Notes for the reading club

Reading Club

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Abstract

The Yellow Book Notes. It is good to write notes!

Contents

1 Preliminary

1.1 Conventions

Metric tensor and Coordinate.– The metric tensor in Minkowski and Euclidean space-time is defined as

$$\eta = \begin{Bmatrix} +1 & & \\ & -1 & \\ & & \dots \end{Bmatrix} \tag{1}$$

and

$$\eta = \begin{Bmatrix} +1 & & \\ & +1 & \\ & & \dots \end{Bmatrix} \tag{2}$$

respectively, where the first index is the time. In the Yellow Book, without specifications, we are working in Eclidean space. The coordinate is defined as $x^\mu = \{t, \vec{x}\}$. So that the norm of a vector in Minkowski space-time is $x^\mu x_\mu = t^2 - r^2$.

γ matrices.– The γ matrices follow the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{3}$$

In Minkowski space time, the γ matrices can be chosen as

$$\begin{aligned} \gamma^0 &= \sigma^x \\ \gamma^1 &= -i\sigma^y, \end{aligned} \tag{4}$$

while in Elidean space, they can be chosen as

$$\begin{aligned} \gamma^0 &= \sigma^x \\ \gamma^1 &= \sigma^y. \end{aligned} \tag{5}$$

Complex Coordinate.– In $2d$ CFT, complex coordinate is powerful as one can utilize the beautiful analytic properties of holomorphic functions. It is defined as

$$\begin{aligned} z &= z^0 + iz^1 \\ \bar{z} &= z^0 - iz^1. \end{aligned} \tag{6}$$

Note that here z and $\bar{z}$ denote coordinates with upper index.

1.2 Free fermions

In Minkowski space time, the Lagrange density for the free fermion reads

$$\mathcal{L} = \frac{g}{2}\left(\psi^1\ i(\partial t + \partial x)\ \psi^1 + \psi^2\ i(\partial t - \partial x)\ \psi^2\right). \tag{7}$$

In terms of $\psi = (\psi^1, \psi^2)$, one can write the theory as

$$\begin{aligned}
\mathcal{L} &= \frac{g}{2}\left(\ \psi^\dagger i\partial_t\psi + \psi^\dagger\ \sigma^z\ i\partial_x\psi\ \right) \\
&= \frac{g}{2}\left(\ \psi^\dagger\sigma^x\sigma^x i\partial_t\psi + \psi^\dagger\ -i\sigma^x\sigma^y\ i\partial_x\psi\ \right) \\
&= \frac{g}{2}\psi^\dagger\sigma^x\left(\sigma^x i\partial_t - i\sigma^y i\partial_x\right)\psi \\
&= \frac{g}{2}\psi^\dagger\gamma^0 i\gamma^\mu\partial_\mu\psi
\end{aligned} \tag{8}$$

where we used

$$\gamma^0 = \sigma^x \quad \gamma^1 = -i\sigma^y \tag{9}$$

Wick rotation It is usually more convenient to work in Euclidean space rather than Minkowski space time. Upon doing the Wick rotation, the action changes as

$$i\ S_M \to -\ S_E. \tag{10}$$

Specifically,

$$\begin{aligned}
i\ S[\psi] &= i\int dxdt\ \frac{g}{2}\psi^\dagger\gamma^0 i\gamma^\mu\partial_\mu\psi \\
&= i^2\int dxdt\ \frac{g}{2}\psi^\dagger\partial_t\psi + i^2\int dxdt\ \frac{g}{2}\psi^\dagger\sigma^x(-i)\sigma^y\partial_x\psi \\
&= -\int dxd\tau\ \frac{g}{2}\psi^\dagger\partial_\tau\psi - \int dxd(-it)\ \frac{g}{2}\psi^\dagger\sigma^x\sigma^y\partial_x\psi \\
&= -\int dxd\tau\ \frac{g}{2}\psi^\dagger\sigma^x\sigma^x\partial_\tau\psi - \int dxd\tau\ \frac{g}{2}\psi^\dagger\sigma^x\sigma^y\partial_x\psi \\
&= -\int dxd\tau\ \frac{g}{2}\psi^\dagger\gamma_E^0\gamma_E^\mu\partial_\mu\psi
\end{aligned} \tag{11}$$

where $\tau = -it$. The Eulidean space action can be written as

$$S_E = \int d^2x\frac{g}{2}\ \psi^\dagger\gamma_E^0\gamma_E^\mu\partial_\mu\psi \tag{12}$$

$1+1d$ *free fermions: Legendre transformation* A lattice version free fermion theory Eq. 2.38 reads

$$\mathcal{L} = \frac{i}{2}\sum_n\left(\psi_n\dot{\psi}_n + \psi_n\psi_{n+1}\right). \tag{13}$$

The canonical momentum corresponding to ψ_n is

$$\pi_n = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_n} = -\frac{i}{2}\psi_n. \tag{14}$$

So that the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \sum_n \pi_n \dot{\psi}_n - \mathcal{L} \\ &= -\frac{i}{2}\sum_n \psi_n \dot{\psi}_n - \frac{i}{2}\sum_n \left(\psi_n \dot{\psi}_n + \psi_n \psi_{n+1}\right) \\ &= -i\sum_n \psi_n \dot{\psi}_n - \frac{i}{2}\sum_n \psi_n \psi_{n+1}.\end{aligned} \tag{15}$$

While it shoud be

$$\mathcal{H} = -\frac{i}{2}\sum_n \psi_n \psi_{n+1}. \tag{16}$$

If we'd like to keep defining the derivative of Grassmann number according to the order of left-to-right, we need to define the Hamiltonian as

$$\mathcal{H} = \sum_n \dot{\psi}_n \pi_n - \mathcal{L} = -\frac{i}{2}\sum_n \psi_n \psi_{n+1} \tag{17}$$

1.3 Free boson

The action for the free boson in the Minkoswki space time reads

$$S = \frac{1}{2} g \int dx dt\ \partial_\mu \phi \partial^\mu \phi, \tag{18}$$

where ϕ is a real scalar field. After Wick rotation $\tau = it$, it becomes

$$\begin{aligned} i\,S &= \frac{i}{2} g \int dx dt\ \partial_t \phi \partial_t \phi - \frac{i}{2} g \int dx dt\ \partial_x \phi \partial_x \phi \\ &= -\frac{1}{2} g \int dx d\tau\ \partial_\tau \phi \partial_\tau \phi - \frac{1}{2} g \int dx dit\ \partial_x \phi \partial_x \phi \\ &= -\frac{1}{2} g \int dx d\tau\ \partial_\mu \phi \partial^\mu \phi \end{aligned} \tag{19}$$

The Euclidean action reads

$$S_E = \frac{1}{2} g \int d^2x\ \partial_\mu \phi \partial^\mu \phi. \tag{20}$$

The two point correlation up to a constant term is

$$\langle \phi(x)\phi(y)\rangle = -\frac{1}{2\pi g}\ln(\rho). \tag{21}$$

1.4 Symmetries at the classical level

The action becomes different after a coordinate transformation. We say it has a symmetry if it remains unchanged and a Noether current can be derived from the symmetry. The coordinate transformation is denoted as

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \tag{22}$$

and the field changes according to

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta F}{\delta \omega_a}(x) \tag{23}$$

where ω_a is a constant and small parameter.

By definition, the change of the action δS disappears for a symmetric transformation. We can get nothing new from this. If we allow ω_a to be arbitrary, the leading contribution to δS becomes

$$\delta S = -\int d^2x j^\mu \partial_\mu \omega_a, \tag{24}$$

where we introduced the the current j^μ. We assume it decreases fast when approaching infinite. So that one obtains

$$\delta S = \int d^2x \ \partial_\mu j^\mu \ \omega_a. \tag{25}$$

This equations holds for all the field configurations. If we require the field configuration to be the one obeying the equation, the action should be invariant for arbitrary coordinate transformation and one finds the conservation of j^μ

$$\partial_\mu j^\mu = 0. \tag{26}$$

Energy-momentum tensor The canonical energy-momentum tensor is defined to be the Noether current of the translation transformation

$$x'^\mu = x^\mu + \epsilon^\nu \delta^\mu_\nu \tag{27}$$

$$T^{\mu\nu} = -\eta^{\mu\nu} L + \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\nu \phi. \tag{28}$$

This definition of $T^{\mu\nu}$ is not guaranteed to be symmetric between the two indices (The requirement of a symmetric $T^{\mu\nu}$ will be clear later).

Another definition that makes the energy-momentum tensor symmetric follows. In the coordinate transformation, if we also consider the variance of the metric tensor (which means the theory is coupled with the dynamical backgroud)

$$\delta g_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \tag{29}$$

the action remains invariant since this is nothing but a reparametrization of the theory (general coordinate covariance). So that one finds

$$\delta S = 0 = -\frac{1}{2} \int d^d x \ \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \left(T^{\mu\nu} + 2 \frac{\delta S}{\delta g_{\mu\nu}} \right). \tag{30}$$

So that one can define the energy-momentum tensor as

$$T^{\mu\nu} = -2 \frac{\delta S}{\delta g_{\mu\nu}} \tag{31}$$

up to a surface term.

Another way to make the energy-momentum tensor symmetric is add a surface term to the canonical one. One can show that with rotation symmetry, such a term can be constructed to make $T^{\mu\nu}$ symmetric.

1.5 Symmetry at the quantum level

All the field configurations contribute to the quantum theory, so that one has no Noether current in general. Still the symmetry has constraints to the quantum theory. For the n—point

correlation functions, one has

$$\begin{aligned}
\langle \phi(x_1')...\phi(x_n')\rangle &= \frac{1}{Z}\int [D\phi]\ \phi(x_1')...\phi(x_n')\ e^{-S[\phi]} && (32)\\
&= \frac{1}{Z}\int [D\phi']\ \phi'(x_1')...\phi'(x_n')\ e^{-S'[\phi']} && (33)\\
&= \frac{1}{Z}\int [D\phi]\ F(\phi(x_1))...F(\phi(x_n))\ e^{-S[\phi]} && (34)\\
&= \langle\ F(\phi(x_1))...F(\phi(x_n))\ \rangle && (35)
\end{aligned}$$

in which we assumed the functional integral measure does not change and the coordinate transformation is a rigid one (ω_a is a constant).

Ward identity As stated above there is no conserved current at the quantum level. The infinitesimal coordinate transformation at the quantum level results in the so-called Ward identity.

We denote the change of fields as

$$\phi'(x) = \phi(x) - i\omega_a\ G_a\ \phi(x). \tag{36}$$

The infinitesimal coordinate transformation (ω_a now is arbitrary) changes the correlation as (We only consider the first order perturbation contribution)

$$\begin{aligned}
\langle \phi'(x_1)...\phi'(x_n)\rangle &= \langle \phi(x_1)...\phi(x_n)\rangle && (37)\\
&= \frac{1}{Z}\int [D\phi'](X+\delta X)e^{-S[\phi]-\int d^dx\partial_\mu j^\mu\omega_a} && (38)\\
&= \frac{1}{Z}\int [D\phi](X+\delta X)e^{-S[\phi]-\int d^dx\partial_\mu j^\mu\omega_a} && (39)\\
&= \langle X\rangle - \int [D\phi]\int d^dx\ X\partial_\mu j^\mu\omega_a e^{-S[\phi]} - \int [D\phi]\delta X e^{-S[\phi]} && (40)
\end{aligned}$$

so that one finds

$$\langle \delta X\rangle = \int d^dx\ \partial_\mu\langle j^\mu\ X\rangle\omega_a(x). \tag{41}$$

As

$$\begin{aligned}
\delta X &= -i\sum_i \phi(x_1)...G_a\phi(x_i)...\phi(x_n)\omega_a(x_i) && (42)\\
&= -i\int d^dx\sum_i \phi(x_1)...G_a\phi(x_i)...\phi(x_n)\delta(x-x_i)\omega_a(x) && (43)
\end{aligned}$$

Since ω_a is arbitrary, one obtains the Ward identity

$$\partial_\mu\langle j^\mu\ X\rangle = -i\sum_i \delta(x-x_i)\ \langle\phi(x_1)...G_a\phi(x_i)...\phi(x_n). \tag{44}$$

So that for each symmetry, there exists a Ward identity, i.e., a constraint to the correlation function. With enough symmetries, one can get all the information of the correlation functions.

1.6 Renormalization group

Dimensional analysis and renormalizability of QFT Let's start with the canonical dimension of fields and couplings in the action,

$$S = \int d^d x \ \mathcal{L}(\phi, \lambda). \tag{45}$$

Since the action is dimensionless, every term in $\mathcal{L}$ has an energy scaling dimension of

$$\Delta(\mathcal{L}) = [\mathcal{L}] = \omega^d \tag{46}$$

which determines the canonical dimension fields and couplings. The renormalizability of a QFT is directly obtained from the energy dimension of Feynman diagrams,

$$\mathcal{D} = d - E_\phi \Delta(\phi) - \Delta(\lambda_i) \tag{47}$$

where E_ϕ is the number of external fields and λ_i the couplings in the theory. A nice discussion about renormalizability can be found online (https://web2.ph.utexas.edu/ vadim/Classes/2022f/notes.html).

Super-renormalizable theories have only couplings with positive dimensions. For such theories, there are finite Feynman diagrams become divergent in the perturbation calculation. Renormalizable theories have couplings with non-negative dimensions, in which a finite number of couplings have zero dimensions. There exists infinite number of divergent Feynman diagrams, but the number of divergent amplitudes is finite. If there is at least one coupling with a negative dimension, the theory is non-renormalizable.

Wilson-Kadanoff RG scheme The renormalization group (RG) builds up the modern understanding of QFT, which is regarded as an *effective field theory*. In the history, many different RG schemes have been developed, which are suitable for very different theories. Most of them are realized in a perturbation way around a known RG fixed point. Here we briefly recall the most popular one, i.e. the Wilson-Kadanoff RG scheme.

In this scheme, a momentum cutoff $\boldsymbol{k} < \Lambda$ is introduced. One first divides modes into fast $\Lambda/s < \boldsymbol{k} < \Lambda$ and slow $\boldsymbol{k} < \Lambda/s$ parts $\phi = \phi_f + \phi_s$. The fast modes are integrated out to result in a new theory

$$e^{-S'(\phi)_{\Lambda/s}} = \int \mathrm{D}\phi_{\Lambda/s<\boldsymbol{k}<\Lambda} \ e^{-S_\Lambda(\phi)} \tag{48}$$

with a smaller cutoff Λ/s. Generally, the action can be divided into three parts

$$S = S_f(\phi_f) + S_s(\phi_s) + S_c(\phi_f, \phi_s). \tag{49}$$

The new theory thus can be written as

$$\begin{aligned} e^{-S'(\phi_s)_{\Lambda/s}} &= \int D\phi_f e^{-S_f - S_s - S_c} \\ &= e^{-S_s} Z_f \ \frac{\int \mathrm{D}\phi_f \ e^{-S_f} e^{-S_c}}{Z_f} \\ &= e^{-S_s} \ Z_f \ \langle e^{-S_c} \rangle_f \end{aligned} \tag{50}$$

where $Z_f = \int \mathrm{D}\phi_f \ e^{-\phi_f}$ is a constant and can be neglected (Note that it does contribute to the total free energy). The new action thus is

$$\begin{aligned} S(\phi_s)_{\Lambda/s} &= -\log\left(\int \mathrm{D}\phi_{\Lambda/s<\boldsymbol{k}<\Lambda} \ e^{-S_\Lambda(\phi)} \right) \\ &= S_s - \log\left(\langle e^{-S_c} \rangle_f \right) \end{aligned} \tag{51}$$

Usually one can not integral out high energy modes exactly, hence cumulant perturbations based on Feynmann diagramm have to be adopted.

This theory can not be compared with the original one, since they have different cutoffs. Another rescaling step

$$\boldsymbol{k} \to s\,\boldsymbol{k} \tag{52}$$

is required to restore the cutoff or energy scale. Since the field operators dependend on length scales, they also need to be rescaled

$$\phi \to s^{\Delta_\phi}\,\phi \tag{53}$$

Now one obtains a new theory $S(\phi,\lambda)_\Lambda$ at the same cutoff but with different parameters, in which we assumed the theory $S(\phi,\lambda)$ remains the same structure.

Keep doing such RG procedures, one can find how the parameters $\lambda_i(s)$ flow in the parameter space along with the RG time s. These RG transformations of the parameters form a semi-group structure. In the whole parameter space, fixed points are special, since they are scale invariant. The parameter near a fixed point λ^* is called relevant or irrelevant when it flows away or close to λ^*, respectively. A RG program is to find all fixed points and analyse how the parameters flow near fixed points. One needs to solve the so-called β equation

$$\beta_i(\lambda_j) = \frac{\partial \lambda_i}{\partial \log(s)}. \tag{54}$$

The zero points of the β function are solutions of fixed points of the RG program

$$\beta_i(\lambda_j^*) = 0. \tag{55}$$

Near the fixed point, usually one can approimate the β function as an linear eigen problem. Eigenvalues of the RG transformation imply how fast λ_i flow to or away from λ^*, which are nothing but the scaling dimensions $\Delta(\tilde{\lambda}_i)$ of the corresponding parameter

$$\frac{\partial \tilde{\lambda}_i}{\partial \log(s)} = \Delta\left(\tilde{\lambda}_i\right)\tilde{\lambda}_i \tag{56}$$

where $\tilde{\lambda}_i$ is a linear combination of the orginal paramters (here we shifted the fixed point to be zero and $\tilde{\lambda}_i$ means the distance to the fixed point λ_i^*). Note that the RG analysis here is also consistent with the renormalizability of a QFT. An irrelevant field ($\Delta(\lambda_i) < 0$) vanishes at IR means it becomes divergent at UV.

There also exist many other RG schemes. For example, one may integrate out all high-energy modes $|k| > \Lambda$. There will be divergence at low dimensions. A popular way to deal with the divergence is to continue the space dimension d to be a real positive number and make perturbation around the upper or lower critical dimension, which is called as $d \mp \epsilon$ expansion in the literature. Another popular and also elegent RG scheme is to introduce a real space short distance cutoff a. The scaling transformation of a is canceled by the change of couplings in the theory. One can use operator product expansion (OPE) to write down the β function. In this approach, one only needs to know the OPE coefficients at a known fixed point rather than doing Feynmann diagram calculations.

Example: poor man's scaling of Kondo effect Example: perturbative RG analysis of ϕ^4 theory
The Ferromagnetic phase transition is usually modeled by a real scalar field theory

$$S = \int d^d x \left\{ \frac{1}{2}(\Delta\phi)^2 + \sum_{n=1,2,4} \left(\frac{\lambda_n}{n!}\phi^n \right) \right\} \tag{57}$$

where the field ϕ can be viewed as flucturations around the mean field solution ϕ_c of the action. Following the Wilson-Kadanoff RG scheme, we identify

$$\begin{aligned} S_f &= \int d^d x \ \left\{ \frac{1}{2} \left(\Delta \phi_f \right)^2 + \frac{\lambda_2}{2} \phi_f^2 \right\} \\ S_s &= \int d^d x \ \left\{ \frac{1}{2} (\Delta \phi_s)^2 + \lambda_1 \phi_s + \frac{\lambda_2}{2} \phi_s^2 \right\} \\ S_c &= \int d^d x \ \left\{ \frac{\lambda_4}{4!} \left(\phi_s + \phi_f \right)^4 \right\}. \end{aligned} \tag{58}$$

At one-loop approximation, using cumulant expansion one can find

$$\langle e^{-S_c} \rangle_f = \exp \left\{ - \langle S_c \rangle_f + \frac{1}{2} \left(\left\langle S_c^2 \right\rangle_f - \langle S_c \rangle_f^2 \right) \right\} \tag{59}$$

In $\langle S_c \rangle_f$ there is a pure slow mode term $\frac{\lambda_4}{4!} \phi_s^4$ and another one

$$\frac{\lambda_4}{4!} C_4^2 \int d^d x \phi_s^2 \left\langle \phi_f(x) \phi_f(x) \right\rangle_f = \int d^d x \left\{ \frac{\left\langle \phi_f(x) \phi_f(x) \right\rangle_f \lambda_4 / 2}{2} \ \phi_s^2 \right\} \tag{60}$$

In $\left\langle S_c^2 \right\rangle_f - \langle S_c \rangle_f^2$ there is one term contributing to the one-loop result

$$\left(\frac{\lambda_4}{4!} C_4^2 \right)^2 \int d^d x \int d^d y \ \left(\phi_s(x) \phi_s(y) \right)^2 \left\langle \phi_f(x) \phi_f(y) \right\rangle_f^2 \tag{61}$$

Example: perturbative RG analysis of BKT transition

2 Conformal symmetry in $d > 2$ dimensions

3 Conformal symmetry in two dimension

In $d = 2$ we have infinitely many *local* conformal transformations. The 6 parameter subgroup of conformal transformations that are everywhere well defined is the *global* conformal group $SL(2, \mathbb{C})/\mathbb{Z}_2$. Locally on the algebra level, it becomes the infinite dimensional Witt algebra. In a quantum theory, one can introduce a central extension term and get the famous Virasoro algegra. It is this infinite dimensional symmetry that ensures fields in a cft have nice local properties.

3.1 Global and local conformal symmetries

Conformal mappings and Witt algebra
Conformal Ward identity
Virasoro algebra: Central extension of Witt algebra

3.2 From correlation functions to OPE

Due to the local nature of field theory, we promote the correlation functions to expansion of non-singular operators, which is termed as operator product expansion (OPE). The idea is basically that far away from the operators inside a bounded region other operators can only feel them as a superposition of single operators (non-singular). The first OPE example follows

from the conformal ward identity, which builds up the OPE between the energy momentum tensor and primary fields

$$T(z)\phi(\omega) \sim \frac{h}{(z-\omega)^2}\phi(\omega) + \frac{1}{z-\omega}\partial\phi(\omega) \tag{62}$$

where on the right hand side, the operators should be understood as to be calculated correlation functions with some other operators located far away from them. Following the conventions defined in the first chapter, one can easily obtain the OPE of $T(z)$ with $\partial\phi$ for free boson

$$T(z)\partial\phi(\omega) \sim \frac{\partial\phi(\omega)}{(z-\omega)^2} + \frac{\partial_\omega^2\phi(\omega)}{z-\omega} \tag{63}$$

and $T(z)$ with ψ for free fermion

$$T(z)\psi(\omega) \sim \frac{\frac{1}{2}\psi(\omega)}{(z-\omega)^2} + \frac{\partial\psi(\omega)}{z-\omega} \tag{64}$$

The OPE can be generalized to arbitrary fields

$$A(z)B(\omega) = \sum_{n=-\infty}^{\Delta(A)+\Delta(B)} \frac{\{AB\}_n(\omega)}{(z-\omega)^n} \tag{65}$$

where $\{AB\}_n(\omega)$ are non-singular fields. Note that the total scaling dimensions can not be changed in OPE.

3.3 Energy-momentum tensor and central charge

The energy-momentum tensor is a quasi-primary field, which does not follow the OPE of T with primaries. There is an aditional term proportional to central charge c in the OPE

$$T(z)T(\omega) \sim \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}. \tag{66}$$

This term also exists in the conformal transormation of T

$$T'(\omega) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z;\omega\} \tag{67}$$

where $\{z;\omega\}$ denotes the Schwarzian derivative. This is consistent with the fact that this term disappears under global conformal transtions which are true symmetry of CFT.

The central charge c is related to the number of degrees of freedom in the theory. This can be reflected in the calculation of free energy density for a cylinder, which is related to the plane via a conformal transormation

$$\omega = \frac{L}{2\pi}\log(z). \tag{68}$$

The energy-momentum tensor becomes

$$T_{cyl}(\omega) = \left(\frac{2\pi}{L}\right)^2 \{T_{pl}(z)\, z^2 - \frac{c}{24}\}. \tag{69}$$

The variation of free energy is a response to the change of metric. One can make another coordinate transformation only along with the circumference direction $\omega^0 \to \omega^0(1+\epsilon)$. Note that this is not a confromal transormation, which will result in the change of the metric tensor. One can find the free energy for a cylinder takes the form of

$$F = f_0 L - \frac{\pi c}{6L}, \tag{70}$$

which indicates that the conformal anomaly reflects the quantum fluctuation effect to the classifcal conformal symmetry.

4 Operator formalism

In this section, we explore the quantization of the cft on a cylinder, which is related to the plane via a conformal transormation.

4.1 Radial quantization

On the plane, one has the freedom to choose the direction of space or time for an Euclidean theory. Here we choose the radial direction to be time and the angle direction to be space. A conformal transformation

$$\xi = \frac{L}{2\pi}\log(z) \tag{71}$$

maps a point z on a complex palne to a point $\xi = t + ix$ on a cylinder with t being the time and $x \in [0, L)$ the space. The Hilbert space defined on the cylinder at a given time t is defined within a circle with a radius $e^{2\pi t/L}$. Naturally the quantum thoery defined on a cylinder can be used to understand the plane.

One can immedialy find many important properties of the radial quantization from the confromal mapping. The time evolution operator, the Hamiltonian, on a cylinder corresponds the dialation operator on the plane and the translation operator, i.e. the momentum, corresponds to the rotation operator on the plane. Such a quantization scheme for a cft is called radial quantization. The time ordering on a cylinder becomes radial ordering on a plane. As a conseconce, the commutation of operators for a quantum theory is related to contour integrals through

$$[A, B] = \oint_0 d\omega \oint_\omega dz\, a(z)b(\omega), \tag{72}$$

where A and B are defined as equal time contour integral of local fields. Note that in the coutour integral, we have assumed that there is no other fields existing between the two integral circles, which means the time difference ϵ here should be infestimal small. In other words, the communicator defined here should be understood as equal-time communicator.

State-field correspondence Following the quantum theory on a cylinder (an operator in inserted at infinit past time to the vaccum state), we define a state corresponding to the field $\phi(z, \bar{z})$

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle. \tag{73}$$

Its dual state is defined as

$$\langle\phi| = \lim_{z,\bar{z}\to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\phi(1/\bar{z}, 1/z). \tag{74}$$

It is clear that states such definied are properly normalized.

4.2 Virasoro agebra

With the radial quantization and state-field correspondence, one can re-express the conformal symmetry, i.e. the Virasoro algera conveniently. We first introduce quantum operators for local fields and the energy-momentum tensor from equal-time contour integral (or equally mode expansion for local fields)

$$\phi_n = \frac{1}{2\pi i}\oint dz\, z^{n+h-1}\phi(z), \tag{75}$$

in which for the energy-momentum tensor $T(z)$ we denote its mode expansion operator as L_n. One then finds operators L_n defind here obey Virasoro algebra using the OPE of $T(z)$

from a straightforward calculation. Again, the communicator between L_n is meaningful as equal-time. One can also obtain the communicator between L_n and ϕ_m

$$[L_n, \phi_m] = (n(h-1) - m)\,\phi_{n+m}. \tag{76}$$

With the Virasoro generators L_n, one can also construct states as

$$L_{-k_1} L_{-k_2} \cdots L_{-k_n} |\phi\rangle. \tag{77}$$

4.3 The Free Boson

4.3.1 Canonical Quantization on the Cylinder

Let $\phi(x,t)$ be a free Boson field defined on a cylinder of circumference L, such that $\phi(x+L,t) = \phi(x,t)$. The Lagrangian of the boson field is

$$\mathcal{L} = \frac{g}{2} \int dx \left\{ (\partial_t \phi)^2 - (\partial_x \phi)^2 \right\}. \tag{78}$$

We can Fourier transform ϕ as

$$\phi(x,t) = \sum_n e^{\frac{2\pi}{L} inx} \phi_n(t), \tag{79}$$

$$\phi_n(t) = \frac{1}{L} \int dx e^{-\frac{2\pi}{L} inx} \phi(x,t). \tag{80}$$

The Lagrangian can be reexpressed as

$$\mathcal{L} = \frac{g}{2} \sum_n \left\{ \dot{\phi}_n \dot{\phi}_{-n} - \left(\frac{2\pi n}{L} \right)^2 \phi_n \phi_{-n} \right\}. \tag{81}$$

The momentum conjugate to ϕ_n becomes

$$\pi_n = gL\dot{\phi}_{-n}, \qquad [\phi_n, \pi_m] = i\delta_{nm}. \tag{82}$$

The Hamiltonian can be expressed as

$$H = \frac{1}{2gL} \sum_n \{ \pi_n \pi_{-n} + (2\pi n g)^2 \phi_n \phi_{-n} \} \tag{83}$$

This corresponds to a sum of decoupled harmonic oscillators with frequencies $\omega = \frac{2\pi}{L}|n|$.

We can introduce creation and annihalation operators, which allow the Hamiltonian to be expressed as

$$H = \frac{1}{2gL} \pi_0^2 + \frac{2\pi}{L} \sum_n (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n). \tag{84}$$

The following commutation relation holds

$$[H, a_{-m}] = \frac{2\pi}{L} m a_{-m}. \tag{85}$$

Applying a_{-m} toan eigenstate with energy E, creates an eigenstate with energy $E + \frac{2\pi m}{L}$. The Fourier modes can be expressed as

$$\phi_n = \frac{i}{n\sqrt{4\pi g}} (a_n - \bar{a}_{-n}) \tag{86}$$

The fields can be expressed as

$$\phi(x,t) = \phi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{\frac{2\pi i}{L}n(x-t)} - \bar{a}_{-n}e^{\frac{2\pi i}{L}n(x+t)}\right). \tag{87}$$

Transforming to Euclidean space, we can define the following conformal coordinates

$$z = e^{\frac{2\pi}{L}(\tau - ix)}, \qquad \bar{z} = e^{\frac{2\pi}{L}(\tau + ix)}. \tag{88}$$

This results in

$$\phi(z,\bar{z}) = \phi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\bar{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + \bar{a}_n \bar{z}^{-n}\right). \tag{89}$$

The field ϕ is not a primary, however the holomorphic field $\partial\phi$ is.

$$i\partial\phi(z) = \frac{\pi_0}{4\pi g z} + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0} a_n z^{-n-1} \tag{90}$$

4.3.2 Vertex Operators

There exists an infinite variety of local fields related to ϕ without introducing a scale. These are called the vertex operators $\mathcal{V}_\alpha$.

$$\mathcal{V}_\alpha =: e^{i\alpha\phi(z,\bar{z})} : \tag{91}$$

The vertex operators have conformal dimensions $h(\alpha) = \bar{h}(\alpha) = \frac{\alpha^2}{8\pi g}$.

4.4 The Fock Space

The eigenstates of H can be labeled by the eigenvalues of π_0. This means that the Fock space is built upon a one-parameter family of vacua $|\alpha\rangle$.

We know that $T(z)$ is given by

$$T(z) = -2\pi g : \partial\phi(z)\partial\phi(z) : \tag{92}$$

$$= \frac{1}{2}\sum_{n,m} z^{-n-m-2} : a_n a_m : . \tag{93}$$

From this we can derive the expression for the Virasoro operators

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m \quad (n \neq 0), \tag{94}$$

$$L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2. \tag{95}$$

This allows for the Hamiltonian to be expressed as

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0) \tag{96}$$

Furthermore the elements of the Fock space $a_{-1}^{n_1}a_{-2}^{n_2}...\bar{a}_{-1}^{m_1}\bar{a}_{-2}^{m_2}...|\alpha\rangle$ are eigenstates of L_0 with conformal dimensions $h = \frac{1}{2}\alpha^2 + \sum_j jn_j$ and $\bar{h} = \frac{1}{2}\alpha^2 + \sum_j jm_j$. The different vacua $|\alpha\rangle$ are related to the absolute vacuum $|0\rangle$ by the vertex operators $\mathcal{V}_\alpha$.

4.4.1 Twisted Boundary Conditions

We can also assume anti-periodic boundary condition. This is compatible with the Lagrangian because it is quadratic in the fields. Changing to anti-periodic boundaries makes the summation index half-integer valued and removes the zero mode. There are now two vacua $|0_+\rangle$ and $|0_-\rangle$.

We have that

$$\langle \phi \partial \phi \rangle = \frac{1}{w} \sum_{n>0} \left(\frac{w}{z}\right)^n. \tag{97}$$

In the periodic case the sum is over integer values and becomes

$$\langle \phi \partial \phi \rangle = \frac{1}{z-w}. \tag{98}$$

In the anti-periodic case the sum is over half integer values and becomes

$$\langle \phi \partial \phi \rangle = \sqrt{\frac{z}{w}} \frac{1}{z-w}. \tag{99}$$

For the vacuum expactation value of the energy-momentum tensor, $\langle T(z) \rangle$, we have in the periodic case $\langle T(z) \rangle = 0$, but in the anti-periodic case $\langle T(z) \rangle = \frac{1}{16z^2}$.

4.4.2 Compactified Boson

We can identify ϕ with $\phi + 2\pi R$ to get the compact boson. In general we can consider the boundary condition

$$\phi(x+L,t) = \phi(x,t) + 2\pi m R, \tag{100}$$

where m represents the winding number of the field. This modifies the mode expansion as

$$\phi(x,t) = \phi_0 + \frac{n}{gRL}t + \frac{2\pi m R}{L}x + \frac{i}{\sqrt{4\pi g}} \sum_{k\neq 0} \frac{1}{k} \left(a_k e^{\frac{2\pi i k}{L}(x-t)} - \bar{a}_{-k} e^{\frac{2\pi i k}{L}(x+t)} \right). \tag{101}$$

After reexpressing in complex coordinates and taking the derivative we get

$$i\partial\phi(z) = \left(\frac{n}{4\pi g R} + \frac{1}{2} m R\right) \frac{1}{z} + \frac{1}{\sqrt{4\pi g} \sum_{k\neq 0} a_k z^{-k-1}}. \tag{102}$$

The virasoro operators L_0 and $\bar{L}_0$ can be expressed as

$$L_0 = \sum_{n>0} a_{-n} a_n + 2\pi g \left(\frac{n}{4\pi g R} + \frac{1}{2} m R \right)^2 \tag{103}$$

$$\bar{L}_0 = \sum_{n>0} \bar{a}_{-n} \bar{a}_n + 2\pi g \left(\frac{n}{4\pi g R} - \frac{1}{2} m R \right)^2 \tag{104}$$

$$\tag{105}$$

4.5 The Free Fermion

The free fermion action is given by

$$S = \frac{1}{2} g \int d^2x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi \tag{106}$$

The central charge of this theory is $c = 1/2$ and ψ has as conformal dimension $h = 1/2$.

4.5.1 Canonical Quantization on a Cylinder

We can take the mode expansion of ψ at $t = 0$ on a cylinder with circumference L. This gives

$$\psi(x) = \sqrt{\frac{2\pi}{L}} \sum_k b_k e^{\frac{2\pi i}{L} kx}. \tag{107}$$

There are two possible types of boundary conditions. With the periodic (Ramond) boundary conditions the index k takes on integer values. With anti-periodic (Neveu-Schwarz) boundary conditions the index k must take half-integer values.

The Hamiltonian can be written as

$$H = \sum_k > 0 \omega_k b_{-k} b_k + E_0, \qquad \omega_k = \frac{2\pi |k|}{L}. \tag{108}$$

4.5.2 Mapping onto the Plane

Mapping ψ to the plane gives

$$\psi_{cyl}(z) = \sqrt{\frac{2\pi z}{L}} \psi_{pl}(z) \tag{109}$$

and thus

$$\psi(z) = \sum_k b_k z^{-k-\frac{1}{2}} \tag{110}$$

This transformation swaps the boundary Conditions

$$\psi(e^{2\pi i} z) = \psi(z) \qquad (Ramond) \tag{111}$$

$$\psi(e^{2\pi i} z) \psi(z) \qquad (Neveu - Schwarz) \tag{112}$$

$$\tag{113}$$

The different sectors will have a different two-point correlation function. For the NS sector we have

$$\langle \psi(z) \psi(w) \rangle = \frac{1}{z - w}. \tag{114}$$

In the R sector we have

$$\langle \psi(z) \psi(w) \rangle = \frac{1}{2} \frac{\sqrt{z/w} + \sqrt{w/z}}{z - w}. \tag{115}$$

Furthermore depending on the boundary conditions the energy-momentum tensor will gain a non-zero expectation value.

$$\langle T(z) \rangle = 0 \qquad (Neveu - Schwarz) \tag{116}$$

$$\langle T(z) \rangle = \frac{1}{16 z^2} \qquad (Ramond) \tag{117}$$

$$\tag{118}$$

4.5.3 Vacuum Energies

The energy momentum tensor on the plane can be written as

$$T(z) = \frac{1}{2} \sum_{n,k} (k + \frac{1}{2}) z^{-n-2} : b_{n-k} b_k :, \tag{119}$$

which naturally leads to

$$L_n = \frac{1}{2}\sum_k (k+\frac{1}{2}) : b_{n-k} b_k : \tag{120}$$

L_0 is given by different expressions depending on the boundary conditions.

$$L_0 = \sum_{k>0} k b_{-k} b_k \qquad (NS) \tag{121}$$

$$L_0 = \sum_{k>0} k b_{-k} b_k + \frac{1}{16} \qquad (R) \tag{122}$$

$$\tag{123}$$

From this we can express the Hamiltonian as

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}). \tag{124}$$

5 Boundary cft

CFT can also be defined in a manifold with boundaries, in which the nice local properties are still applied from the CFT defined on a plane. A boundary cft can be expressed as a cft defined on the upper half plane with a real axis at its bottom, at which fields obey conformal boundaries. A boundary changing operator is inserted at the origin point if the left and right half plane have different boundaries. This whole subject is related to a name, Cardy. He adopted the method of mirror image to map the anti-holomorphic part of the theory to the lower half plane, which resultes in a chiral cft defined on the whole plane. Now the conformal symmetry realized on the plane can be used to study boundary cfts. For example, the two-point correlation functions becomes four-point chiral correlation functions in a boundary cft. The classification of boundary cfts is also closely realted to surface critical behaviors. Different surface critical behaviors are naturally understood as different conformal boundaries.

6 Normal ordering

For free fields the OPE of the field with itself contains only one term with a constant prefactor. It can be regularized by normal ordering the fields, or equivalently, subtracting its expectation value. Using the former prescription for $T(z)T(w)$ only kills the $\propto c$ term. So clearly we need a more elaborate definition of normal ordering. We shall define proper normal ordering for general fields as subtracting all the singular terms from the OPE. We will write this normal ordering as

$$(AB)(z). \tag{125}$$

Concretely, given the OPE

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{126}$$

we have that

$$(AB)(w) = \{AB\}_0(w). \tag{127}$$

Equivalently, we can compute the normal ordering of fields using contour integral methods:

$$(AB)(w) = \frac{1}{2\pi i}\oint \frac{dz}{z-w} A(z)B(w). \tag{128}$$

The contraction of fields contains only the singular terms of the OPE:

$$\overbrace{A(z)B}(w) = \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}. \tag{129}$$

We now want to express the modes of the normal ordered field in terms of the modes of the input fields. Given fields A and B and points $|z| > |x| > |w|$ we write

$$A(z) = \sum_n (z-x)^{-n-h_A} A_n(x) \tag{130a}$$

$$B(w) = \sum_n (w-x)^{-n-h_B} B_n(x). \tag{130b}$$

Contour integrating ultimately results in:

$$(AB)_m = \sum_{n \leq -h_A} A_n B_{m-n} + \sum_{n > -h_A} B_{m-n} A_n, \tag{131}$$

where we defined the modes of (AB) as:

$$(AB)(z) = \sum_n z^{-n-h_A-h_B} (AB)_n. \tag{132}$$

Some warnings are in place:

1. Normal ordering is not commutative: $(AB)(z) \neq (BA)(z)$.
2. Normal ordering is not associative: $((AB)C)(z) \neq (A(BC))(z)$.
3. With this definition of normal ordering, Wick's theorem needs to be revisited. This is done in Appendix 6.B of the Book.

7 Conformal families and Operator algebra

> There's nothing stronger than family.
>
> D. T.

The goal of this section is to introduce the notion of conformal blocks and associated to this the method of conformal bootstrapping as a way to solve CFTs, ie. compute the correlation functions, explicitly. Before that we revisit the notion of descendant fields and conformal families.

In the following we mostly only care about the holomorphic part of fields.

7.1 Descendant fields

A descendant is generated from a primary by acting with the Virasoro operators:

$$\phi^{(-n)}(w) = (L_{-n}\phi)(w) = \frac{1}{2\pi i} \oint_w dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w), \tag{133}$$

in particular:

$$\phi^{(0)}(w) = h\phi(w), \qquad \phi^{(-1)} = \partial\phi(w). \tag{134}$$

Consider following correlation function of states that are part of the same family:

$$\langle (L_{-n}\phi)(w)X\rangle\,, \tag{135}$$

where X denotes a string of primary fields: $X = \phi_1(w_1)...\phi_N(w_N)$. After a computation one finds:

$$\langle (L_{-n}\phi)(w)X\rangle = \mathcal{L}_{-n}\langle \phi(w)X\rangle \qquad (n \geq 1). \tag{136}$$

With the differential operator

$$\mathcal{L}_{-n} = \sum_i \left\{ \frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i} \right\}. \tag{137}$$

In other words, knowing all the correlation functions between primaries, $\langle \phi(w)X\rangle$, is sufficient to compute the correlation functions that involve descendants by applying the differential operators $\mathcal{L}_{-n}$. More generally, for descendants of the form

$$\phi^{(-k,-n)}(w) = (L_{-k}L_{-n}\phi)(w), \tag{138}$$

and so on, we find in a similar way that

$$\left\langle (L_{-k_1}...L_{-k_n}\phi)(w)X\right\rangle = \mathcal{L}_{-k_1}...\mathcal{L}_{-k_n}\langle \phi(w)X\rangle \qquad (n \geq 1) \tag{139}$$

7.2 Conformal families

A *conformal family* is a set of states that transform according to a representation of the conformal group. A family contains a primary and its descendants. We will denote the conformal family associated with the primary ϕ by $[\phi]$. First descendants of a primary are sometimes called *secondary fields*. Another way to say that a conformal family transforms under itself is to say that the OPE of $T(z)$ with any member of the family will be composed solely of other members within the same family. Concretely:

$$T(z)\phi^{(-n)}(w) = \frac{cn(n^2-1)/12}{(z-w)^{n+2}}\phi(w) + \sum_{k=1}^{n}\frac{n+k}{(z-w)^{k+2}}\phi^{(k-n)}(w) + \sum_{k\geq 0}(z-w)^{k-2}\phi^{(-k,-n)}(w) \tag{140}$$

7.3 The operator algebra

The two - and three point functions of a CFT are fixed by conformal invariance. However, we need additional dynamical information to compute the three-point fusion coefficients C_{ijk} (for example using a conformal bootstrap approach). This information is contained in the *operator algebra*. The OPE which also includes the regular terms of all primary fields with each other. Using the operator algebra we can reduce all correlation functions to two-point correlation functions.

First we choose a basis of fields such that $C_{\alpha\beta} = \delta_{\alpha\beta}$ in

$$\left\langle \phi_\alpha(w,\overline{w})\phi_\beta(z,\overline{z})\right\rangle = \frac{C_{\alpha\beta}}{(w-z)^{2h}(\overline{w}-\overline{z})^{2\overline{h}}}. \tag{141}$$

This implies that states belonging to different conformal families are always orthogonal. From scale invariance it follows that:

$$\phi_1(z,\overline{z})\phi_2(0,0) = \sum_p \sum_{\{k,\overline{k}\}} C_{12}^{p\{k,\overline{k}\}} z^{h_p-h_1-h_2+K}\overline{z}^{\overline{h}_p-\overline{h}_1-\overline{h}_2+\overline{K}}\phi_p^{(k,\overline{k})}(0,0). \tag{142}$$

We introduced the notation $K = \sum_i k_i$.

Writing

$$C_{12}^{p\{0,0\}} \equiv C_{12}^{p} = C_{p12}, \tag{143}$$

we find that

$$C_{12}^{p\{k,\bar{k}\}} = C_{12}^{p} \beta_{12}^{p\{k\}} \overline{\beta}_{12}^{p\{\bar{k}\}}. \tag{144}$$

This means that descendants fields are correlated to a given third field only if the primary is correlated. An the holomorphic and antiholomorphic parts factorize.

An example is given in the Book. Even in a relatively simple case, finding the three point function is not straightforward!

In conclusion, given the central charge, the conformal dimensions and the three-point coefficients C_{pnm}, one can - in principle - determine the operator algebra. Using the operator algebra, all the n-point correlation functions can be computed and the entire theory is solved.

7.4 Conformal blocks

Let us illustrate how the four-point functions can be reduced to three-point functions using the machinery introduced in the previous sections.

We consider the four-point function

$$\left\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\right\rangle. \tag{145}$$

For sake of simplicity, we shall carry out a global conformal transformation to put $z_4 = 0$, $z_1 = \infty$, $z_2 = 1$, $z_3 = x$. We define:

$$G_{34}^{21}(x,\overline{x}) = \left\langle h_1,\bar{h}_1 \middle| \phi_2(1,1)\phi_3(x,\overline{x}) \middle| h_4,\bar{h}_4 \right\rangle. \tag{146}$$

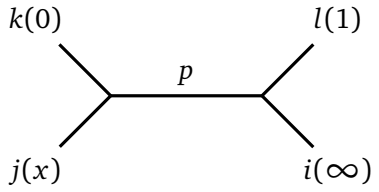
Note the order of the indices!

Using operator algebra techniques, we can write this function as:

$$G_{34}^{21}(x,\overline{x}) = \sum_p C_{34}^{p} C_{12}^{p} A_{34}^{21}(p|x,\overline{x}). \tag{147}$$

The sum over p is a sum over intermediate conformal families that play the role of mediating channels in the scattering from fields from $(0,x)$ towards $(1,\infty)$. These functions $A_{34}^{21}(p|x,\overline{x})$ are called *partial waves*. They can be depicted by:

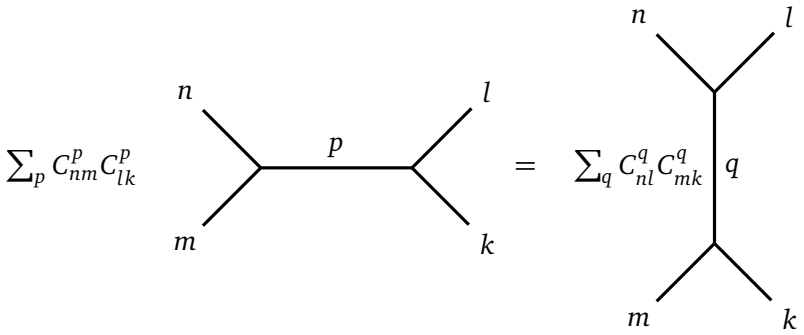
$A_{kl}^{ji}(p|x,\overline{x}) =$



The partial wave factorizes in a holomorphic and antiholomorphic part, according to:

$$A_{34}^{21}(p|x,\overline{x}) = \mathcal{F}_{34}^{21}(p|x)\overline{\mathcal{F}}_{34}^{21}(p|\overline{x}), \tag{148}$$

we call these functions $\mathcal{F}$, the *conformal blocks*. There is a recipe to compute these conformal blocks, even though it is a pain to compute these in practice. Physically speaking, these conformal blocks are the part in the four-point function that is fixed by conformal invariance. They depend on the anharmonic ratios via a series expansion. The remaining elements are the three-point coefficients, which are *not* fixed by conformal invariance.



7.5 Crossing symmetry and the conformal bootstrap

What happens if we would choose instead of $z_4 = 0$, $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ a different order of the fields? Following identities can be obtained relatively easy:

$$G^{21}_{34}(x,\overline{x}) = G^{41}_{32}(1-x,1-\overline{x}), \tag{149}$$

and

$$G^{21}_{34}(x,\overline{x}) = \frac{1}{x^{2h_3}\overline{x}^{2\bar{h}_3}} G^{24}_{31}(1/x,1/\overline{x}). \tag{150}$$

These identities are specific instances of the *crossing symmetry* of the functions G. Explicitly we can write the first identity as

$$\sum_p C^p_{21} C^p_{34} \mathcal{F}^{21}_{34}(p|x)\overline{\mathcal{F}}^{21}_{34}(p|\overline{x}) = \sum_q C^q_{41} C^q_{32} \mathcal{F}^{41}_{32}(p|1-x)\overline{\mathcal{F}}^{41}_{32}(p|1-\overline{x}), \tag{151}$$

which has a aesthetically pleasing pictorial interpretation: Crossing symmetry imposes N^4 constrains on the $N^3 + N$ parameters C^p_{mn} and h_p. Exploiting crossing symmetry to compute these parameters is called *conformal* bootstrap.

A Central extensions of Lie algebras

In this section $\mathfrak{g}, \mathfrak{h}, \ldots$ denote (possibly infinite) Lie algebras over some field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. This section is mainly based on Wikipedia and [**?**].

A.1 Extensions

Definition: A *Lie algebra extension* is a short exact sequence of Lie algebras:

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g}. \tag{152}$$

One calls $\mathfrak{e}$ an extension of $\mathfrak{g}$ by $\mathfrak{h}$. By exactness of the sequence one has $\mathfrak{g} \cong \mathfrak{e}/\operatorname{Im}\iota$.

Definition: A *central extension* is an extension $\mathfrak{e}$ of $\mathfrak{g}$ by $\mathfrak{h}$, such that $\operatorname{Im}\iota$ is contained in the center of $\mathfrak{e}$, $\iota(\mathfrak{h}) \subseteq Z(\mathfrak{e})$.

Notice that for a central extension $\mathfrak{h}$ is necessarily abelian. We now introduce a notion of trivial central extensions as follows:

Definition: A Lie algebra extension

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \tag{153}$$

splits if there exists a Lie algebra morphism $\beta : \mathfrak{g} \mapsto \mathfrak{e}$ such that $\pi \circ \beta = \mathrm{id}_{\mathfrak{e}}$. β is called a splitting map.

A central extension

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g}. \tag{154}$$

that splits is trivial in the sense that it is equivalent[1] to one where $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{h}$.

Let us now consider a central extension and a map (not necesserily a Lie algebra homomorphism) $\beta : \mathfrak{g} \to \mathfrak{e}$ such that $\pi \circ \beta = \mathrm{id}_{\mathfrak{e}}$. From this map construct $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ as follows:

$$\Theta(x, y) := [\Theta(x), \Theta(y)] - \Theta([x, y]). \tag{155}$$

This map is:

1. Antisymmetric.
2. Bilinear.
3. Satisfies $\Theta(x, [y, z]) + \Theta(y, [z, x]) + \Theta(z, [x, y]) = 0$.

Given Θ one can now show that there is an isomorphism between the vector spaces $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{h}$ that is given by:

$$\Psi : \mathfrak{g} \oplus \mathfrak{h} \mapsto \mathfrak{e} : (x, y) \mapsto \beta(x) + y. \tag{156}$$

A Lie bracket on $\mathfrak{g} \oplus \mathfrak{h}$ is given by:

$$[x \oplus z, y \oplus z']_{\mathfrak{e}} := [x, y]_{\mathfrak{g}} + \Theta(x, y). \tag{157}$$

Lemma: In the above construction β is a splitting map if and only if

$$\Theta(x, y) = \mu([x, y]), \tag{158}$$

for some $\mu \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$.

Now comes the classification of central extensions of Lie algebras:

Theorem: Every central extension comes from a map Θ that satisfies the above properties (1-3). Conversely, every central extension gives rise to a map Θ that satisfies the above properties (1-3).

A.2 Lie algebra cohomology

The classification of Lie algebra extensions is very satisfying. It smells a lot like a cohomological classification. Indeed, the extensions are classified by functions depending on two variables satisfying the condition (3) that is exactly the one needed to fulfill the Jacobi identity of the central extension. Moreover, the central extension is trivial if the *2-cocycle* Θ is trivial in the following sense: $\Theta(x, y) = \mu([x, y])$. This is reminiscent of considering 2-cocycles to be trivial if they are equal to a coboundary. Let us put this on a bit more rigorous footing.

Definitions:

1. $Z^2(\mathfrak{g}, \mathfrak{h}) = \left\{\Theta \in \Lambda^2(\mathfrak{g}, \mathfrak{h}) | \Theta : (3)\right\}$.

[1] To do: introduce the notion of equivalent extensions.

2. $B^2(\mathfrak{g},\mathfrak{h}) = \{\Theta : \mathfrak{g}\times\mathfrak{g} \mapsto \mathfrak{h} | \exists \mu \in \mathrm{Hom}(\mathfrak{g},\mathfrak{h}) : \Theta(-,-) = \mu([-,-])\}$.

3. $H^2(\mathfrak{g},\mathfrak{h}) := Z^2(\mathfrak{g},\mathfrak{h})/B^2(\mathfrak{g},\mathfrak{h})$.

H^2 is of course called the second cohomology group. We thus obtain the following reformulation of the classification of central extensions:

Theorem: The equivalence classes of central extensions

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \tag{159}$$

are in one-to-one correspondence with the elements of $H^2(\mathfrak{g},\mathfrak{h})$.

For completeness, let us introduce a notion of cochain complexes for Lie algebras. A cochain f is a alternating multilinear map f:

$$f : \Lambda^n\mathfrak{g} \mapsto \mathfrak{h}. \tag{160}$$

Here, $\mathfrak{h}$ is considered a $\mathfrak{g}$-module or -representation.

The differential of an n-cochain is given by

$$\begin{aligned}(df)(x_1,\ldots,x_{n+1}) = &\sum_i (-1)^{i+1} x_i f(x_1,\ldots,\hat{x}_i,\ldots,x_{n+1}) + \\ &\sum_{i<j} (-1)^{i+j} f\left([x_i,x_j],x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1}\right),\end{aligned} \tag{161}$$

so for example, with trivial action we obtain

$$(df)(x_1,x_2) = f([x_1,x_2]), \tag{162}$$

and

$$\begin{aligned}(df)(x_1,x_2,x_3) &= -f([x_1,x_2],x_3) + f([x_1,x_3],x_2) - f([x_2,x_3],x_1) \\ &= -f([x_1,x_2],x_3) - f([x_3,x_1],x_2) - f([x_2,x_3],x_1) \\ &= f(x_3,[x_1,x_2]) + f(x_2,[x_3,x_1]) + f(x_1,[x_2,x_3]).\end{aligned} \tag{163}$$

So clearly, $Z^2(\mathfrak{g},\mathfrak{h})$ defined above is the group of 2-cocycles satisfying $d\Theta = 0$ and $B^2(\mathfrak{g},\mathfrak{h})$ the set of coboundaries: $\Theta = d\mu$.