Complex and Repeated Roots of the Characteristic Equation

Lecture 4

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Complex Roots of the Characteristic Equation

From the previous lecture, in order to find the general solution to a second-order linear homogeneous differential equation with constant coefficients

$$ay'' + by' + cy = 0,$$

it suffices to find two solutions whose Wronskian determinant is nonzero. We also know that $y = e^{rt}$ is a solution if and only if r is a root of the characteristic equation

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Consider the case where the roots r_1, r_2 of the characteristic equation are complex numbers. If we are satisfied with complex-valued solutions, then we are finished; the general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ with c_1, c_2 arbitrary complex numbers.

Complex Roots of the Characteristic Equation

What if we are only interested in real-valued solutions? For example, the motion of a mass on a spring with friction is governed by the differential equation

$$my'' + \gamma y' + ky = 0,$$

with $m, \gamma, k > 0$ and y a real-valued function representing the position of the mass. If $\gamma^2 < 4km$, then the roots of the characteristic equation are complex. If the differential equation has real coefficients, then complex roots are always a complex conjugate pair $r_1, r_2 = \lambda \pm i\mu$.

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In the zero friction case $\gamma = 0$, it is not too hard to guess the solutions $y_1 = \cos(\sqrt{\frac{k}{m}}t)$ and $y_2 = \sin(\sqrt{\frac{k}{m}}t)$. Can something similar work in the general case?

The Complex Exponential Function

We can define the complex exponential function using a power series

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for } z \text{ complex.}$$

One can check that this satisfies the usual exponential rule $e^{z_1+z_2} = e^{z_1}e^{z_2}$, so if $z = \lambda + i\mu$ is written in terms of its real and imaginary parts, we have $e^{\lambda+i\mu} = e^{\lambda}e^{i\mu}$.

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Moreover,

$$\begin{aligned} e^{i\mu} &= \sum_{n=0}^{\infty} \frac{(i\mu)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{2k+1}}{(2k+1)!} \\ &= \cos\mu + i \sin\mu \quad \text{(Euler's formula).} \end{aligned}$$

Real-valued Solutions

If $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ are the two roots of the characteristic equation, we can use Euler's formula $e^{i\mu} = \cos\mu + i\sin\mu$ to write the two associated solutions as

$$
\begin{aligned}
y_1 &= e^{(\lambda+i\mu)t} = e^{\lambda t}e^{i\mu t} = e^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right) \\
y_2 &= e^{(\lambda-i\mu)t} = e^{\lambda t}e^{-i\mu t} = e^{\lambda t}\left(\cos(\mu t) - i\sin(\mu t)\right).
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From here, we can produce the real-valued solutions

$$\frac{y_1 + y_2}{2} = e^{\lambda t}\cos(\mu t) \quad \text{and} \quad \frac{y_1 - y_2}{2i} = e^{\lambda t}\sin(\mu t).$$

Real-valued Solutions

Next, we compute the Wronskian

$$
\begin{aligned}
W(e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)) &= e^{\lambda t}\cos(\mu t)\left(\lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t)\right) \\
&\quad - \left(\lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t)\right) e^{\lambda t}\sin(\mu t) \\
&= \mu e^{2\lambda t}\left(\cos^2(\mu t) + \sin^2(\mu t)\right) \\
&= \mu e^{2\lambda t},
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which is nonzero as long as $\mu \neq 0$.

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which is nonzero as long as $\mu \neq 0$.

We conclude that $y = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$ with c_1, c_2 arbitrary real constants is the general real-valued solution.

Example

Solve the following initial value problem and describe the behavior as $t \to \infty$

$$y'' + 2y' + 4y = 0, \quad y(0) = 2, \; y'(0) = 3.$$

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The characteristic equation is $r^2 + 2r + 4 = 0$ with roots

$$r_1, r_2 = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm i\sqrt{3}.$$

Thus, the general solution is

$$y = c_1 e^{-t} \cos(\sqrt{3}t) + c_2 e^{-t} \sin(\sqrt{3}t).$$

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The general solution is $y = c_1 e^{-t}\cos(\sqrt{3}t) + c_2 e^{-t}\sin(\sqrt{3}t)$. Plugging in $t = 0$, we obtain $2 = y(0) = c_1$. Taking the derivative and plugging in $t = 0$, we obtain

$$\begin{aligned} y' &= -2e^{-t}\cos(\sqrt{3}t) - 2\sqrt{3}e^{-t}\sin(\sqrt{3}t) \\ &\quad - c_2 e^{-t}\sin(\sqrt{3}t) + c_2\sqrt{3}e^{-t}\cos(\sqrt{3}t) \\ 3 = y'(0) &= -2 + c_2\sqrt{3}, \end{aligned}$$

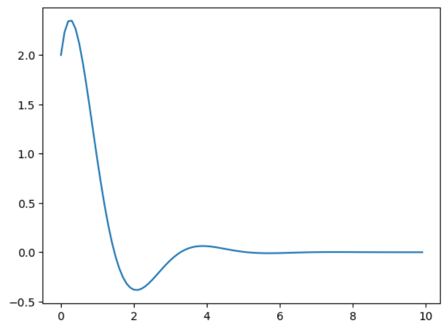
so $c_2 = 5/\sqrt{3}$.

Example

Therefore, the solution to the initial value problem is

$$y = 2e^{-t}\cos(\sqrt{3}t) + \frac{5}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t).$$

Oscillations with decaying amplitude as $t \to \infty$.



Repeated Roots of the Characteristic Equation

The quadratic $ar^2 + br + c = 0$ has a repeated root if and only if $b^2 - 4ac = 0$. In this case the repeated root is $r_1 = r_2 = -\frac{b}{2a}$. We have one solution $y_1 = e^{-\frac{bt}{2a}}$, but how do we find another linearly independent solution?

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Idea: We know that $ce^{-\frac{bt}{2a}}$ is a solution for any constant c, so maybe writing $y = v(t)e^{-\frac{bt}{2a}}$ will be helpful. In fact, this method called reduction of order works for any linear homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

as long as one nontrivial solution $y_1(t)$ is known.

Reduction of Order

Assume that $y_1(t)$ is a solution (not identically zero for all t) and write $y(t) = v(t)y_1(t)$. Then,

$$\begin{aligned} y' &= v'y_1 + vy_1' \\ y'' &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

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Plugging into the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

we obtain

$$\begin{aligned} (v''y_1 + 2v'y_1' + vy_1'') + p(t)(v'y_1 + vy_1') + q(t)vy_1 &= 0 \\ y_1v'' + (2y_1' + p(t)y_1)v' + (y_1'' + p(t)y_1' + q(t)y_1)v &= 0 \\ y_1v'' + (2y_1' + p(t)y_1)v' &= 0. \end{aligned}$$

Reduction of Order

The equation from the previous slide

$$y_1 v'' + (2y_1' + p(t)y_1)v' = 0$$

is actually a first-order differential equation in terms of the unknown function $w = v'$. We've reduced from second-order to first-order hence the name of the method.

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Denote $P(t) := \dfrac{2y_1' + p(t)y_1}{y_1}$. Then, this differential equation is

$$w' + P(t)w = 0,$$

which has the general solution $v' = w = c_2 e^{-\int P(t)\, dt}$. Thus,

$$v = c_2 \int e^{-\int P(t)\, dt}\, dt + c_1$$

$$y = vy_1 = c_2 y_1 \int e^{-\int P(t)\, dt}\, dt + c_1 y_1.$$

Repeated Roots Second Solution

Returning to the case $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$, we have $y_1 = e^{-\frac{bt}{2a}}$, $y_1' = -\frac{b}{2a}e^{-\frac{bt}{2a}}$, and $p(t) = \frac{b}{a}$, so

$$P(t) = \frac{2y_1' + p(t)y_1}{y_1} = \frac{2\left(-\frac{b}{2a}e^{-\frac{bt}{2a}}\right) + \frac{b}{a}e^{-\frac{bt}{2a}}}{e^{-\frac{bt}{2a}}} = 0.$$

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Thus,

$$\begin{aligned} v' &= c_2 e^{\int 0 \; dt} = c_2 \\ v &= \int c_2 \; dt = c_2 t + c_1 \\ y &= vy_1 = c_2 t e^{-\frac{bt}{2a}} + c_1 e^{-\frac{bt}{2a}}. \end{aligned}$$

The Wronskian determinant is

$$\begin{aligned} &W(e^{-\frac{bt}{2a}}, te^{-\frac{bt}{2a}}) \\ &\quad = e^{-\frac{bt}{2a}}\left(e^{-\frac{bt}{2a}} - \frac{bt}{2a}e^{-\frac{bt}{2a}}\right) - \left(-\frac{b}{2a}e^{-\frac{bt}{2a}}\right) te^{-\frac{bt}{2a}} \\ &\quad = e^{-\frac{bt}{a}}, \end{aligned}$$

which is never zero. Thus, the general solution is

$$y = c_1 e^{-\frac{bt}{2a}} + c_2 te^{-\frac{bt}{2a}}.$$

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The characteristic equation is $r^2 - 4r + 4 = (r-2)^2 = 0$, so $r_1 = r_2 = 2$ is a repeated root. This means that the general solution is

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$$y = c_1 e^{2t} + c_2 t e^{2t}.$$

Plugging in $t = 0$ yields $1 = y(0) = c_1$. The derivative is

$$y' = 2e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t}.$$

Plugging in $t = 0$, we obtain $1 = y'(0) = 2 + c_2$, so $c_2 = -1$. We conclude that the solution to the IVP is

$$y = e^{2t} - te^{2t}.$$

QUANTUM FORMALISM

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