QUANTUM FORMALISM
SINCE 2020
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Writing and Understanding Proofs

Lecture 1
Types of Proof

Dr Max Arnott

Quantum Formalism (QF) Free Bootcamp
Brought to you by Zaiku Group.

QF's CORE MISSION:
Make Abstract Mathematics Accessible.

Goals for the series

In these (at least) two lectures, we aim to:

- Understand some of the main types of proof.
- Learn about mathematically formal writing.
- Know how to verify a proof and check for common mistakes.
- Examine some good proofs ... and some not so good ones.

In this session, we cover the main styles of proof: direct proof, proof by contradiction, proof by induction, proof by contrapositive, and counterexamples.

For each of these proof types, we explain their logic, and work through an example. At the end, we leave some exercises.

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Notation

A *proposition* is a statement which can be said to be either true or false (e.g., '2 is even', '$3 \times 5 = 10$', 'the sky is blue').

Let A and B be propositions.

- $\neg A$ denotes the logical *negation* of A. E.g., $\neg$(*'all multiples of* 4 *are even'*) is 'there exists an odd multiple of 4'.
- $A \wedge B$ denotes A *and* B. $A \vee B$ denotes A *or* B.
- $A \implies B$ denotes that A *implies* B, meaning that 'if A is true, then B must also be true'. E.g., '[x is even] $\wedge$ [x is a multiple of 3] $\implies$ [x is a multiple of 6]'.
  Clearly, we always have that $A \implies A \vee B$ and $A \wedge B \implies A$.
- $A \iff B$ denotes that both $A \implies B$ and $B \implies A$, so that A is true *if and only if* B is true. In this case, A and B are *logically equivalent*. E.g., '$x^2 = 4 \iff x = \pm 2$'.

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Important to know:
Let A and B be sets. Then '$A = B$' is logically equivalent to '$x \in A \iff$ x $\in B$'.

$$\forall A \forall B \left(A = B \iff \forall x \left(x \in A \iff x \in B\right)\right),$$

axiom of extensionality.
So, when asked to prove that $A = B$, we need to show that 'all elements of A are elements of B', and 'all elements of B are elements of A'.
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Direct proof example

Theorem

Let $A_1, \ldots, A_k$ be closed sets for some topology on a set X. Then $\bigcup_{i=1}^k A_i$ is closed.

Proof.

By definition of closed sets, we have that A_i^c $(:= X \setminus A_i)$ is open for every $i \in \{1, \ldots, k\}$.

In a topology, every intersection of finitely many open sets is open. Thus $\bigcap_{i=1}^k A_i^c$ is open.

We apply one of De Morgan's laws to the above to see that

$$\bigcap_{i=1}^{k} A_i^c = \bigcap_{i=1}^{k} X \setminus A_i = X \setminus \bigcup_{i=1}^{k} A_i = (\bigcup_{i=1}^{k} A_i)^c$$

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In a topology, complements of open sets are closed. Therefore, $((\bigcup_{i=1}^k A_i)^c)^c = \bigcup_{i=1}^k A_i$ is closed, as required. □

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Proof by contradiction

Suppose that you are asked to prove a proposition P. The *law of excluded middle* tells us that $P \vee \neg P$ must be true.

Therefore, if we can show that $\neg P$ is *not* true, it will follow that P must be true.

We prove that $\neg P$ is false by showing that we can use it to derive a contradiction.

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Theorem

$\sqrt{2}$ is irrational.

Proof.

Suppose, for the sake of contradiction, that $\sqrt{2}$ is rational. Then $\sqrt{2}$ can be expressed as a fraction $\frac{p}{q}$ of integers p and q. Moreover, we can assume that these integers have no common factors (apart from 1), and $q \neq 0$.
Squaring both sides of the equation $\sqrt{2} = \frac{p}{q}$ gives us that $2 = \frac{p^2}{q^2}$.
Multiplying both sides by q^2 then gives $2q^2 = p^2$, so that p^2 is even. It follows that p is even. Therefore, there exists some integer k such that $p = 2k$.
Substituting this back into the equation: $2q^2 = (2k)^2$ tells us that $2q^2 = 4k^2$, and hence $q^2 = 2k^2$. This implies that q^2 is even, so q must be even as well.
The fact that both p and q are even contradicts the claim that p and q have no common factors apart from 1. Therefore, the assumption that $\sqrt{2}$ is rational must be false.
Hence, $\sqrt{2}$ is irrational. $\square$

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Proof by induction can be used whenever we are asked to prove that a claim holds *for every* $n \in \mathbb{N}$. There are four main parts to every induction proof.

Basis: We prove that the claim holds for the special case of $n = 1$.

Assumption: We assume that the claim is true for some arbitrary $n = k \in \mathbb{N}$.

Induction: We show that, given our assumption, the claim also holds for the case $n = k + 1$.

Conclusion: Because the claim holds for $n = k + 1$ whenever it holds for $n = k$, and we have shown that it holds for $n = 1$, it must hold also for $n = 2$...and because it holds for $n = 2$, it must be true for $n = 3$...Therefore, the claim holds for every $n \in \mathbb{N}$.

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Theorem

Every set S with $|S| = n \in \mathbb{N}$ has exactly 2^n many subsets (i.e., $|\mathcal{P}(S)| = 2^n$).

Proof.

- *Basis :* If S has 1 element, we can write $S = \{x\}$. The subsets of S are $\{x\}$ and $\emptyset$. So S has 2^1-many subsets as required.
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Proof by contrapositive

Suppose that I am asked to prove a proposition of the form '$(P \implies Q)$'. The *contrapositive* statement to this proposition is '$(\neg Q \implies \neg P)$'. The statements '$(P \implies Q)$' and '$(\neg Q \implies \neg P)$' are *logically equivalent*. So, we can choose to prove the contrapositive statement instead.

Example for motivation: the claim '*if I am water, then I am wet*', clearly implies the contrapositive claim '*if I am not wet, then I am not water*', and vice versa.

Common mistake: do not get the *contrapositive* $(\neg Q \implies \neg P)$ of $(P \implies Q)$ confused with the *reverse implication* $(Q \implies P)$. The reverse implication is not necessarily equivalent. In the above example, this would correspond to '*if I am wet, then I am water*'.

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Theorem

Let $(X, \mathcal{T})$ be a topological space, and let $(A, \mathcal{T}_A)$ be a subspace of X with the subspace topology. If $(X, \mathcal{T})$ is Hausdorff, then $(A, \mathcal{T}_A)$ is Hausdorff.

Proof.

We prove the contrapositive. So assume that $(A, \mathcal{T}_A)$ is not Hausdorff and take two points $a_1, a_2 \in A$ such that, whenever $A_1, A_2 \in \mathcal{T}_A$ with $a_1 \in A_1$ and $a_2 \in A_2$, we must have that $A_1 \cap A_2 \neq \emptyset$. We will show that $(X, \mathcal{T})$ is not Hasudorff either. Now, write $\mathcal{T}_A$ by its definition as

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\} .$$

Since $A \subset X$, we have that $a_1, a_2 \in X$. Let $X_1 X_2 \in \mathcal{T}$ with $a_1 \in X_1$ and $a_2 \in X_2$. We must show that $X_1 \cap X_2 \neq \emptyset$. Now, $(A \cap X_1)$ and $(A \cap X_2)$ belong to $\mathcal{T}_A$ by the definition of the subspace topology. To complete the proof, notice that since $a_1 \in (X_1 \cap A) \in \mathcal{T}_A$ and $a_2 \in (X_2 \cap A) \in \mathcal{T}_A$,

$$X_1 \cap X_2 \supseteq (X_1 \cap A) \cap (X_2 \cap A) \neq \emptyset . \qquad \square$$

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Exercises

Complete your own 'proof four ways' (direct, contrapositive, induction, contradiction) for the following claims:

Theorem

Let $n \in \mathbb{N}$. If n is a multiple of 4, then n is a multiple of 2.

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Let $n \in \mathbb{N}$. Then $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

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Let $n \in \mathbb{Z}$. Then $n^2 + 2n + 1 \geq 0$.

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Counter example (dis)proofs

To prove a proposition of the form $\neg(\forall x, P(x))$, is to disprove the claim $(\forall x, P(x))$. This can be done by proving the proposition $(\exists x, \neg P(x))$. To achieve this, all that is needed is to find an example of some x for which $P(x)$ does not hold. Such an x is called a *counter example* for the proposition $(\forall x, P(x))$.

Example for motivation: if I want to disprove the claim '*all cars are blue*', I can do this by showing that '*there is a car that is not blue*'. Any non-blue car would be a counterexample.

Common mistake: oftentimes, people mistakenly use that $\neg(\forall x, P(x))$ is equivalent to $(\forall x, \neg P)$. This would be like trying to disprove the above claim '*all cars are blue*', by showing that '*all cars are not blue*' - a proposition which is not true!

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Infinite sets always have the same cardinality.

Disproof.

The sets $\mathbb{R}$ and $\mathbb{N}$ have different cardinalities. (See the video on Cantor's diagonal argument for this). $\square$

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