Second-Order Linear Homogeneous Equations and the Wronskian

Lecture 3

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Second-Order Equations

Nonlinear equations are too difficult to solve. Second-order linear equations can be placed in the standard form

$$y'' + p(t)y' + q(t)y = g(t).$$

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The easiest to solve are linear homogeneous equations with constant coefficients

$$ay'' + by' + cy = 0,$$

where a, b, c are real constants.

Example

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Will something similar work for general a, b, c?

Characteristic Equation

Guess $y = e^{rt}$ as a solution to

$$ay'' + by' + cy = 0.$$

Plugging in, we obtain $(ar^2 + br + c)e^{rt} = 0$, so r is a root of

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This quadratic is called the characteristic equation associated to the ODE.

If r_1, r_2 are real distinct roots of the characteristic equation, then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution to $ay'' + by' + cy = 0$ (we'll see why soon).

Example

Solve the initial value problem

$$y'' + 3y' - 4y = 0, \quad y(0) = 2,\ y'(0) = 17$$

The characteristic equation $r^2 + 3r - 4 = 0$ factors as $(r+4)(r-1) = 0$, so the roots are $r_1 = -4$ and $r_2 = 1$. This gives a general solution of

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To solve the IVP, we take the derivative and plug in the initial conditions.

$$\begin{aligned} y &= c_1 e^{-4t} + c_2 e^t \\ y' &= -4c_1 e^{-4t} + c_2 e^t \end{aligned}$$

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This is a system of two linear equations with two unknowns. Subtracting the two equations yields $-15 = 5c_1$ so $c_1 = -3$. Then, plugging back into the first equation, we obtain $2 = -3 + c_2$ or $c_2 = 5$.

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This gives $y = -3e^{-4t} + 5e^{t}$ as the solution to the IVP.

Theorem (Principle of Superposition)

If y_1, y_2 are solutions to

$$y'' + p(t)y' + q(t)y = 0,$$

then so is $y = c_1 y_1 + c_2 y_2$ for any constants c_1, c_2.

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Proof.

If $y = c_1 y_1 + c_2 y_2$, then $y' = c_1 y_1' + c_2 y_2'$ and $y'' = c_1 y_1'' + c_2 y_2''$, so

$$\begin{aligned} y'' + p(t)y' + q(t)y \\ &= c_1 \left(y_1'' + p(t)y_1' + q(t)y_1 \right) + c_2 \left(y_2'' + p(t)y_2' + q(t)y_2 \right) \\ &= 0 + 0 = 0. \end{aligned}$$

This shows that $y = c_1 y_1 + c_2 y_2$ is also a solution. □

Theorem (Second-Order Linear Existence and Uniqueness)

If $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval $\alpha < t < \beta$ containing t_0, then for any real numbers y_0, y_0' the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0,\ y'(t_0) = y_0'$$

has a unique solution also defined on the interval $\alpha < t < \beta$.

Solving IVPs

In order for $y = c_1 y_1 + c_2 y_2$ to be the general solution, we need to show that we can solve all initial value problems with an appropriate choice of c_1, c_2. Using $y' = c_1 y_1' + c_2 y_2'$, we can write the initial condition $y(t_0) = y_0$, $y'(t_0) = y_0'$ as

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned}$$

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or in matrix form

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

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Linear algebra tells us that this has a unique solution for c_1, c_2 if and only if

$$\det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0.$$

The Wronskian Determinant

The determinant from the previous slide is called Wronskian determinant of y_1, y_2 at t_0 with notation

$$W(y_1, y_2)(t_0) := y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0).$$

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Example

The Wronskian determinant of the solutions $y_1 = e^t$, $y_2 = e^{-t}$ from the first example is

$$W(e^t, e^{-t})(t_0) = e^{t_0}(-e^{-t_0}) - e^{-t_0}e^{t_0} = -2 \neq 0$$

for all t_0.

Example

In general, if r_1 and r_2 are distinct real numbers, then the Wronskian determinant of $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ is

$$\begin{aligned} W(e^{r_1 t}, e^{r_2 t})(t_0) &= e^{r_1 t_0}(r_2 e^{r_2 t_0}) - (r_1 e^{r_1 t_0}) e^{r_2 t_0} \\ &= (r_2 - r_1) e^{(r_1 + r_2) t_0} \end{aligned}$$

which is never zero for any value of t_0.

Theorem (Abel's Theorem)

If $p(t)$ and $q(t)$ are continuous on an open interval $\alpha < t < \beta$ and y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$ on $\alpha < t < \beta$, then

$$W(y_1, y_2)(t) = Ce^{-\int p(t)\, dt}.$$

In particular, $W(y_1, y_2)$ is either always zero on $\alpha < t < \beta$ or never zero on $\alpha < t < \beta$.

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Proof.

Since y_1, y_2 are solutions, we have

$$\begin{aligned} y_1'' + p(t)y_1' + q(t)y_1 &= 0 \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0 \end{aligned}$$

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Multiply the first equation by $-y_2$ and the second by y_1 and add

$$-y_1''y_2 + y_1y_2'' + p(t)\left(-y_1'y_2 + y_1y_2'\right) = 0.$$

Proof.

From the previous slide, we have the equation

$$-y_1''y_2 + y_1y_2'' + p(t)\left(-y_1'y_2 + y_1y_2'\right) = 0.$$

The term in parentheses is $W(y_1, y_2) = y_1y_2' - y_1'y_2$, and its derivative is

$$\begin{aligned}\frac{d}{dt}W(y_1, y_2) &= y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2 \\ &= y_1y_2'' - y_1''y_2.\end{aligned}$$

which also appears in the above equation.

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Thus, the above equation is a first order separable equation

$$\frac{d}{dt}W(y_1, y_2) + p(t)W(y_1, y_2) = 0$$

with solution

$$W(y_1, y_2)(t) = Ce^{-\int p(t)\ dt}.$$

Theorem

Suppose y_1 and y_2 are solutions to

$$y'' + p(t)y' + q(t)y = 0$$

defined on an open interval $\alpha < t < \beta$ where $p(t)$ and $q(t)$ are continuous. Then, the family $y = c_1 y_1 + c_2 y_2$ with c_1, c_2 arbitrary constants includes all solutions on the interval if and only if there is a point t_0 in the interval such that $W(y_1, y_2)(t_0) \neq 0$. In this case, we call $y = c_1 y_1 + c_2 y_2$ the general solution on this interval.

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