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Mathematical Tools of Quantum Mechanics: Introduction to Hilbert Spaces

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Notation

- the closure of Y: $\overline{Y}$;
- the linear span of a set Y: $\operatorname{lin} Y$;
- $\mathcal{H}$ always denotes a Hilbert space;

Preliminaries: Cardinality of a set

Loose definition

The cardinality of a set A, denoted by $|A|$, is the number of elements of the set.

- $|\emptyset| = 0$;
- $|\mathbb{N}|$ is countable infinite;
- we say that a set A is countable when A is finite or $|A|$ is countable infinite;
- a set A is uncountable (infinite) if $|A| > |\mathbb{N}|$.

Recap

A Hilbert space $\mathcal{H}$:

- is a vector space over a field $\mathbb{F}(=\mathbb{C})$;
- it has an inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H}\mapsto\mathbb{C}$;
- it is a normed space with the norm induced by the inner product $\|x\| = \sqrt{\langle x, x\rangle}$;
- it is a Banach space (i.e. is complete).

Orthogonality and orthonormality

Definition

Let E be an inner product space and $x, y \in E$. We say that x, y are orthogonal when $\langle x, y \rangle = 0$. We write $x \perp y$.
A family $(e_\alpha)_{\alpha \in A}$ is called an orthogonal system if $e_\alpha \perp e_\beta$ when $\alpha \neq \beta$.
When we also have that $\|e_\alpha\| = 1$ for each α, then we say that $(e_\alpha)_{\alpha \in A}$ is an orthonormal system.

Note

When we can index by $\mathbb{N}$ an orthogonal (respectively orthonormal) system, in some literature they call it an orthogonal (respectively orthonormal) sequence.

Definition

Let $\mathcal{H}$ a Hilbert space. An orthogonal (orthonormal) system $B = (e_\alpha)$ such that $\overline{\operatorname{lin} B} = E$ is called an orthogonal (orthonormal) basis.

Remark

Orthogonality implies linear independence.
If $x = \alpha y \neq 0$, $\langle y, x \rangle = \langle \alpha y, y \rangle = \alpha \langle y, y \rangle \neq 0$.

Theorem:Base

Let $\mathcal{H}$ be a Hilbert space. Then $\mathcal{H}$ has an orthonormal basis. In particular, every orthonormal system can be expanded until we obtain a basis for $\mathcal{H}$.
(The proof depends on Zorn's lemma.)

Corollary

Let B_1 and B_2 be bases for $\mathcal{H}$. Then $|B_1| = |B_2|$.

Dimension of a Hilbert space

Definition

The dimension of a Hilbert space is the cardinality of its basis.

Remark

- Finite Dimension: $\mathcal{H}$ has a finite basis. For example $\mathbb{C}^n$.
- Countably Infinite Dimension: $\mathcal{H}$ has a countable infinite basis. For example, ℓ^2.
- Uncountably Infinite Dimension: The cardinality of the basis is strictly bigger than $|\mathbb{N}|$. For example, $L^2(\mathbb{R})$.

Definition:

A Hilbert space $\mathcal{H}$ is separable if it has a countable orthonormal basis (i.e. an orthonormal basis indexed by $\mathbb{N}$ or finite).

A small detour: Separability in a wider context.

Dense Subspace of a Normed Space:

Let E be a normed space and D be a subspace of E. D is said to be dense in E if

$$\overline{D} = E$$

I.E., for x in E, there exists a sequence (x_n) in D such that $x_n \to x$.

Separable Normed Space:

A normed space E is said to be separable if there exists a countable dense subset $D \subset E$.

Theorem

Let $\mathcal{H}$ be a Hilbert space and (e_n) and orthonormal basis of $\mathcal{H}$. Then, for every $x \in \mathcal{H}$,

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \quad \text{and} \quad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|.$$

Proof: N.Young. An introduction to Hilbert Space, P.37

Gram-Schmidt Orthonormalization Process:

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.
Suppose (v_n) is a set of linearly independent vectors in $\mathcal{H}$. Then the following is an orthonormal system

$$
\begin{aligned}
u_1 &= \frac{v_1}{\|v_1\|} \\
u_2 &= \frac{v_2 - \langle v_2, u_1 \rangle u_1}{\|v_2 - \langle v_2, u_1 \rangle u_1\|} \\
&\vdots \\
u_k &= \frac{v_k - \sum_{i=1}^{k-1} \langle v_k, u_i \rangle u_i}{\|v_k - \sum_{i=1}^{k-1} \langle v_k, u_i \rangle u_i\|} \quad \text{for } k = 1, 2, \ldots
\end{aligned}
$$

Functionals

Definition

Let V be a vector space over a field $\mathbb{C}$. A functional on V is a mapping $F : V \to \mathbb{C}$.
A linear functional on V is a mapping $F : V \to \mathbb{C}$ that satisfies:

$$F(\alpha v + \beta w) = \alpha F(v) + \beta F(w) \quad (v, w \in V, \alpha, \beta \in \mathbb{C}).$$

The space of all linear functionals on V is written by $V^{\#}$ and it is known as the algebraic dual of V.
Let E be a normed space. A functional $F : E \to \mathbb{C}$ is continuous on $x \in E$ if, for every sequence in E $(x_n) \to x$ in $(E, \|\cdot\|)$,

$$F(x_n) \to F(x) \text{ in } (\mathbb{C}, |\cdot|).$$

We say that F is continuous on E if F is continuous on every $x \in E$. We write E^* for the set of all continuous linear functionals on the space E.

Characterization of continuity for linear functionals

Theorem

Let $(E, \|\cdot\|)$ be a normed space over $\mathbb{C}$ and F a linear functional on E. TFAE:

- F is continuous;
- F is continuous at 0;
- $\sup\{|F(x)| : x \in E, \|x\| \leq 1\} < \infty$. I.E. if it is bounded in the unit ball of E.

Dual spaces

Theorem

Given a normed space E, E^* is always a Banach space with respect to the pointwise operations and the norm $\|F\| = \sup\{|F(x)| : x \in E, \|x\| \leq 1\}$. (Proof: N.Young. Introduction to Hilbert Space, P.61)

Definition

We call $(E^*, \|\cdot\|)$ the dual space of E or the Banach dual of E.

Remark

Given a normed space E, E^{**} is a Banach space. When E is a Banach space E can be embedded in E^{**}.

Example

For Banach spaces in general E does not need to be the same as E^*.

Example

$E = \ell^1$ the space of all sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{i=1}^{\infty} |x_i|$ is finite. The dual is $E^* = \ell^\infty$, which is the space of all sequences $x = (x_1, x_2, \ldots)$ such that $\sup |x_i|$ is finite.

Riesz-representation Theorem and Dirac notation

Riesz-representation Theorem or Riesz-Fréchet theorem

Let $\mathcal{H}$ be a Hilbert space and let F be a continuous linear functional on $\mathcal{H}$. There exists a unique $y \in \mathcal{H}$ such that

$$F(x) = \langle x, y \rangle \quad (x \in \mathcal{H}).$$

Moreover $\|y\| = \|F\|$.

Dirac Notation

- Ket $|y\rangle$ - it is denoting that we are in $\mathcal{H}$.
- Bra $\langle x|$ - it is denoting that we are in $\mathcal{H}^*$.

When you write: $\langle x|y\rangle = \langle x, y\rangle$.

Tensor products of vector spaces. A brief introduction: Bilinear forms

Definition

Let X, Y, Z be vector spaces. A mapping $A : X \times Y \to Z$ is bilinear if

$$A(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y);$$
$$A(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 A(x, y_1) + \beta_2 A(x, y_1),$$

for $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$.
We write $B(X \times Y, Z)$ for the vector space of bilinear mappings from $X \times Y$ to Z.

Tensor products of vector spaces

Let $x \in X$ and $y \in Y$. We define $x \otimes y$ as:

$$x \otimes y : B(X \times Y) \longrightarrow \mathbb{C}$$
$$(x \otimes y)(A) = \langle A, x \otimes y \rangle = A(x, y)$$

We define $X \otimes Y := lin\{x \otimes y : x \in X, y \in Y\} \subset B(X \times Y)^{\#}$.
For $u \in X \otimes Y$, then $u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ for $n \in \mathbb{N}$, $x_i \in X$, $y_i \in Y$, Then

$$u(A) = \left\langle A, \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \right\rangle = \sum_{i=1}^{n} \lambda_i A(x_i, y_i).$$

The representation of u might not be unique.

Definition

We define rank u as the smallest number n for which there is a representation of u that contains n terms. Tensors of rank 1 are elementary tensors.

Some properties of tensor products

Some properties of tensor products

- $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$;
- $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$;
- $\lambda(x \otimes y) = (\lambda x) \otimes y = x \otimes (\lambda y)$;
- $0 \otimes y = y \otimes 0 = 0$.

Some properties of tensor products

L.I. subspaces and basis of tensor products

Let X, Y vector spaces:

- E, F linearly independent subspaces of X, Y respectively. Then $\{x \otimes y : x \in E, y \in F\}$ is a linearly independent subset of $X \otimes Y$.
- E, F basis for X, Y respectively. Then $\{x \otimes y : x \in E, y \in F\}$ is a basis for $X \otimes Y$.

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Thank you for your attention

Recommended Bibliography:

Megginson, R.E., An introduction to Banach space theory
Young, N., An introduction to Hilbert space
Ryan, R.A., Introduction to Tensor Products of Banach spaces