quantumformalism.com

Zaiku Group
Quantum Formalism

Mathematical Tools of Quantum Mechanics: Introduction to Hilbert Spaces

Dr. M.Eugenia Celorrio Ramirez

Vector spaces

Let $\mathbb{F}$ be a field and V be a set. Consider the following two operations:

$$
\begin{aligned}
+ : V \times V &\to V, & \cdot : \mathbb{F} \times V &\to V \\
(u, v) &\mapsto u + v & (\lambda, v) &\mapsto \lambda \cdot v.
\end{aligned}
$$

Suppose that V satisfies the following:

- $u + v = v + u$, for every $u, v \in V$;
- $u + (v + w) = (u + v) + w$, for every $u, v, w \in V$;
- there exists a zero vector, 0_V such that $0_V + v = v$, for every $v \in V$;
- for every $v \in V$, there exists a vector $-v$ such that $-v + v = 0_V$;
- $1 \cdot v = v$, for every $v \in V$;
- $\lambda \cdot (\mu v) = (\lambda\mu) \cdot v$, for every $v \in V$, $\lambda \in \mathbb{F}$;
- $\lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v$, for every $u, v \in V$, $\lambda, \mu \in \mathbb{F}$;
- $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$, for every $v \in V$, $\lambda, \mu \in \mathbb{F}$.

Definition

Let V be a set that satisfies all of the above. Then we say that V is a $\mathbb{F}$*-vector space* or a *vector space over the field* $\mathbb{F}$

We will focus in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Examples

The following are $\mathbb{R}$-vector spaces

- The set of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.
- The set of $M_n(\mathbb{R})$, square $n \times n$ matrices with entries on $\mathbb{R}$, with the usual addition and scalar multiplication.
- The set of all functions from $\mathbb{R}$ to $\mathbb{R}$ where the addition is

  $$(f+g)(x) = f(x) + g(x) \quad (f, g : \mathbb{R} \to \mathbb{R})$$

  and the scalar multiplication is

  $$(\lambda f)(x) = \lambda f(x) \quad (\lambda \in \mathbb{F}, f : \mathbb{R} \to \mathbb{R})$$

Linear subspaces

Definition

Let V be a vector space and let $W \subset V$ be a subset. We say that W is a subspace of V if it satisfies the following:

- $0_V \in W$;
- for every $\lambda \in \mathbb{F}$ and $w \in W$, $\lambda \cdot w \in W$;
- for very $w, v \in W$, $w + v \in W$,

where $+$ and $\cdot$ are the restriction of the addition and scalar multiplication from V to W.

Inner products

Definition

Let V be a vector space over $\mathbb{C}$. A mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is an *inner product* if it satisfies the following, for every $x, y, z \in V$ and $\lambda, \mu \in \mathbb{C}$:

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;
- $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$.

Inner products are conjugate linear in the second variable

Take $x, y \in V$ and $\alpha \in \mathbb{C}$
$\langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle} = \overline{\alpha \langle y, x \rangle} = \overline{\alpha}\,\overline{\langle y, x \rangle} = \overline{\alpha} \langle x, y \rangle$. $\square$

Remark

A vector space with an inner product is called an *inner product space* or a *pre-Hilbert space.*

Norms and normed spaces

Definition

Let E be a vector space. A *norm* is a mapping

$$\|\cdot\| : E \to \mathbb{R}$$

that satisfies the following conditions:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$;
- $\|\lambda x\| = |\lambda|\|x\|$, for every $x \in E$ and $\lambda \in \mathbb{C}$;
- $\|x + y\| \leq \|x\| + \|y\|$, for every $x, y \in E$.

A normed space is a vector space together with a norm.

Remark

Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Consider $\|x\| = \sqrt{\langle x, x \rangle}$. Then $(E, \|\cdot\|)$ is a normed space.

Proof

We need to verify:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$. Follows from the definition of inner product.
- $\|\lambda x\| = |\lambda| \|x\|$, for every $x \in E$ and $\lambda \in \mathbb{C}$.

$$\|\lambda x\|^2 = \langle \lambda x, \lambda x \rangle = \lambda \bar{\lambda} \langle x, x \rangle$$

- $\|x+y\| \leq \|x\| + \|y\|$, for every $x, y \in E$.

Theorem: Cauchy-Schwarz's Inequality

Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space. The, for any $x, y \in E$ we have

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Furthermore, $|\langle x, y\rangle| = \|x\|\|y\|$ if and only if x and y are linearly dependent.

Proof

If x and y are linearly dependent, there exists $\alpha \in \mathbb{C}$ such that $x = \alpha y$, and so $\|x\|\|y\| = |\alpha|\|y\|^2$. Also

$$|\langle x, y\rangle| = |\langle \alpha y, y\rangle = |\alpha||\langle y, y\rangle| = |\alpha|\|y\|^2,$$

so the result holds.

$$|\langle x, y\rangle| \leq \|x\|\|y\| \iff |\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y, \rangle.$$

Let's consider x, y linearly independent. (In particular $y \neq 0$). Then, for any $\alpha \in \mathbb{C}$,

$$\begin{aligned} 0 &< \langle x + \alpha y, x + \alpha y\rangle = \langle x, x\rangle + \alpha\langle y, x\rangle + \overline{\alpha}\langle x, y\rangle + |\alpha|^2\langle y, y, \rangle \\ &= \langle x, x\rangle + 2Re(\alpha\langle y, x\rangle) + |\alpha|^2\langle y, y\rangle \end{aligned}$$

Take $\alpha = -\frac{\langle x,y\rangle}{\langle y,y\rangle}$

$$0 < \langle x, x\rangle - 2Re\left(\frac{\langle x, y\rangle}{\langle y, y\rangle}\overline{\langle x, y\rangle}\right) + \left|\frac{-\langle x, y\rangle}{\langle y, y\rangle}\right|^2 \langle y, y\rangle$$
$$= \frac{\langle x, x\rangle\langle y, y\rangle}{\langle y, y\rangle} - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle}$$

which is equivalent to $|\langle x, y\rangle|^2 < \langle x, x\rangle\langle y, y\rangle = \|x\|\|y\|$. $\square$

- $\|x+y\| \leq \|x\| + \|y\|$, for every $x, y \in E$.
  For $x, y \in E$

$$\begin{aligned}
\|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x, x\rangle + 2Re(\langle y, x\rangle) + \langle y, y\rangle \\
&\leq \langle x, x\rangle + 2|\langle y, x\rangle| + \langle y, y\rangle \\
&\leq \|x\| + 2\|y\|\|x\| + \|y\| = (\|x\| + \|y\|)^2. \quad \square
\end{aligned}$$

Parallelogram law

Theorem: Parallelogram Law

Let E be an inner product space. Then for $x, y \in E$

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Convergent sequences

Reminder

Let $A \neq \emptyset$ be a set. A sequence is a list of objects of A with a defined order $(a_1, a_2, \cdots)$. Mathematically, it is defined as a map $f : \mathbb{N} \to A$, where $f(n) = a_n$.

Definition

A sequence (x_n) in a normed space E is said to be convergent to a limit $x \in E$ if, for every $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$, the following inequality holds:

$$\|x_n - x\| < \varepsilon.$$

Cauchy sequences

Definition

A sequence (x_n) in a normed space E is said to be a Cauchy sequence if, for every $\varepsilon > 0$, there exists a positive integer N such that for all $m, n \geq N$, the following inequality holds:

$$\|x_m - x_n\| < \varepsilon.$$

Examples

Consider $\mathbb{Q}$ with the norm $|\cdot|$ and the sequence (x_n) where $x_n = 10^{-n}\lfloor 10^n \pi \rfloor$. Then (x_n) is Cauchy but it is nor convergent. It would be convergent in $\mathbb{R}$.

Completeness

Definition

A normed space E is said complete when for every Cauchy sequence (x_n) there exists an element $x \in E$ such that $(x_n) \to x$. A complete normed space is called a Banach space.

Definition: Hilbert spaces

A complete inner product space is called a *Hilbert space.*

Recap

A Hilbert space $\mathcal{H}$:

- is a vector space over a field $\mathbb{F}(=\mathbb{R},\mathbb{C})$;
- it has an inner product $\langle\cdot,\cdot\rangle : H \times H \mapsto \mathbb{C}$;
- it is a normed space with the norm induced by the inner product $\|x\| = \sqrt{\langle x, x\rangle}$;
- it is complete.

So, every Hilbert space is a Banach space, but not the other way around.

Characterization of Hilbert Spaces

Theorem

Let $(E, \|\cdot\|)$ be a Banach space. Then E is a Hilbert space (i.e. there exists an inner product on E that induces the norm $\|\cdot\|$) if and only if $\|\cdot\|$ satisfies the parallelogram law. In this case

$$\langle x, y\rangle = \frac{1}{4}\sum_{n=0}^{3} i^n \|x + i^n y\|^2 \quad (x, y \in E)$$

defines such inner product on E.

Examples of Hilbert spaces

- $\mathbb{R}^n$ with the standard dot product (inner product) defined in the following way: for $x, y \in \mathbb{R}^n$

$$\langle x, y \rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

- $\mathbb{C}^n$ with the standard dot product (inner product) defined in the following way: for $x, y \in \mathbb{C}^n$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

where $\overline{y_i}$ is the complex conjugate of y_i.

- The space $\ell^2(\mathbb{C})$ consists of sequences of complex numbers $(x_1, x_2, x_3, \ldots)$ which are square sumable, meaning that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty,$$

with the inner product given by:

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

where $\overline{y_i}$ represents the complex conjugate of y_i.
- The space $L^2(\mathbb{R})$, which is the space of Lebegue measurable functions $f : \mathbb{R} \to \mathbb{C}$ which are square-integrable. The inner product in this space is given by:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

quantumformalism.com

Dr M.Eugenia Celorrio Ramirez
Quantum Formalism

Thank you for your attention

Recommended Bibliography:
Megginson, R.E., An introduction to Banach space theory
Young, N., An introduction to Hilbert space
Allan, G.R. and Dales, H.G., Introduction to Banach spaces and algebras