11.7. NORMAL SHOCK WAVES

A body moving in compressible fluid creates disturbances that propagate through the fluid.

When amplitude of these waves infinitesimally small (change of flow properties across the wave infinitesimally small) → weak waves

When amplitude of these waves finite (change of flow properties across the wave finite)→ shock waves

Across a shock wave, the gas is compressed instantaneously → irreversible process; entropy rises.

Shock waves:

1) Oblique shock waves (shock wave inclined with respect to flow direction)

2) Normal shock waves (shock wave normal to flow direction) → will analyze this.

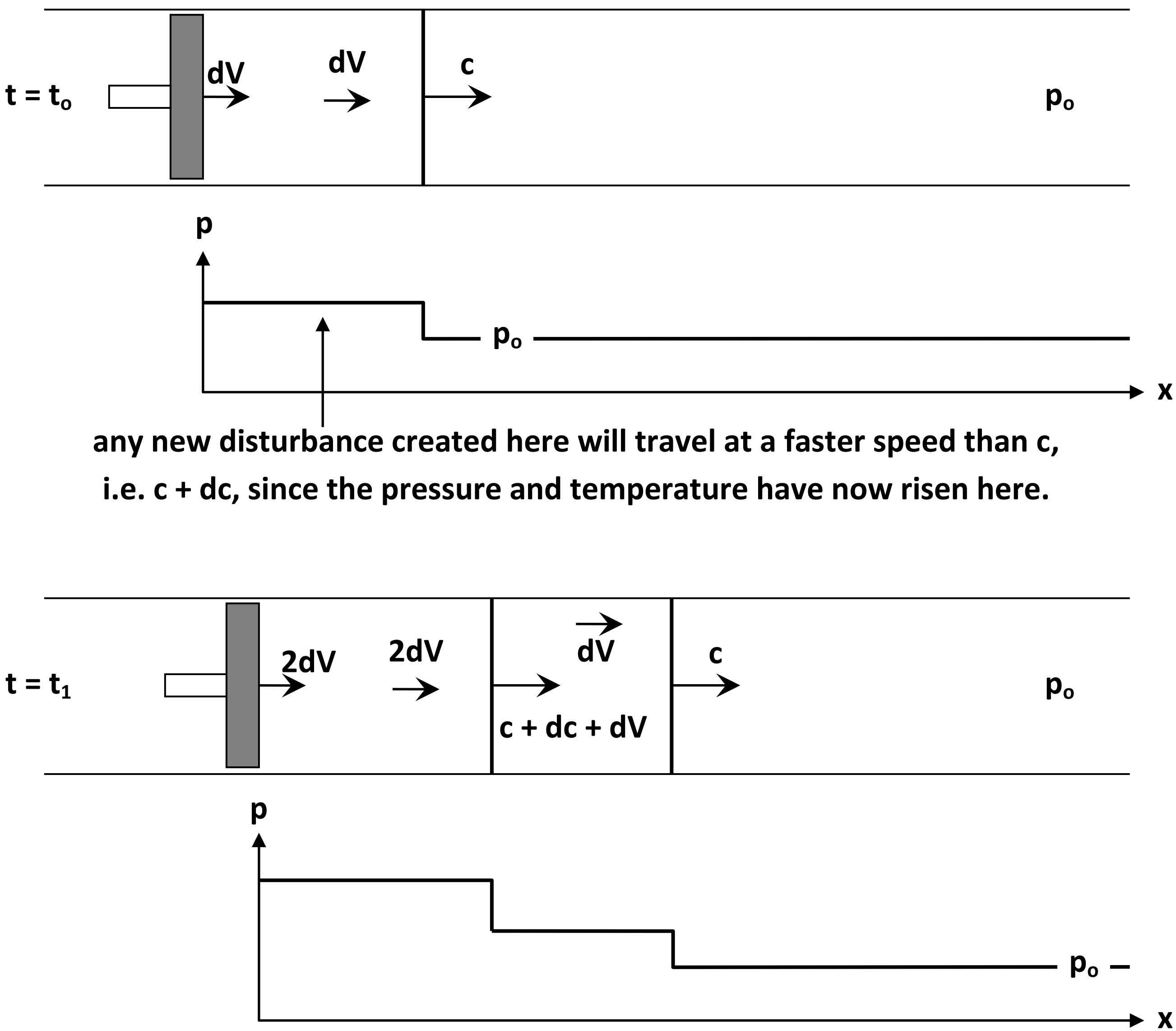
11.7.1. DEVELOPMENT OF WAVES

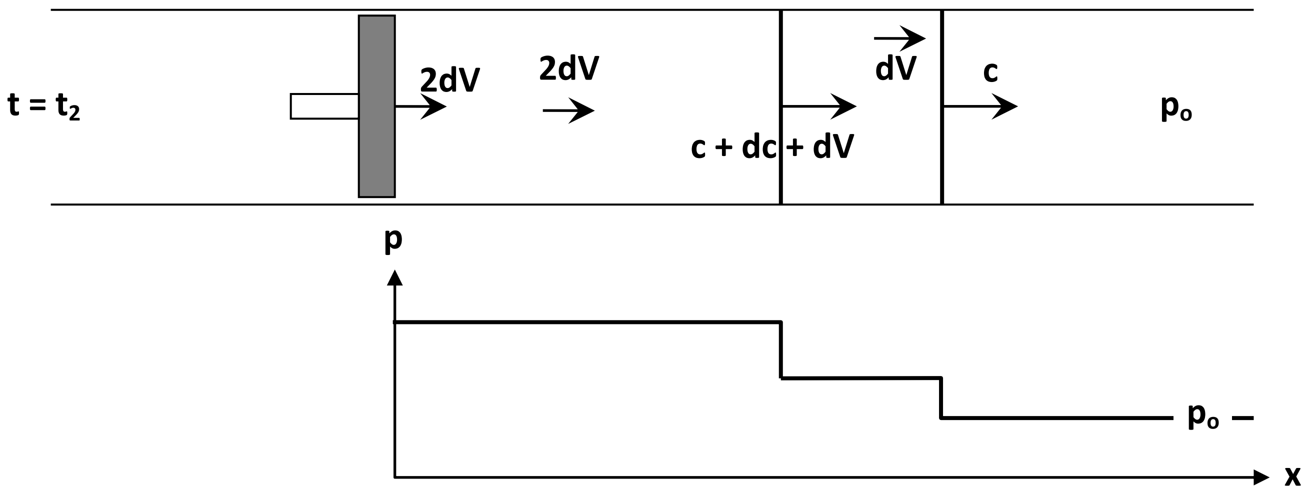
Series of compression and expansion waves propagating in fluid.

11.7.1.1. Development of Compression Waves

Across a compression wave, the flow decelerates and the pressure increases.

Figure 11.22. Development of compression waves

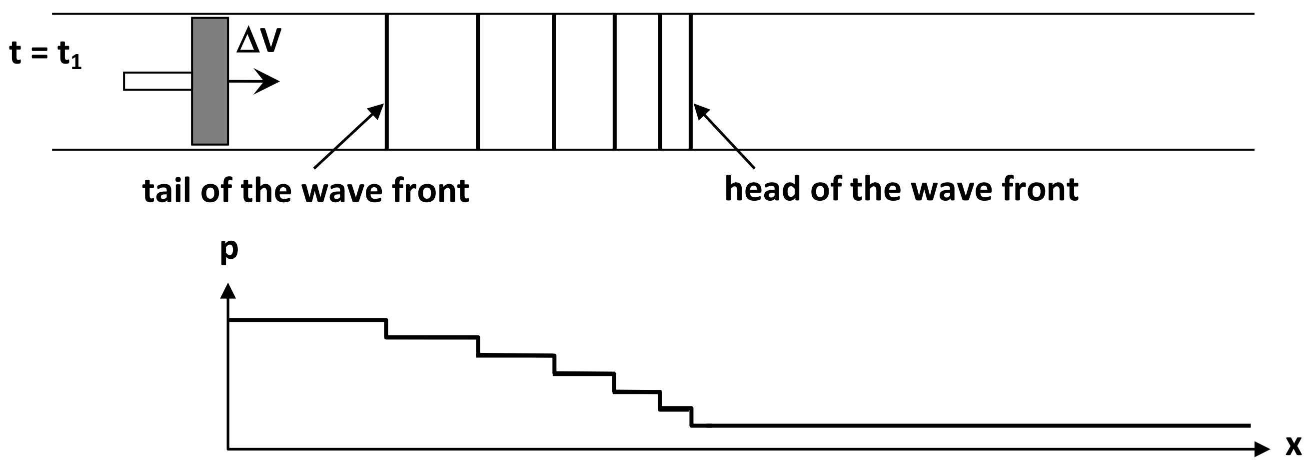


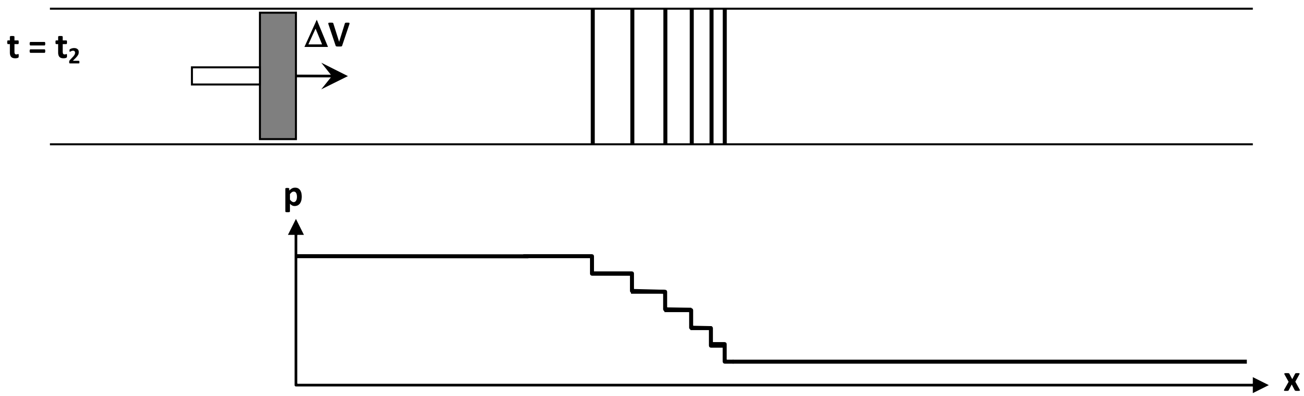


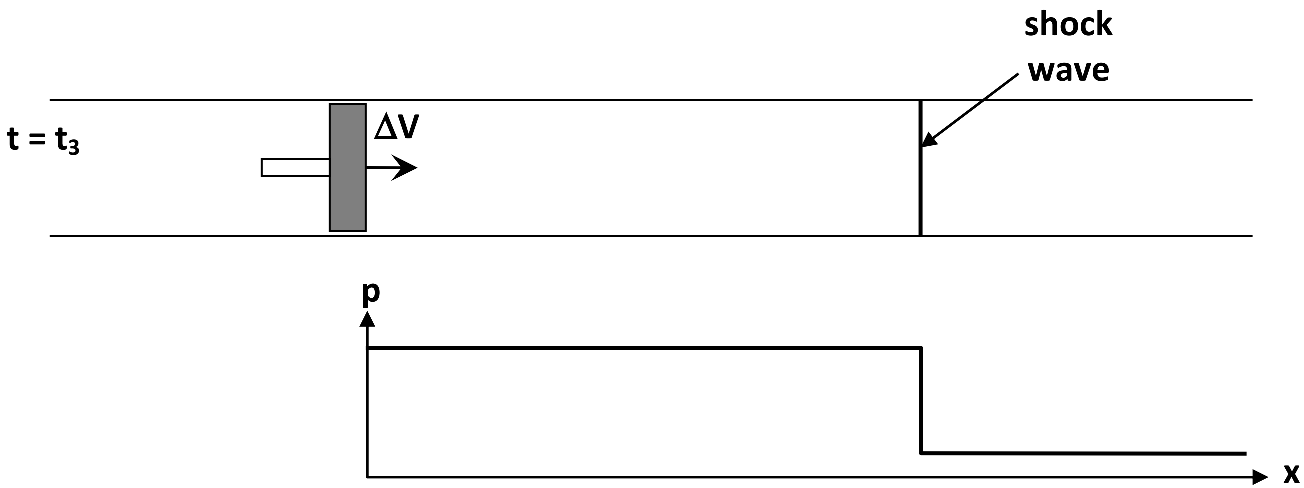
The wave traveling behind has a higher wave speed → compression waves getting closer and closer.

Figure 11.23. Formation of a Shock Wave

Piston accelerated from rest to finite velocity, ΔV. Series of waves created (wave front).





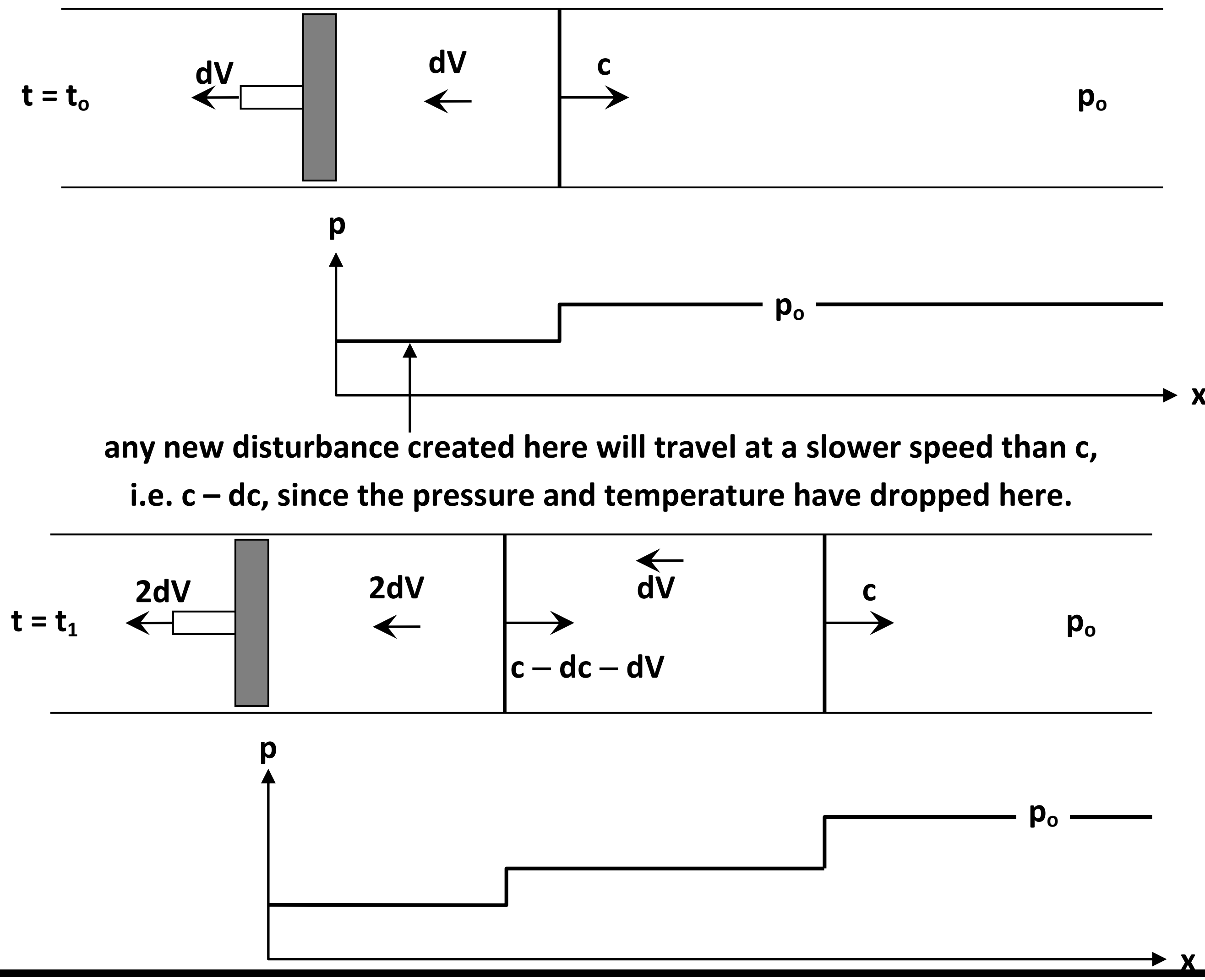


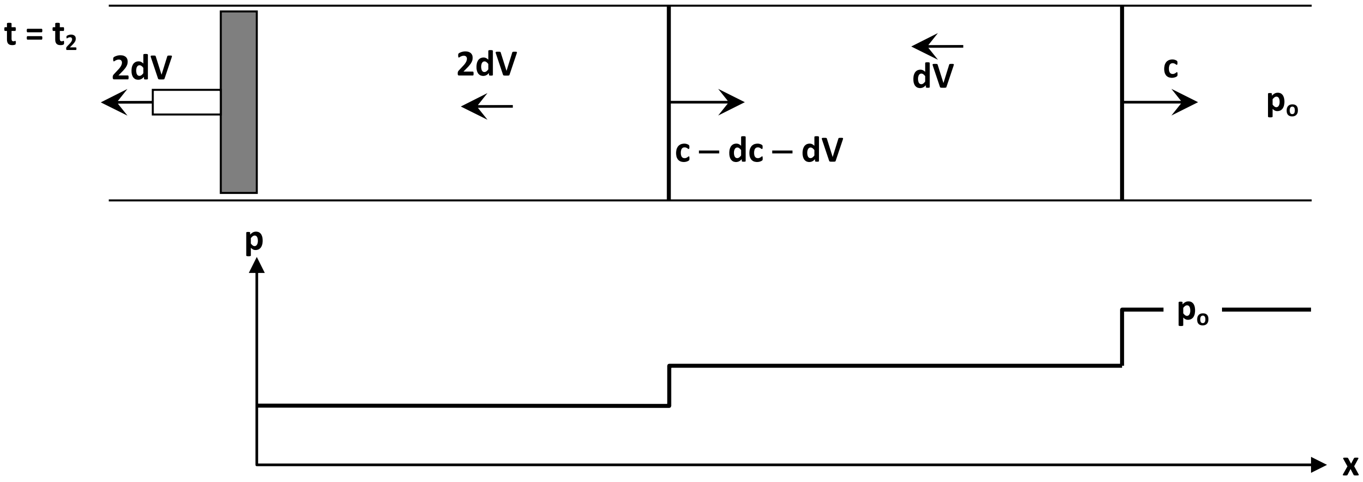
Thickness of shock wave ≈ 0.25 μm → extremely large p and T gradients across.

11.7.1.2. Development of Expansion Waves

Across an expansion wave, the flow accelerates and the pressure decreases.

Figure 11.24. Development of expansion waves





Since the expansion waves can not catch up with one another, a shock wave can not form with expansion waves.

11.7.2. Governing Equations Across a Shock Wave:

1-D, steady-state, adiabatic flow with no friction. Property change across a shock wave is irreversible.

Assumptions:

- **Shock wave is perpendicular to flow**
- **Thickness of shock wave small → assume shock over constant cross-section**
- **Exclude B/L effects on the shock → frictionless duct**
- **No external work**
- **Body forces negligible**

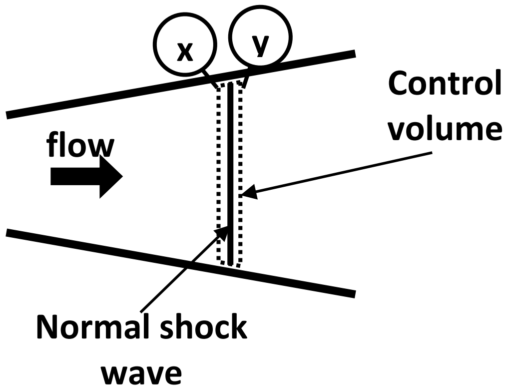


Figure 11.25

Properties change across the schock wave.

x: upstream of shock wave

y: downstream of shock wave

Governing equations on the infinitesimally thin control volume:

Continuity:

$$\dot{m} = \rho_x V_x A = \rho_y V_y A$$

$$Mass\ flux: G = \frac{\dot{m}}{A} = \rho_x V_x = \rho_y V_y = constant \qquad (1)$$

Momentum:

$$p_x A - p_y A = \dot{m}(V_x - V_y)$$

$$p_x - p_y = G(V_x - V_y)$$

Substituting the continuity (1) above yields

$$p_x - p_y = \rho_x {V_x}^2 - \rho_y {V_y}^2 \qquad (2)$$

Energy:

$$h_{ox} = h_x + \frac{{V_x}^2}{2} = h_y + \frac{{V_y}^2}{2} = h_{oy} \qquad (3)$$

Second Law:

Irreversible flow across shock wave

$$s_y > s_x \qquad (4)$$

Equation of State:

Perfect gas,

$$p = \rho RT \qquad (5)$$

FANNO LINE (section 11.9.2):

Suppose the shock upstream state and mass flux through the duct are known.

The downstream of the shock wave is to be determined.

Using only the continuity equation, energy equation and the equation of state (ideal gas law), the possible downstream states can be determined (since we did not use the momentum equation, we do not have a unique downstream state, yet). When these states are marked on the h-s diagram, the resulting locus curve is known as the Fanno Line.

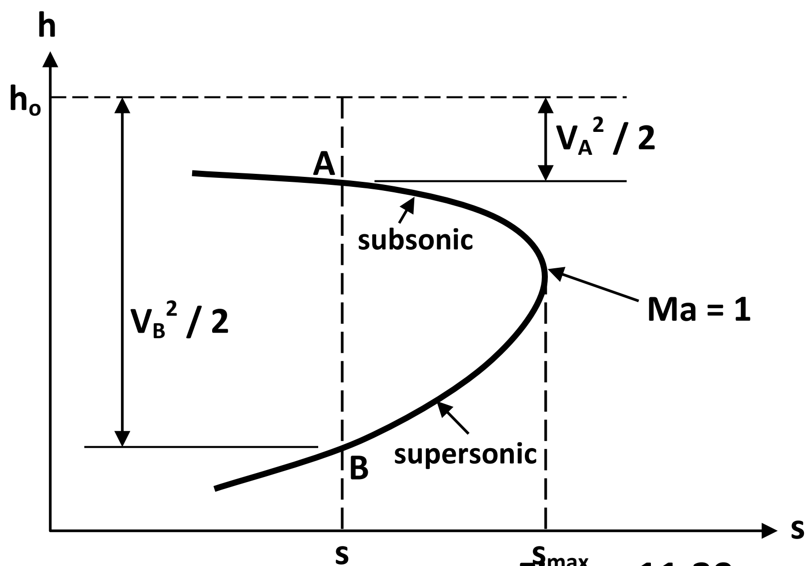


Figure 11.39

The Fanno Line represents <u>all downstream states for a known upstream state for adiabatic flow (h_o = const.) with friction</u> (there can be friction since the momentum equation was not used), <u>where the mass flux is fixed</u> (for a different mass flux, the curve will shift). Note that one of the points on the curve represents the shock upstream state, x.

It can be shown that on the Fanno line:

- **the maximum entropy state corresponds to Ma = 1**
- **the upper curve represents subsonic flow and the lower curve represents supersonic flow**

Since the entropy must increase in adiabatic flow with friction:

- **a subsonic flow accelerates due to friction (s↑), approaching Ma = 1,**
- **a supersonic flow decelerates due to friction (s↑), again approaching Ma = 1**
- **But a subsonic flow can never become supersonic or a supersonic flow can never become subsonic due only to friction. The limit is choking.**

RAYLEIGH LINE (section 11.10.2):

Suppose the shock upstream state and mass flux through the duct are known.

The downstream of the shock wave is to be determined.

Using only the continuity equation, momentum equation and the equation of state (ideal gas law), the possible downstream states can be determined (since we did not use the energy equation, we do not have a unique exit state, yet). When these states are marked on the h-s diagram, the resulting locus curve is known as the Rayleigh Line.

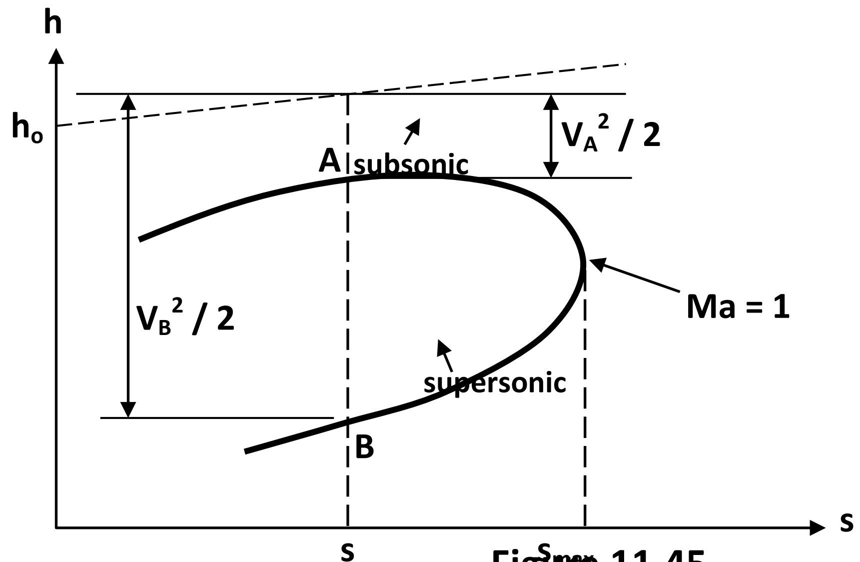


Figure 11.45

The Rayleigh Line represents _all downstream states for a known upstream state for frictionless flow with heat transfer_ (heat transfer is possible since the energy equation was not used), _where the mass flux is fixed_ (for a different mass flux, the curve will shift). Note that one of the points on the curve represents the shock upstream state, x. Since heat transfer is allowed, the total enthalpy also changes.

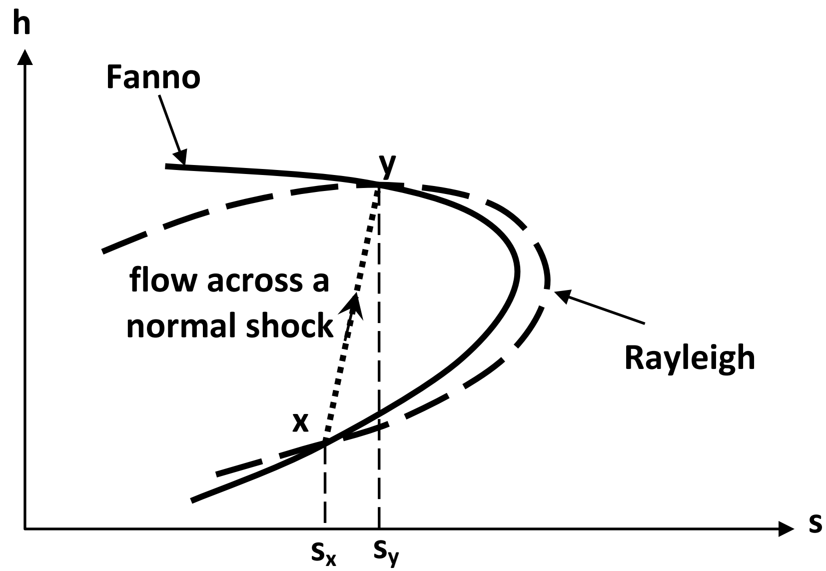
It can be shown that on the Rayleigh line:

- **the maximum entropy state corresponds to Ma = 1**
- **the upper curve represents subsonic flow and the lower curve represents supersonic flow**

The entropy may increase or decrease depending on whether heat is transferred to or from the control volume in the frictionless flow. Thus:

- **With heating (s↑), a subsonic flow accelerates, approaching Ma = 1,**
- **With heating (s↑), a supersonic flow decelerates, again approaching Ma = 1**
- **But further heating does not make a subsonic flow become supersonic or a supersonic flow become subsonic. _Cooling_ must follow for the flow to cross Ma = 1 smoothly.**

Now suppose, for a given mass flux and a given shock upstream state, the Fanno and the Rayleigh lines are plotted on the same graph.



It turns out that two points intersect. These points represent states at which the flow is both adiabatic and frictionless. The flow accross the shock is adiabatic and frictionless. Then, one of these points is the shock upstream state and the other, the shock downstream state.

The entropy values of the two points are not the same. Since the flow across a shock wave is irreversible, the downstream state must have the higher entropy.

The upstream state is supersonic and the downstream state is subsonic. Thus, shock waves can happen only in supersonic flow and the flow becomes subsonic once it crosses a shock wave.

11.7.3. Relations for the Flow of a Perfect Gas Across a Shock Wave:

Using the five equations (continuity, momentum, energy, entropy and equation of state), the downstream properties are expressed with respect to Ma_x (upstream Ma). These relations are tabulated in Appendix C.4 for air (k = 1.4).

Upstream and Downstream Mach Numbers:

Substitute ideal gas law (5) into continuity (1):

$$\frac{T_y}{T_x} = \frac{p_y}{p_x}\frac{V_y}{V_x}$$

Knowing that $Ma = \frac{V}{c} = \frac{V}{\sqrt{kRT}}$, the above equation becomes

$$\frac{T_y}{T_x} = \left(\frac{p_y}{p_x}\right)^2 \left(\frac{Ma_y}{Ma_x}\right)^2 \qquad (6)$$

Energy equation (3) can be reexpressed as

$$c_p T_x + \frac{{V_x}^2}{2} = c_p T_y + \frac{{V_y}^2}{2} \rightarrow T_{ox} = T_x + \frac{{V_x}^2}{2c_p} = T_y + \frac{{V_y}^2}{2c_p} = T_{oy}$$

Stagnation temperature remains constant across the shock wave!

Using $Ma^2 = \frac{V^2}{kRT}$, and $c_p = kR/(k-1)$, the energy equation becomes

$$\frac{T_y}{T_x} = \frac{1 + \frac{k-1}{2}{Ma_x}^2}{1 + \frac{k-1}{2}{Ma_y}^2} \qquad (7)$$

Combining equations (6) and (7),

$$\frac{p_y}{p_x} = \frac{Ma_x}{Ma_y}\sqrt{\frac{1 + \frac{k-1}{2}{Ma_x}^2}{1 + \frac{k-1}{2}{Ma_y}^2}} \qquad (8)$$

Using the ideal gas law and the Ma definition in the momentum equation (2), the momentum equation is rearranged as

$$\frac{p_y}{p_x} = \frac{1 + k{Ma_x}^2}{1 + k{Ma_y}^2} \qquad (9)$$

Equating (8) and (9) and solving for Ma_y yields

$$Ma_y = \sqrt{\frac{(k-1){Ma_x}^2 + 2}{2k{Ma_x}^2 - (k-1)}} \qquad (10)$$

<u>So if you know the Mach number upstream of the shock wave, you can determine the downstream Mach number</u>.

<u>Pressure Ratio Across a Shock Wave</u>:

Substitute (10) in (9)

$$\frac{p_y}{p_x} = \frac{2k}{k+1} Ma_x{}^2 - \frac{k-1}{k+1} \qquad (11)$$

<u>Temperature Ratio Across a Shock Wave</u>:

Substitute (10) in (7)

$$\frac{T_y}{T_x} = \frac{\left(1 + \frac{k-1}{2} Ma_x{}^2\right)\left(\frac{2k}{k-1} Ma_x{}^2 - 1\right)}{\frac{(k+1)^2}{2(k-1)} Ma_x{}^2} \qquad (12)$$

<u>Density Ratio Across a Shock Wave</u>:

From equation of state

$$\frac{\rho_y}{\rho_x} = \frac{p_y}{p_x} \frac{T_x}{T_y}$$

Substituting equations (11) and (12) above

$$\frac{\rho_y}{\rho_x} = \frac{(k+1)^2 {Ma_x}^2}{2 + (k-1){Ma_x}^2} \quad (13)$$

<u>Velocity Ratio Across a Shock Wave</u>:

Using equations (1) and (13)

$$\frac{V_y}{V_x} = \frac{\rho_x}{\rho_y} = \frac{2 + (k-1){Ma_x}^2}{(k+1)^2 {Ma_x}^2} \quad (14)$$

<u>Stagnation Pressure Ratio Across a Shock Wave</u>:

$$\frac{p_{oy}}{p_{ox}} = \frac{p_{oy}}{p_y}\frac{p_y}{p_x}\frac{p_x}{p_{ox}}$$

The stagnation to static pressure ratio was derived earlier in section 11.6.4 (in isentropic flow):

$$\frac{p_o}{p} = \left(1 + \frac{k-1}{2} Ma^2\right)^{\frac{k}{k-1}}$$

Combining the above two equations and substituting (11) yields

$$\frac{p_{oy}}{p_{ox}} = \left(\frac{\frac{k+1}{2}{Ma_x}^2}{1+\frac{k-1}{2}{Ma_x}^2}\right)^{\frac{k}{k-1}} \left(\frac{2k}{k+1}{Ma_x}^2 - \frac{k-1}{k+1}\right)^{\frac{1}{1-k}} \qquad (15)$$

<u>Critical Area Ratio Across a Shock Wave</u>:

Critical area is used as reference in isentropic flow.

Flow across a shock wave not isentropic; critical area changes, $A_x^* \neq A_y^*$

Recall

$$\frac{\dot{m}\sqrt{RT_o}}{Ap_o} = \sqrt{k}\, Ma\left(1+\frac{k-1}{2}Ma^2\right)^{\frac{k+1}{2(k-1)}}$$

derived in section 11.6.4.

When the critical area is reached, $A = A^*$ and Ma = 1

$$\frac{\dot{m}\sqrt{RT_o}}{A^*p_o} = \sqrt{k}\,\left(1+\frac{k-1}{2}\right)^{\frac{k+1}{2(k-1)}} = \ constant$$

Thus,

$$\frac{\dot{m}\sqrt{RT_{ox}}}{A_x{}^{*}p_{ox}} = \frac{\dot{m}\sqrt{RT_{oy}}}{A_y{}^{*}p_{oy}}$$

From the energy equation, it was shown that $T_{ox} = T_{oy}$. Substituting (15) above

$$\frac{A_y{}^{*}}{A_x{}^{*}} = \frac{p_{ox}}{p_{oy}} = \left(\frac{\frac{k+1}{2}Ma_x{}^2}{1+\frac{k-1}{2}Ma_x{}^2}\right)^{\frac{k}{1-k}}\left(\frac{2k}{k+1}Ma_x{}^2 - \frac{k-1}{k+1}\right)^{\frac{1}{k-1}}$$

<u>Entropy Change Across a Shock Wave</u>:

For a perfect gas,

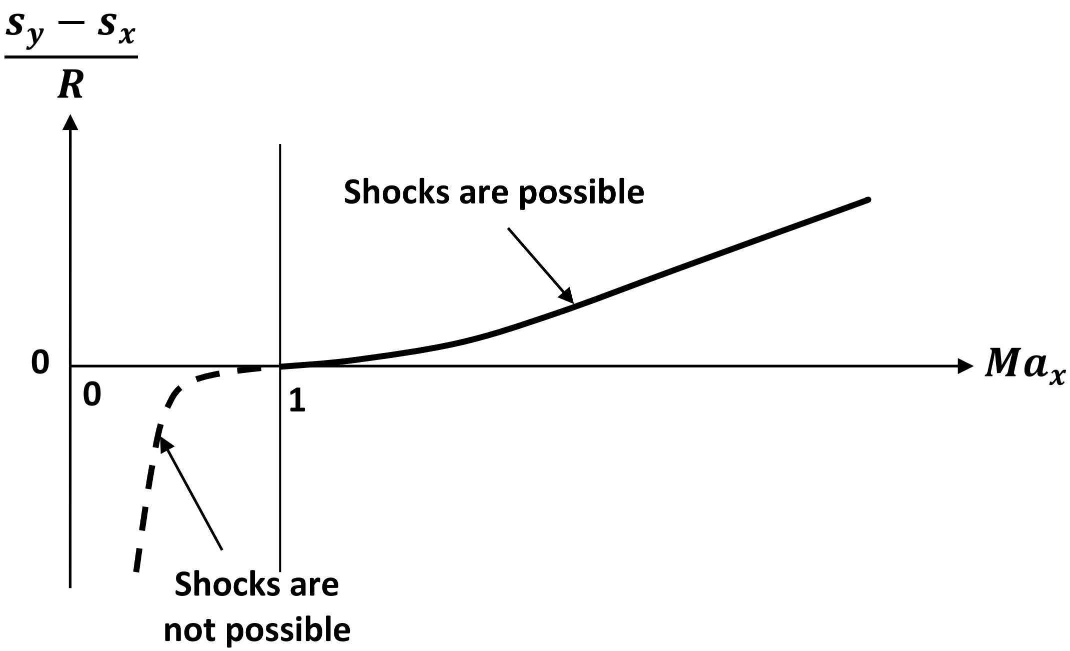
$$s_y - s_x = c_p \ln\frac{T_y}{T_x} - \mathrm{R}\ln\frac{p_y}{p_x}$$

Using the previously derived temperature and pressure ratios along with stagnation properties, the above equation can be rearranged as

$$\frac{s_y - s_x}{R} = -\ln\frac{p_{oy}}{p_{ox}}$$

$$\rightarrow \frac{s_y - s_x}{R} = \frac{k}{k-1}\ln\left(\frac{2k}{(k+1){Ma_x}^2} + \frac{k-1}{k+1}\right) + \frac{1}{1-k}\ln\left(\frac{2k}{k+1}{Ma_x}^2 - \frac{k-1}{k+1}\right)$$

When the entropy change is plotted against the upstream Mach number (for 1 ≤ k ≤ 1.67):



Once again, it is seen that <u>shocks are possible only in supersonic flow</u>

Problem 11.28:

Air stream with Ma = 1.8 goes through normal shock wave. Stagnation temperature and pressure before shock: 150 kPa and 350 K. Determine

a) T and p after the shock, b) Ma and V after the shock, c) T_o and p_o after the shock, d) Δs across the shock.

Problem 11.29:

Supersonic air stream entering a diverging duct slows down due to a shock wave.

Determine

a) pressure after the normal shock wave,

b) pressure at the exit of the duct.