Lecture20_QHardware_class15

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Lecture20...

Quantum Hardware – Optical Models

Class XV

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Based on the book, Quantum Computation and Quantum Information, by M. Nielsen and I. Chuang

Optical Cavity Quantum Electrodynamics (QED) – Two-level atoms

In the previous lecture, we discussed the energy levels of an atom. We saw that when solving the Schrödinger differential equation we make use of special functions to obtain exact solutions, even to the simplest type of atoms. The solutions is better written in terms of special functions as spherical harmonics

$$Y_{lm}(\theta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(\cos\theta)e^{im\varphi}$$

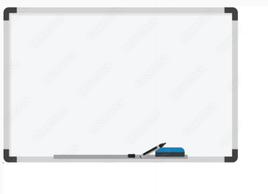
where P_{lm} are the Legendre functions

$$P_{lm}(x) = \frac{(1-x^2)^{m/2}}{2^l l!}\frac{d^{m+l}}{dx^{m+l}}(x^2-1)^l$$

with $-l \leq m \leq l$.

Today we will keep studying some consequences of this atomic model on the whiteboard. With the help of properties linked to the spherical harmonics, we will find the Hamiltonian from the atom in the two-level subspace in which the difference of energy is simply, $E_{i+1} - E_i = \hbar w_0$.

WHITEBOARD



TWO-LEVEL ATOMS

- LET'S SHOW THAT $\hat{r}$ CAN BE EXPRESSED IN TERMS OF SPHERICAL HARMONICS

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WE KNOW FROM SPHERICAL COORDINATES

$x = r \sin\theta \cos\varphi$
$y = r \sin\theta \sin\varphi$ $\Big\rangle$ SPHERICAL COORDINATES

THE SPHERICAL HARMONICS CAN BE WRITTEN IN FOLLOWING

$$Y_{\ell m} = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_{\ell m}(\cos\theta)\, e^{im\varphi} \quad (7.63)$$

$$P_{\ell m}(x) = \frac{(1-x^2)^{m/2}}{2^\ell \ell!} \frac{d^{m+\ell}}{dx^{m+\ell}} (x^2-1)^\ell \quad (7.64)$$

$\left.\begin{matrix} n=0 \\ n=1 \end{matrix}\right\} \to$ TWO-LEVEL ATOMS

$\ell = +1, 0 \quad , \quad -\ell \leq m \leq \ell$

$\ell = 1, \quad m = -1, 1$

$Y_{1,1} \quad , \quad Y_{1,-1}$

$\left\{\begin{matrix} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\varphi} \\ Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi} \end{matrix}\right.$ $\Big\rangle$ COHEN P. 682

$\cdot\, Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \left(\cos\varphi + i \sin\varphi\right)$

$\cdot\, Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \left(\cos\varphi - i \sin\varphi\right)$ (*)

$$Y_{11} - Y_{1-1} = 2\sqrt{\frac{3}{8\pi}} \sin\theta\cos\varphi$$

$$Y_{1-1} + Y_{11} = -2i\sqrt{\frac{3}{8\pi}} \sin\theta \sin\varphi$$

$$Y_{1-1} + Y_{11} = -2i\sqrt{\frac{3}{8\pi}} \sin\theta \sin\varphi$$

So, $\begin{cases} x = \frac{r}{2}\sqrt{\frac{8\pi}{3}}\left[Y_{1-1} - Y_{11}\right] \\ y = -\frac{r}{2i}\sqrt{\frac{8\pi}{3}}\left[Y_{1-1} + Y_{11}\right] \end{cases}$ > COHEN p. 837

i) $-x + iy = -\sqrt{\frac{8\pi}{3}}\, r\, Y_{1-1} \Rightarrow -r Y_{1-1} = \sqrt{\frac{3}{8\pi}}(-x+iy)$

ii) $x + iy = -\sqrt{\frac{8\pi}{3}}\, r\, Y_{11} \Rightarrow -r Y_{11} = \sqrt{\frac{3}{8\pi}}(x+iy)$ (**)

FROM Eqs. (*)

- $Y_{1-1} Y_{11} = -\frac{3}{8\pi}\sin^2\theta$ AND $Y_{11} Y_{-11} = -\frac{3}{8\pi}\sin^2\theta$

AND FROM (**)

- $-r\, Y_{1-1} Y_{11} = \sqrt{\frac{3}{8\pi}}(-x+iy) Y_{11}$
- $-r\, Y_{11} Y_{1-1} = \sqrt{\frac{3}{8\pi}}(x+iy) Y_{1-1}$

AND AFTER SUMMING THE LAST TWO EQUATIONS

$$-2r\, Y_{1-1} Y_{11} = \sqrt{\frac{3}{8\pi}}\left[(-x+iy)Y_{11} + (x+iy)Y_{1-1}\right] = -2r\left(\frac{-3}{8\pi}\right)\sin^2\theta$$

$$\theta = \pi/2$$

$$2r\left(\frac{3}{8\pi}\right) = \sqrt{\frac{3}{8\pi}}\left[(-x+iy)Y_{11} + (x+iy)Y_{1-1}\right]$$

$$r = \sqrt{\frac{2\pi}{3}}\left[(-x+iy)Y_{11} + (x+iy)Y_{1-1}\right] \quad (7.59)$$

$x \to r_x$ $\quad y \to r_y$

$x \to r_x \quad , \quad y \to r_y$

$$\hat{r} = \sqrt{\frac{2\pi}{3}} \left[(-r_x + i r_y)\, Y_{11}(\theta,\varphi) + (r_x + i r_y)\, Y_{1-1}(\theta,\varphi) \right] \qquad (7.59)$$

– IN THIS BASIS

$$\langle l_1, m_1 | \hat{r} | l_2, m_2 \rangle = \sqrt{\frac{2\pi}{3}} \left[(-r_x + i r_y) \langle l_1, m_1 | Y_{11} | l_2, m_2 \rangle \right.$$
$$\left. + (r_x + i r_y) \langle l_1, m_1 | Y_{1-1} | l_2, m_2 \rangle \right]$$

THE RELEVANT TERMS ARE

$$\langle l_1, m_1 | Y_{1,m} | l_2, m_2 \rangle = \int d\Omega \; Y^*_{l_1 m_1} \, Y_{1m} \, Y_{l_2 m_2} \qquad (7.60)$$

EXERCISE 7.16

• SHOW THAT (7.60) IS NON-ZERO ONLY WHEN $m_2 - m_1 = \pm 1$, AND $\Delta l = \pm 1$ (HOMEWORK)!

HINT: COHEN PAG 1046 AND 1310

– AN ATOM COUPLED TO THE EXTERNAL WORLD NEVER HAS PERFECTLY DEFINED ENERGY EIGENSTATES

– SMALL PERTURBATIONS SUCH AS FLUCTUATING ELECTRIC POTENTIALS OR EVEN INTERACTIONS WITH VACUUM, CAUSE ENERGY LEVEL TO BE SMEARED OUT AND BECOME A DISTRIBUTION WITH FINITE WIDTH

– IN SOME SITUATIONS IT'S POSSIBLE TO ARRANGE CIRCUMSTANCES SUCH THAT THE TWO-LEVEL ATOM APPROXIMATION WORKS WELL

– THE WHOLE POINT OF THIS PROCEDURE IS THAT IN THE

APPROXIMATION, IF $|\psi_1\rangle$ AND $|\psi_2\rangle$ ARE TWO SELECTED LEVELS, THEN THE MATRIX ELEMENTS OF $\hat{r}$ ARE

$$r_{ij} = \langle \psi_i | \hat{r} | \psi_j \rangle$$

→ REMEMBER FROM OUR STUDIES ABOUT THE SHO

$$r_{n'n} = \langle n' | r | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\, \delta_{n', n-1} + \sqrt{n+1}\, \delta_{n', n+1} \right)$$

- $r_{00} = \langle 0 | r | 0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(0 + \overset{=0}{\delta_{0,1}} \right) = \underline{0}$
- $r_{01} = \langle 0 | r | 1 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(1\, \overset{=1}{\delta_{00}} + \sqrt{2}\, \overset{=0}{\delta_{0,2}} \right) = \underline{\sqrt{\frac{\hbar}{2m\omega}}}$
- $r_{10} = \langle 1 | r | 0 \rangle = \underline{\sqrt{\frac{\hbar}{2m\omega}}}$
- $\underline{r_{11}} = \langle 1 | r | 1 \rangle = 0$, $r \in (x, y, z)$

USING THESE FOUR EXPRESSIONS IN EQ. (7.59)

$$(7.59) \Rightarrow r = \sqrt{\frac{2\pi}{3}} \left[(-x + iy)\, Y_{11} + (x + iy)\, Y_{1-1} \right]$$

$$r_{ij} = \sqrt{\frac{2\pi}{3}} \left[(-x_{ij} + i y_{ij})\, Y_{11} + (x_{ij} + i y_{ij})\, Y_{1-1} \right]$$

$$r_{ij} = \sqrt{\frac{2\pi}{3}} \left\{ + \sqrt{\frac{\hbar}{2m\omega}} \left[-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] Y_{11} + \sqrt{\frac{\hbar}{2m\omega}} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] Y_{1-1} \right\}$$

$$r_{ij} = \sqrt{\frac{2\pi}{3}} \sqrt{\frac{\hbar}{2m\omega}} \left\{ \begin{pmatrix} 0 & -1+i \\ -1+i & 0 \end{pmatrix} \underline{Y_{11}} + \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \underline{Y_{1-1}} \right\}$$

$$\ldots \sqrt{\ldots 2m\omega}\left(\begin{pmatrix} -1+i & 0 \end{pmatrix} / \underline{11} \qquad \begin{pmatrix} 1+i & 0 \end{pmatrix} / \underline{1-1} \right)$$

$$\underline{Y_{11}} = -\sqrt{+\frac{3}{8\pi}}\,\sin\theta\, e^{i\varphi}$$

$$\underline{Y_{1-1}} = \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\varphi}$$

$$r_{ij} = \sqrt{\frac{\pi\hbar}{3m}}\sqrt{\frac{3}{8\pi}}\,\sin\theta\left\{ \begin{pmatrix} 0 & 1-i \\ 1-i & 0 \end{pmatrix} e^{i\varphi} + \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} e^{-i\varphi} \right\}$$

$$e^{\pm i\varphi} = \cos\varphi \pm i\sin\varphi$$

$$r_{ij} = \bar{r}_0 \left\{ 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cos\varphi + 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\varphi \right\}$$

$$\underline{\bar{r}_0} = \sqrt{\frac{\hbar}{8m\omega}}\,\sin\theta$$

$$r_{ij} = 2\bar{r}_0 \left\{ (\cos\varphi + \sin\varphi)\, X \right\},$$ $X \to$ PAULI X GATE

$$r_0 = 2\underline{\bar{r}_0}(\cos\varphi + \sin\varphi) = 2\sqrt{\frac{\hbar}{8m\omega}}\,\sin\theta\,(\cos\varphi + \sin\varphi)$$

$$r_{ij} = r_0 \underline{X}$$

BUT IN EQ. (7.65) WE HAVE

$$r_{ij} = \langle \psi_i | \hat{r} | \psi_j \rangle \approx r_0 Y$$

(7.65)

(Y-PAULI GATE

COMMENT IN THE BOOK; THAT WE OBTAIN Y AS OPPOSED TO X DOESN'T MATTER – IT IS A MATTER OF CONVENTION AND CONVENIENCE FOR LATER CALCULATIONS

— THIS EXPRESSION WILL BE RELEVANT IN DESCRIBING INTERACTIONS BETWEEN THE ATOM AND INCIDENT ELECTRIC FIELD

— THE HAMILTONIAN OF THE ATOM ITSELF IN A n-LEVEL SUBSPACE IS

$$H_{ATOM}^{GENERAL} = -\sum_n E_n |n\rangle\langle n| \quad , \quad E_n = \hbar\omega(n - 1/2)$$

$$\begin{cases} E_0 = -\dfrac{\hbar\omega}{2} \\ E_1 = \dfrac{\hbar\omega}{2} \end{cases}$$

IN A TWO-LEVEL SUBSPACE WE HAVE

$$H_{ATOM} = -E_0|0\rangle\langle 0| - E_1|1\rangle\langle 1|$$

- $\langle 0|H_{ATOM}|0\rangle = -E_0 = \dfrac{\hbar\omega_0}{2}$, • $\langle 1|H_{ATOM}|1\rangle = -E_1 = -\dfrac{\hbar\omega_0}{2}$
- $\langle 0|H_{ATOM}|1\rangle = \langle 1|H_{ATOM}|0\rangle = 0$

$$H_{ATOM} = \begin{pmatrix} \dfrac{\hbar\omega_0}{2} & 0 \\ 0 & -\dfrac{\hbar\omega_0}{2} \end{pmatrix}$$

$$H_{ATOM} = \hbar\omega_0/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar\omega}{2} Z \qquad (7.66)$$

NEXT

$$(7.56) \quad \vec{E} = i\vec{\epsilon} E_0 \left[a e^{ikr} - a^\dagger e^{-ikr}\right] \quad , \quad \vec{E} \to \vec{r} \to \hat{r}$$

$$H_I = H_{ij} = \langle\psi_i|E|\psi_j\rangle \longrightarrow \langle\psi_i|\hat{r}|\psi_j\rangle$$