorphous materials, Langer has introduced the concept of dynamically generated shear transformation zones (STZ), where the plastic activity occurs. Although most papers in this series deal with uniform situations, he investigated the possible development of heterogeneities in [11], where the response of a 2D material to an applied deviatoric stress is considered. In contrast to the present study, the plastic strain tensor is described there without any assumption as to its orientation. The results in that paper suggest that for a plastic strain tensor with the symmetry of the forcing, the deviatoric stress induced by the STZ has an eight-lobe pattern ($\sim \cos(4\theta)$), with an algebraic power law decrease $\sim 1/r^4$. Within our model, we obtain the same symmetry but a different power law $\sim 1/r^2$. The origin of this discrepancy seems to lie in the fact that Langer considers potential displacement fields (*i.e.* that are the gradient of some scalar quantity), thereby restricting the set of solutions available. In both Eshelby's and our approach, no such constraint is imposed, and we find at leading order a non-potential displacement field.

Obviously, the computations of this section can be easily generalized to a localized plastic event of arbitrary symmetry $\boldsymbol{\epsilon}^{\mathrm{pl}}_{xy} = \boldsymbol{\epsilon}_0 a^D \delta(\mathbf{r})$ in an arbitrary dimension, as long as the medium is unbounded.

We now turn to the most important part of our paper, namely the effect of the finite-size geometry on the stress distortion due to a localized event. In this study we will keep the focus on the 2D case with localized shear events.

4 Effects of a shear plastic event in a medium bounded by walls

As in the analysis above, we consider localized plastic events with the symmetry of the forcing (simple shear) in 2D. Such an event, $\epsilon^{\mathrm{pl}}_{xy} = \epsilon_0 a^2 \delta(\mathbf{r}-\mathbf{r}')$ is still equivalent

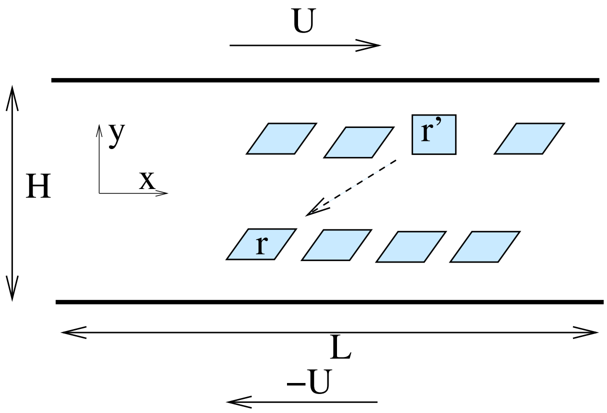

Fig. 4. A shear strain $\frac{2U}{H}$ is applied to a two-dimensional elasto-plastic medium. A plastic event occurs at position $\mathbf{r}'$. The elastic deformation it induces adds up on the elastic loading.

to a set of two dipoles of forces, as depicted in Figure 2. Indeed the argument that led to this picture can be repeated in a finite geometry. Therefore, the corresponding propagator can be viewed as the sum of the propagators for the four forces, which brings us back to considerations pertaining to the Green's function for the effect of a single force on the elastic *finite* medium.

We focus on the case of an imposed shear strain represented in Figure 4, where two solid walls adhering perfectly to the medium are shifted horizontally.

Therefore, the boundary condition for the total displacement field is

$$\mathbf{u}(x, \pm H/2) = \pm \mathbf{U} \ . \tag{20}$$

We proceed with the same decomposition as in the infinite medium case. The displacement field is the sum of the homogeneous elastic loading and of the perturbation due to the plastic event (again we denote by $\mathbf{u}^1, \boldsymbol{\sigma}^1$ the displacement and deviatoric stress tensor induced by the plastic event). The boundary condition for the perturbation in the imposed strain regime is

$$\mathbf{u}^1(x, \pm H/2) = \mathbf{0} \ . \tag{21}$$

The total response of the elasto-plastic medium is then

$$\begin{aligned} u_x(x,y) &= \frac{2Uy}{H} + u^1_x \ , \\ u_y(x,y) &= u^1_y \ , \\ \sigma_{xy}(x,y) &= \frac{2\mu U}{H} + \sigma^1_{xy}(x,y) \ . \end{aligned} \tag{22}$$

4.1 Event close to a wall

We focus here on the modification of the previous picture if the event occurs close to one of the walls, say the bottom one at $y_{\mathrm{wall}} = -H/2$, with the other wall far away. We thus seek the consequences at $\mathbf{r}$ of an event occurring at $\mathbf{r}'$ with

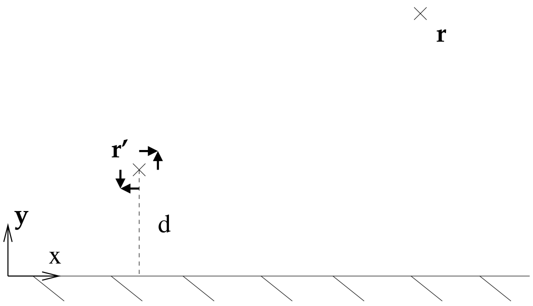

Fig. 5. An event close to the bottom wall: $H \gg |\mathbf{r} - \mathbf{r}'| \gg d$.

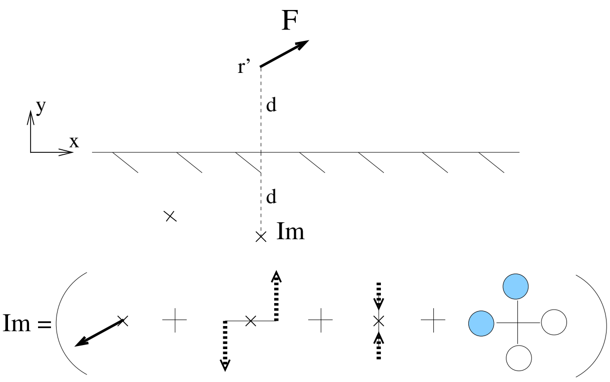

Fig. 6. Point force at $\mathbf{r}'$ and its elastic image: a force $-\mathbf{F}$, two dipoles of forces of strength $2dF_x$ and $-2dF_y$, respectively, and two dipoles of potential.

$|H/2 + y'| \ll |\mathbf{r} - \mathbf{r}'| \ll H$, see Figure 5. Practically, we consequently deal with a semi-infinite geometry.

Again, the effect of a plastic event is the sum of those of the four forces represented in Figure 5. Hence, we first study the displacement field induced by a single force $\mathbf{F}$ at position $\mathbf{r}'$ in the vicinity of the wall. It is classically obtained as the sum of the displacement fields *in an infinite medium* for the force and its *image* (so that the displacement field is zero on the wall) The image is depicted in Figure 6: its locus is the image of $\mathbf{r}'$ with respect to the wall, and it is the sum of i) a force $-\mathbf{F}$, ii) two dipoles of forces (the first one two opposite forces $\pm F_x \mathbf{e}_y$ located at a distance $2d\mathbf{e}_x$ apart, the second one two opposite forces $\pm F_y \mathbf{e}_y$ located at a distance $2d\mathbf{e}_y$ apart), and iii) two dipoles of potential (the first one of strength $\pm F_x d/\mu$ at a distance $2d\mathbf{e}_x$, the second one of strength $\mp F_y d/\mu$ at a distance $2d\mathbf{e}_y$), where $d = |y' + H/2|$ is the distance between the event and the wall (we refer the reader to the more explicit description in [14] where this result is derived). Its long-range behaviour, $|\mathbf{r} - \mathbf{r}'| \gg d$, is dictated by the dipoles of forces in Figure 6, yielding a displacement field that scales as $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$.

Returning to the whole plastic event (a set of dipoles as in Fig. 2), one could expect a similar cancellation of the first-order terms, and thus a faster decay than in the absence of the wall (*e.g*, a far-field displacement scaling as $1/|\mathbf{r} - \mathbf{r}'|^2$). However, inspection of the structure of the image shows that such is not the case (see Fig. 7): the dipole of horizontal forces is duplicated in its direct image,

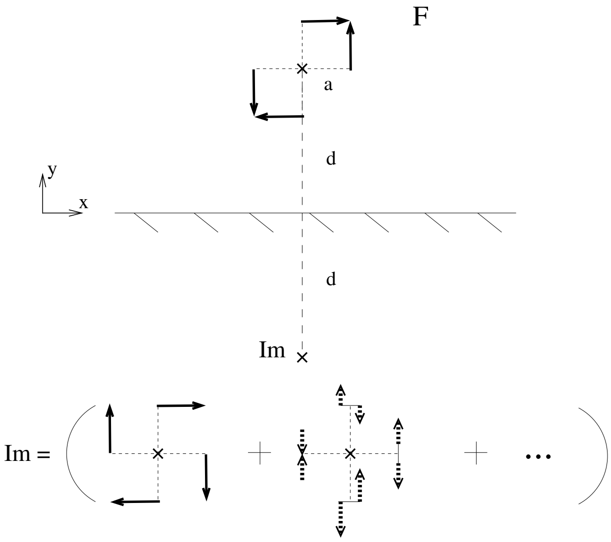

Fig. 7. Plastic event represented by two force dipoles and its *image* with respect to the wall, which can be constructed by summing those of the four forces. The contributions represented here (not at scale) are the direct images (solid arrows, all of amplitude F) and the induced force dipoles (dashed arrows, of amplitude $2(d-a)F$ and $2(d+a)F$ for the top and bottom one, of amplitude $2dF$ for the two others). Inspection shows that the net dominant long-range behaviour is twice that produced by the same event in an infinite medium.

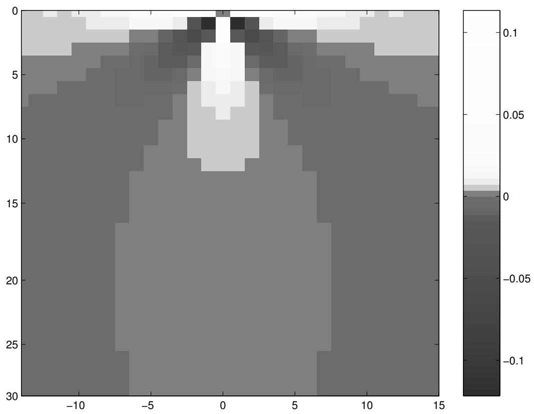

Fig. 8. Perturbation of the shear stress field for a plastic event (plastic deformation of amplitude $\epsilon_0 = 1$) occurring next to the top wall.

and the set of dipoles of vertical forces it generates yield a dipole of strength $4aF$. The contributions of the (original) dipole of vertical forces are weaker due to cancellations. Altogether, one is left with a dominant term that has the same symmetry than in an infinite medium (and a similar $1/r$ decay for the displacement) with an amplitude twice as strong. We have analytically checked that the stress decay is consistent with the above picture, as for a localized event close to the wall we obtain a propagator that is directly related to its equivalent in the absence of the wall equation (18):

$$G_{\text{wall}}(r, \theta) = 2G^{\infty}(r, \theta)(1 + O(d/r)) \ , \qquad (23)$$

where clearly $x - x' = r\cos(\theta)$, $y - y' = r\sin(\theta)$. This picture is also consistent with numerical calculations of the propagator for an event next to the wall in a finite geometry, which yields the picture in Figure 8 (calculation to be described in Sect. 4.2 below).

A plastic strain in the vicinity of the wall therefore yields an elastic field similar in structure but stronger in amplitude than its counterpart in an infinite medium.

4.2 Medium of finite thickness

In this subsection, we turn to a medium of finite thickness H. We focus again on an imposed strain situation, and therefore on the problem corresponding to the system of equation (6) together with the no-displacement boundary conditions (21). We indicate ways of calculating the propagator in this geometry but mostly emphasize consequences of a single event on integral quantities.

4.2.1 Finite H, infinite L

A first method to treat the case of a medium of finite thickness and infinite length, consists in the systematic construction of a series of images so as to cancel the displacements on both walls. Following this strategy, Pozrikidis [14] performed a full analytical calculation for the case of the deformation field induced by a point force. Formally, the perturbation due to a plastic event can then be deduced by adding up the consequences of each of the four forces it consists of. This leads to expressions that although exact are heavy to deal with and somewhat opaque. We therefore turn to other methods in the following, focusing on a geometry periodic in the x-direction.

4.2.2 Finite H, L periodic: first method

We now focus on a system of thickness H and of finite extent L in the x-direction, and consider periodic boundary conditions in that direction. This is equivalent to analyzing, in an unbounded geometry, the effect of a periodic array of plastic events of a given amplitude at positions $(x' + kL, y')$ with $k \in \mathbb{Z}$ (Fig. 9), or formally $\hat{\epsilon}^{\rm pl}_{xy}(\boldsymbol{r}) = a^2\epsilon_0 \sum_{k\in\mathbb{Z}} \delta(\mathbf{r} - (\boldsymbol{r'} + kL\boldsymbol{x}))$. The resulting displacement field $\mathbf{u}^1$ can then be viewed as the sum of the displacement field induced by a similar periodic array of plastic events *in an infinitely thick medium* $\mathbf{u}^\infty$ and of a correction term $\mathbf{v}^H$ due to the finite size H. $\mathbf{u}^1 = \mathbf{u}^\infty + \mathbf{v}^H$ is a function of $x - x', y$ and y'. The displacement field $\mathbf{u}^\infty$ can be expressed using the propagator for a single event (15):

$$\mathbf{u}^\infty(x, y) = a^2\epsilon_0 \sum_{k\in\mathbb{Z}} \mathbf{P}^\infty(x - x' - kL, y') \ . \qquad (24)$$

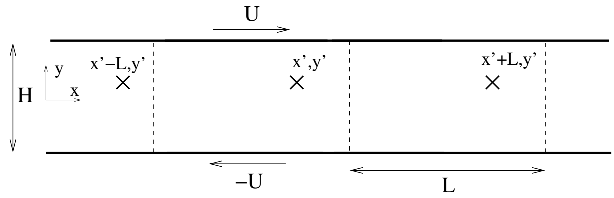

Fig. 9. A periodic array of plastic events at position $x'+kL, y'$.

Given the periodicity in the x-direction, $\mathbf{u}$ and $\mathbf{v}$ can be decomposed in Fourier series:

$$\mathbf{u}^\infty(x, y) = \mathbf{U}_0^\infty(y) + \sum_{n=1}^{\infty} \left(\mathbf{U}_{cn}^\infty(y)\cos(2n\pi x/L) + \mathbf{U}_{sn}^\infty(y)\sin(2n\pi x/L) \right) ,$$

$$\mathbf{v}^H(x, y) = \mathbf{V}_0^H(y) + \sum_{n=1}^{\infty} \left(\mathbf{V}_{cn}^H(y)\ \cos(2n\pi x/L) + \ \mathbf{V}_{sn}^H(y)\ \sin(2n\pi x/L) \right) ,$$

The correction displacement field $\mathbf{v}^H$ is the solution of incompressible linear elasticity with no source (*i.e.* the set of equations (3,4) with no plasticity) but with the boundary conditions required to grant that $\mathbf{u}^1$ is zero on the walls:

$$\mathbf{v}^H(x, \pm H/2) + \mathbf{u}^\infty(x, \pm H/2) = 0 \ . \qquad (25)$$

The functions $\mathbf{V}_{cn}^H$, $\mathbf{V}_{sn}^H$ can be calculated independently (*i.e.* mode by mode) with the boundary conditions: $\mathbf{V}_{cn}^H(\pm H/2) = -\mathbf{U}_{cn}^\infty(\pm H/2)$, $\mathbf{V}_{sn}^H(\pm H/2) = -\mathbf{U}_{sn}^\infty(\pm H/2)$. We skip this full calculation (similar to the low Reynolds number hydrodynamic study in [17]), since we display in the next section an exact expression for the displacement that is more amenable to numerical simulation.

Instead, we focus here on the zeroth mode $\mathbf{V}_0^H(y)$. From equation (19),

$$U_{0x}^\infty(y) = \ \mathrm{Sign}\,(y - y')a^2\epsilon_0/L \ ; \qquad U_{0y}^\infty(y) = 0 \ , \qquad (26)$$

so that

$$V_{0x}^H(y) = -2a^2\epsilon_0\frac{y}{HL} \ ; \qquad V_{0y}^H(y) = 0. \qquad (27)$$

This suffices to deduce consequences of the plastic event in terms of integrals over constant y lines:

$$\int_{-L/2}^{L/2} u_x^1(x, y)\mathrm{d}x = -2\left(a^2\epsilon_0\frac{y}{H}\right) + \mathrm{Sign}(y - y')a^2\epsilon_0 \ ,$$

$$\int_{-L/2}^{L/2} u_y^1(x, y)\mathrm{d}x = 0 \ .$$

Then, linear elasticity implies that the overall variation of the force on the top plate due to the plastic event is

$$\delta F = \int_{-L/2}^{L/2} \sigma_{xy}{}^1\mathrm{d}x = -2\frac{\mu\epsilon_0 a^2}{H} \ . \qquad (28)$$

The corresponding drop of the average shear stress in the medium is, obviously, $\delta\langle\sigma\rangle = \delta F/L$.

Thus, *a single localized plastic event (of a given amplitude $\epsilon_0 a^2$) relaxes the net force exerted by the medium on the walls by a quantity scaling as $\frac{1}{H}$. Remarkably, this quantity is independent of the position of the event in the medium.* A corollary is that a finite density of plastic events ϕ, will relax this total force by an amount $\sim (\phi HL)\delta F$, corresponding to a relaxation of the average stress $\delta\langle\sigma\rangle = \delta F/L$ independent of the size of the system.

Note that the integrals over a period L calculated above for the consequences of a periodic array of plastic events, are equal to the integrals over x from $-\infty$ to $+\infty$ in a finite H infinite L geometry for the consequences of a single plastic event. For example, in the latter geometry $\delta F = \int_{-\infty}^{\infty} {\sigma_{xy}}^1 \mathrm{d}x = -2\frac{\mu\epsilon_0 a^2}{H}$, which clarifies in what sense equation (19) obtained in the previous section for an infinite medium corresponds to the limit $H \to \infty$. The global relaxation due to a single plastic event is consequently directly related to the finite size of the system.

An alternative derivation of this finite-size dependence of the force (or average stress) relaxation is proposed in Appendix A, which yields a somewhat complementary physical insight.

4.2.3 Finite H, L periodic: second method

Although we aim to focus in this paper on the qualitative aspects presented previously, we propose here an explicit formula for the stress perturbation induced by a localized plastic event in a finite-size geometry. The actual expression is presented in reciprocal space, which may appear at first somewhat cumbersome, but turns out to be convenient for numerical calculations. For the sake of readability, we provide here only the principles of this two-step derivation and the resulting formulae, the details of the calculations being presented in Appendix B.

First step: we formally extend the actual system ($0 < y < H$, and L periodic along x) by the following two operations: first, an antisymmetric image system is constructed that extends to $-H < y < 0$, then the $2H$ thick resulting system is repeated periodically in the y-direction. If the plastic strain in the original system is $\epsilon^{\mathrm{pl}}(x, y)$, our construction yields a system without walls that is $2H$ periodic in the y-direction, with, in the upper half period $H > y > 0$, a plastic strain $\epsilon^{\mathrm{pl}*}(x, y) = \epsilon^{\mathrm{pl}}(x, y)$, and in the lower half $-H < y < 0$ an *antisymmetric image* plastic strain $\epsilon^{\mathrm{pl}*}(x, y) = -\epsilon^{\mathrm{pl}}(x, -y)$. This construction ensures that the overall displacement generated by these strains has an x-component that is symmetric by reflection by the planes $y = 0$ and $y = H$, and an y-component that is antisymmetric in the same operations. We have therefore generated a solution that satisfies the condition of a zero y-component of the displacement on the $y = 0$ and $y = H$ planes (loci of the walls in the original system).

Second step: we now want to cancel the remaining displacements along x without modifying the above result, and without adding sources in the system $0 < y < H$. This can be achieved by adding on the planes $y = 0$ and $y = H$ appropriate force fields f_x directed along x (again asymmetric and $2H$ periodic along y). Given the symmetry and periodicity of the system, it is clear that the y displacement on the walls is not modified by this addition.

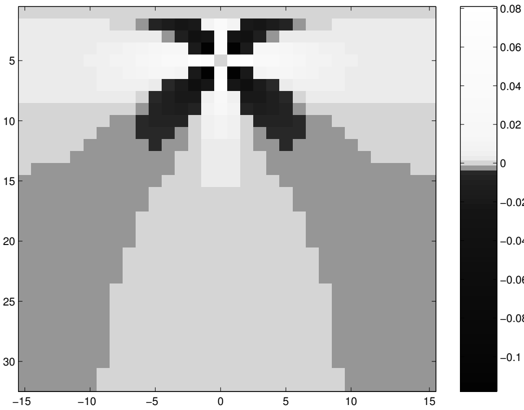

Fig. 10. Perturbation of the shear stress field for a plastic event (plastic deformation of amplitude $\epsilon_0 a^2 = 1$) occurring in the vicinity the wall. The discretization corresponds to $H/a = L/a = 32$.

When this is achieved, we have in the upper half $0 < y < H$, a solution to (6) that satisfies the no-displacement boundary condition on the walls (21). The corresponding stress field can be expressed in Fourier series:

$$\sigma^1(x, y) = \sum_{m,n\in\mathcal{Z}} e^{ip_m x} e^{iq_n y} \hat{\sigma}^*(m, n)$$

with $p_m = \frac{2\pi m}{L}$ and $q_n = \frac{2\pi n}{2H}$. It is the sum of the term directly generated by the plastic strains $\hat{\epsilon}^*$ and that due to the added force fields:

$$\begin{aligned}\hat{\sigma}^*(m, n) &= 2\mu\{\hat{G}^\infty(m, n)\hat{\epsilon}^{\mathrm{pl}*}(m, n) \\ &\quad + \frac{1}{2}(ip_m \hat{O}_{xy}(m, n) + iq_n \hat{O}_{xx}(m, n))\hat{f}_x(m, n)\}, \\ \hat{\sigma}^*(0, 0) &= 0\ ,\end{aligned} \tag{29}$$

The propagator G^∞ and the Oseen tensor $\mathbf{O}$ are the direct counterparts for periodic systems of those defined in Section 3. Of course, the force field f_x in the above expression is itself proportional to the plastic strain; the corresponding formulae are given in Appendix B.

Inverting back to real space allows to compute numerically the response to a localized event. We have checked that for a large system the stress created by a plastic event far from the walls according to this formula is close to that obtained analytically in Section 3 for an infinite medium. Similarly, for an event directly neighbour to the wall, we recover the results of Section 4.1 for a semi-infinite medium. Figure 10 represents an intermediate situation: the event occurs in the vicinity of the wall.

5 Conclusions and perspectives

Starting from a general elasto-plastic model, we have computed in different 2D geometries the modification of the shear stress resulting from a localized plastic event with a symmetry of simple shear. We have first calculated the corresponding perturbation in an infinite system forced with a symmetry of shear. The stress field has the $\cos(4\theta)$ angular dependence displayed in Figure 3 and decays algebraically, $\sim \frac{1}{r^2}$ in two dimensions. Then, we have shown that the stress field perturbation due to a plastic event occurring close to a wall has a modified near-field structure but decays far away with the same law and pattern, although with an amplitude twice as large. Eventually, we have proposed two ways of calculating the perturbation of the stress field due to a plastic event occurring in a finite medium. The first one allowed us to demonstrate in a simple way that a plastic event of a given strain amplitude relaxes the average stress by an amount which is independent of its position (*i.e.* distance to the walls) and inversely proportional to the size of the system. The second one allowed us to derive explicit expressions that permit calculation of the whole stress field in a finite-size geometry.

The extension of our results to a three-dimensional situation is rather straightforward, and the qualitative statements are obviously similar. Similarly, we have focused on a shear plastic strain and on the shear component of the induced stress, but the same steps can be taken if one seeks to describe the whole tensorial stress field generated by plastic events of arbitrary symmetry.

Eventually, we have focused, in Section 4, on a situation where the walls were kept fixed during the plastic event imposing a zero-displacement field on the plates. The extension of our results to situations where the total force on the plates is kept fixed and the plates can globally translate (imposing a constant displacement on their walls) is immediate: one simply needs to add to our solution a simple shear displacement (corresponding to a stress $-\delta F/L$ with the notations of Sect. 4).

This study is meant to be a first step towards the modeling of the flow of an elasto-plastic material. The next one consists in plugging in a plastic law that describes the onset and evolution of the localized plastic events. Coupling such a local plastic behaviour to the long-range elasticity described here should yield interesting collective behaviours and hopefully insights in the flow mechanisms. For such an endeavour, our quantification of finite-size effects is important for the steady-state average balance between the stress released by the plastic events and that imposed by the elastic loading. Also the geometry and decay law of the elastic perturbation is to be considered when addressing the emergence of a collective/cooperative organization at low shear rates. Important questions regarding possible spatial and temporal heterogeneities in such flows (see Refs. [2–5,16]) emerge in light of the present results:

– We have shown that a localized plastic event relaxes the average stress but also modifies the stress pattern in *all* the system, decreasing the stress of some elements and increasing that of others. Does this allow shear banding with a limited zone flowing in coexistence with a non-flowing region?

– The average stress released during an event of given strain amplitude is independent of the position of the event in the medium, and is the same on all the lines. Yet the geometry of the propagator is clearly modified by the proximity of a wall. Do these elements favor a localization of the flow at the wall?

We intend to address these issues in future reports.

Appendix A. Finite-size effects

We propose an alternative simple approach which gives a different physical insight into the effects of finite size evidenced in Section 4.2. The following argument holds equally for an infinite $L \to \infty$ geometry or for an L periodic medium, and is here presented for the latter. We consider a plastic event occurring at (x', y') described by $\epsilon^{\mathrm{pl}}_{xy}(x, y) = \epsilon_0 a^2 \delta(x - x', y - y')$ (and its repeated images along x due to the L periodicity). The stress perturbation is

$$\sigma^1_{xy}(x, y) = \int_{-L/2}^{L/2} \mathrm{d}x_1 \int_{-H/2}^{H/2} \mathrm{d}y_1\, G^{HL}(x - x_1, y, y_1) \epsilon^{\mathrm{pl}}_{xy}(x_1, y_1), \tag{A.1}$$

which actually defines the propagator G^{HL} for the present periodic medium. The corresponding force release on the bottom of a layer at height y is

$$\delta F = \int_{-L/2}^{L/2} \mathrm{d}x\ \sigma^1_{xy}(x, y) = a^2 \epsilon_0 \int_{-L/2}^{L/2} \mathrm{d}x\ G^{HL}(x - x', y, y'). \tag{A.2}$$

We remark that this can be rewritten:

$$\delta F = \int_{-L/2}^{L/2} \mathrm{d}x_1 \int_{-H/2}^{H/2} \mathrm{d}y_1\, G^{HL}(x - x_1, y, y_1)(\epsilon_0 a^2 \delta(y_1 - y')), \tag{A.3}$$

which means (see (A.1)) that the variation of the integrated stress over a layer at height y (δF) induced by an event occurring at position x', y', is equal to the stress variation at a site (x', y) induced by a continuous sum of events on a layer y'.

Representing these events by pairs of dipoles, the sum reduces to that of the x components as the y components cancel out upon summation on the line, so that we are looking for the stress generated at (x', y) by the two lines of forces at $y' - a$ and $y' + a$ depicted in Figure 11. The displacement fields resulting from each of these sums separately are easily calculated and depicted on the right of Figure 11. For example, the second one is the solution of the following problem: on each side of the plane $y = y' - a$, the medium is purely elastic, with no displacement at the walls, and at $y = y' - a$ we require a continuous displacement and a jump in the stress of amplitude $\mu\epsilon_0 a$.

The total effect of the continuous line of events is the displacement field presented on the left of Figure 12. The slopes of the profile for $y > y' + a$, and $y < y' - a$ are the same, and correspond to $\delta F = -2\frac{\mu\epsilon_0 a^2}{H}$, as derived differently in the main text, see equation (28). Remarkably, this

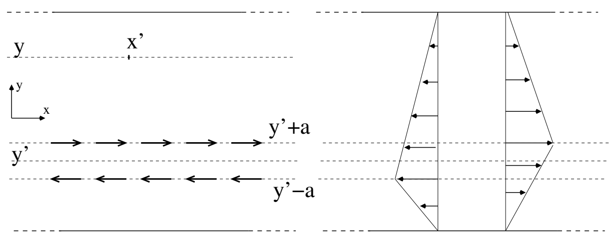

Fig. 11. Left: a continuous line of events at height y' is equivalent, using the representation of Figure 2 for a single event, to two lines of horizontal forces at $y'+a$ and $y'-a$. Right: displacement profiles resulting from the sum of negative forces at $y'-a$ (left), and from the sum of positive forces at height $y'+a$ (right).

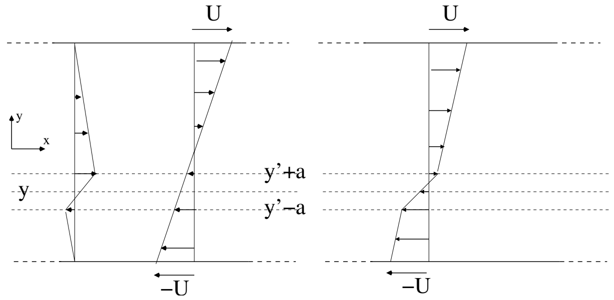

Fig. 12. Left: displacement profile due to the line of events at y' (obtained by summing the two profiles of Fig. 11 right), and schematic description of the elastic loading. Right: total response to the global forcing and a line of plastic event (a homogeneous fracture), obtained by summing the two profiles on the left.

value, which corresponds to the force release on the top wall, is independent of the position of the plastic event.

We remark that the deformation profile obtained here is very similar to the one obtained by Kabla and Debrégeas in their explicit simulation of a shear foam [16] (which in their case was the line-averaged consequence of a single local event, equivalent to the present local consequence of a line of events according to the argument below (A.3)).

As a follow-up along the same picture, the cumulative displacement after loading plus a continuous line of plastic events is represented on the right of Figure 12: the total shear in the system is then the homogeneous shear due to the forcing minus the shear release due to the plastic events.

Appendix B. Stress field induced by a plastic deformation in a finite medium

In this appendix we provide the explicit derivation of the stress field change due to a plastic activity in a finite system, follow the strategy outlined in Section 4.2.3. The aim is to obtain the response of the system depicted in Figure 9 to a strain solicitation $\epsilon^{\rm pl}(x,y)$ with "stick" boundary conditions on the wall similar to equation (21). For this calculation, we choose to take the origin of the y-

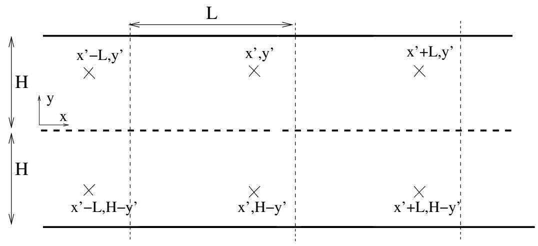

Fig. 13. Auxiliary system corresponding to the real system of Figure 9: it is periodic in both directions and symmetric in the y-direction. The driving plastic strain field $\epsilon^{\rm pl*}$ is antisymmetric.

axis on the bottom wall so that the walls (and thus the boundary conditions) are at $y=0$ and $y=H$.

a. Geometry of the auxiliary system

To solve this problem, an auxiliary system with no walls is constructed. Its geometry is that of the original system, symmetrized with respect to $y=0$, and periodized along y. The new system has thus a period L in the x-direction, and a period $2H$ in the y-direction, see Figure 13. Fields in this new system are described by stars. The symmetry with respect to the plane $y=0$ is made in such a way that the plastic strain is antisymmetric $\epsilon^{\rm pl*}(x,-y)=-\epsilon^{\rm pl*}(x,y)$, which is equivalent, in the force dipole representations, to having the force fields in the lower half to be the symmetric of that in the upper part $\boldsymbol{F}^*(x,y)=\boldsymbol{F}^*(x,-y)$. The symmetry and periodicity of the problem ensure that the displacement field $\mathbf{u}^*$ generated by this plastic strain satisfies

$$u_y^*(x,0)=0; u_y^*(x,H)=0\ . \tag{B.1}$$

b. Response to a plastic event

This new elastic system is submitted to a plastic strain $\epsilon^{\rm pl*}(x,y)=\epsilon^{\rm pl}(x,y)$ if $y>0$, and $\epsilon^{\rm pl*}(x,y)=-\epsilon^{\rm pl}(x,-y)$ if $y<0$. For reasons explained in the text (Sect. 4.2.3), a force field f_x^* is added at the location of the walls in the real system, to cancel the displacements $u_x(x,0)$ and $u_x(x,H)$ created by the plastic strains,

$$f_x^*(x,y)=f_0^*(x)\delta(y)+f_H^*(x)\delta(y-H)\ . \tag{B.2}$$

This "no displacement on planes $y=0$ and $y=H$" condition is implemented in reciprocal space. We introduce notations for the Fourier series:

$$\begin{aligned}\boldsymbol{u}^*(x,y)&=\sum_{m,n\in\mathcal{Z}}e^{i\frac{2\pi mx}{L}}e^{i\frac{2\pi ny}{2H}}\hat{\boldsymbol{u}}^*(m,n)\ ,\\ \hat{\boldsymbol{u}}^*(m,n)&=\frac{1}{2HL}\int{\rm d}x{\rm d}y\,\boldsymbol{u}^*(x,y)\,e^{-i\frac{2\pi mx}{L}}e^{-i\frac{2\pi ny}{2H}}\ ,\\ p_m&=\frac{2\pi m}{L}\ ,\quad q_n=\frac{2\pi n}{2H}\ ,\quad q^2=p_m^2+q_n^2\ .\end{aligned} \tag{B.3}$$

From our calculations for an infinite medium in Section 3, it is logical to write the total displacement using propagators for plastic events and the Oseen propagator for simple forces:

$$\hat{\boldsymbol{u}}^*(m,n) = \hat{\boldsymbol{P}}^\infty(m,n)\hat{\epsilon}^{\rm pl*}(m,n) + \hat{\boldsymbol{O}}(m,n)\hat{f}_x(m,n) \ . \qquad (B.4)$$

The equations for $\boldsymbol{P}^\infty$ and the Oseen tensor $\boldsymbol{O}$ in the present representation (Fourier series) are formally exactly similar to those obtained in (8) and (15) in terms of Fourier transforms (with $q_x \to p_m$ and $q_y \to q_n$).

The condition $u^*(x,0) = 0; u^*(x,H) = 0$ allows to compute $\hat{f}_x^*(m,n)$:

$$\hat{f}_x^*(m,n) = \frac{\sum_1 \frac{2i}{q^4} p'_n (q_m^2 - p'^2_n)\hat{\epsilon}^{\rm pl*}(m,n')}{\sum_1 \frac{p'^2_n}{\mu q^4}},$$

for odd n;

$$\hat{f}_x^*(m,n) = \frac{\sum_2 \frac{2i}{q^4} p'_n (q_m^2 - p'^2_n)\hat{\epsilon}^{\rm pl*}(m,n')}{\sum_2 \frac{p'^2_n}{\mu q^4}},$$

for even n and $m \neq 0$;

$$\hat{f}_x^*(0,n) = 0 \quad \text{for even } n.$$

$\sum_1$ describes the sum over odd values of n', and $\sum_2$ the sum over even values of n'.

The shear stress in this geometry can then be derived from the displacement field:

$$\hat{\sigma}^*(m,n) = 2\mu\left(\frac{1}{2}[ip_m \hat{u}_y^*(m,n) + iq_n \hat{u}_x^*(m,n)] - \hat{\epsilon}^{\rm pl*}(m,n)\right). \qquad (B.5)$$

For symmetry reasons, $\hat{\sigma}^*(0,0) = 0$.

c. Back to the real system

The displacement field $\mathbf{u}^*(x,y)$ that we have constructed and computed satisfies the elasticity equations for $H > y > 0$ with the plastic strain $\epsilon^{\rm pl}(x,y)$ as only source in that area, and verifies the boundary conditions $u^*(x,0) = 0$ and $u^*(x,H) = 0$. It is therefore the solution to our initial problem for the real system: for $H > y > 0$, $\mathbf{u}^1(x,y) = \mathbf{u}^*(x,y)$, and $\sigma^1(x,y) = \sigma^*(x,y)$. Hence, the expression of the stress given in equation (29).

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