First Order Differential Equations

Lecture 1

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Terminology

Ordinary Differential Equation (ODE): unknown function is of a single-variable (these lectures)

e.g. $\frac{dv}{dt} = g - \frac{\gamma}{m} v$

Partial Differential Equation (PDE): unknown function is of two or more variables e.g. Schrödinger equation, wave equation

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The order of a differential equation is the order of the highest derivative that appears.

Linear ODEs

Writing an ODE as $F(t, y, y', \ldots, y^{(n)}) = 0$, it is linear if F is a linear function of $y, y', \ldots, y^{(n)}$ (not necessarily t).

Examples

$\dfrac{dv}{dt} = g - \dfrac{\gamma}{m} v$ (first order, linear)

$y''' + 2e^t y'' + y' = t^4$ (third order, linear)

$\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L} \sin\theta = 0$ (second order, non-linear)

Big Questions

1. Do solutions exist?
2. Are they unique? How many conditions are needed for a unique solution?
3. Can we find the solution(s)?

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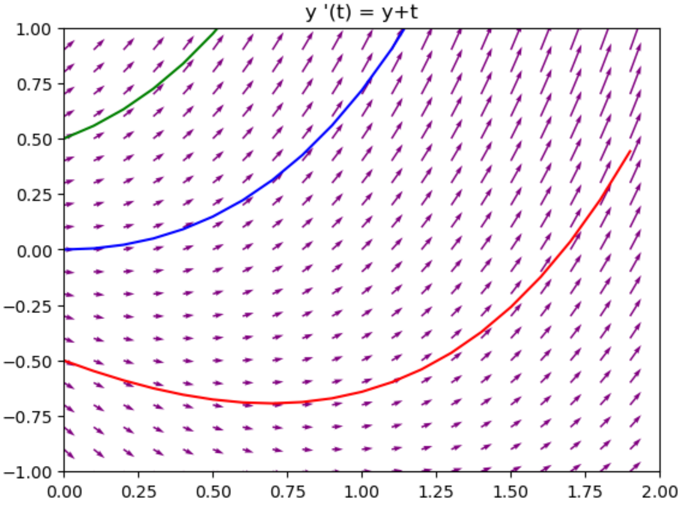
Techniques

1. Grab bag of tricks for some simpler ODEs.
2. Linearization and other techniques to approximate ODEs with the above.
3. Computers!

We'll first study equations of the form $\dfrac{dy}{dt} = f(t, y)$.



Separable Equations

Easy example: $\dfrac{dy}{dt} = f(t)$

Integrate both sides to find the solution $y(t) = \displaystyle\int f(t)dt + C$.

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Slightly harder example: $\dfrac{dy}{dt} = f(t)g(y)$

An equation of this form is called separable. To solve, we divide by $g(y)$ and then integrate both sides.

$$\begin{aligned} \frac{1}{g(y)}\frac{dy}{dt} &= f(t) \\ \int \frac{1}{g(y)}\frac{dy}{dt}\, dt &= \int f(t)\, dt \\ \int \frac{1}{g(y)} dy \ &= \int f(t)\, dt \end{aligned}$$

Example

Solve $\frac{dy}{dt} = ry$ (continuously compounded interest)

$$\begin{aligned}
\frac{dy}{dt} &= ry \\
\frac{1}{y}\frac{dy}{dt} &= r \\
\int \frac{1}{y}\,dy &= \int r\,dt \\
\ln|y| &= rt + C \\
y &= Ce^{rt}
\end{aligned}$$

$$P(t)\frac{dy}{dt} + Q(t)y = R(t)$$

If $P(t) \neq 0$, then we can divide by $P(t)$ to put the equation into standard form

$$\frac{dy}{dt} + p(t)y = q(t)$$

Idea to solve: multiply the entire equation by some function $\mu(t)$ so that left-hand side is a product rule expression $(fg)' = fg' + f'g$. The function $\mu(t)$ is called an integrating factor, because it allows us to solve the equation by integrating both sides.

First Order Linear ODEs

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)q(t)$$

We want the left-hand side to equal $\dfrac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)\dfrac{dy}{dt} + \dfrac{d\mu}{dt}y$, so we need $\frac{d\mu}{dt} = \mu(t)p(t)$

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This is a separable equation, so we can solve as before.

$$\begin{aligned}
\frac{1}{\mu}\frac{d\mu}{dt} &= p(t) \\
\int \frac{1}{\mu}\,d\mu &= \int p(t)\,dt \\
\ln|\mu| &= \int p(t)\,dt \\
\mu(t) &= e^{\int p(t)\,dt}
\end{aligned}$$

Example

Solve the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2$$

Convert to standard form $y' + \frac{2}{t}y = 4t$ so $p(t) = 2/t$.

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$$\mu(t) = e^{\int p(t)\ dt}$$
$$\mu(t) = e^{\int \frac{2}{t}\ dt}$$
$$\mu(t) = e^{2\ln|t|} = t^2$$

Multiplying the standard form equation by $\mu(t) = t^2$, we obtain

$$t^2y' + 2ty = 4t^3.$$

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Solve the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2$$

From the previous slide

$$\begin{aligned} t^2y' + 2ty &= 4t^3 \\ \int (t^2y)'\,dt &= \int 4t^3\,dt \\ t^2y &= t^4 + C \\ y &= t^2 + \frac{C}{t^2}. \end{aligned}$$

First Order Linear ODEs

Example

Solve the initial value problem

$$ty' + 2y = 4t^2, \quad y(1) = 2$$

The general solution is $y = t^2 + \frac{C}{t^2}$. To solve the initial value problem, we plug in $t = 1$ and solve for C:

$$2 = y(1) = 1 + C$$

so $C = 1$ and $y = t^2 + \frac{1}{t^2}$ is the solution to the IVP (only valid for $t > 0$).