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[10]The parameter k has no relevance when velocities are measured, but it can be useful in other determinations. If in Eq. (7) the extrema $z=\pm a/2$ are used, the values of ϵ at these points are given by

$$\epsilon\left(\pm\frac{a}{2}\right)=\mp\frac{24}{125}\sqrt{5}\mu_0\frac{m}{a^2}v. \tag{19}$$

These values are proportional to the velocity v, and the magnetic dipolar moment $\mathbf{m}$. Once the velocity is known by means of the geometrical properties of the coil, or by analyzing the induced signal, this equation can be used to determine the value of $\mathbf{m}$, which is not very easy to obtain using other methods. The same is valid if the value of k given by least-squares analysis in Eq. (14) is used.

High friction limit of the Kramers equation: The multiple time-scale approach

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The purpose of the paper is to give a pedagogical introduction to the multiple time-scale technique, on the example of the high-friction limit of the Kramers equation. We begin with a discussion of the standard perturbation technique as presented in van Kampen's reference book, which will be shown to fail in the long-time limit. Application of the multiple time-scale technique avoids these difficulties and leads to a uniform expansion in powers of the inverse of the friction. Analogy with the Chapman–Enskog expansion is discussed.

I. INTRODUCTION

From a pedagogical point of view, the theory of Brownian motion is of fundamental interest in a graduate course on statistical mechanics. Starting with quite simple assumptions, a microscopic stochastic model—namely the Langevin model—can be introduced to describe the thermal motion of the suspended Brownian particles. It is noteworthy that this simple treatment already contains all the basic features of nonequilibrium thermodynamics, such as diffusive motion, the fluctuation–dissipation theorem, the Einstein relation between mobility and diffusion, and the Green–Kubo relation for the diffusion coefficient.[1] A complete description, however, requires a more precise characterization of the stochastic processes involved in Brownian motion, culminating in the Fokker–Planck evolution equation for the probability distribution of the Brownian particle.[1,2] Two levels of description can be used. First, if both the velocity and position of the particle are considered, the probability distribution $P(\mathbf{r},\mathbf{v},t)$ evolves according to the Kramers equation

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{r}+\frac{\mathscr{F}(\mathbf{r})}{M}\cdot\nabla_{\mathbf{v}}\right]P(\mathbf{r},\mathbf{v},t)$$

$$=\xi\nabla_{\mathbf{v}}\cdot\left\{\left(\mathbf{v}+\frac{k_BT}{M}\nabla_{\mathbf{v}}\right)P(\mathbf{r},\mathbf{v},t)\right\}, \tag{1}$$

where M denotes the mass of the Brownian particle, ξ the friction, k_BT the bath temperature, and $\mathscr{F}(\mathbf{r})$ an external force field. This description was first given by Klein[3] and Kramers.[4] On the other hand, if only the spatial evolution of the particle is considered, one may rather use the Smoluchowski equation, governing the time evolution of the probability density $\rho(\mathbf{r},t)$ in configuration space:

$$\frac{\partial}{\partial t}\rho(\mathbf{r},t)=\frac{1}{M\xi}\nabla_{\mathbf{r}}\cdot\{k_BT\nabla_{\mathbf{r}}-\mathscr{F}(\mathbf{r})\}\rho(\mathbf{r},t). \tag{2}$$

This description, first put forward by Smoluchowski,[5] predates that of Kramers and Klein.

However, it took a long time to clarify the link between both descriptions. Kramers[4] already pointed out that the Smoluchowski description should be obtained from the Kramers equation in the high-friction limit. This can indeed be intuitively understood: If the friction is high, the velocity relaxes after a short time, of order $1/\xi$, and the evolution on

larger time scales is only determined by the spatial distribution. However, a rigorous perturbative derivation of the Smoluchowski equation from the Kramers one was given by Wilemski only in 1976.[6] Another systematic expansion procedure was later suggested by Titulaer.[7] The latter is closely related to the Chapman–Enskog perturbation scheme to solve the Boltzmann equation.[8] This method is indeed natural in the present case, in view of the formal analogy between the collisional terms occurring in the Boltzmann and in the Kramers equations. In both equations, the right-hand side characterizes the effect of collisions on the time evolution of the distribution function. The singular nature of the expansion is due to the presence of the small parameter $1/\xi$ (replaced by the Knudsen number in the case of the Boltzmann equation) multiplying the time derivative in the Kramers equation. This makes a straightforward application of the perturbation theory impossible. In his reference textbook[1] (pp. 234–236), van Kampen proposes an alternative, pedagogical method, which is closer in spirit to the Hilbert perturbation scheme.[8] This procedure leads indeed to the Smoluchowski equation. The purpose of this paper is to show that this pedagogical approach is however incomplete and leads to inconsistencies in the expansion procedure. In Sec. II, we will show that hidden secular terms are present in this description, which diverge as time goes to infinity. These terms make the asymptotic expansion nonuniform in the small parameter $1/\xi$. In Sec. III, we propose an alternative approach, based on the multiple-time scale method. This procedure eliminates the secular divergences and construct a uniform expansion in the small parameter (the inverse friction in our case).[9] The multiple time-scale method allows one to recover quite easily all the features of the Chapman–Enskog procedure of Titulaer.[7] Moreover, it gives a clear interpretation of the contribution of all orders in $1/\xi$ in the time derivative, as it is formally introduced in the Chapman–Enskog expansion.[8]

II. THE HILBERT EXPANSION

In this section, we first reproduce and then discuss the expansion procedure as given in van Kampen's reference book, in order to point out the limitation of this "Hilbert" perturbation scheme. We will restrict ourselves to the one-dimensional case to simplify the analysis.

First, we introduce new dimensionless variables

$$\tau = t\frac{v_T}{l}; \quad V = \frac{v}{v_T}; \quad X = \frac{x}{l}; \quad F = \mathscr{F}\frac{l}{Mv_T^2} \tag{3}$$

and a dimensionless friction

$$\xi_d = \xi\frac{l}{v_T}, \tag{4}$$

where $v_T = \sqrt{k_B T/M}$ is the thermal velocity and l is a characteristic length scale of the system (such as the Brownian particle diameter). The Kramers equation then takes the following form:

$$\frac{\partial}{\partial V}\left(V + \frac{\partial}{\partial V}\right)P(X,V;\tau) = \frac{1}{\xi_d}\left[\frac{\partial}{\partial \tau} + V\frac{\partial}{\partial X} + F(X)\frac{\partial}{\partial V}\right]P(X,V;\tau). \tag{5}$$

Let us insert into (5) the "naive" expansion

$$P = P^{(0)} + \xi_d^{-1}P^{(1)} + \xi_d^{-2}P^{(2)} + \cdots. \tag{6}$$

By identifying terms of the same order, we find the following equations:

$$\begin{aligned}
&\mathscr{L}_{\mathrm{FP}}P^{(0)} = 0,\\
&\mathscr{L}_{\mathrm{FP}}P^{(1)} = \left[\frac{\partial}{\partial \tau} + V\frac{\partial}{\partial X} + F(X)\frac{\partial}{\partial V}\right]P^{(0)},\\
&\mathscr{L}_{\mathrm{FP}}P^{(2)} = \left[\frac{\partial}{\partial t} + V\frac{\partial}{\partial X} + F(X)\frac{\partial}{\partial V}\right]P^{(1)},
\end{aligned} \tag{7}$$

where we introduced a "Fokker–Planck" operator $\mathscr{L}_{\mathrm{FP}}$ defined as

$$\mathscr{L}_{\mathrm{FP}} = \frac{\partial}{\partial V}\left(V + \frac{\partial}{\partial V}\right). \tag{8}$$

The zeroth order equation imposes a Maxwellian velocity distribution

$$P^{(0)}(X,V;\tau) = \Phi(X;\tau)e^{-1/2V^2} \tag{9}$$

and the function Φ has to be determined.

The first-order term then gives

$$\mathscr{L}_{\mathrm{FP}}P^{(1)} = \frac{\partial \Phi}{\partial \tau}e^{-1/2V^2} + V\left\{\frac{\partial \Phi}{\partial X} - F\Phi\right\}e^{-1/2V^2}. \tag{10}$$

By integrating both sides over V, one obtains a "solubility condition"

$$\frac{\partial \Phi}{\partial \tau} = 0. \tag{11}$$

This can be understood in terms of linear algebra. The eigenfunctions of $\mathscr{L}_{\mathrm{FP}}$ are the functions $H_n(V/\sqrt{2})\exp(-V^2/2)$, where H_n is the nth Hermite polynomial. Since the Maxwellian is associated with a null eigenvalue, the orthogonality between the eigenfunctions imposes that all terms multiplying the Maxwellian on the right-hand side (rhs) of equations like (10) must vanish. This is precisely the "solubility condition" (11).

Equation (10) can now be solved to yield

$$P^{(1)}(X,V;\tau) = -V\left\{\frac{\partial \Phi}{\partial X} - F\Phi\right\}e^{-1/2V^2} + \Psi(X;\tau)e^{-1/2V^2}, \tag{12}$$

where Ψ has to be determined.

By inserting this solution into (7), one obtains

$$\begin{aligned}
\mathscr{L}_{\mathrm{FP}}P^{(2)} = &\left\{\frac{\partial \Psi}{\partial \tau} - \frac{\partial}{\partial X}\left(\frac{\partial \Phi}{\partial X} - F\Phi\right)\right\}e^{-1/2V^2} + \left\{\frac{\partial \Psi}{\partial X} - F\Psi\right\}\\
&\times Ve^{-1/2V^2} - \left\{F\left(\frac{\partial \Phi}{\partial X} - F\Phi\right) - \frac{\partial}{\partial X}\right.\\
&\left.\times\left(\frac{\partial \Phi}{\partial X} - F\Phi\right)\right\}(1-V^2)e^{-1/2V^2},
\end{aligned} \tag{13}$$

where the terms have been grouped into Hermite polynomials $H_n(V/\sqrt{2})$. According to the solubility condition, the first term in the rhs of (13) must be set to zero, so that

$$\frac{\partial \Psi}{\partial \tau} - \frac{\partial}{\partial X}\left(\frac{\partial \Phi}{\partial X} - F\Phi\right) = 0. \tag{14}$$

Following van Kampen, one can then collect the results to obtain the distribution function to order $1/\xi_d^2$:

$$P(X,V;\tau)=e^{-1/2V^2}\left[\Phi(X)-\xi_d^{-1}V\left(\frac{\partial\Phi}{\partial X}-F\Phi\right)\right.$$
$$\left.+\xi_d^{-1}\Psi(X;\tau)+\mathcal{O}(\xi_d^{-2})\right]. \tag{15}$$

After integrating over the velocity, one obtains the spatial probability density $\rho(X;\tau)$ as

$$\rho(X;\tau)=\sqrt{2\pi}[\Phi(X)+\xi_d^{-1}\Psi(X;\tau)+\mathcal{O}(\xi_d^{-2})] \tag{16}$$

and Eq. (14) then yields the (dimensionless) Smoluchowski equation

$$\frac{\partial\rho(X,\tau)}{\partial\tau}=\frac{1}{\xi_d}\frac{\partial}{\partial X}\left(\frac{\partial\rho(X,\tau)}{\partial X}-F(X)\rho(X,\tau)\right). \tag{17}$$

However, a careful inspection of Eq. (14) shows that Ψ diverges as time goes to infinity. Indeed, Eq. (11) shows that Φ is independent of time. Then, the first term in (14) does not depend on time either and the function Ψ is proportional to time τ

$$\Psi(X;\tau)\sim\tau. \tag{18}$$

The standard expansion (6) of the solution thus leads to secular divergences and cannot be correct in the long-time limit. In other words, a naive expansion is not uniformly convergent for small $1/\xi_d$. The limit of time going to infinity cannot be inverted with the limit of friction going to infinity, so that taking small corrections in $1/\xi_d$ into account leads to an upper bound for the time τ. In fact, this procedure implicitly expands time dependent terms like τ/ξ_d as ξ_d^{-1} terms, and secular divergences appear in the long-time limit.

III. THE MULTIPLE TIME-SCALE APPROACH

In the limit $\xi^{-1}\to 0$, the Kramers argument presented above indicates a time-scale separation. There is first a very short period ($t\sim\xi^{-1}$) during which the velocity of the Brownian particle thermalizes. Then the dynamical evolution is controlled by the time dependence of the spatial distribution on longer time scales. This separation of time scales in the limit $\xi^{-1}\to 0$ suggests the application of the multiple time-scale analysis. This method replaces the physical distribution function $P(X,V;\tau)$ by an auxiliary function, $P(X,V;\tau_0,\tau_1,\tau_2,\ldots)$, depending on several time scales. Accordingly, the time derivative in the physical evolution equation (5) is replaced by the sum of time derivatives on each time scale,

$$\frac{\partial}{\partial\tau}\to\frac{\partial}{\partial\tau_0}+\xi_d^{-1}\frac{\partial}{\partial\tau_1}+\xi_d^{-2}\frac{\partial}{\partial\tau_2}+\cdots. \tag{19}$$

The auxiliary distribution function $P(X,V;\tau_0,\tau_1,\tau_2,\ldots)$ is then expanded in powers of the small parameter ξ_d^{-1} and inserted into the evolution equation, where terms of the same order are identified. The physical solution of the system is eventually obtained by restricting the different time variables to the so-called "physical line"

$$\tau_0=\tau;\quad \tau_1=\xi_d^{-1}\tau;\quad \tau_2=\xi_d^{-2}\tau;\ldots \tag{20}$$

so that

$$P(X,V;\tau)=P^{(0)}(X,V;\tau,\xi_d^{-1}\tau,\xi_d^{-2}\tau,\ldots)+\xi_d^{-1}$$
$$\times P^{(1)}(X,V;\tau,\xi_d^{-1}\tau,\xi_d^{-2}\tau,\ldots)+\cdots \tag{21}$$

will be the solution of the Kramers equation (5). Equation (20) indicates that the dependence of the distribution function on τ_n characterizes the evolution on the time scale $\tau\sim\xi_d^n$ $(n=0,1,2,\ldots)$.

The crucial difference with the standard perturbation method is that outside the physical line (20), the auxiliary distribution function has no physical meaning. Then appropriate boundary conditions can be imposed to require the expansion to be uniform in the small parameter ξ_d^{-1}. This freedom will be used to eliminate secular divergences.

The identification of different powers of ξ_d^{-1} in the Kramers equation gives the following relations:

$$\mathcal{L}_{\rm FP}P^{(0)}=0,$$
$$\mathcal{L}_{\rm FP}P^{(1)}=\left[\frac{\partial}{\partial\tau_0}+V\frac{\partial}{\partial X}+F(X)\frac{\partial}{\partial V}\right]P^{(0)}, \tag{22}$$
$$\mathcal{L}_{\rm FP}P^{(2)}=\left[\frac{\partial}{\partial\tau_0}+V\frac{\partial}{\partial X}+F(X)\frac{\partial}{\partial V}\right]P^{(1)}+\frac{\partial}{\partial\tau_1}P^{(0)}.$$

The zeroth-order equation still imposes a Maxwellian velocity distribution

$$P^{(0)}(X,V;\tau_0,\tau_1,\ldots)=\Phi(X;\tau_0,\tau_1,\ldots)e^{-1/2V^2}, \tag{23}$$

where the function Φ now depends on all time variables.

The first-order term thus obeys to the following equation:

$$\mathcal{L}_{\rm FP}P^{(1)}=\frac{\partial\Phi}{\partial\tau_0}e^{-1/2V^2}+V\left\{\frac{\partial\Phi}{\partial X}-F\Phi\right\}e^{-1/2V^2}. \tag{24}$$

The solubility condition then requires the function Φ to be independent of the τ_0 time scale

$$\frac{\partial\Phi}{\partial\tau_0}=0, \tag{25}$$

and the first correction for the distribution function is now given by

$$P^{(1)}(X,V;\tau_0,\tau_1,\ldots)$$
$$=-V\left\{\frac{\partial\Phi(X;\tau_1,\ldots)}{\partial X}-F\Phi(X;\tau_1,\ldots)\right\}e^{-1/2V^2}$$
$$+\Psi(X;\tau_0,\tau_1,\ldots)e^{-1/2V^2}. \tag{26}$$

The equation for $P^{(2)}$ becomes

$$\mathcal{L}_{\rm FP}P^{(2)}=\left\{\frac{\partial\Psi}{\partial\tau_0}+\frac{\partial\Phi}{\partial\tau_1}-\frac{\partial}{\partial X}\left(\frac{\partial\Phi}{\partial X}-F\Phi\right)\right\}e^{-1/2V^2}$$
$$+\left\{\frac{\partial\Psi}{\partial X}-F\Psi\right\}Ve^{-1/2V^2}-\left\{F\left(\frac{\partial\Phi}{\partial X}-F\Phi\right)\right.$$
$$\left.-\frac{\partial}{\partial X}\left(\frac{\partial\Phi}{\partial X}-F\Phi\right)\right\}(1-V^2)e^{-1/2V^2}. \tag{27}$$

The solubility condition then requires

$$\frac{\partial\Psi}{\partial\tau_0}=-\left(\frac{\partial\Phi}{\partial\tau_1}-\frac{\partial}{\partial X}\left(\frac{\partial\Phi}{\partial X}-F\Phi\right)\right). \tag{28}$$

The rhs of this equation does not depend on τ_0 [see Eq. (25)], and one must impose the condition

$$\frac{\partial \Psi}{\partial \tau_0}=0 \tag{29}$$

to eliminate the secular divergence as τ_0 grows to infinity. This leads to a closed equation for Φ:

$$\frac{\partial \Phi}{\partial \tau_1}-\frac{\partial}{\partial X}\left(\frac{\partial \Phi}{\partial X}-F\Phi\right)=0. \tag{30}$$

This result has to be compared with Eq. (14) of the standard Hilbert procedure, where the time dependence of the density was only contained in the first-order correction [through Ψ, see Eq. (16)]. A correct expansion thus modifies the perturbative scheme and leads to different couplings between the successive corrections to the probability distribution.

The physically relevant equation for ρ is then obtained by restricting the different variables τ_i to the physical line (20). The time derivative of ρ can now be written

$$\begin{aligned}\frac{\partial}{\partial \tau}\rho(X,\tau)&=\sqrt{2\pi}\,\frac{\partial}{\partial \tau_0}\left[\Phi(X;\tau_1,\ldots)+\xi_d^{-1}\Psi(X;\tau_1,\ldots)\right.\\&\quad\left.+\mathcal{O}(\xi_d^{-2})\right]_{|\tau_0=\tau,\tau_1=\tau/\xi_d,\ldots}+\frac{1}{\xi_d}\sqrt{2\pi}\,\frac{\partial}{\partial \tau_1}\\&\quad\times\left[\Phi(X;\tau_1,\ldots)+\xi_d^{-1}\Psi(X;\tau_1,\ldots)\right.\\&\quad\left.+\mathcal{O}(\xi_d^{-2})\right]_{|\tau_0=\tau,\tau_1=\tau/\xi_d,\ldots}\\&=\frac{1}{\xi_d}\sqrt{2\pi}\,\frac{\partial}{\partial \tau_1}\Phi(X;\tau_1,\ldots)_{|\tau_0=\tau,\tau_1=\tau/\xi_d,\ldots}\\&\quad+\mathcal{O}(\xi_d^{-2}).\end{aligned} \tag{31}$$

Using (30) and (16), we eventually recover the (dimensionless) Smoluchowski equation

$$\frac{\partial \rho(X,\tau)}{\partial \tau}=\frac{1}{\xi_d}\frac{\partial}{\partial X}\left(\frac{\partial \rho(X,\tau)}{\partial X}-F(X)\rho(X,\tau)\right)+\mathcal{O}(\xi_d^{-2}). \tag{32}$$

IV. DISCUSSION

Let us make a few remarks on the present derivation.

First, as mentioned in the Introduction, the Smoluchowski equation governs the spatial evolution of the Brownian particle. This evolution is then expected to occur on a diffusion time scale $t_D\sim l^2/D$, where $D=k_BT/M\xi$ is the diffusion coefficient, or in dimensionless variables, $\tau_D=(v_T/l)t_D\sim\xi_d$. This feature emerges in a natural way from the multiple time-scale analysis. Indeed, according to Eq. (30), we have shown that the spatial density probability evolves on the τ_1 time scale, which precisely characterizes the evolution of the system on time scale $\tau\sim\xi_d\sim\tau_D$ [see discussion after Eq. (21)]!

As indicated above, the $t\gg\xi^{-1}$ limit of $P^{(0)}$ is just the Maxwellian, which can be interpreted here as the "local equilibrium approximation" of the Fokker–Planck operator, $\mathscr{L}_{\mathrm{FP}}$. As shown in Eq. (25), this solution does not evolve on the time scale τ_0, where this approximated form for $P^{(0)}$ remains valid. This property confirms the relevance and the deep physical meaning of the local equilibrium approximation, as a state of the system which exists over a large time scale.

Finally, let us remark that the formal analogy between the Kramers equation and the Boltzmann equation can serve as an introduction to perturbative expansion methods in kinetic theory. It should be noted that the multiple time-scale method is indeed very close in spirit to the Chapman–Enskog procedure. Indeed, the latter introduces an *ad-hoc* formal expansion of the time derivative in powers of the small parameter ϵ:

$$\frac{\partial f}{\partial t}=\sum_{n=0}^{\infty}\epsilon^n\frac{\partial^{(n)}f}{\partial t} \tag{33}$$

(in kinetic theory, the Knudsen number is the small parameter). This is a subtle point of the procedure, which stems from the expansion in powers of ϵ of the functional dependence of the distribution function on the "hydrodynamic" fields.[7,8] The explicit expressions for the operators $\partial^{(n)}/\partial t$ are computed in the course of the calculation thanks to solubility conditions. Equation (33) is formally equivalent to the multiple time-scale expansion of the time derivative (19). In this sense, the application of the operator $\partial^{(n)}/\partial t$ of the Chapman–Enskog method characterizes the dynamical evolution of the system on the time scale $t\sim\epsilon^{-n}$. For the purpose of a pedagogical approach, the multiple time-scale approach seems, however, more natural to us, since it gives a clear understanding of each term of the expansion. Moreover, it explicitly shows the crucial importance of ϵ-dependent corrections in the time derivative (33) to obtain a consistent expansion. Absence of such terms would inevitably lead to inconsistencies identical to those occurring in the naive expansion of the Kramers equation.

To conclude, the study of the Kramers equation in the high friction limit gives a pedagogical and instructive presentation of the multiple time-scale method—pedagogical, since it can be shown explicitly that the method avoids all the troubles of more naive perturbation expansions; instructive, since it allows a deep insight into the underlying physics. This technique has already been successfully applied to a variety of problems in physics,[9] such as the Lorentz model,[10] adiabatic systems[11] or a microscopic (exact) approach to Brownian motion[12] to cite some recent work. For such complicated situations, it describes *in a systematic way* the dynamical evolution of the system on each successive time scale. This property constitutes the real power of the method.

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A geometric approach to nonlocality in the Bohm model of quantum mechanics

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The nonlocality of Bohm's model of quantum mechanics is demonstrated geometrically. This is done by using the properties that in the Bohm model the particle trajectories in configuration space do not cross, yet the predictions of quantum mechanics must be reproduced. This approach provides a simple demonstration of how a model of quantum mechanics that assumes "elements of reality" for unmeasured observables must be nonlocal. While it is tempting to try to use this type of nonlocality for superluminal signalling, we discuss the fact that any attempt to do this will result in failure. © *1997 American Association of Physics Teachers.*

I. INTRODUCTION

In 1952, a deterministic, realistic model of quantum mechanics (in which particles always have real trajectories) was proposed by Bohm.[1] In the last decade, there has been much interest in this model, partly due to the widespread availability of computers which have enabled the trajectories predicted by this model to be computed and plotted.[2,3] There have, as a result, been numerous publications investigating, developing, and extending the model, which has resulted in two books devoted specifically to it[4,5] as well as an article in a popular science journal.[6] It has also been suggested that it be more widely taught as an antidote to complacency in asking fundamental questions in quantum mechanics, since unlike the standard approach, Bohm's model is deterministic and references to the observer can be eliminated.[7] One of the above books has also demonstrated how the Bohm model can be presented to students as a natural transition from classical mechanics (in the Hamilton–Jacobi formalism) to quantum mechanics.[5]

One interesting characteristic of the Bohm model of quantum mechanics is its explicit nonlocality for entangled two-particle states. This feature of the Bohm model is what inspired Bell to discover a proof that any deterministic model which assumes realism and reproduces quantum mechanics must be explicitly nonlocal.[8] In previous discussions of the Bohm model of quantum mechanics, the nonlocal aspects of the theory have been demonstrated algebraically.[4,5] In this article it is shown from a general principle which is true in the Bohm model, and the fact that the Bohm model reproduces the predictions of quantum mechanics, that the explicit nonlocality in this model can be demonstrated with a simple geometric method.

II. THE BOHM MODEL OF QUANTUM MECHANICS

To derive the Bohm model of quantum mechanics, first, take the Schrödinger equation for a single particle and make the substitution

$$\Psi = Re^{iS/\hbar}, \tag{1}$$

where R and S are real functions of space and time for the particle. Then separate the real and imaginary parts of the equation in order to obtain the following two equations for a single-particle state:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0, \tag{2}$$

$$\frac{\partial R^2}{\partial t} + \nabla_1\left(R^2\frac{\nabla S}{m}\right) = 0. \tag{3}$$

Here, ∇ is the partial derivative with respect to the position space of the particle, and m is the particle's mass.

Bohm noticed the similarity of Eq. (2) to the Hamilton–Jacobi equation for classical mechanics. By analogy with the Hamilton–Jacobi equation, the momentum of the particle is then defined to be

$$\mathbf{p} = \nabla S. \tag{4}$$

Equation (3) then denotes conservation of probability. This is Bohm's model of quantum mechanics for a single particle. It gives the same predictions as quantum mechanics, since it is derived from the Schrödinger equation.

Note that since Ψ is a single-valued function of the position of the particle, therefore by Eq. (1), S is as well. It then follows by Eq. (4) that the momentum $\mathbf{p}$ is also a single-valued function of position. Since the momentum is a single-

