Nonhomogeneous Second-order Linear Differential Equations

Lecture 5

Thomas Silverman

Theorem (Nonhomogeneous Superposition Principle)

Suppose that Y_1 and Y_2 are both solutions to the nonhomogeneous differential equation

$$y'' + p(t)y' + q(t)y = g(t).$$

Then, $Y_1 - Y_2$ is a solution to the complementary homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

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Proof.

Set $y = Y_1 - Y_2$. We have $y' = Y_1' - Y_2'$ and $y'' = Y_1'' - Y_2''$. Plugging into the left-hand side of the differential equation, we obtain

$$\begin{aligned}(Y_1'' - Y_2'') &+ p(t)(Y_1' - Y_2') + q(t)(Y_1 - Y_2) \\ &= (Y_1'' + p(t)Y_1' + q(t)Y_1) - (Y_2'' + p(t)Y_2' + q(t)Y_2) \\ &= g(t) - g(t) = 0.\end{aligned}$$

General Nonhomogeneous Solution

The Nonhomogeneous Superposition Principle implies that the general solution to

$$y'' + p(t)y' + q(t)y = g(t).$$

has the form $y = c_1 y_1 + c_2 y_2 + Y$, where $y_c = c_1 y_1 + c_2 y_2$ is the general solution to the complementary homogeneous equation and Y is any particular solution to the nonhomogeneous equation.

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In the constant coefficient case, we know how to find y_c, so all that remains is to find a particular solution Y. We will discuss two methods:

1. Undetermined Coefficients
2. Variation of Parameters.

Example 1

Find a particular solution to

$$y'' + 2y' + y = 2e^{-3t}.$$

Undetermined Coefficients

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Find a particular solution to

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Guess $Y = Ae^{-3t}$. Then, $Y' = -3Ae^{-3t}$ and $Y'' = 9Ae^{-3t}$. Plugging into the differential equation, we obtain

$$\begin{aligned}(9A - 6A + A)e^{-3t} &= 2e^{-3t}\\ 4A &= 2\\ A &= 1/2.\end{aligned}$$

Thus, $Y = \frac{1}{2}e^{-3t}$ is a particular solution.

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Following the previous example, we might guess $Y = A\cos(2t)$. Then, $Y' = -2A\sin(2t)$ and $Y'' = -4A\cos(2t)$. Plugging into the differential equation, we obtain

$$-4A\cos(2t) - 4A\sin(2t) + A\cos(2t) = 5\cos(2t).$$

Because of the $\sin(2t)$ term on the left-hand side, this equation has no solution.

Example 2

Find a particular solution to

$$y'' + 2y' + y = 5\cos(2t).$$

To fix our guess, instead try $Y = A\cos(2t) + B\sin(2t)$. Then, $Y' = -2A\sin(2t) + 2B\cos(2t)$ and $Y'' = -4A\cos(2t) - 4B\sin(2t)$. Plugging into the differential equation, we obtain

$$(-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) = 5\cos(2t).$$

This gives the system of two linear equations

$$\begin{aligned} -3A + 4B &= 5 \\ -4A - 3B &= 0. \end{aligned}$$

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$$\begin{aligned} -3A + 4B &= 5 \\ -4A - 3B &= 0. \end{aligned}$$

Multiplying the first equation by 3 and the second equation by 4 and adding yields $-25A = 15$ or $A = -3/5$.
Plugging into the second equation, we then obtain $\frac{12}{5} - 3B = 0$ or $B = 4/5$.

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We conclude that $Y = -\dfrac{3}{5}\cos(2t) + \dfrac{4}{5}\sin(2t)$ is a particular solution.

Example 3

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Guess $Y = At^2$. Then, $Y' = 2At$ and $Y'' = 2A$. Plugging into the differential equation, we obtain

$$2A + 4At + At^2 = t^2.$$

Because of the t^1 and t^0 terms on the left-hand side, this equation has no solution.

Example 3

Find a particular solution to

$$y'' + 2y' + y = t^2.$$

This time to fix our guess, we can try $Y = At^2 + Bt + C$. Then, $Y' = 2At + B$ and $Y'' = 2A$. Plugging into the differential equation, we obtain

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$$2A + (4At + 2B) + (At^2 + Bt + C) = t^2.$$

Collecting various powers of t yields the system of linear equations

$$\begin{aligned} 2A + 2B + C &= 0 \\ 4A + B &= 0 \\ A &= 1. \end{aligned}$$

The value of $A = 1$ is clear, the second equation shows $B = -4$, and finally the first equation yields $C = -2A - 2B = 6$. This means that $Y = t^2 - 4t + 6$ is a particular solution.

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Our usual superposition calculations show that if Y_1 solves

$$ay'' + by' + cy = g_1(t)$$

and Y_2 solves

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then $Y_1 + Y_2$ solves

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Applying this principle to the solutions to Examples 2 and 3, we obtain the particular solution $Y = -\dfrac{3}{5}\cos(2t) + \dfrac{4}{5}\sin(2t) + t^2 - 4t + 6.$

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Guess $Y = Ae^{-t}$. Then, $Y' = -Ae^{-t}$ and $Y'' = Ae^{-t}$. Plugging into the differential equation, we obtain

$$(A - 2A + A)e^{-t} = e^{-t}.$$

This equation has no solution.

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$$(A - 2A + A)e^{-t} = e^{-t}.$$

This equation has no solution.

To fix our guess, we might try $Y = Ate^{-t}$ instead. Then, $Y' = A(-t + 1)e^{-t}$ and $Y'' = A(t - 2)e^{-t}$. Plugging into the differential equation, we obtain

$$(A - 2A + A)te^{-t} + (-2A + 2A)e^{-t} = e^{-t}.$$

This still doesn't work.

Example 5

Find a particular solution to

$$y'' + 2y' + y = e^{-t}.$$

Looking at the solution to the complementary homogeneous differential equation, we can see the problem. The characteristic equation is $r^2 + 2r + 1 = (r+1)^2 = 0$, so $r_1 = r_2 = -1$ is a repeated root and the general solution is

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$$y_c = c_1 e^{-t} + c_2 t e^{-t}.$$

This is why our guesses were giving zero when we tried to plug them into the left-hand side of the differential equation. To fix the problem, we just need to multiply by t one more time to obtain an expression $Y = At^2 e^{-t}$ that does not appear as a term in the complementary homogeneous solution.

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Guess $Y = At^2e^{-t}$. Then, $Y' = A(-t^2 + 2t)e^{-t}$ and $Y'' = A(t^2 - 4t + 2)e^{-t}$. Plugging into the differential equation, we obtain

$$(At^2 - 2At^2 + At^2)e^{-t} + (-4At + 4At)e^{-t} + 2Ae^{-t} = e^{-t}.$$

This reduces to $2Ae^{-t} = e^{-t}$, so $A = 1/2$.

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This reduces to $2Ae^{-t} = e^{-t}$, so $A = 1/2$.

We obtain the particular solution $Y = \dfrac{1}{2}t^2e^{-t}$ and the general solution

$$y = c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}.$$

Table of Guesses

$g(t)$	$Y(t)$
$P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_0$	$t^s \left(A_n t^n + A_{n-1} t^{n-1} + \ldots + A_0\right)$
$P_n(t) e^{\alpha t}$	$t^s \left(A_n t^n + \ldots + A_0\right) e^{\alpha t}$
$P_n(t) e^{\alpha t} \cos(\beta t)$ or $P_n(t) e^{\alpha t} \sin(\beta t)$	$t^s \left(A_n t^n + \ldots + A_0\right) e^{\alpha t} \cos(\beta t)$ $+ t^s \left(B_n t^n + \ldots + B_0\right) e^{\alpha t} \sin(\beta t)$

Here s is the minimal nonnegative integer such that no term in Y solves the complementary homogeneous equation.

Variation of Parameters

This method works for any second-order linear differential equation

$$y'' + p(t)y' + q(t) = g(t).$$

Assume the solution to the complementary homogeneous equation $y_c = c_1 y_1 + c_2 y_2$ is known. Just as in the method of reduction of order from the previous lecture, we will try replacing the constants c_1, c_2 with functions $u_1(t), u_2(t)$, i.e. “varying the parameters”.

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$$y = u_1(t)y_1 + u_2(t)y_2$$
$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Since we only have one equation and two unknowns u_1, u_2, it is reasonable to assume that we can impose one more condition and still find a particular solution.

Variation of Parameters

We'll impose the condition $u_1' y_1 + u_2' y_2 = 0$ to make the formula for y', and hence the formula for y'', simpler. Then, we have

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ y' &= u_1 y_1' + u_2 y_2' \\ y'' &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''. \end{aligned}$$

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Plugging in to the differential equation $y'' + p(t)y' + q(t) = g(t)$, we obtain

$$\begin{aligned} g(t) =& (u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(t)(u_1y_1' + u_2y_2') \\ &+ q(t)(u_1y_1 + u_2y_2) \\ g(t) =& u_1(y_1'' + p(t)y_1' + q(t)y_1) + u_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &+ u_1'y_1' + u_2'y_2'. \end{aligned}$$

Variation of Parameters

From the previous slide, we obtain the following system of equations (using that y_1 and y_2 solve the complementary homogeneous equation for the second equation)

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$$

or in matrix form

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

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Using the inverse matrix, we compute

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(y_1, y_2)(t)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

The matrix equation from the previous slide gives the following equations for u_1' and u_2'

$$u_1' = -\frac{y_2 g(t)}{W(y_1, y_2)(t)}$$
$$u_2' = \frac{y_1 g(t)}{W(y_1, y_2)(t)}.$$

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Integrating, we obtain

$$\begin{aligned} u_1 &= -\int \frac{y_2(t) g(t)}{W(y_1, y_2)(t)}\, dt + c_1 \\ u_2 &= \int \frac{y_1(t) g(t)}{W(y_1, y_2)(t)}\, dt + c_2. \end{aligned}$$

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\end{aligned}
$$

The solution to the differential equation $y = u_1 y_1 + u_2 y_2$ is therefore

$$
y = c_1 y_1 + c_2 y_2 - y_1 \int \frac{y_2(t) g(t)}{W(y_1, y_2)(t)}\, dt + y_2 \int \frac{y_1(t) g(t)}{W(y_1, y_2)(t)}\, dt.
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Example 6

Solve the following differential equation

$$y'' + y = \sec t.$$

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The function $g(t) = \sec t$ does not appear in our table of guesses for the method of undetermined coefficients, so we'll need to use variation of parameters. First, we solve the complementary homogeneous differential equation $y'' + y = 0$. The characteristic equation is $r^2 + 1 = 0$, so the roots are $r_1, r_2 = \pm i$. This yields a complementary solution of $y_c = c_1 \cos t + c_2 \sin t$.

Example 6

Solve the following differential equation

$$y'' + y = \sec t.$$

Plugging $y_1 = \cos t$, $y_1' = -\sin t$, $y_2 = \sin t$, $y_2' = \cos t$, and $g(t) = \sec t$ into the earlier system of equations

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t), \end{aligned}$$

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we obtain

$$\begin{aligned} u_1' \cos t + u_2' \sin t &= 0 \\ -u_1' \sin t + u_2' \cos t &= \sec t. \end{aligned}$$

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Multiplying the first equation by $\sin t$ and the second by $\cos t$ and adding, we obtain

$$\begin{aligned} u_2'(\sin^2 t + \cos^2 t) &= \cos t \sec t \\ u_2' &= 1 \\ u_2 &= t + c_2. \end{aligned}$$

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Then, plugging $u_2' = 1$ into the first equation $u_1' \cos t + u_2' \sin t = 0$, we obtain

$$\begin{aligned} u_1' \cos t + \sin t &= 0 \\ u_1' &= -\tan t \\ u_1 &= \ln |\cos t| + c_1. \end{aligned}$$

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$$\begin{aligned} u_1' \cos t + \sin t &= 0 \\ u_1' &= -\tan t \\ u_1 &= \ln|\cos t| + c_1. \end{aligned}$$

We conclude that the solution to the differential equation is

$$y = c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t.$$

This would have been very difficult to guess.

Comparison of Methods

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   - Only works for certain functions $g(t)$.
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   - Only works in the constant coefficient case.
   - Only works for certain functions $g(t)$.
   - Computations only involve linear algebra.
2. Variation of Parameters
   - Works for any second-order linear differential equation.
   - Works for any function $g(t)$.
   - Computations involve taking integrals.

QUANTUM FORMALISM

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