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Zaiku Group
Quantum Formalism

Mathematical Tools of Quantum Mechanics: Introduction to Hilbert Spaces

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(a) Identity - $I \in \mathcal{B}(\mathcal{H}) : x \mapsto x$ for all $x \in \mathcal{H}$. For any $T \in \mathcal{B}(\mathcal{H}; \mathcal{J})$, and any $S \in \mathcal{B}(\mathcal{J}; \mathcal{H})$, we have $IS = S,\ TI = T$.
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(b) Hermitian - $T \in \mathcal{B}(\mathcal{H}) : T^* = T$; These operators are also known as *self-adjoint*. Examples: we always have that I and 0 are hermitian. The spectrum of a Hermitian operator is always a subset of $\mathbb{R}$. Examples: The standard Pauli operations are hermitian.

(c) Isometry - $T \in \mathcal{B}(\mathcal{H}; \mathcal{K}) : \|Tx\|_{\mathcal{K}} = \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$. Examples: rotations and reflections in $\mathbb{R}^2$ are always isometric.
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In quantum computing, the set of unitary operators on the state space $\mathbb{C}^{2^n}$ form the possible operations applicable to the n qubits of the system.
The set of all unitary operators on a Hilbert space forms a *group* with operator composition and identity I (*exercise (medium) : prove this!*).
The spectrum of a unitary operator is always a subset of the unit circle of $\mathbb{C}$, i.e.

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Finite-Rank - $T \in \mathcal{B}(\mathcal{H};\mathcal{K})$ with finite-dimensional image. Such operators are always expressible as linear combinations of rank-one operators.

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(i) Projection - Recall that for every closed subspace $\mathcal{L}$ of $\mathcal{H}$, there exists another subspace $\mathcal{L}^{\perp}$ for which $\mathcal{L} \oplus \mathcal{L}^{\perp} = \mathcal{H}$. The projection $P_{\mathcal{L}} \in \mathcal{B}(\mathcal{H})$ is the function which maps a sum $l + l' \mapsto l$ for every $l \in \mathcal{L}$ and $l' \in \mathcal{L}^{\perp}$. Projections are *idempotent*, i.e., $P_{\mathcal{L}}P_{\mathcal{L}} = P_{\mathcal{L}}$.
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(j) Tensor Product -If $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, we write $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ for the operator which acts on pure tensors $h \otimes k$ for $h \in \mathcal{H}$ and $k \in \mathcal{K}$ by $h \otimes k \mapsto Th \otimes Tk$, and extend by linearity. Notice that not every bounded operator on $\mathcal{H} \otimes \mathcal{K}$ is formed in this way. (Intuition for QIS people : Not every gate on a n-qubit system can be deconstructed using only 1-qubit gates, else we would have no entangling gates.)

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(*Exercises - (medium) : Show that if T is compact, then so is STR for every $S \in \mathcal{B}(\mathcal{K}; \mathcal{J})$ and every $R \in \mathcal{B}(\mathcal{L}; \mathcal{H})$. Next, show that if T is compact, then so is T^*.*)Every finite-rank operator is compact.

(i) Projection - Recall that for every closed subspace $\mathcal{L}$ of $\mathcal{H}$, there exists another subspace $\mathcal{L}^{\perp}$ for which $\mathcal{L} \oplus \mathcal{L}^{\perp} = \mathcal{H}$. The projection $P_{\mathcal{L}} \in \mathcal{B}(\mathcal{H})$ is the function which maps a sum $l + l' \mapsto l$ for every $l \in \mathcal{L}$ and $l' \in \mathcal{L}^{\perp}$. Projections are *idempotent*, i.e., $P_{\mathcal{L}} P_{\mathcal{L}} = P_{\mathcal{L}}$.
(*Exercises - (easy) : Show that $P_{\mathcal{L}} P_{\mathcal{L}^{\perp}} = 0$. Show that $\|P_{\mathcal{L}}\| = 1$ if* $\{0\} \subsetneq \mathcal{L} \subset \mathcal{H}$.)

(j) Tensor Product -If $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$, we write $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ for the operator which acts on pure tensors $h \otimes k$ for $h \in \mathcal{H}$ and $k \in \mathcal{K}$ by $h \otimes k \mapsto Th \otimes Tk$, and extend by linearity. Notice that not every bounded operator on $\mathcal{H} \otimes \mathcal{K}$ is formed in this way. (Intuition for QIS people : Not every gate on a n-qubit system can be deconstructed using only 1-qubit gates, else we would have no entangling gates.)

Thank you for listening!