The Postulates of Quantum Mechanics

Part II

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Example

EXPERIMENT: THE POLARIZATION OF LIGHT

- Let us start discussing a simple optical experiment whose subject is the polarization of light
- This will permit us to introduce the fundamental concepts which concern the measurement and evolution of physical quantities

EXPERIMENT

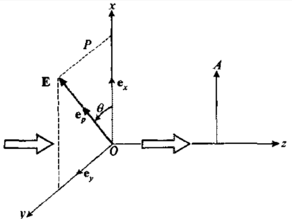

FIGURE 2

A simple measurement experiment relating to the polarization of a light wave. A beam of light propagates along the direction Oz and crosses successively the polarizer P and the analyzer A. θ is the angle between Ox and the electric field of the wave transmitted by P. The vibrations transmitted by A are parallel to Ox.

Experiment: The polarization of light

- Ψ) The experiment consists of directing a polarized plane monochromatic light wave onto an analyzer A
- Ψ) Oz designates the direction of propagation of this wave and $\hat{e}_p$, the unit vector describing its polarization
- Ψ) The analyzer A transmits light polarized parallel Ox and absorbs light polarized parallel to Oy
- Ψ) The classical description (an intense light beam) of this experiment is the polarized plane wave is characterized by an electric field of the form:

$$\vec{E}(\vec{r}, t) = E_0 \hat{e}_p e^{i(kz - wt)} \tag{1}$$

where E_0 is a constant and the light intensity I is proportional to $|E_0|^2$.

Experiment: The polarization of light

Ψ) After this passage through the analyzer A, the plane wave is polarized along $0x$:

$$\vec{E}'(\vec{r},t) = E_0'\hat{e}_x e^{i(kz-wt)} \tag{2}$$

and its intensity I' proportional to $|E_0'|^2$ is given by: $I' = I\cos^2\theta$.

QUANTUM LEVEL

Ψ) When I is weak enough for the photons to reach the analyzer one by one. What will happen?

Ψ) Let us place a photon detector behind this analyzer

Experiment: The polarization of light

Ψ) First of all, the detector never registers a fraction of a photon

Ψ) The photon crosses the analyzer or it is entirely absorbed by it

Ψ) Next, we can not predict with certainty whether a given incident photon will pass or be absorbed, we only know the corresponding probabilities

Ψ) Finally, if we send out a large number N of photons one after the other. The result will correspond to the classical law where about $N\cos^2\theta$ photons will be detected after the analyzer

Experiment: The polarization of light

We will retain the following ideas from this description:

Ψ) The measurement device can give certain results which we call eigen or proper results

Ψ) In this experiment there are only two possible results: (i) the photons cross the analyzer (ii) they are stopped;

Ψ) To each of these eigen results corresponds an eigenstate

Ψ) Here, the two eigenstates are characterized by $\hat{e}_p = \hat{e}_x$ or $\hat{e}_p = \hat{e}_y$

Ψ) If $\hat{e}_p = \hat{e}_x$, we know with certainty that the photon will traverse the analyzer, if $\hat{e}_p = \hat{e}_y$, it will be stopped

Ψ) The correspondence between eigen results and the eigenstates is therefore the following: if the particle is, before the measurement, in one of the eigenstates, the result of this measurement is the associated eigen result

EXPERIMENT: THE POLARIZATION OF LIGHT

- Ψ) When the state before the measurement is arbitrary, only probabilities of obtaining the different eigen results can be predicted
- Ψ) To find these probabilities, one decomposes the state of the particles into a linear combination of the various eigenstates
- Ψ) For an arbitrary $\hat{e}_p$

$$\hat{e}_p = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta \tag{3}$$

- Ψ) The probability of obtaining a given eigen result is proportional to the square of the absolute value of the coefficient of the corresponding eigenstate
- Ψ) The probability factor is determined by the condition that the sum of all these probabilities must be 1
- Ψ) From Eq. (3), each photon has probability $\cos^2\theta$ of traversing the analyzer, and $\sin^2\theta$ of being absorbed

Experiment: The polarization of light

This rule is called in Quantum Mechanics by:

THE PRINCIPLE OF SPECTRAL DECOMPOSITION

- Ψ) Note that, there has been an abrupt change in the state of the particles
- Ψ) Before the measurement, this state was defined by a vector $\vec{E}(\vec{r}, t)$, which was collinear to $\hat{e}_p$
- Ψ) After the measurement we possess additional information (the photon passed) which is incorporated by describing the state by a different vector, which is now collinear with $\hat{e}_x$
- Ψ) This expresses the fact that the measurement disturbs the microscopic system (here, the photon) in a fundamental way

Spectral Decomposition

THE PRINCIPLE OF SPECTRAL DECOMPOSITION: CASE OF DISCRETE SPECTRUM

- $\Psi)$ Consider a system whose state is characterized at a given time by the ket $|\psi\rangle$, assumed to be normalized to 1 ($\langle\psi|\psi\rangle = 1$)
- $\Psi)$ We want to predict the result of measuring a physical quantity $\mathcal{A}$ associated with the observable A. This prediction is of a probabilistic sort
- $\Psi)$ We are now going to give the rules that allow us to calculate the probability of obtaining any given eigenvalue of A

Let us assume that the spectrum of A is discrete

- $\Psi)$ If all the eigenvalues a_n of A are non-degenerate
- $\Psi)$ There is associated with each of them a unique (to within a constant factor) eigenvector $|u_n\rangle$

$$A|u_n\rangle = a_n|u_n\rangle \tag{4}$$

The Principle of Spectral Decomposition: Case of Discrete Spectrum

Ψ) Since A is an observable, the set of the $|u_n\rangle$ is normalized

Ψ) It constitutes a basis in $\mathcal{E}$, and the set $|\psi\rangle$ can be written

$$|\psi\rangle = \sum_n c_n |u_n\rangle \tag{5}$$

Ψ) We postulate that the probability $\mathcal{P}(a_n)$ of finding a_n when $\mathcal{A}$ is measured is:

$$\mathcal{P}(a_n) = |c_n|^2 = |\langle u_n|\psi\rangle|^2 \tag{6}$$

The Principle of Spectral Decomposition: Case of Discrete Spectrum

Fourth Postulate (case of the discrete non-degenerate spectrum): When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the corresponding observable A is:

$$\mathcal{P}(a_n) = |\langle u_n|\psi\rangle|^2$$

where $|u_n\rangle$ is the normalized eigenvector of A associated with the eigenvalue a_n

The Principle of Spectral Decomposition: Case of Discrete Spectrum

Ψ) If now, some of the eigenvalues a_n are degenerate, several orthonormalized eigenvectors $|u_n^i\rangle$ correspond to them: $A|u_n^i\rangle = a_n|u_n^i\rangle$, where $i = 1, 2, ..., g_n$, $|\psi\rangle$ can still be expanded in the orthonormal basis $\{|u_n^i\rangle\}$:

$$|\psi\rangle = \sum_n \sum_{i=1}^{g_n} c_n^i |u_n^i\rangle \tag{7}$$

Ψ) In this case, the probability $\mathcal{P}(a_n)$ becomes

$$\mathcal{P}(a_n) = \sum_{i=1}^{g_n} \left|c_n^i\right|^2 = \sum_{i=1}^{g_n} \left|\langle u_n^i|\psi\rangle\right|^2 \tag{8}$$

Ψ) Eq. (6) is then seen as a special case of Eq. (8), which can therefore be considered to be the general formula

The Principle of Spectral Decomposition: Case of Discrete Spectrum

Fourth Postulate (case of the discrete spectrum): When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining the eigenvalue a_n of the corresponding observable A is:

$$\mathcal{P}(a_n) = \sum_{i=1}^{g_n} \left| \langle u_n^i | \psi \rangle \right|^2$$

where g_n is the degree of degeneracy of a_n and $\{u_n^i\}(i = 1, 2, ..., g_n)$ is an orthonormal set of vectors which form a basis in the eigenspace $\mathcal{E}_n$ associated with the eigenvalue a_n of A.

The Principle of Spectral Decomposition: Case of Discrete Spectrum

Ψ) For this postulate to make sense, it is necessary that, if the eigenvalue a_n is degenerate, the probability $\mathcal{P}(a_n)$ must be independent of the choice of the $\{|u_n^i\rangle\}$ basis in $\mathcal{E}_n$

Ψ) To verify this, consider the vector:

$$|\psi_n\rangle = \sum_{i=1}^{g_n} c_n^i |u_n^i\rangle \tag{9}$$

where the coefficients c_n^i are the same as those appearing in the expansion of Eq. (7) of $|\psi\rangle$

THE PRINCIPLE OF SPECTRAL DECOMPOSITION: CASE OF DISCRETE SPECTRUM

- Ψ) $|\psi_n\rangle$ is part of $|\psi\rangle$ which belongs to $\mathcal{E}_n$, that is, the projection of $|\psi\rangle$ onto $\mathcal{E}_n : |\psi_n\rangle = P_n|\psi\rangle$
- Ψ) Where $P_n = \sum_i^{g_n} |u_n^i\rangle\langle u_n^i|$, is the projector onto $\mathcal{E}_n$
- Ψ) From this expression, it is clear that a change in the basis in $\mathcal{E}_n$ does not affect $\mathcal{P}(a_n)$
- Ψ) This probability is written

$$\mathcal{P}(a_n) = \langle\psi|P_n^\dagger P_n|\psi\rangle = \langle\psi|P_n|\psi\rangle \tag{10}$$

we used the fact ($P_n^\dagger = P_n$), P_n is Hermitian and that it is a projector ($P_n^2 = P_n$)

The Principle of Spectral Decomposition: Case of a Continuous Spectrum

- Ψ) Now let us assume that the spectrum of A is continuous and non-degenerate
- Ψ) The system, orthonormal in the extended sense, of eigenvectors $|v_\alpha\rangle$ of A

$$A|v_\alpha\rangle = \alpha|v_\alpha\rangle \tag{11}$$

forms a continuous basis in $\mathcal{E}$, in terms of which $|\psi\rangle$ can be expanded

$$|\psi\rangle = \int d\alpha(\alpha)|v_\alpha\rangle \tag{12}$$

since the possible results of a measurement of $\mathcal{A}$ form a continuous set

The Principle of Spectral Decomposition: Case of a Continuous Spectrum

- Ψ) We must define a probability density, just as we did for the interpretation of the wave function of a particle
- Ψ) Just remember, following the de Broglie hypothesis, we might apply the ideas of a particle wavelength: $\lambda = \frac{2\pi}{|\vec{k}|} = \frac{h}{|\vec{p}|}$. To all material particles
- Ψ) Thus we must substitute the classical concept of a trajectory by the quantum state of a particle such as an electron is characterized by a wave function $\psi(\vec{r}, t)$
- Ψ) Since the possible positions of the particles form a continuum, the probability $d\mathcal{P}(\vec{r}, t) = c|\psi(\vec{r}, t)|^2 d^3r$
- Ψ) In our case, the probability $d\mathcal{P}(\alpha)$ of obtaining a value included between α and $\alpha + d\alpha$ is given by: $d\mathcal{P}(\alpha) = \rho(\alpha)d\alpha$, with $\rho(\alpha) = |c(\alpha)|^2 = |\langle v_\alpha|\psi\rangle|^2$

The Principle of Spectral Decomposition: Case of a Continuous Spectrum

Fourth Postulate (Case of Continuous Non-Degenerate Spectrum):

When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $d\mathcal{P}(\alpha)$ of obtaining a result included between α and $\alpha + d\alpha$ is equal to:

$$d\mathcal{P}(\alpha) = |\langle v_\alpha|\psi\rangle|^2 d\alpha$$

where $|v_\alpha\rangle$ is the eigenvector corresponding to the eigenvalue α of the observable A associated with $\mathcal{A}$

The Principle of Spectral Decomposition: Case of a Continuous Spectrum

Homework: Since $|\psi\rangle$ is normalized, verify explicitly that each of the cases considered above has a total probability equal to 1.

- Ψ) An important consequence: consider two kets $|\psi\rangle$ and $|\psi'\rangle$ such that $|\psi'\rangle = e^{i\theta}|\psi\rangle$, where θ is a real number.
- Ψ) If $|\psi\rangle$ is normalized, so is $|\psi'\rangle$: $\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle = 1.$
- Ψ) The probabilities predicted for an arbitrary measurement are the same for $|\psi\rangle$ and $|\psi'\rangle$ since, for any $|u_n\rangle$: $|\langle u_n^i|\psi'\rangle|^2 = |e^{i\theta}\langle u_n^i|\psi\rangle|^2 = |\langle u_n^i|\psi\rangle|^2$
- Ψ) Similarly, we can replace $|\psi\rangle$ by: $|\psi''\rangle = \alpha e^{i\theta}|\psi\rangle$. without changing any of the physical results

The Principle of Spectral Decomposition: Case of a Continuous Spectrum

Ψ) There appear, in both, the numerator and denominator of an expression like

$$\mathcal{P}(a_n) = \frac{1}{\langle\psi|\psi\rangle} \sum_{i=1}^{g_n} \left|c_n^i\right|^2$$

or

$$\rho(\alpha) = \frac{1}{\langle\psi|\psi\rangle} \left|c(\alpha)\right|^2$$

the factors of $|\alpha|^2$ cancel each other.

The Principle of Spectral Decomposition: Case of a Continuous Spectrum

Ψ) Therefore, two proportional state vectors represent the same physical state

Ψ) Care must be taken to interpret this result correctly. For example, let us assume: $|\psi\rangle = \lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle$. Where λ_1 and λ_2 are complex numbers.

Ψ) It is true that $e^{i\theta_1}|\psi_1\rangle$ represents, for all θ_1, the same physical state as $|\psi_1\rangle$, and $e^{i\theta_2}|\psi_2\rangle$ represents the same physical state as $|\psi_2\rangle$. But, in general:

$$|\varphi\rangle = \lambda_1 e^{i\theta_1}|\psi_1\rangle + \lambda_2 e^{i\theta_2}|\psi_2\rangle \tag{13}$$

does not describe the same state as $|\psi\rangle$ since,

$$\mathcal{P}(a_n) = |\langle u_n|\varphi\rangle|^2 = |\lambda_1|^2\mathcal{P}_1(a_n) + |\lambda_2|^2\mathcal{P}_2(a_n) + 2Re\left\{\lambda_1\lambda_2^\star\langle u_n|\psi\rangle\langle u_n|\psi\rangle^\star\right\}.$$

This is not valid to the special case: $\theta_1 = \theta_2 + 2n\pi, \;\; n = 0, 1, ..N$ (Homework)

Wave Packet

Reduction of the Wave Packet

- Ψ) We have introduced the concept of wave packet reduction in the example of the measurement of the polarization of photons. We are now going to generalize it considering the case of a discrete spectrum
- Ψ) Assume that we want to measure, at a given time, the physical quantity $\mathcal{A}$
- Ψ) If the ket $|\psi\rangle$, which represents the state of the system immediately before the measurement is known
- Ψ) The fourth postulate allows us to predict the probabilities of obtaining the various possible results
- Ψ) But when the measurement is performed it is obvious that only one of these possible results is obtained
- Ψ) Immediately after this measurement, we can not speak about the probability of having obtained this to that value since We know which one was obtained

Reduction of the Wave Packet

- Ψ) Thus, with this additional information, it is understandable that the state of the system after the measurement should be different from $|\psi\rangle$
- Ψ) Let us first consider the case where the measurement of $\mathcal{A}$ yields a simple eigenvalue a_n of the observable A
- Ψ) We postulate that the state of the system immediately after this measurement is the eigenvector $|u_n\rangle$ associated with a_n: $|\psi\rangle_{(a_n)} \Longrightarrow |u_n\rangle$
- Ψ) When the eigenvalue a_n given by the measurement is degenerate, we can generalize this equation as follows

Reduction of the Wave Packet

Ψ) If the expansion of the state $|\psi\rangle$ immediately before the measurement is written: $|\psi\rangle = \sum_n \sum_i^{g_n} c_n^i |u_n^i\rangle$.

Ψ) The modification of the state vector due to the measurement is written: $|\psi\rangle_{(a_n)} \Longrightarrow \frac{1}{\sqrt{\sum_{i=1}^{g_n} |c_n^i|^2}} \sum_{i=1}^{g_n} c_n^i |u_n^i\rangle$

Ψ) Where $\sum_{i=1}^{g_n} |c_n^i|^2$ is the vector $|\psi_n\rangle$ defined in Eq. (9) (fourth postulate-discrete case) $\Rightarrow$ that is, the projection of $|\psi\rangle$ onto the eigensubspace associated with a_n

Ψ) Using the projector operator (P_n): $|\psi_n\rangle = P_n|\psi\rangle = \sum_{i=1}^{g_n} |u_n^i\rangle\langle u_n^i|\psi\rangle$, it is easy to show (homework) that

$$|\psi\rangle_{(a_n)} \Longrightarrow \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

we can now postulate

Reduction of the Wave Packet

If the measurement of the quantity $\mathcal{A}$ on the system in the state $|\psi\rangle$ give the result a_n, the state of the system immediately after the measurement is the normalized projector

$$\frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

of $|\psi\rangle$ onto the eigensubspace associated with a_n

Reduction of the Wave Packet

Ψ) The state of the system immediately after the measurement is therefore always an eigenvector of A with the eigenvalue a_n

Ψ) We stress the fact that it is not an arbitrary ket of the subspace $\mathcal{E}_n$, but part of the $|\psi\rangle$ which belongs to $\mathcal{E}_n$

Ψ) Eq. $|\psi\rangle_{(a_n)} \Longrightarrow |u_n\rangle$ can be seen as a special case of

$$|\psi\rangle_{(a_n)} \Longrightarrow \frac{1}{\sqrt{\sum_{i=1}^{g_n} |c_n^i|^2}} c_n^i |u_n^i\rangle$$

Ψ) When $g_n = 1$, the summation over i disappears and thus

$$\frac{1}{|c_n|} c_n |u_n\rangle = e^{iArg c_n} |u_n\rangle$$

this ket indeed describes the same physical state as $|u_n\rangle$

Time Evolution

Time Evolution of Systems

Ψ) We already presented the Schrödinger equation for a particle

Ψ) Now, we write it in the general case

Time Evolution of Systems

Sixth Postulate: The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

where $H(t)$ is called the Hamiltonian operator of the system and it is the observable associated with the total energy of the system.

Summary

THE POSTULATES OF QUANTUM MECHANICS

- Ψ) Any state of a quantum system at time t_0 is characterized by a vector (ket) $|\psi(t_0)\rangle$ that belongs to the Hilbert space. This vector in Hilbert space describes completely the physical state of the system. Everything that can be said about the system is contained in $|\psi(t_0)\rangle$
- Ψ) Every measurable physical quantity is described by a self-adjoint operator (called observable) acting in the Hilbert space of the system
- Ψ) The only possible result of a measurement of a physical quantity is one of the eigenvalues of the self-adjoint operator associated with it

The Postulates of Quantum Mechanics

Ψ) The probability of finding one of the eigenvalues (for example, a_n) associated with the observed quantity is given by (in the discrete case):

$$\mathcal{P}(a_n) = \sum_{i=1}^{g_n} |\langle u_n^i|\psi\rangle|^2 = \langle\psi|P_n|\psi\rangle$$

where P_n is the projector over the eigensubspace of the Hilbert space with eigenvalue a_n. $|u_n^i\rangle$ is one of the eigenstates with this eigenvalue with degeneracy g_n.

Ψ) After a measurement generating the eigenvalue a_n, the state of the system collapses to

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

The Postulates of Quantum Mechanics

Ψ) The evolution of the state vector while no experiments are carried out is governed by the Schrödinger equation:

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

where $H(t)$ is the Hamiltonian operator of the system.
In the Heisenberg description, the observable A varies with time using the equation

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A]$$

while the quantum state remains constant.