Existence and Uniqueness Theorems

Lecture 2

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Theorem (First-Order Linear Existence and Uniqueness)

If $p(t)$ and $g(t)$ are continuous on an open interval $\alpha < t < \beta$ containing t_0, then for any real number y_0 the initial value problem

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

has a unique solution also defined on the interval $\alpha < t < \beta$.

Proof.

Direct construction using the method of integrating factors from the previous lecture. □

Theorem (Second-Order Linear Existence and Uniqueness)

If $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval $\alpha < t < \beta$ containing t_0, then for any real numbers y_0, y_0' the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0,\ y'(t_0) = y_0'$$

has a unique solution also defined on the interval $\alpha < t < \beta$.

The proof of this theorem is much more challenging and will not be presented here.

Theorem (General First-Order Existence and Uniqueness)

If f and $\dfrac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing (t_0, y_0), then the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution defined on a possibly smaller interval $t_0 - h < t < t_0 + h$.

If f is continuous but $\dfrac{\partial f}{\partial y}$ is not, then the solution exists but may not be unique.

The proof of this theorem will also be omitted. The main tool is the contraction mapping theorem.

Example 1

We solved the initial value problem

$$y' + \frac{2}{t}y = 4t, \quad y(1) = 2$$

in the previous lecture, and obtained the solution $y = t^2 + \dfrac{1}{t^2}$ defined on the interval $t > 0$.

Examining the First-Order Linear Existence and Uniquness Theorem, we can see that we were guaranteed a unique solution defined on this interval since $g(t) = 4t$ is continuous everywhere and $p(t) = 2/t$ is continuous for $t \neq 0$.

Example 2

Solve the initial value problem

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The denominator factors as $y^3 - 4y = y(y+2)(y-2)$, so f is continuous except at $y = 0, \pm 2$.

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The partial derivative is $\dfrac{\partial f}{\partial y} = \dfrac{-(3y^2-4)(x+3x^2)}{(y^3-4y)^2}$, which is also continuous except at $y = 0, \pm 2$.

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$$\frac{dy}{dx} = \frac{x + 3x^2}{y^3 - 4y}, \quad y(0) = 1.$$

The only discontinuities for f and $\dfrac{\partial f}{\partial y}$ are at $y = 0, \pm 2$, so we can make the rectangle in the Existence and Uniqueness Theorem infinitely wide in the x-direction. However, if we actually solve this IVP, we will see that the solution is not defined on all of $\mathbb{R}$.

Example 2

Solve the initial value problem

$$\frac{dy}{dx} = \frac{x + 3x^2}{y^3 - 4y}, \quad y(0) = 1.$$

The equation is separable, so we solve by separating the variables and integrating

$$(y^3 - 4y)\frac{dy}{dx} = x + 3x^2$$
$$\int y^3 - 4y \; dy = \int x + 3x^2 \; dx$$
$$\frac{y^4}{4} - 2y^2 = \frac{x^2}{2} + x^3 + C$$

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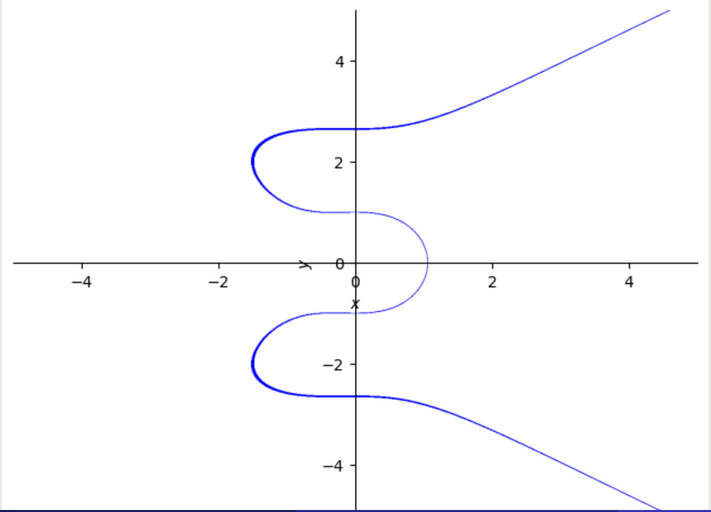
Plugging in $x = 0$, $y = 1$ yields $\frac{1}{4} - 2 = 0 + C$ or $C = -\frac{7}{4}$, so the solution to the IVP is $\dfrac{y^4}{4} - 2y^2 = \dfrac{x^2}{2} + x^3 - \dfrac{7}{4}$.

Example 2

$$\frac{y^4}{4} - 2y^2 = \frac{x^2}{2} + x^3 - \frac{7}{4}$$

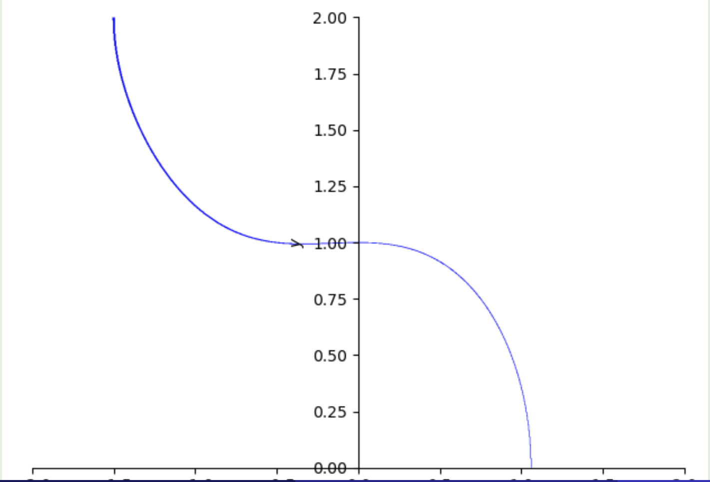
To see where this solution is well-defined, we examine the graph.



Example 2

$$\frac{y^4}{4} - 2y^2 = \frac{x^2}{2} + x^3 - \frac{7}{4}$$

Only this portion between $y = 0$ and $y = 2$ is part of the solution to the IVP.



Example 3

Solve the initial value problem

$$y' = y^2, \quad y(0) = 1.$$

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This ODE is also nonlinear, so we use the general theorem with $f(t, y) = y^2$. We have $\dfrac{\partial f}{\partial y} = 2y$, so f and $\dfrac{\partial f}{\partial y}$ are continuous everywhere. The theorem therefore tells us that every initial value problem has a unique solution defined on some interval.

Example 3

Solve the initial value problem

$$y' = y^2, \quad y(0) = 1.$$

This equation is separable, so we use separation of variables again.

$$\begin{aligned} \frac{1}{y^2} y' &= 1 \\ \int \frac{1}{y^2}\, dy &= \int 1\, dt \\ -\frac{1}{y} &= t + C \\ y &= -\frac{1}{t + C} \end{aligned}$$

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$C = -1$ gives the solution $y = \dfrac{1}{1 - t}$ to the IVP.

Example 3

Solve the initial value problem

$$y' = y^2, \quad y(0) = 1.$$

Note that the solution $y = \dfrac{1}{1-t}$ is only defined for $-\infty < t < 1$. However, unlike in the linear Example 1, there is no indication that $t = 1$ is special in any way from the differential equation or the initial condition. Additionally, the point where the discontinuity in the solution occurs will change depending on the initial condition.

Moreover, $y = -\dfrac{1}{t+C}$ is not the general solution to the ODE, since the function $y(t) = 0$ for all t is also a solution.

Example 4

Solve the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0.$$

Here, $f(t, y) = y^{1/3}$ is continuous everywhere, but $\dfrac{\partial f}{\partial y} = \dfrac{1}{3} y^{-2/3}$ has a discontinuity at $y = 0$, so the theorem guarantees that a solution to the IVP exists but it may not be unique. Separation of variables yields

$$\begin{aligned} y^{-1/3} y' &= 1 \\ \int y^{-1/3}\, dy &= \int 1\, dt \\ \frac{3}{2} y^{2/3} &= t + C \end{aligned}$$

Plugging in $t = 0$, $y = 0$, we obtain $C = 0$.

Example 4

Solve the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0.$$

From the previous slide, we have that the solution to the IVP satisfies

$$\frac{3}{2} y^{2/3} = t.$$

To invert this expression and solve for y, we must take a square root, so the negative square root is also a possibility. Thus, we have found two solutions

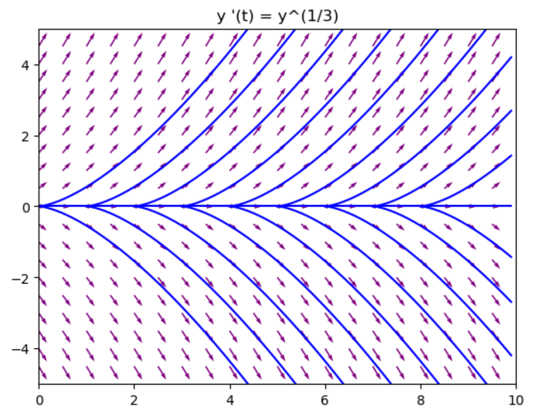
$$y = \pm \left(\frac{2}{3} t \right)^{3/2}$$

to the IVP. However, $y(t) = 0$ for all t also solves the IVP.

Example 4

This allows us to produce an infinite family of solutions to the IVP

$$y(t) = \begin{cases} 0 & \text{for } t \leq t_0 \\ \pm\left(\frac{2}{3}(t - t_0)\right)^{3/2} & \text{for } t > t_0. \end{cases}$$



Properties of First Order Linear ODEs

1. Has a ”general solution”.
2. Can be solved directly (at least in terms of integrals).
3. The possible discontinuities/singularities of the solution can be detected by finding discontinuities of $p(t)$ and $g(t)$.

All three of these statements are false for general nonlinear first order differential equations $\frac{dy}{dt} = f(t, y)$.

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