Note for 2d CFT

Reading Club

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Abstract

The Yellow Book Notes. It is good to write notes!

Contents

1 Preliminary

1.1 Conventions

This part summrises conventions used in this note.

Metric tensor and Coordinate.— The metric tensor in Minkowski and Euclidean space-time is defined as

$$\eta_{\alpha\beta} = \begin{Bmatrix} +1 & & \\ & -1 & \\ & & \ddots \end{Bmatrix} \tag{1}$$

and

$$g_{\alpha\beta} = \begin{Bmatrix} +1 & & \\ & +1 & \\ & & \ddots \end{Bmatrix} \tag{2}$$

respectively, where the first index is the time. In the Yellow Book, without specifications, we are working in Eclidean space. The coordinate is defined as $x^\mu = (t, x^\alpha)$. So that the norm of a vector in Minkowski space-time is $x^\mu x_\mu = t^2 - r^2$.

γ **matrices.–** The γ matrices follow the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{3}$$

In Minkowski space time , the γ matrices can be chosen as

$$\gamma^0 = \sigma^x$$
$$\gamma^1 = i\sigma^y$$
$$\gamma^3 = -\gamma^0\gamma^1 = \sigma^z$$

where γ^3 plays the role of γ^5. In Elidean space, they can be chosen as

$$\gamma^0 = \sigma^x$$
$$\gamma^1 = \sigma^y. \tag{4}$$

Below in the fermon case, we are going to show how we fix the convention for the Euclidean space gamma matrix. We also usually a slashed notation for $\not{\partial} := \gamma^\mu \partial_\mu$.

Complex Coordinate
In CFT, there are two independent Virasoro algebras. Fields have holomorphic (resp. antiholomorphic) part $\phi_R(x-t)$ (reps. $\phi_L(x+t)$). It is usually more convenient to use imaginary time $\tau = it$ and complex coordinates

$$x^z = x^0 + ix^1$$
$$x^{\bar{z}} = x^0 - ix^1$$
$$\partial_z = \frac{1}{2}(\partial_0 - i\partial_1)$$
$$\partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1). \tag{5}$$

Here x^z and $x^{\bar{z}}$ denote coordinates with upper index. They are usually denoted by z and $\bar{z}$. One can simply treat z and $\bar{z}$ as ordinary upper coordinates and write down the fields with the new coordinate. For example, the right and left mover scalar fields become $\phi(z)$ and $\bar{\phi}(\bar{z})$ respectively. In terms of the new coordinate the metric tensor is

$$g_{\mu\nu} = \begin{Bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{Bmatrix} \qquad g^{\mu\nu} = \begin{Bmatrix} 0 & 2 \\ 2 & 0 \end{Bmatrix} \tag{6}$$

Note that one has the freedom to interprete x^0 as time or space, which only change conventions rather than physics. For example, the correlation functions of physical operators always contain the chiral and anti-chiral parts. In particular, in Euclidean space the theory has rotation symmetry.

1.2 Free fermions

In Minkowski space time ($d = 1 + 1$), the free Dirac spinor has two componenet. Its action reads [1,2]

$$
\begin{aligned}
S &= \frac{1}{4\pi K} \int dx dt \ \left(i\psi_+^\dagger (\partial t - \partial x)\psi_+ + i\psi_-^\dagger (\partial t + \partial x) \psi_- \right). \\
&= \frac{1}{4\pi K} \int dx dt \ \left(i\psi_+^\dagger \partial_- \psi_+ + i\psi_-^\dagger \partial_+ \psi_- \right).
\end{aligned}
\tag{7}
$$

The equation of motion has the solution for the chiral fermions

$$
\begin{aligned}
\psi_+ &= \psi_+(t + x) \\
\psi_- &= \psi_-(t - x)
\end{aligned}
\tag{8}
$$

They correspond to left and right moving particles as the speed of light. In terms of $\psi = (\psi_+, \psi_-)^t$, one can write the theory as

$$
\begin{aligned}
S &= \frac{1}{4\pi K} \int dx dt \ i\psi^\dagger (\partial_t - \sigma^z \partial_x) \psi \\
&= \frac{1}{4\pi K} \int dx dt \ i\psi^\dagger (\sigma^x \sigma^x \partial_t + \sigma^x i\sigma^y \partial_x) \psi \\
&= \frac{1}{4\pi K} \int dx dt \ \psi^\dagger i\gamma^0 \gamma^\mu \partial_\mu \psi \\
&= \frac{1}{4\pi K} \int dx dt \ \overline{\psi} i\slashed{\partial} \psi
\end{aligned}
\tag{9}
$$

where we used

$$
\gamma^0 = \sigma^x \quad \gamma^1 = i\sigma^y \quad \text{and} \quad \overline{\psi} = \psi^\dagger \gamma^0
\tag{10}
$$

A mass term $m\overline{\psi}\psi = m\left(\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+ \right)$ can be added into the action

$$
S = \frac{1}{4\pi K} \int dx dt \ \overline{\psi} \left(i\slashed{\partial} - m \right) \psi.
\tag{11}
$$

The massless fermions are Weyl spinors without a mixing between the left and right moving parts.

Wick rotation

It is usually more convenient to work in Euclidean space rather than Minkowski space time. We first assume the Wick rotation acts as $t \to i\tau$. The left mover under this convention is antiholomorphic $\psi_+ \sim f(\tau - ix)$. The action follows

$$
i S_M \to - S_E.
\tag{12}
$$

Specifically, in terms of spinors one has

$$
\begin{aligned}
-i\,S &= -i\frac{1}{4\pi K}\int dx\,d(i\tau)\,i\left(\psi_+^\dagger\,(\partial_{i\tau}-\partial_x)\,\psi_+ + \psi_-^\dagger\,i(\partial_{i\tau}+\partial_x)\,\psi_-\right) \\
&= \frac{1}{4\pi K}\int dx\,d\tau\left(\psi_+^\dagger\,(\partial_\tau-i\partial_x)\,\psi_+ + \psi_-^\dagger\,(\partial_\tau+i\partial_x)\,\psi_-\right) \\
&= \frac{1}{4\pi K}\int d^2x\,2\left(\psi_+^\dagger\,\partial\psi_+ + \psi_-^\dagger\,\overline{\partial}\psi_-\right) \\
&= \frac{1}{4\pi K}\int d^2z\left(\psi_+^\dagger\,\partial\psi_+ + \psi_-^\dagger\,\overline{\partial}\psi_-\right)
\end{aligned}
\tag{13}
$$

The equation of motion tells us that, ψ_+ is antiholomorphic $\psi_+ \sim f(\overline{z})$, consistent with the Wick rotation assumption. Respectively ψ_- is a holomorphic field.

If we choose $t \to -i\tau$, one can find $t+x \to -i(\tau+ix)$, which means that ψ_+ now becomes a holomorphic field $\psi_+ \sim f(z)$. The action becomes

$$
\begin{aligned}
-i\,S &= -i\frac{1}{4\pi K}\int dx\,d(-i\tau)\,i\left(\psi_+^\dagger\,(\partial_{-i\tau}-\partial_x)\,\psi_+ + \psi_-^\dagger\,i(\partial_{-i\tau}+\partial_x)\,\psi_-\right) \\
&= \frac{1}{4\pi K}\int dx\,d\tau\left(\psi_+^\dagger\,(\partial_\tau+i\partial_x)\,\psi_+ + \psi_-^\dagger\,(\partial_\tau-i\partial_x)\,\psi_-\right) \\
&= \frac{1}{4\pi K}\int d^2x\,2\left(\psi_+^\dagger\,\overline{\partial}\psi_+ + \psi_-^\dagger\,\partial\psi_-\right) \\
&= \frac{1}{4\pi K}\int d^2z\left(\psi_+^\dagger\,\overline{\partial}\psi_+ + \psi_-^\dagger\,\partial\psi_-\right)
\end{aligned}
\tag{14}
$$

It is clear that under this convention, ψ_+ is a holomorphic field. To align with the boson theory, in this note we choos the latter one

$$
\begin{aligned}
&t \to -i\tau \\
&\psi_+ \sim f(z) \\
&\psi_- \sim f(\overline{z}).
\end{aligned}
\tag{15}
$$

It is essential to exercise caution when employing both complex and real coordinates in the subsequent discussions. Specifically, we will be utilizing both $\psi_+(t+x)$ and $\psi_+(z)$, although strictly speaking, the latter should be expressed as $\psi_+(-i(\tau+ix)) = \psi_+(-iz)$.

In terms of spinors one can find that

$$
\begin{aligned}
S &= \frac{1}{4\pi K}\int dx\,d\tau\left(\psi^\dagger\partial_\tau\psi + \psi^\dagger i\sigma^z\partial_x\psi\right) \\
&= \frac{1}{4\pi K}\int dx\,d\tau\left(\psi^\dagger\sigma^x\sigma^x\partial_\tau\psi + \psi^\dagger\sigma^x\sigma^y\partial_x\psi\right) \\
&= \frac{1}{4\pi K}\int dx\,d\tau\,\overline{\psi}\,\slashed{\partial}\psi
\end{aligned}
\tag{16}
$$

Note that the integral meausre also contriutes a $1/2$ factor. This fixes our convention of γ matrices in Euclidean space

$$
\begin{aligned}
&\gamma^0 = \sigma^x \\
&\gamma^1 = \sigma^y
\end{aligned}
\tag{17}
$$

The two point correlation function reads

$$
\begin{aligned}
\langle \psi_+^\dagger(z)\,\psi_+(w) \rangle &= K\frac{1}{z-w} \\
\langle \psi_-^\dagger(\bar{z})\,\psi_-(\bar{\omega}) \rangle &= K\frac{1}{\bar{z}-\bar{w}} \\
\langle \psi_-(\bar{z})\,\psi_+(w) \rangle &= 0
\end{aligned}
\tag{18}
$$

Hamiltonian formalism—
The canonical momentum reads $\frac{\partial L}{\partial \dot{\psi}} = i\psi^\dagger$ (or $-i\psi^\dagger$?). One can write down the Hammiltonian

$$
\begin{aligned}
H &= \int dx \left(-i\psi^\dagger \gamma^0 \gamma^1 \partial_x \psi \right) \\
&= \int dx \left(i\psi^\dagger \sigma^z \partial_x \psi \right) \\
&= \int dx \left(i\psi_+^\dagger \partial_x \psi_+ - i\psi_-^\dagger \partial_x \psi_- \right) \\
&= \int dk \left(-k\psi_+^\dagger(k)\psi_+(k) + k\psi_-^\dagger(k)\psi_-^\dagger(k) \right)
\end{aligned}
\tag{19}
$$

which demonstrates that the left (ψ_+) and right (ψ_-) moving spinors have a dispersion relation $\omega \sim -k$ and $\omega \sim k$ respectively.

A lattice version free fermion theory Eq. 2.38 reads

$$
\mathcal{L} = \sum_n \left(i\psi_n^\dagger \dot{\psi}_n + \psi_n^\dagger \psi_{n+1} + \psi_{n+1}^\dagger \psi_n \right).
\tag{20}
$$

The canonical momentum corresponding to ψ_n is

$$
\pi_n = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_n} = i\psi_n^\dagger.
\tag{21}
$$

So that the Hamiltonian is

$$
\begin{aligned}
\mathcal{H} &= \sum_n \pi_n \dot{\psi}_n - \mathcal{L} \\
&= -\sum_n \left(\psi_n^\dagger \psi_{n+1} + \psi_{n+1}^\dagger \psi_n \right).
\end{aligned}
\tag{22}
$$

Note that in the above we have adopted

$$
\delta\mathcal{L} = \pi\,\delta\dot{\psi}
\tag{23}
$$

which fixes the transformation to be

$$
\mathcal{H} = \pi\dot{\psi} - \mathcal{L}.
\tag{24}
$$

Equivalently we can also define the derivative as

$$
\delta\mathcal{L} = \delta\dot{\psi}\,\pi
\tag{25}
$$

Now we have

$$
\mathcal{H} = \dot{\psi}\,\pi - \mathcal{L}.
\tag{26}
$$

Symmetries and conserved current

Global continuous $U(1)$ symmetry

The free fermion has the global $U(1)$ symmetry defined by $\psi \to e^{i\alpha}\psi = \psi + i\alpha\psi$. The functional derivative is then $\frac{\delta F}{\delta \alpha} = i\psi$. One can find the corresponding Noether's curret reads

$$j_\nu^\mu \sim -i\overline{\psi}\gamma^\mu(i\psi) = \overline{\psi}\gamma^\mu\psi. \tag{27}$$

The charge density is the total fermion charge density $j_\nu^0 \sim \psi^\dagger\psi = \psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-$.

The free fermion theory also admits an axial $U(1)$ symmetry $\psi \to e^{i\alpha\gamma^5}\psi = \psi + i\alpha\gamma^5\psi$. Its corresponding Noether's current writes

$$j_a^\mu \sim \overline{\psi}\gamma^\mu\gamma^5\psi \tag{28}$$

whose charge density reads $j_a^0 \sim \psi^\dagger\gamma^5\psi = \psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-$. It is the difference in the density between the left and right movers.

1.3 Free boson

The free (non-) compact boson in the Minkoswki space time reads [1]

$$S = \frac{1}{8\pi K}\int \, dxdt \, \partial_\mu\varphi\partial^\mu\varphi, \tag{29}$$

where φ is a real scalar field. After Wick rotation $\tau = it$, it becomes

$$
\begin{aligned}
iS &= \frac{i}{8\pi K}\int \, dxdt \, \partial_t\varphi\partial_t\varphi - \frac{i}{2}g\int \, dxdt \, \partial_x\varphi\partial_x\varphi \\
&= -\frac{1}{8\pi K}\int \, dxd\tau \, \partial_\tau\varphi\partial_\tau\varphi - \frac{1}{2}g\int \, dxdit \, \partial_x\varphi\partial_x\varphi \\
&= -\frac{1}{8\pi K}\int \, dxd\tau \, \partial_\mu\varphi\partial^\mu\varphi
\end{aligned} \tag{30}
$$

The Euclidean action reads

$$S_E = \frac{1}{8\pi K}\int \, d^2x \, \partial_\mu\varphi\partial^\mu\varphi. \tag{31}$$

To find the two-point correlation function, let's rewrite the action as

$$S = \frac{1}{2}\int \, d^2xd^2y \, \varphi(x)A(x,y)\varphi(y), \tag{32}$$

where $A(x,y) = -\frac{1}{4\pi K}\partial^2$. The two-point correlation function satisfies

$$-\frac{1}{4\pi K}\partial^2\langle\varphi(x)\varphi(y)\rangle = \delta(x-y). \tag{33}$$

Integrating both sides, one can find

$$
\begin{aligned}
1 &= -\frac{1}{4\pi K}\int_0^\rho 2\pi rdr\left(\frac{1}{r}\frac{\partial}{\partial r}\left(rG'(r)\right)\right) \\
&= -\frac{1}{2K}\rho G'(\rho).
\end{aligned} \tag{34}
$$

One then finds that the two point correlation up to a constant term is

$$\langle \varphi(x)\varphi(y) \rangle = -2K \log(r), \tag{35}$$

where r is the distance between the two points. This is consistent with the fact that the scaling dimension of φ is 0. The above result is actually not well defined as the two-point correlation has both short and long distance divergence. Let's rewritte the Euclidean action as

$$S_E = \frac{1}{4\pi K} \int \frac{d^2k}{(2\pi)^2} \, \varphi_k \left(\frac{k^2}{2} \right) \varphi_{-k}. \tag{36}$$

One finds the real-space two-point correlation function

$$G(r) = \langle \varphi(x)\varphi(y) \rangle = 4\pi K \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikr \cos(\theta)}}{k^2} \tag{37}$$

which is divergent at IR and UV. One can remove the $k = 0$ and high energy modes ($k > 2\pi/a$) to redefine a convergent correlation function

$$G(r) - G(0) = 4\pi K \int \frac{d^2k}{(2\pi)^2} \frac{e^{ikr \cos(\theta)} - 1}{k^2}. \tag{38}$$

The large k contribution results in

$$G(r) - G(0) \sim -2K \log(r) \tag{39}$$

A class of useful operators are the so-called (normal ordered) vertex operator $V_\alpha(x) := e^{i\alpha \varphi(x)}$. Its two point correlation reads

$$\langle V_\alpha(x) V_{-\beta}(y) \rangle = e^{\alpha^2 \langle \varphi(x)\varphi(y) \rangle} \sim \frac{\delta_{\alpha,\beta}}{r^{2\alpha^2 K}}. \tag{40}$$

One can find that the exponents of vertex operators in the correlation functions must sum up to zero. This is usually called neutral conditon for vertex operators. This actually reflects the $U(1)$ symmetry of the free scalar boson.

One can also write down the Hamiltonian for the free boson. The canonical momentum corresponding to φ is $\pi = \frac{\partial L}{\partial \dot{\varphi}} = \frac{\dot{\varphi}}{4\pi K}$. One finds the Hamiltonian is

$$H = \int dx \left(2\pi K \pi^2 + \frac{1}{8\pi K}(\partial_x \varphi)^2 \right). \tag{41}$$

Complex coordinate—.
The theory can be defined on a cylinder in which the coordinate is $\omega = \tau + i\sigma$ where σ is defined on a circle with a unit radius. We can use another complex coordinate

$$z = e^\omega = e^\tau e^{i\sigma} \tag{42}$$

to rewritte the theory on a plane parameterized with z.

Let's look at the same theory in terms of the complex coordinate, which is heavily used in the $2d$ CFT. We first write down the action as

$$S = \frac{1}{8\pi K} \int d^2x \sqrt{g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi. \tag{43}$$

With the complex coordinate introduced previously, one can immedialy write down the action as

$$S = \frac{1}{8\pi K} \int dz d\bar{z} \frac{1}{2} \left(2\partial \varphi \bar{\partial} \varphi + 2\bar{\partial} \varphi \partial \varphi \right)$$
$$= \frac{1}{4\pi K} \int dz d\bar{z} \left(\partial \varphi \bar{\partial} \varphi \right) \tag{44}$$

The chiral two-point correlation can be deduced from the physical one to be $\langle \varphi(z)\varphi(0) \rangle \sim = -K \log(z)$.

8

1.4 Maxwell equation

We define the vector-pontential or gauge field and current as

$$
\begin{aligned}
A^\mu &= (\phi, \mathbf{A}) \\
j^\mu &= (\rho, \mathbf{j})
\end{aligned}
\tag{45}
$$

and the field strength

$$
F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu.
\tag{46}
$$

The electric field $\mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$ and magnetic fields are related to $F_{\mu\nu}$ via

$$
\begin{aligned}
F_{0i} &= \mathbf{E} \\
F_{ij} &= -\epsilon_{ijk}B_k.
\end{aligned}
\tag{47}
$$

Note that $\mathbf{E}$ and $\mathbf{B}$ are *not* vectors, and ϵ_{ijk} is the Levi-Civita symbol (not a tensor). The Maxwell equation now can be expressed in a compact form

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= j^\nu \\
\epsilon_{\mu\nu\rho\sigma}\partial_\rho F_{\mu\nu} &= 0
\end{aligned}
\tag{48}
$$

where the current j^μ is a vector $j^\mu := (\rho, j^i)$.

1.5 Symmetries at the classical level

The action becomes different after a coordinate transformation. We say it has a symmetry if it remains unchanged and a Noether current can be derived from the symmetry. The coordinate transformation is denoted as

$$
x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}
\tag{49}
$$

and the field changes according to

$$
\phi'(x') = \phi(x) + \omega_a \frac{\delta F}{\delta \omega_a}(x)
\tag{50}
$$

where ω_a is a constant and small parameter.

By definition, the change of the action δS disappears for a symmetric transformation. We can get nothing new from this. If we allow ω_a to be arbitrary, the leading contribution to δS becomes

$$
\delta S = -\int d^2x\, j^\mu \partial_\mu \omega_a,
\tag{51}
$$

where we introduced the the current j^μ. We assume it decreases fast when approaching infinite. So that one obtains

$$
\delta S = \int d^2x\, \partial_\mu j^\mu\, \omega_a.
\tag{52}
$$

This equations holds for all the field configurations. If we require the field configuration to be the one obeying the equation, the action should be invariant for arbitrary coordinate transformation and one finds the conservation of j^μ

$$
\partial_\mu j^\mu = 0.
\tag{53}
$$

Energy-momentum tensor The canonical energy-momentum tensor is defined to be the Noether current of the translation transformation

$$
x'^{\mu} = x^{\mu} + \epsilon^{\nu} \delta^{\mu}_{\nu} \tag{54}
$$

$$
T^{\mu\nu} = -\eta^{\mu\nu} L + \frac{\partial L}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi. \tag{55}
$$

This definition of $T^{\mu\nu}$ is not guaranteed to be symmetric between the two indices (The requirement of a symmetric $T^{\mu\nu}$ will be clear later).

Another definition that makes the energy-momentum tensor symmetric follows. In the coordinate transformation, if we also consider the variance of the metric tensor (which means the theory is coupled with the dynamical background)

$$
\delta g_{\mu\nu} = -\partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} \tag{56}
$$

the action remains invariant since this is nothing but a reparametrization of the theory (general coordinate covariance). So that one finds

$$
\delta S = 0 = -\frac{1}{2} \int d^{d}x \, \left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right) \left(T^{\mu\nu} + 2\frac{\delta S}{\delta g_{\mu\nu}}\right). \tag{57}
$$

So that one can define the energy-momentum tensor as

$$
T^{\mu\nu} = -2\frac{\delta S}{\delta g_{\mu\nu}} \tag{58}
$$

up to a surface term.

Another way to make the energy-momentum tensor symmetric is add a surface term to the canonical one. One can show that with rotation symmetry, such a term can be constructed to make $T^{\mu\nu}$ symmetric.

1.6 Symmetry at the quantum level

All the field configurations contribute to the quantum theory, so that one has no Noether current in general. Still the symmetry has constraints to the quantum theory. For the $n-$point correlation functions, one has

$$
\langle \phi(x'_1) \cdots \phi(x'_n) \rangle = \frac{1}{Z} \int [D\phi] \, \phi(x'_1) \cdots \phi(x'_n) \, e^{-S[\phi]} \tag{59}
$$

$$
= \frac{1}{Z} \int [D\phi'] \, \phi'(x'_1) \cdots \phi'(x'_n) \, e^{-S'[\phi']} \tag{60}
$$

$$
= \frac{1}{Z} \int [D\phi] \, F(\phi(x_1)) \cdots F(\phi(x_n)) \, e^{-S[\phi]} \tag{61}
$$

$$
= \langle \, F(\phi(x_1)) \cdots F(\phi(x_n)) \, \rangle \tag{62}
$$

in which we assumed the functional integral measure does not change and the coordinate transformation is a rigid one (ω_a is a constant).

Ward identity As stated above there is no conserved current at the quantum level. The infinitesimal coordinate transformation at the quantum level results in the so-called Ward identity.

We denote the change of fields as

$$
\phi'(x) = \phi(x) - i\omega_a \, G_a \, \phi(x). \tag{63}
$$

The infinitesimal coordinate transformation (ω_a now is arbitrary) changes the correlation as (We only consider the first order perturbation contribution)

$$
\begin{aligned}
\langle \phi'(x_1)\cdots\phi'(x_n)\rangle &= \langle \phi(x_1)\cdots\phi(x_n)\rangle && (64)\\
&= \frac{1}{Z}\int [D\phi'](X+\delta X)e^{-S[\phi]-\int d^d x\,\partial_\mu j^\mu \omega_a} && (65)\\
&= \frac{1}{Z}\int [D\phi](X+\delta X)e^{-S[\phi]-\int d^d x\,\partial_\mu j^\mu \omega_a} && (66)\\
&= \langle X\rangle - \int [D\phi]\int d^d x\, X\partial_\mu j^\mu \omega_a e^{-S[\phi]} - \int [D\phi]\delta X e^{-S[\phi]} && (67)
\end{aligned}
$$

so that one finds

$$
\langle \delta X\rangle = \int d^d x\, \partial_\mu \langle j^\mu X\rangle \omega_a(x). \tag{68}
$$

As

$$
\begin{aligned}
\delta X &= -i\sum_i \phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\omega_a(x_i) && (69)\\
&= -i\int d^d x\sum_i \phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\delta(x-x_i)\omega_a(x) && (70)
\end{aligned}
$$

Since ω_a is arbitrary, one obtains the Ward identity

$$
\partial_\mu \langle j^\mu X\rangle = -i\sum_i \delta(x-x_i)\,\langle \phi(x_1)\cdots G_a\phi(x_i)\cdots\phi(x_n)\rangle. \tag{71}
$$

So that for each symmetry, there exists a Ward identity, i.e., a constraint to the correlation function. With enough symmetries, one can get all the information of the correlation functions.

1.7 Renormalization group

Dimensional analysis and renormalizability of QFT Let's start with the canonical dimension of fields and couplings in the action,

$$
S = \int d^d x\, \mathcal{L}(\phi,\lambda). \tag{72}
$$

Since the action is dimensionless, every term in $\mathcal{L}$ has an energy scaling dimension of

$$
\Delta(\mathcal{L}) = [\mathcal{L}] = \omega^d \tag{73}
$$

which determines the canonical dimension fields and couplings. The renormalizability of a QFT is directly obtained from the energy dimension of Feynman diagrams,

$$
\mathcal{D} = d - E_\phi \Delta(\phi) - \Delta(\lambda_i) \tag{74}
$$

where E_ϕ is the number of external fields and λ_i the couplings in the theory. A nice discussion about renormalizability can be found online (https://web2.ph.utexas.edu/ vadim/ Classes/2022f/notes.html).

Super-renormalizable theories have only couplings with positive dimensions. For such theories, there are finite Feynman diagrams become divergent in the perturbation calculation. Renormalizable theories have couplings with non-negative dimensions, in which a finite number of couplings have zero dimensions. There exists infinite number of divergent Feynman

diagrams, but the number of divergent amplitudes is finite. If there is at least one coupling with a negative dimension, the theory is non-renormalizable.

Wilson-Kadanoff RG scheme The renormalization group (RG) builds up the modern understanding of QFT, which is regarded as an *effective field theory*. In the history, many different RG schemes have been developed, which are suitable for very different theories. Most of them are realized in a perturbation way around a known RG fixed point. Here we briefly recall the most popular one, i.e. the Wilson-Kadanoff RG scheme.

In this scheme, a momentum cutoff $k < \Lambda$ is introduced. One first divides modes into fast $\Lambda/s < k < \Lambda$ and slow $k < \Lambda/s$ parts $\phi = \phi_f + \phi_s$. The fast modes are integrated out to result in a new theory

$$e^{-S'(\phi)_{\Lambda/s}} = \int D\phi_{\Lambda/s < k < \Lambda} \; e^{-S_\Lambda(\phi)} \tag{75}$$

with a smaller cutoff Λ/s. Generally, the action can be divided into three parts

$$S = S_f(\phi_f) + S_s(\phi_s) + S_c(\phi_f, \phi_s). \tag{76}$$

The new theory thus can be written as

$$
\begin{aligned}
e^{-S'(\phi_s)_{\Lambda/s}} &= \int D\phi_f e^{-S_f - S_s - S_c} \\
&= e^{-S_s} Z_f \frac{\int D\phi_f \; e^{-S_f} e^{-S_c}}{Z_f} \\
&= e^{-S_s} Z_f \left\langle e^{-S_c} \right\rangle_f
\end{aligned}
\tag{77}
$$

where $Z_f = \int D\phi_f \; e^{-\phi_f}$ is a constant and can be neglected (Note that it does contribute to the total free energy). The new action thus is

$$
\begin{aligned}
S(\phi_s)_{\Lambda/s} &= -\log\left(\int D\phi_{\Lambda/s < k < \Lambda} \; e^{-S_\Lambda(\phi)} \right) \\
&= S_s - \log\left(\langle e^{-S_c} \rangle_f \right)
\end{aligned}
\tag{78}
$$

Usually one can not integral out high energy modes exactly, hence cumulant perturbations based on Feynmann diagramm have to be adopted.

This theory can not be compared with the original one, since they have different cutoffs. Another rescaling step

$$k \to s\, k \tag{79}$$

is required to restore the cutoff or energy scale. Since the field operators dependend on length scales, they also need to be rescaled

$$\phi \to s^{\Delta_\phi} \phi \tag{80}$$

Now one obtains a new theory $S(\phi, \lambda)_\Lambda$ at the same cutoff but with different parameters, in which we assumed the theory $S(\phi, \lambda)$ remains the same structure.

Keep doing such RG procedures, one can find how the parameters $\lambda_i(s)$ flow in the parameter space along with the RG time s. These RG transformations of the parameters form a semi-group structure. In the whole parameter space, fixed points are special, since they are scale invariant. The parameter near a fixed point λ^* is called relevant or irrelevant when it flows away or close to λ^*, respectively. A RG program is to find all fixed points and analyse how the parameters flow near fixed points. One needs to solve the so-called β equation

$$\beta_i(\lambda_j) = \frac{\partial \lambda_i}{\partial \log(s)}. \tag{81}$$

The zero points of the β function are solutions of fixed points of the RG program

$$\beta_i(\lambda_j^*) = 0. \tag{82}$$

Near the fixed point, usually one can approimate the β function as an linear eigen problem. Eigenvalues of the RG transformation imply how fast λ_i flow to or away from λ^*, which are nothing but the scaling dimensions $\Delta(\tilde{\lambda}_i)$ of the corresponding parameter

$$\frac{\partial \tilde{\lambda}_i}{\partial \log(s)} = \Delta\left(\tilde{\lambda}_i\right) \tilde{\lambda}_i \tag{83}$$

where $\tilde{\lambda}_i$ is a linear combination of the orginal paramters (here we shifted the fixed point to be zero and $\tilde{\lambda}_i$ means the distance to the fixed point λ_i^*). Note that the RG analysis here is also consistent with the renormalizability of a QFT. An irrelevant field ($\Delta(\lambda_i) < 0$) vanishes at IR means it becomes divergent at UV.

There also exist many other RG schemes. For example, one may integrate out all high-energy modes $|k| > \Lambda$. There will be divergence at low dimensions. A popular way to deal with the divergence is to continue the space dimension d to be a real positive number and make perturbation around the upper or lower critical dimension, which is called as $d \mp \epsilon$ expansion in the literature. Another popular and also elegent RG scheme is to introduce a real space short distance cutoff a. The scaling transformation of a is canceled by the change of couplings in the theory. One can use operator product expansion (OPE) to write down the β function. In this approach, one only needs to know the OPE coefficients at a known fixed point rather than doing Feynmann diagram calculations.

Example: poor man's scaling of Kondo effect

Example: perturbative RG analysis of ϕ^4 theory

The Ferromagnetic phase transition is usually modeled by a real scalar field theory

$$S = \int d^d x \left\{ \frac{1}{2}(\partial \phi)^2 + \sum_{n=1,2,4} \left(\frac{\lambda_n}{n!} \phi^n \right) \right\} \tag{84}$$

where the field ϕ can be viewed as flucturations around the mean field solution ϕ_c of the action. Following the Wilson-Kadanoff RG scheme, we identify

$$
\begin{aligned}
S_f &= \int d^d x \left\{ \frac{1}{2}(\Delta \phi_f)^2 + \frac{\lambda_2}{2} \phi_f^2 \right\} \\
S_s &= \int d^d x \left\{ \frac{1}{2}(\Delta \phi_s)^2 + \lambda_1 \phi_s + \frac{\lambda_2}{2} \phi_s^2 \right\} \\
S_c &= \int d^d x \left\{ \frac{\lambda_4}{4!}(\phi_s + \phi_f)^4 \right\}.
\end{aligned}
\tag{85}
$$

At one-loop approximation, using cumulant expansion one can find

$$\langle e^{-S_c} \rangle_f = \exp\left\{ -\langle S_c \rangle_f + \frac{1}{2}\left(\langle S_c^2 \rangle_f - \langle S_c \rangle_f^2 \right) \right\} \tag{86}$$

In $\langle S_c \rangle_f$ there is a pure slow mode term $\frac{\lambda_4}{4!} \phi_s^4$ and another one

$$\frac{\lambda_4}{4!} C_4^2 \int d^d x \, \phi_s^2 \langle \phi_f(x) \phi_f(x) \rangle_f = \int d^d x \left\{ \frac{\langle \phi_f(x) \phi_f(x) \rangle_f \lambda_4/2}{2} \phi_s^2 \right\} \tag{87}$$

In $\langle S_c^2 \rangle_f - \langle S_c \rangle_f^2$ there is one term contributing to the one-loop result

$$\left(\frac{\lambda_4}{4!} C_4^2 \right)^2 \int d^d x \int d^d y \, (\phi_s(x)\phi_s(y))^2 \left\langle \phi_f(x)\phi_f(y) \right\rangle_f^2 \tag{88}$$

Field theoretical renormalization of the ϕ^4 theory

In higher order perturbative RG analysis of an interacting field theory, it is usually more convenient to consult the field theoretical renormalization. We briefly review how to renormalize the ϕ^4 theory

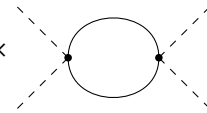$$S = \int \frac{1}{2}(\partial \phi)^2 + \frac{1}{2}m_0^2 \phi^2 + \frac{1}{4!}\lambda_0 \phi^4 \tag{89}$$

to the two-loop level in $d \leq 4$ dimension. Higher order terms (ϕ^p with $p > 4$) are neglected, since they are non-renormalizable, meaning one loses the ability to track information at high momentums. Or put it in another way, they are irrelevant in the Wilsonian RG. We ignore the composite field [1] renormalization here.

- One-loop renormalization

At one-loop, we only need to perform first order contribution

$$\frac{\lambda_0}{2} \times$$

for the $\Gamma^{(2)}$ vertex, and the diagram

$$-\frac{(\lambda_0)^2}{2} \times \qquad + 2 \text{ permutations}$$

for $\Gamma^{(4)}$. Regularized by Λ, one can write down the one-loop resultes

$$\Gamma^{(2)} = k^2 + (m_0)^2 + \frac{\lambda_0}{2} \int \frac{1}{q^2 + (m_0)^2}$$

$$\Gamma^{(4)} = \lambda_0 - \frac{(\lambda_0)^2}{2} \int \frac{1}{q^2 + (m_0)^2} \frac{1}{(k_1 + k_2 - q)^2 + (m_0)^2} + 2 \text{ permutations.} \tag{90}$$

At large momentum they behave as

$$\begin{aligned} \left[\Gamma^{(2)} \right] &\sim \Lambda^{d-2} \\ \left[\Gamma^{(4)} \right] &\sim \Lambda^{d-4} \end{aligned} \tag{91}$$

so that $\Gamma^{(2)}$ is divergent at $d \leq 4$, $\Gamma^{(4)}$ is logarithmic dimvergent at $d = 4$. All the other higher order vertex functions are convergent when $\Lambda \to \infty$.

Keeping terms consistently at one loop, one can introduce renormalized physical parameters in terms of the bare ones

$$\begin{aligned} (m_1)^2 &= (m_0)^2 + \frac{\lambda_0}{2} \int \frac{1}{q^2 + (m_0)^2} \\ \lambda_1 &= \lambda_0 - \frac{3(\lambda_0)^2}{2} \int \frac{1}{\left(q^2 + (m_0)^2 \right)^2} \end{aligned} \tag{92}$$

---

[1] The composite field is related to the dimension of ϕ^2 or the critical exponent ν.

or one can reverse the logic and express bare parameters in terms of physical ones

$$(m_0)^2 = (m_1)^2 - \frac{\lambda}{2} \int \frac{1}{q^2 + (m_1)^2}$$

$$\lambda_0 = \lambda_1 + \frac{3(\lambda_1)^2}{2} \int \frac{1}{\left(q^2 + (m_1)^2\right)^2}. \tag{93}$$

Now if the renormalized parameters m_1 and λ are finite, the vertex functions

$$\Gamma^{(2)}(k, m_1, \lambda_1) = k^2 + (m_1)^2$$

$$\Gamma^{(4)}(k_i, m_1, \lambda_1) = \lambda_1 - \frac{(\lambda_1)^2}{2} \int \frac{1}{q^2 + (m_1)^2} \frac{1}{(k_1 + k_2 - q)^2 + (m_1)^2} + 2 \text{ permutations}$$

$$+ \frac{(\lambda_1)^2}{2} \int \frac{1}{\left(q^2 + (m_1)^2\right)^2}. \tag{94}$$

do not suffer from the UV divergence at one loop.

One can see that the one-loop corrections renormalize the critical point (the mass) and the coupling constant λ_0. The scaling relations remain the same though, meaning that the scaling dimensions are still the canonical ones. At the two loop, to remove all UV divergences one has to introduce the field renormalization factor Z_ϕ and redefine fields in the action, which give rise to the anormalous correction to critical exponents.

- Two-loop renormalization

At the two-loop level, we need to consider more diagrams
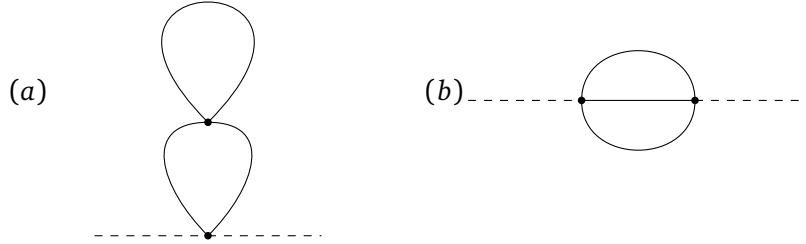
for $\Gamma^{(2)}$ and
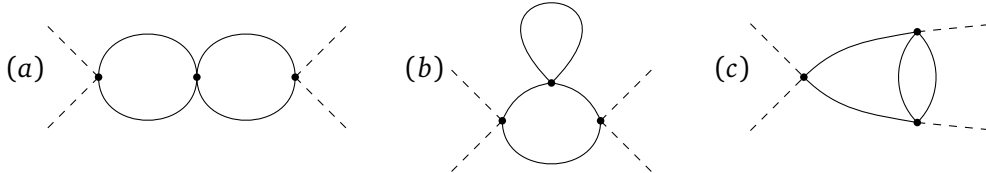
for $\Gamma^{(4)}$ vertex.

Let's first consider the renormalization of $\Gamma^{(2)}$. Again we choose the renormalized mass to be

$$(m_1)^2 = \Gamma^{(2)}(k = 0) = f(m_0, \lambda, \Lambda). \tag{95}$$

One can write the vertex function in terms of renormalized mass m_1. In two-loop diagrams one can directly replace $(m_0)^2$ with $(m_1)^2$. However, in one-loop contributions one has to

expand the bare diagram in terms of the renormalized ones to two-loop level, which resultes in a diagram cancelling the first two-loop one. Finally one finds

$$(m_0)^2 = (m_1)^2 - \frac{\lambda_0}{2} \int \frac{1}{(m_1)^2 + q^2} + \frac{(\lambda_0)^2}{6} \int \frac{1}{(q_1^2 + m_1^2)(q_2^2 + m_1^2)((q_1 + q_2)^2 + m_1^2)} \quad (96)$$

and

$$\Gamma^{(2)}(k) = k^2 + m_1^2 - \frac{\lambda_0^2}{6} \int \frac{1}{(q_1^2 + m_1^2)(q_2^2 + m_1^2)((k - q_1 - q_2)^2 + m_1^2)}$$
$$+ \frac{\lambda_0^2}{6} \int \frac{1}{(q_1^2 + m_1^2)(q_2^2 + m_1^2)((q_1 + q_2)^2 + m_1^2)}. \quad (97)$$

Note that $\Gamma^{(2)}$ is convergent at $d < 4$ dimensions. At $d = 4$, there is a logarithmic divergence.

Similarly in $\Gamma^{(4)}$, one can expand the one-loop diagrams to two-loop level and replace the bare mass with the renormalized one. The second two-loop diagram is cancelled due to the renormalization of mass. Now $\Gamma^{(4)}$ contains only a logarithmic divergence when $d \to 4$. To cure the logarithmic divergence at $d = 4$, one can introduce the renormalized coupling

$$\lambda = \Gamma^{(4)}(k_i = 0, m_1^2, \Lambda)$$
$$= \lambda_0 - \frac{3}{2}\lambda_0^2 \int \frac{1}{(q^2 + m_1^2)^2} + \frac{3}{4}\lambda_0^3 \left(\int \frac{1}{(q^2 + m_1^2)^2} \right)^2 +$$
$$3\lambda_0^3 \int \frac{1}{(q_1^2 + m_1^2)^2 (q_2^2 + m_1^2)((q_1 + q_2)^2 + m_1^2)}. \quad (98)$$

Or inversely, one can write down the bare coupling in terms of the renormalized one

$$\lambda_0 = \lambda + \frac{3}{2}\lambda^2 \int \frac{1}{(q^2 + m_1^2)^2} + \frac{15}{4}\lambda^3 \left(\int \frac{1}{(q^2 + m_1^2)^2} \right)^2 -$$
$$3\lambda^3 \int \frac{1}{(q_1^2 + m_1^2)^2 (q_2^2 + m_1^2)((q_1 + q_2)^2 + m_1^2)}. \quad (99)$$

And one can also write down the vertex function $\Gamma^{(4)}$ in terms of λ and m_1. Now $\Gamma^{(4)}(k_i, m_1^2, \lambda)$ is convergent at $d \le 4$.

Turning back $\Gamma^{(2)}$, one can replace λ_0 with λ safely at the two-loop level. The bare mass should be replaced by

$$(m_0)^2 = (m_1)^2 - \frac{\lambda}{2} \int \frac{1}{(m_1)^2 + q^2} - \frac{3}{4}\lambda^2 \int \frac{1}{(q^2 + m_1^2)^2} \int \frac{1}{(m_1)^2 + q^2} +$$
$$\frac{(\lambda)^2}{6} \int \frac{1}{(q_1^2 + m_1^2)(q_2^2 + m_1^2)((q_1 + q_2)^2 + m_1^2)}. \quad (100)$$

To cure the logarithmic divergent divergence, we introduce the field renormalization

$$\Gamma_R^{(2)} = Z_\phi(\lambda, m_1, \Lambda)\Gamma^{(2)}(k, m_1^2, \Lambda), \quad (101)$$

where

$$Z_\phi = 1 + \lambda^2 z_2$$
$$z_2 = \frac{1}{6}\frac{\partial}{\partial k^2} \int \frac{1}{(q_1^2 + m_1^2)(q_2^2 + m_1^2)((k - q_1 - q_2)^2 + m_1^2)}. \quad (102)$$

Now we redefine the mass as

$$m^2 = Z_\phi m_1^2. \tag{103}$$

This renormalization means we can choose

$$
\begin{aligned}
G^{(2)} &= Z_\phi G_R^{(2)} \\
g &= Z_\phi^2 \lambda \\
G_c^{(4)} &= Z_\phi^2 G_{cR}^{(4)} \\
\Gamma^{(4)} &= Z_\phi^{-2} \Gamma_R^{(4)}
\end{aligned}
\tag{104}
$$

Note that the replacement of λ with g generates terms with four loops, which means we can safely express bare quantities in terms of g and m^2. The vertex functions are finite at the two-loop level when $d \leq 4$ with the normalization conditions

$$
\begin{aligned}
\Gamma_R^{(E)}(k_i, m^2, g) &= Z_\phi^{E/2} \Gamma^{(E)}(k_i, m_0^2, \lambda, \Lambda) \\
\Gamma_R^{(2)}(0, m^2, g) &= m^2 \\
\frac{\partial}{\partial k^2} \Gamma_R^{(2)}(k, m^2, g)|_{k^2=0} &= 1 \\
\Gamma_R^{(4)}(0, m^2, g) &= g
\end{aligned}
\tag{105}
$$

One should be clear that the normalization condition is not unique. Actually there can be infinite number of normalization approaches. Moreover, the perturbative renormalization is an asymptopic series. In the above normalization, the mass is finite. For the critical theory, especially for the calculation of critical exponents, it is usually more convenient to choose the masless normalization $m^2 = 0$ at a finite (symmetric) momentum

$$
\begin{aligned}
\Gamma_R^{(2)}(0, 0, g) &= 0 \\
\frac{\partial}{\partial k^2} \Gamma_R^{(2)}(k, 0, g)\bigg|_{k^2=\kappa^2} &= 1 \\
\Gamma_R^{(4)}(k_i, 0, g)|_{\text{SP}} &= g.
\end{aligned}
\tag{106}
$$

Another approach is to look at small mass and large k/m. Of course these approaches are equivalent to each other.

- Renormalization for the massless ϕ^4 and critical exponents

The physical or renormalized parameters are running upon the renormalization, which means for a given bare theory one can obtain different renormalized theories with different UV cutoffs. One can also invert the logic and write down different bare theories corresponding to the same renormalized one. In either case, one can write down flow equations for the parameters in terms of the cutoff Λ or the running dimensionless coupling u. At the fixed point, the flow equation disappears, meaning the physics there does not depend on the UV cutoff and one can safely sends $\Lambda \to \infty$. The solution of the flow equations can be conveniently used to calculate critical exponents.

To study critical behaivors, we first rewrite the parameters as

$$
\begin{aligned}
\lambda_0 &= \kappa^\epsilon u_0 \\
g &= \kappa^\epsilon u
\end{aligned}
\tag{107}
$$

where κ is the symmetric momentum used in the massless theory renormalization, u_0 and u are dimensionless. As stated above the bare theory does not depend on the renormalization

$$\Gamma_R^{(N)}(k_i; u, \kappa) = Z_\phi^{N/2} \Gamma(k_i; u_0, \Lambda) \tag{108}$$

which gives rise to the RG equation

$$\left(\kappa \frac{\partial}{\partial \kappa} + \beta(u) \frac{\partial}{\partial u} - \frac{N}{2} \gamma_\phi \right) \Gamma_R^{(N)}(k_i; u, \kappa) = 0 \tag{109}$$

where

$$\beta(u) = \kappa \left(\frac{\partial u}{\partial \kappa} \right)_{\lambda, \Lambda} = -\epsilon \left(\frac{\partial \ln(u_0)}{\partial u} \right)^{-1}$$

$$\gamma_\phi = \kappa \left(\frac{\partial \ln(Z_\phi)}{\partial \kappa} \right)_{\lambda, \Lambda} = \beta(u) \frac{\partial \ln(Z_\phi)}{\partial u}. \tag{110}$$

The solution of the RG equation reads

$$\Gamma_R^{(N)}(k_i; u, \kappa) = \exp \left\{ -\frac{N}{2} \int_1^\rho \frac{\gamma_\phi(u(x))}{x} \right\} \Gamma_R^{(N)}(k_i; u(\rho), \rho\kappa). \tag{111}$$

At the fixed point, it reduces to a simpler form

$$\Gamma_R^{(N)}(k_i; u_c, \kappa) = \kappa^{\frac{1}{2} N \gamma_\phi(u_c)} \check{\Gamma}(k_i). \tag{112}$$

Since the canonical dimension of $\Gamma_R^{(N)}$ is fixed to be $N + d - \frac{1}{2} N d$, one can rewrite the vertex function as

$$\Gamma_R^{(N)}(k_i; u_c, \kappa) = \kappa^{N + d - \frac{1}{2} N d} f_\Gamma(k_i/\kappa). \tag{113}$$

One can find the scaling of the vertex function reads

$$\Gamma_R^{(N)}(\rho k_i; u_c, \kappa) = \rho^{N + d - \frac{1}{2} N d - \frac{1}{2} N \gamma_\phi(u_c)} \Gamma_R^{(N)}(k_i; u_c, \kappa). \tag{114}$$

It is clear now $\gamma_\phi(u_c)$ is the so-called anormalous exponent

$$\eta = \gamma_\phi(u_c). \tag{115}$$

for the ϕ operator.

Now let's derive the renormalization at the two-loop level. Following similar calculations in the last section, one can express the bare quantites in terms of the renormalized ones, or the other way around. At the two-loop level the mass normalization condition determines the critical bare μ_c^2

$$\begin{aligned}
\mu_c^2 &= -\frac{\lambda_0}{2} \int \frac{1}{q^2 + \mu_c^2} + \frac{\lambda_0^2}{4} \int \frac{1}{q^2 + \mu_c^2} \int \frac{1}{\left(q^2 + \mu_c^2 \right)^2} \\
&\quad + \frac{\lambda_0^2}{6} \int \frac{1}{\left(q_1^2 + \mu_c^2 \right) \left(q_2^2 + \mu_c^2 \right) \left((q_1 + q_2)^2 + \mu_c^2 \right)} \\
&= -\frac{\lambda_0}{2} \int \frac{1}{q^2} + \frac{\lambda_0^2}{6} \int \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \\
&= C_\mu^1 \lambda_0 + C_\mu^2 \lambda_0^2
\end{aligned} \tag{116}$$

The second normalization condition gives

$$Z_\phi = 1 + \frac{\lambda_0^2}{6} \frac{\partial}{\partial k^2} \left(\int \frac{1}{\left(q_1^2 + \mu_c^2\right)\left(q_2^2 + \mu_c^2\right)\left((k - q_1 - q_2)^2 + \mu_c^2\right)} \right)\Bigg|_{k^2 = \kappa^2}$$

$$= 1 + \frac{\lambda_0^2}{6} \frac{\partial}{\partial k^2} \left(\int \frac{1}{q_1^2 q_2^2 (k - q_1 - q_2)^2} \right)\Bigg|_{k^2 = \kappa^2} \tag{117}$$

$$= 1 + C_Z \lambda_0^2$$

and the last one

$$g = Z_\phi^2 \left(\lambda_0 - \frac{3}{2}\lambda_0^2 \int \frac{1}{\left(q^2 + \mu_c^2\right)\left((k_1 + k_2 - q)^2 + \mu_c^2\right)} + \frac{3}{4}\lambda_0^3 \left(\int \frac{1}{\left(q^2 + \mu_c^2\right)\left((k_1 + k_2 - q)^2 + \mu_c^2\right)} \right)^2 + \right.$$

$$\left. 3\lambda_0^3 \int \frac{1}{\left(q_1^2 + \mu_c^2\right)\left((k_1 + k_2 - q_1)^2 + \mu_c^2\right)\left(q_2^2 + \mu_c^2\right)\left((q_1 + q_2 - k_3)^2 + \mu_c^2\right)} \right)$$

$$= Z_\phi^2 \left(\lambda_0 - \frac{3}{2}\lambda_0^2 \int \frac{1}{q^2 (k_1 + k_2 - q)^2} + \frac{3}{4}\lambda_0^3 \left(\int \frac{1}{q^2 (k_1 + k_2 - q)^2} \right)^2 + \right.$$

$$\left. 3\lambda_0^3 \int \frac{1}{q_1^2 (k_1 + k_2 - q_1)^2 q_2^2 (q_1 + q_2 - k_3)^2} \right)$$

$$= \left(1 + 2C_Z \lambda_0^2\right)\left(\lambda_0 - \frac{3}{2}\lambda_0^2 C_2 + \frac{3}{4}\lambda_0^3 C_2^2 + 3\lambda_0^3 C_3 \right)$$

$$= \lambda_0 - \frac{3}{2}C_2\lambda_0^2 + \left(2C_Z + \frac{3}{4}C_2^2 + 3C_3\right)\lambda_0^3 \tag{118}$$

or one can express the bare parameter in terms of the renormalized one

$$\lambda_0 = g + \frac{3}{2}C_2\lambda_0^2 - \left(2C_Z + \frac{3}{4}C_2^2 + 3C_3\right)\lambda_0^3$$

$$= g + \frac{3}{2}C_2 g^2 + \left(\frac{15}{4}C_2^2 - 2C_Z - 3C_3\right)g^3 \tag{119}$$

where we have used

$$C_z = \frac{1}{6} \frac{\partial}{\partial k^2} \left(\int \frac{1}{q_1^2 q_2^2 (k - q_1 - q_2)^2} \right)\Bigg|_{k^2 = \kappa^2}$$

$$C_2 = \int \frac{1}{q^2 (k_1 + k_2 - q)^2} \tag{120}$$

$$C_3 = \int \frac{1}{q_1^2 (k_1 + k_2 - q_1)^2 q_2^2 (q_1 + q_2 - k_3)^2}$$

Using the dimensional expansion to the $O(\epsilon)$ order, they read

$$C_z \kappa^{2\epsilon} = -\frac{1}{48\epsilon}\left(1 + \frac{5}{4}\epsilon\right)$$

$$C_2 \kappa^{\epsilon} = \frac{1}{\epsilon}\left(1 + \frac{1}{2}\epsilon\right) \tag{121}$$

$$C_3 \kappa^{2\epsilon} = \frac{1}{2\epsilon^2}\left(1 + \frac{3}{2}\epsilon\right)$$

Similarly the field renormalization constant can also be replaced by

$$Z_\phi = 1 + C_Z g^2 \tag{122}$$

at the two-loop level.

In terms of dimensionless parameters, one can rewrite the above results as

$$Z_\phi = 1 + C_Z \kappa^{2\epsilon} u^2$$
$$u_0 = u + \frac{3}{2} C_2 \kappa^\epsilon u^2 + \left(\frac{15}{4} C_2^2 - 2C_Z - 3C_3 \right) \kappa^{2\epsilon} u^3 \tag{123}$$

One finds the β equation

$$\beta(u) = -\epsilon \left(\frac{\partial \ln(u_0)}{\partial u} \right)^{-1} = -\epsilon u \left(1 - \frac{3}{2} C_2 u - \left(3C_2^2 - 4C_z - 6C_3 \right) u^3 \right) \tag{124}$$

whose zeros contain two fixed points, the Gaussian one with $u_c = 0$ and the Wilson-Fisher fixed point

$$u_c = \frac{2}{3} \epsilon. \tag{125}$$

The anormalous exponent reads

$$\gamma_\phi(u_c) = \frac{\epsilon^2}{54}. \tag{126}$$

At this order, $\gamma_\phi(u_c)$ is still far away from the high-termerature series expansion result 0.041 at $d = 3$ or the 2d Ising CFT exact result $1/4$. At the next order of dimensional expansion, it becomes better. However, it is known ϵ expansion is not convergent. Resummation techniques are necessary to obtain accurate critical exponents.

- Renormalization for the massive ϕ^4 and critical exponents

Though technically easier for massless renormalization, there is no fundamental difficult to stop one to renormalize an massive theory and calculate critical exponents using the massive ϕ^4. Now the coupling constants become

$$\lambda_0 = u_0 m^\epsilon$$
$$g = u m^\epsilon. \tag{127}$$

One can use the fact that the divergent integrals when sending $\Lambda \to \infty$ have nothing to do with the mass and become poles in the ϵ expansion. Take the one-loop renormalization for the coupling as an example

$$\lambda_0/m^\epsilon = m^{-\epsilon} \left(\lambda_1 + \frac{3}{2} \lambda_1^2 \int^\Lambda \frac{1}{(q^2 + m^2)^2} \right)$$
$$= u_1 + \frac{3}{2} u_1^2 m^\epsilon \int^\Lambda \frac{1}{(q^2 + m^2)^2} \tag{128}$$

which becomes

$$u_0 = u_1 + \frac{3}{2} u_1^2 \int \frac{1}{(q^2 + 1)^2}$$
$$= u_1 + \frac{3}{2} u_1^2 \left(\frac{1}{\epsilon} - \frac{1}{2} + O(\epsilon) \right) \tag{129}$$

when $\Lambda \to \infty$. Similarly, one can obtain the ϵ expansion for the bare coupling, β equation and the anormalous exponent at the two-loop level. To the leading order of the ϵ expansion, one can find

$$u_c = \frac{2}{3}\epsilon$$
$$\gamma_\phi = \frac{1}{54}\epsilon^2 \tag{130}$$

consistent with the massless renormalization.

Example: Momentum shell perturbative RG analysis of BKT transition

The sine-Gordon theory reads

$$S = \int \frac{1}{8\pi K}(\nabla\varphi)^2 + g\cos(\beta\varphi) \tag{131}$$

where $K = 1/2$, $\beta = 2$ and φ is compactified as $\varphi \sim \varphi + 2\pi$. The symmetry for the φ scalar is spoiled down from $O(2)_\theta \times O(2)_\varphi$ to $O(2)_\theta \times Z_2^\varphi$ where $Z_2^\varphi : \varphi \to \varphi + \pi$. In the following we are going to use perturbative Wilsonian RG to analyse the effect of the marginal field g ($\Delta_g = 2 - \beta^2 K = 0$) close to the free compact boson $K = 1/2$.

To the second order approximation, the effective action can be written down

$$
\begin{aligned}
S_{\Lambda'}^{\text{eff}}(\varphi) =&\, S_0(\varphi) - \text{Log}\left(1 - <S_1(\varphi,h)>_h + \frac{1}{2}<S_1(\varphi,h)S_1(\varphi,h)>_h\right) \\
=&\, S_0(\varphi) + <S_1(\varphi,h)>_h \\
&- \frac{1}{2}<S_1(\varphi,h)S_1(\varphi,h)>_h + \frac{1}{2}<S_1(\varphi,h)>_h<S_1(\varphi,h)>_h\,.
\end{aligned} \tag{132}
$$

where we denote slow or fast field as φ or h respectively.

- First order term $\langle S_1(\phi,h)\rangle_h$

$$
\begin{aligned}
\langle S_1(\phi,h)\rangle_h &= g\langle\cos\beta(\varphi+h)\rangle = \frac{g}{2}\sum_\sigma <e^{i\sigma\beta(\phi+h)}>_h \\
&= \sum_\sigma \frac{g}{2}e^{i\sigma\beta\varphi}e^{-\frac{1}{2}\beta^2<h(x)^2>_h} \\
&= \frac{ge^{i\sigma\beta\phi}}{2}e^{-\beta^2 K\log\left(\frac{\Lambda}{\Lambda'}\right)} \\
&= g\cos(\beta\phi)\left(\frac{\Lambda}{\Lambda'}\right)^{-\beta^2 K} \\
&= g\cos(\beta\phi)\left(1 - \beta^2 K\,\mathrm{d}l\right)
\end{aligned} \tag{133}
$$

After rescaling the coordinates with a factor $\frac{\Lambda}{\Lambda'} = 1 + \mathrm{d}l$, the effective action becomes

$$S^{\text{eff}}(\phi) = S_0(\phi) + g\left(1 + \left(d - \beta^2 K\right)\mathrm{d}l\right)\int\cos(\beta\phi). \tag{134}$$

One concludes that g is relevant or irrelevant when $\beta^2 K < 2$ or $\beta^2 K > 2$, respectively.

- Second order perturbation $\left(<S_1{}^2> - <S_1><S_1>\right)$

One can find

$$\langle S_1(\phi,h)S_2(\phi,h)\rangle_h$$
$$= \frac{g^2}{4}\langle e^{i\sigma_1\beta(\phi_1+h_1)}e^{i\sigma_2\beta(\phi_2+h_2)}\rangle_h$$
$$= \frac{g^2}{4}e^{i\sigma_1\beta\phi_1+i\sigma_2\beta\phi_2} < e^{i\beta\sigma_1 h_1}e^{i\beta\sigma_2 h_2} >_h$$
$$= \frac{g^2}{4}e^{i\sigma\beta(\phi_1-\phi_2)}e^{\beta^2(<h_1 h_2>-<h^2>)} + \frac{g^2}{4}e^{i\sigma\beta(\phi_1+\phi_2)}e^{-\beta^2(<h_1 h_2>+<h^2>)} \qquad (135)$$
$$= \frac{g^2}{2}\cos(\beta(\phi_1-\phi_2))e^{+\frac{\beta^2}{2}\langle(h_1-h_2)^2\rangle_h}$$
$$+ \frac{g^2}{2}\cos(\beta(\phi_1+\phi_2))e^{-\frac{\beta^2}{2}\langle(h_1-h_2)^2\rangle_h}$$

and another term

$$\langle S_1(\phi,h)\rangle_h \langle S_2(\phi,h)\rangle_h = g^2\cos(\beta\phi_1)\cos(\beta\phi_2)e^{-\beta^2\langle h^2\rangle_h}$$
$$= \frac{g^2}{2}(1-2\beta^2 K dl)(\cos(\beta(\phi_1-\phi_2)+\cos(\beta(\phi_1+\phi_2))) \qquad (136)$$

So that to second order perturbation, the effective action is

$$S_{\Lambda'}^{\text{eff}}(\phi)$$
$$= S_0(\phi) + < S_1(\phi,h) >_h - \frac{1}{2}(< S_1(\phi,h)S_1(\phi,h) >_h - < S_1(\phi,h) >_h < S_1(\phi,h) >_h)$$
$$= S_0(\phi) + g\cos(\beta\phi)e^{-\frac{\beta^2}{2}\langle h^2\rangle_h} - \frac{g^2}{4}e^{-\beta^2\langle h^2\rangle_h}\int d^2 x_1 d^2 x_2 \qquad (137)$$
$$\left[\left(e^{\beta^2\langle h_1 h_2\rangle_h}-1\right)\cos(\beta(\phi_1-\phi_2)) + \left(e^{-\beta^2\langle h_1 h_2\rangle_h}-1\right)\cos(\beta(\phi_1+\phi_2))\right]$$

Note that the integral is from Λ' to Λ, so that the neutrality condition is not applied here. Now we assume the two point correlation $e^{\beta^2\langle h_1 h_2\rangle_h}$ to be short ranged, which allow us to do approximation and only keep the leading order of the relative distance r. With a hard cutoff used above, the two point correlation ($\langle h_1 h_2\rangle_h$) has a long fluctuating tail. One can add a smooth factor ($e^{-k/\Lambda}$) for the fast modes to suppress the integral to obtain a short range correlation functions. Due to short range nature of the two point correlation, one can introduce the new coordinate

$$R = \frac{x+y}{2}, \; r = x-y \qquad (138)$$

The last term is approximated to be

$$\cos(\beta(\phi_1+\phi_2))] = \cos(2\beta\phi(R)) + \text{higher order terms} \qquad (139)$$

which is less relevant compared with the first one since its scaling dimension $4\beta^2 K$.

Another term reads

$$\cos(\beta(\phi_1-\phi_2)) = e^{-\beta^2\langle\varphi^2\rangle_\varphi} : \cos(\beta\; r\cdot\nabla\phi) := e^{-\beta^2\langle\varphi^2\rangle_\varphi}\left(1-\frac{\beta^2}{2}(r\cdot\nabla\varphi)^2\right) \qquad (140)$$

22

One finds

$$
-\frac{g^2}{4}\int e^{-\beta^2\langle h^2\rangle_h}\left(e^{\beta^2\langle h_1 h_2\rangle_h}-1\right)\cos\left(\beta\left(\phi_1-\phi_2\right)\right)
$$

$$
-\frac{g^2}{4}\int e^{-\beta^2\langle\varphi^2\rangle_\Lambda}\left(e^{\beta^2\langle h_1 h_2\rangle_h}-1\right):\cos\left(\beta\left(\phi_1-\phi_2\right)\right):
$$

$$
-\frac{g^2}{4}\sim\int\{C(\Lambda)\}\times\{\beta^2 K f(r)\mathrm{d}l\}\times\left(1-\frac{\beta^2}{2}(r\cdot\nabla\varphi)^2\right)
$$

$$
\sim f(\Lambda)g^2\beta^4 K\mathrm{d}l\int\mathrm{d}^2 R\,(\partial\varphi)^2
$$

(141)

in which we used

$$
e^{\beta^2\langle h_1 h_2\rangle_h}-1=\beta^2 K f(r)\mathrm{d}l \tag{142}
$$

and

$$
\int\mathrm{d}^2 r\,f(r)(r\cdot\nabla\phi(R))^2=\int\mathrm{d}^2 r\,f(r)\left(r\cos\left(\theta_{r\cdot\nabla\phi}\right)|\nabla\varphi(R)|\right)^2
$$

$$
=\int\mathrm{d}r\mathrm{d}\theta\,rf(r)\,r^2\cos^2\left(\theta_{r\cdot\nabla\phi}\right)(\nabla\phi(R))^2 \tag{143}
$$

$$
\sim(\nabla\phi(R))^2\int\mathrm{d}r\,\tilde{f}(r).
$$

Note that the normalization of the cos term combing with the fast mode contribution makes the second perturbation does not depends on Λ'. One does not need to do rescaling for the cutoff. The second order term renormalize the parameter K.

In summary, to the second order perturbation the effective action reads

$$
S^{\mathrm{eff}}(\phi)=S_0(\phi)+g(1-\beta^2 K\mathrm{d}l)\int_{\Lambda'}d^2 x\cos(\beta\varphi)+f(\Lambda)g^2\beta^4 K\mathrm{d}l\int\mathrm{d}^2 R\,(\partial\varphi)^2
$$

$$
=\left(\frac{1}{8\pi K}+f(\Lambda)g^2\beta^4 K\mathrm{d}l\right)\int_\Lambda(\partial\varphi)^2+g\left(1+\left(2-\beta^2 K\right)\mathrm{d}l\right)\int_\Lambda\cos(\beta\varphi)
$$

(144)

Now one can write down the β function of K and g

$$
\frac{\partial g}{\partial l}=(2-\beta^2 K)g
$$

$$
\frac{\partial K}{\partial l}=-C\,\beta^4 K^2\,g^2 \tag{145}
$$

We assume g is small and $K\sim 1/2$

$$
\frac{\partial\tilde{g}}{\partial l}=-\tilde{K}\tilde{g}
$$

$$
\frac{\partial\tilde{K}}{\partial l}=-\tilde{g}^2 \tag{146}
$$

where $\tilde{K}:=4K-2$ and $\tilde{g}$ is normalized to simplify the notation.

2 Conformal symmetry in two dimension

In $d = 2$ we have infinitely many *local* conformal transformations. The 6 parameter subgroup of conformal transformations that are everywhere well defined is the *global* conformal group $SL(2, \mathbb{C})/\mathbb{Z}_2$. Locally on the algebra level, it becomes the infinite dimensional Witt algebra. In a quantum theory, one can introduce a central extension term and get the famous Virasoro algegra. It is this infinite dimensional symmetry that ensures fields in a cft have nice local properties.

Any holomorphic or antiholomorphic transformations are allowed by the symmetry. Since the symmetry is represented by two independent Virasoro algebras (chiral and antichiral), in the following we only present the chiral part. This can also be shown from the the Noether current $T^{\mu\nu}$ corresponding to the coordinate translation symmetry. Recall that as a Noether current one has $\partial_\mu T^{\mu\nu} = 0$ and with rotation symmetry $T^{\mu\nu}$ is symmetric and traceless. In terms of the original coordinate

$$
\begin{aligned}
T^{zz} &= T^{00} + 2iT^{01} - T^{11} \\
T^{z\bar{z}} &= T^{00} - iT^{01} + iT^{10} + T^{11} \\
T^{\bar{z}\bar{z}} &= T^{00} - 2iT^{01} - T^{11}.
\end{aligned}
\tag{147}
$$

One can find that

$$
\begin{aligned}
\partial_z T^{zz} &= 0 \\
T^{z\bar{z}} &= 0 \\
\partial_{\bar{z}} T^{\bar{z}\bar{z}} &= 0.
\end{aligned}
\tag{148}
$$

So that T^{zz} and $T^{\bar{z}\bar{z}}$ are antiholomorphic and holomorphic functions respectively. We introduce $T(z) := T_{zz}(z)$ and $\overline{T}(\bar{z}) := T_{\bar{z}\bar{z}}(\bar{z})$ in the following.

2.1 From correlation functions to OPE

Due to the local nature of field theory, we promote the correlation functions to expansion of non-singular operators, which is termed as operator product expansion (OPE). The idea is basically that far away from the operators inside a bounded region other operators can only feel them as a superposition of single operators (non-singular). The first OPE example follows from the conformal ward identity, which builds up the OPE between the energy momentum tensor and primary fields

$$
T(z)\phi(\omega) \sim \frac{h}{(z-\omega)^2}\phi(\omega) + \frac{1}{z-\omega}\partial\phi(\omega)
\tag{149}
$$

where on the right hand side, the operators should be understood as to be calculated correlation functions with some other operators located far away from them. Following the conventions defined in the first chapter, one can easily obtain the OPE of $T(z)$ with $\partial\phi$ for free boson

$$
T(z)\partial\phi(\omega) \sim \frac{\partial\phi(\omega)}{(z-\omega)^2} + \frac{\partial_\omega^2\phi(\omega)}{z-\omega}
\tag{150}
$$

and $T(z)$ with ψ for free fermion

$$
T(z)\psi(\omega) \sim \frac{\frac{1}{2}\psi(\omega)}{(z-\omega)^2} + \frac{\partial\psi(\omega)}{z-\omega}
\tag{151}
$$

The OPE can be generalized to arbitrary fields

$$A(z)B(\omega) = \sum_{n=-\infty}^{\Delta(A)+\Delta(B)} \frac{\{AB\}_n(\omega)}{(z-\omega)^n} \tag{152}$$

where $\{AB\}_n(\omega)$ are non-singular fields. Note that the total scaling dimensions can not be changed in OPE.

2.2 Energy-momentum tensor and central charge

The energy-momentum tensor is a quasi-primary field, which does not follow the OPE of T with primaries. There is an aditional term proportional to central charge c in the OPE

$$T(z)T(\omega) \sim \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}. \tag{153}$$

This term also exists in the conformal transormation of T

$$T'(\omega) = \left(\frac{dw}{dz}\right)^{-2} T(z) + \frac{c}{12}\{z;\omega\} \tag{154}$$

where $\{z;\omega\}$ denotes the Schwarzian derivative. This is consistent with the fact that this term disappears under global conformal transitions which are true symmetry of CFT.

The central charge c is related to the number of degrees of freedom in the theory. This can be reflected in the calculation of free energy density for a cylinder, which is related to the plane via a conformal transormation

$$\omega = \frac{L}{2\pi} \log(z). \tag{155}$$

The energy-momentum tensor becomes

$$T_{cyl}(\omega) = \left(\frac{2\pi}{L}\right)^2 \{T_{pl}(z)\,z^2 - \frac{c}{24}\}. \tag{156}$$

The variation of free energy is a response to the change of metric. One can make another coordinate transformation only along with the circumference direction $\omega^0 \to \omega^0(1+\epsilon)$. Note that this is not a confromal transormation, which will result in the change of the metric tensor. One can find the free energy for a cylinder takes the form of

$$F = f_0 L - \frac{\pi c}{6L}, \tag{157}$$

which indicates that the conformal anomaly reflects the quantum fluctuation effect to the classifcal conformal symmetry.

3 Operator formalism

In this section, we explore the quantization of the cft on a cylinder, which is related to the plane via a conformal transormation.

3.1 Radial quantization

On the plane, one has the freedom to choose the direction of space or time for an Euclidean theory. Here we choose the radial direction to be time and the angle direction to be space. A conformal transformation

$$\xi = \frac{L}{2\pi} \log(z) \tag{158}$$

maps a point z on a complex palne to a point $\xi = t + ix$ on a cylinder with t being the time and $x \in [0, L]$ the space. The Hilbert space defined on the cylinder at a given time t is defined within a circle with a radius $e^{2\pi t/L}$. Naturally the quantum theory defined on a cylinder can be used to understand the plane.

One can immedialy find many important properties of the radial quantization from the confromal mapping. The time evolution operator, the Hamiltonian, on a cylinder corresponds the dialation operator on the plane and the translation operator, i.e. the momentum, corresponds to the rotation operator on the plane. Such a quantization scheme for a cft is called radial quantization. The time ordering on a cylinder becomes radial ordering on a plane. As a conseconce, the commutation of operators for a quantum theory is related to contour integrals through

$$[A, B] = \oint_0 d\omega \oint_\omega dz \, a(z)b(\omega), \tag{159}$$

where A and B are defined as equal time contour integral of local fields. Note that in the coutour integral, we have assumed that there is no other fields existing between the two integral circles, which means the time difference ϵ here should be infestimal small. In other words, the communicator defined here should be understood as equal-time communicator.

State-field correspondence Following the quantum theory on a cylinder (an operator in inserted at infinit past time to the vaccum state), we define a state corresponding to the field $\phi(z, \bar{z})$

$$|\phi\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z})|0\rangle. \tag{160}$$

Its dual state is defined as

$$\langle\phi| = \lim_{z, \bar{z} \to 0} \bar{z}^{-2h} z^{-2\bar{h}} \langle 0|\phi(1/\bar{z}, 1/z). \tag{161}$$

It is clear that states such definied are properly normalized.

3.2 Virasoro agebra

With the radial quantization and state-field correspondence, one can re-express the conformal symmetry, i.e. the Virasoro algera conveniently. We first introduce quantum operators for local fields and the energy-momentum tensor from equal-time contour integral (or equally mode expansion for local fields)

$$\phi_n = \frac{1}{2\pi i} \oint dz \, z^{n+h-1} \phi(z), \tag{162}$$

in which for the energy-momentum tensor $T(z)$ we denote its mode expansion operator as L_n. One then finds operators L_n defind here obey Virasoro algebra using the OPE of $T(z)$

from a straightforward calculation. Again, the communicator between L_n is meaningful as equal-time. One can also obtain the communicator between L_n and ϕ_m

$$[L_n, \phi_m] = (n(h-1) - m)\phi_{n+m}. \tag{163}$$

With the Virasoro generators L_n, one can also construct states as

$$L_{-k_1} L_{-k_2} \cdots L_{-k_n} |\phi\rangle. \tag{164}$$

3.3 The Free Boson

Canonical Quantization on the Cylinder–. Rewrite this part using the current operator to simplify the convention. Let $\phi(x, t)$ be a free Boson field defined on a cylinder of circumference L, such that $\phi(x + L, t) = \phi(x, t)$. The Lagrangian of the boson field is

$$\mathcal{L} = \frac{1}{4\pi K} \int dx \left\{ (\partial_t \phi)^2 - (\partial_x \phi)^2 \right\}. \tag{165}$$

Note that to connect to the convention in the previous part $g = \frac{1}{2\pi K}$. We can Fourier transform ϕ as

$$\phi(x, t) = \sum_n e^{\frac{2\pi}{L} i n x} \phi_n(t), \tag{166}$$

$$\phi_n(t) = \frac{1}{L} \int dx e^{-\frac{2\pi}{L} i n x} \phi(x, t). \tag{167}$$

The Lagrangian can be reexpressed as

$$\mathcal{L} = \frac{1}{4\pi K} \sum_n \left\{ \dot\phi_n \dot\phi_{-n} - \left(\frac{2\pi n}{L}\right)^2 \phi_n \phi_{-n} \right\}. \tag{168}$$

The momentum conjugate to ϕ_n becomes

$$\pi_n = g L \dot\phi_{-n}, \qquad [\phi_n, \pi_m] = i \delta_{nm}. \tag{169}$$

The Hamiltonian can be expressed as

$$H = \frac{1}{2gL} \sum_n \{\pi_n \pi_{-n} + (2\pi n g)^2 \phi_n \phi_{-n}\} \tag{170}$$

This corresponds to a sum of decoupled harmonic oscillators with frequencies $\omega = \frac{2\pi}{L}|n|$.

We can introduce creation and annihalation operators, which allow the Hamiltonian to be expressed as

$$H = \frac{1}{2gL} \pi_0^2 + \frac{2\pi}{L} \sum_n (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n). \tag{171}$$

The following commutation relation holds

$$[H, a_{-m}] = \frac{2\pi}{L} m a_{-m}. \tag{172}$$

Applying a_{-m} toan eigenstate with energy E, creates an eigenstate with energy $E + \frac{2\pi m}{L}$. The Fourier modes can be expressed as

$$\phi_n = \frac{i}{n\sqrt{4\pi g}} (a_n - \bar{a}_{-n}) \tag{173}$$

The fields can be expressed as

$$\phi(x,t) = \phi_0 + \frac{1}{gL}\pi_0 t + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n e^{\frac{2\pi i}{L}n(x-t)} - \overline{a}_{-n}e^{\frac{2\pi i}{L}n(x+t)}\right). \tag{174}$$

Transforming to Euclidean space, we can define the following conformal coordinates

$$z = e^{\frac{2\pi}{L}(\tau - ix)}, \qquad \overline{z} = e^{\frac{2\pi}{L}(\tau + ix)}. \tag{175}$$

This results in

$$\phi(z,\overline{z}) = \phi_0 - \frac{i}{4\pi g}\pi_0 \ln(z\overline{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\left(a_n z^{-n} + \overline{a}_n \overline{z}^{-n}\right). \tag{176}$$

The field ϕ is not a primary, however the holomorphic field $\partial\phi$ is.

$$i\partial\phi(z) = \frac{\pi_0}{4\pi g z} + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}a_n z^{-n-1} \tag{177}$$

Vertex Operators–. There exists an infinite variety of local fields related to ϕ without introducing a scale. These are called the vertex operators $\mathcal{V}_\alpha$.

$$\mathcal{V}_\alpha =: e^{i\alpha\phi(z,\overline{z})} : \tag{178}$$

The vertex operators have conformal dimensions $h(\alpha) = \overline{h}(\alpha) = \frac{\alpha^2}{8\pi g} = \frac{\alpha^2 K}{4}$.

3.4   The Fock Space

The eigenstates of H can be labeled by the eigenvalues of π_0. This means that the Fock space is built upon a one-parameter family of vacua $|\alpha\rangle$.

We know that $T(z)$ is given by

$$T(z) = -2\pi g : \partial\phi(z)\partial\phi(z) : \tag{179}$$

$$= \frac{1}{2}\sum_{n,m}z^{-n-m-2} : a_n a_m : . \tag{180}$$

From this we can derive the expression for the Virasoro operators

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}}a_{n-m}a_m \quad (n\neq 0), \tag{181}$$

$$L_0 = \sum_{n>0}a_{-n}a_n + \frac{1}{2}a_0^2. \tag{182}$$

This allows for the Hamiltonian to be expressed as

$$H = \frac{2\pi}{L}(L_0 + \overline{L}_0) \tag{183}$$

Furthermore the elements of the Fock space $a_{-1}^{n_1}a_{-2}^{n_2}\cdots\overline{a}_{-1}^{m_1}\overline{a}_{-2}^{m_2}\cdots|\alpha\rangle$ are eigenstates of L_0 with conformal dimensions $h = \frac{1}{2}\alpha^2 + \sum_j jn_j$ and $\overline{h} = \frac{1}{2}\alpha^2 + \sum_j jm_j$. The different vacua $|\alpha\rangle$ are related to the absolute vacuum $|0\rangle$ by the vertex operators $\mathcal{V}_\alpha$.

Twisted Boundary Conditions—.
We can also assume anti-periodic boundary condition. This is compatible with the Lagrangian

because it is quadratic in the fields. Changing to anti-periodic boundaries makes the summation index half-integer valued and removes the zero mode. There are now two vacua $|0_+\rangle$ and $|0_-\rangle$.

We have that

$$\langle \phi \partial \phi \rangle = \frac{1}{w} \sum_{n>0} \left(\frac{w}{z}\right)^n. \tag{184}$$

In the periodic case the sum is over integer values and becomes

$$\langle \phi \partial \phi \rangle = \frac{1}{z-w}. \tag{185}$$

In the anti-periodic case the sum is over half integer values and becomes

$$\langle \phi \partial \phi \rangle = \sqrt{\frac{z}{w}}\frac{1}{z-w}. \tag{186}$$

For the vacuum expactation value of the energy-momentum tensor, $\langle T(z)\rangle$, we have in the periodic case $\langle T(z)\rangle = 0$, but in the anti-periodic case $\langle T(z)\rangle = \frac{1}{16z^2}$.

Compactified Boson
We can identify ϕ with $\phi + 2\pi R$ to get the compact boson. In general we can consider the boundary condition

$$\phi(x+L,t) = \phi(x,t) + 2\pi m R, \tag{187}$$

where m represents the winding number of the field. This modifies the mode expansion as

$$\phi(x,t) = \phi_0 + \frac{n}{gRL}t + \frac{2\pi m R}{L}x + \frac{i}{\sqrt{4\pi g}}\sum_{k\neq 0}\frac{1}{k}\left(a_k e^{\frac{2\pi ik}{L}(x-t)} - \bar{a}_{-k}e^{\frac{2\pi ik}{L}(x+t)}\right). \tag{188}$$

After reexpressing in complex coordinates and taking the derivative we get

$$i\partial\phi(z) = \left(\frac{n}{4\pi gR}+\frac{1}{2}mR\right)\frac{1}{z} + \frac{1}{\sqrt{4\pi g}\sum_{k\neq 0}a_k z^{-k-1}}. \tag{189}$$

The virasoro operators L_0 and $\overline{L}_0$ can be expressed as

$$L_0 = \sum_{n>0}a_{-n}a_n + 2\pi g\left(\frac{n}{4\pi gR}+\frac{1}{2}mR\right)^2 \tag{190}$$

$$\overline{L}_0 = \sum_{n>0}\bar{a}_{-n}\bar{a}_n + 2\pi g\left(\frac{n}{4\pi gR}-\frac{1}{2}mR\right)^2 \tag{191}$$

3.5 The Free Fermion

The free fermion action is given by

$$S = \frac{1}{2}g\int d^2x\,\Psi^\dagger\gamma^0\gamma^\mu\partial_\mu\Psi \tag{192}$$

The central charge of this theory is $c = 1/2$ and ψ has as conformal dimension $h = 1/2$.

Canonical Quantization on a Cylinder
We can take the mode expansion of ψ at $t = 0$ on a cylinder with circumference L. This gives

$$\psi(x) = \sqrt{\frac{2\pi}{L}}\sum_k b_k e^{\frac{2\pi i}{L}kx}. \tag{193}$$

There are two possible types of boundary conditions. With the periodic (Ramond) boundary conditions the index k takes on integer values. With anti-periodic (Neveu-Schwarz) boundary conditions the index k must take half-integer values.

The Hamiltonian can be written as

$$H = \sum_{k>0} \omega_k b_{-k} b_k + E_0, \qquad \omega_k = \frac{2\pi |k|}{L}. \tag{194}$$

Mapping onto the Plane

Mapping ψ to the plane gives

$$\psi_{cyl}(z) = \sqrt{\frac{2\pi z}{L}} \psi_{pl}(z) \tag{195}$$

and thus

$$\psi(z) = \sum_k b_k z^{-k-\frac{1}{2}} \tag{196}$$

This transformation swaps the boundary Conditions

$$\begin{aligned} \psi(e^{2\pi i} z) &= -\psi(z) && (Ramond) \\ \psi(e^{2\pi i} z) &= \psi(z) && (Neveu - Schwarz) \end{aligned} \tag{197}$$

Note that the fermion field has a dimension, which changes the boundary condition when mapped from the cylinder to the plane.

The different sectors will have a different two-point correlation function. For the NS sector we have

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{z-w}. \tag{198}$$

In the R sector we have

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{2} \frac{\sqrt{z/w} + \sqrt{w/z}}{z-w}. \tag{199}$$

Furthermore depending on the boundary conditions the energy-momentum tensor will gain a non-zero expectation value.

$$\langle T(z) \rangle = 0 \qquad (Neveu - Schwarz) \tag{200}$$

$$\langle T(z) \rangle = \frac{1}{16z^2} \qquad (Ramond) \tag{201}$$

Vacuum Energies

The energy momentum tensor on the plane can be written as

$$T(z) = \frac{1}{2} \sum_{n,k} (k + \frac{1}{2}) z^{-n-2} : b_{n-k} b_k :, \tag{202}$$

which naturally leads to

$$L_n = \frac{1}{2} \sum_k (k + \frac{1}{2}) : b_{n-k} b_k : \tag{203}$$

L_0 is given by different expressions depending on the boundary conditions.

$$L_0 = \sum_{k>0} k b_{-k} b_k \qquad (NS) \tag{204}$$

$$L_0 = \sum_{k>0} k b_{-k} b_k + \frac{1}{16} \qquad (R) \tag{205}$$

$$\tag{206}$$

From this we can express the Hamiltonian as

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}). \tag{207}$$

3.6 Normal ordering

For free fields the OPE of the field with itself contains only one term with a constant prefactor. It can be regularized by normal ordering the fields, or equivalently, subtracting its expectation value. Using the former prescription for $T(z)T(w)$ only kills the $\propto c$ term. So clearly we need a more elaborate definition of normal ordering. We shall define proper normal ordering for general fields as subtracting all the singular terms from the OPE. We will write this normal ordering as

$$(AB)(z). \tag{208}$$

Concretely, given the OPE

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{209}$$

we have that

$$(AB)(w) = \{AB\}_0(w). \tag{210}$$

Equivalently, we can compute the normal ordering of fields using contour integral methods:

$$(AB)(w) = \frac{1}{2\pi i} \oint \frac{dz}{z-w} A(z)B(w). \tag{211}$$

The contraction of fields contains only the singular terms of the OPE:

$$\overbrace{A(z)B}(w) = \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}. \tag{212}$$

We now want to express the modes of the normal ordered field in terms of the modes of the input fields. Given fields A and B and points $|z| > |x| > |w|$ we write

$$A(z) = \sum_n (z-x)^{-n-h_A} A_n(x) \tag{213a}$$

$$B(w) = \sum_n (w-x)^{-n-h_B} B_n(x). \tag{213b}$$

Contour integrating ultimately results in:

$$(AB)_m = \sum_{n \leq -h_A} A_n B_{m-n} + \sum_{n > -h_A} B_{m-n} A_n, \tag{214}$$

where we defined the modes of (AB) as:

$$(AB)(z) = \sum_n z^{-n-h_A-h_B} (AB)_n. \tag{215}$$

Some warnings are in place:

1. Normal ordering is not commutative: $(AB)(z) \neq (BA)(z)$.

2. Normal ordering is not associative: $((AB)C)(z) \neq (A(BC))(z)$.

3. With this definition of normal ordering, Wick's theorem needs to be revisited. This is done in Appendix 6.B of the Book.

3.7 Conformal families and Operator algebra

> There's nothing stronger than family.
>
> D. T.

The goal of this section is to introduce the notion of conformal blocks and associated to this the method of conformal bootstrapping as a way to solve CFTs, i.e. compute the correlation functions, explicitly. Before that we revisit the notion of descendant fields and conformal families. In the following we mostly only care about the holomorphic part of fields.

Descendant fields
A descendant is generated from a primary by acting with the Virasoro operators:

$$\phi^{(-n)}(w) = (L_{-n}\phi)(w) = \frac{1}{2\pi i} \oint_w dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w), \tag{216}$$

in particular:

$$\phi^{(0)}(w) = h\phi(w), \qquad \phi^{(-1)} = \partial\phi(w). \tag{217}$$

Consider following correlation function of states that are part of the same family:

$$\langle (L_{-n}\phi)(w)X \rangle, \tag{218}$$

where X denotes a string of primary fields: $X = \phi_1(w_1)\cdots\phi_N(w_N)$. After a computation one finds:

$$\langle (L_{-n}\phi)(w)X \rangle = \mathcal{L}_{-n}\langle \phi(w)X \rangle \qquad (n \geq 1). \tag{219}$$

With the differential operator

$$\mathcal{L}_{-n} = \sum_i \left\{ \frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i} \right\}. \tag{220}$$

In other words, knowing all the correlation functions between primaries, $\langle \phi(w)X \rangle$, is sufficient to compute the correlation functions that involve descendants by applying the differential operators $\mathcal{L}_{-n}$. More generally, for descendants of the form

$$\phi^{(-k,-n)}(w) = (L_{-k}L_{-n}\phi)(w), \tag{221}$$

and so on, we find in a similar way that

$$\left\langle (L_{-k_1}\cdots L_{-k_n}\phi)(w)X \right\rangle = \mathcal{L}_{-k_1}\cdots\mathcal{L}_{-k_n}\langle \phi(w)X \rangle \qquad (n \geq 1) \tag{222}$$

Conformal families
A *conformal family* is a set of states that transform according to a representation of the conformal group. A family contains a primary and its descendants. We will denote the conformal

family associated with the primary ϕ by $[\phi]$. First descendants of a primary are sometimes called *secondary fields*. Another way to say that a conformal family transforms under itself is to say that the OPE of $T(z)$ with any member of the family will be composed solely of other members within the same family. Concretely:

$$
T(z)\phi^{(-n)}(w) = \frac{cn(n^2-1)/12}{(z-w)^{n+2}}\phi(w) + \sum_{k=1}^{n}\frac{n+k}{(z-w)^{k+2}}\phi^{(k-n)}(w) + \sum_{k\geq 0}(z-w)^{k-2}\phi^{(-k,-n)}(w)
$$
(223)

The operator algebra

The two and three-point functions of a CFT are fixed by conformal invariance. However, we need additional dynamical information to compute the three-point fusion coefficients C_{ijk} (for example using a conformal bootstrap approach). This information is contained in the *operator algebra*. The OPE which also includes the regular terms of all primary fields with each other. Using the operator algebra we can reduce all correlation functions to two-point correlation functions.

First we choose a basis of fields such that $C_{\alpha\beta} = \delta_{\alpha\beta}$ in

$$
\left\langle \phi_\alpha(w,\overline{w})\phi_\beta(z,\overline{z})\right\rangle = \frac{C_{\alpha\beta}}{(w-z)^{2h}(\overline{w}-\overline{z})^{2\overline{h}}}.
$$
(224)

This implies that states belonging to different conformal families are always orthogonal. From scale invariance it follows that:

$$
\phi_1(z,\overline{z})\phi_2(0,0) = \sum_{p}\sum_{\{k,\overline{k}\}} C_{12}^{p\{k,\overline{k}\}} z^{h_p-h_1-h_2+K}\overline{z}^{\overline{h}_p-\overline{h}_1-\overline{h}_2+\overline{K}}\phi_p^{(k,\overline{k})}(0,0).
$$
(225)

We introduced the notation $K = \sum_i k_i$.

Writing

$$
C_{12}^{p\{0,0\}} \equiv C_{12}^p = C_{p12},
$$
(226)

we find that

$$
C_{12}^{p\{k,\overline{k}\}} = C_{12}^p \beta_{12}^{p\{k\}}\overline{\beta}_{12}^{p\{\overline{k}\}}.
$$
(227)

This means that descendants fields are correlated to a given third field only if the primary is correlated. An the holomorphic and antiholomorphic parts factorize.

An example is given in the Book. Even in a relatively simple case, finding the three point function is not straightforward!

In conclusion, given the central charge, the conformal dimensions and the three-point coefficients C_{pnm}, one can, in principle, determine the operator algebra. Using the operator algebra, all the n-point correlation functions can be computed and the entire theory is solved.

Conformal blocks

Let us illustrate how the four-point functions can be reduced to three-point functions using the machinery introduced in the previous sections.

We consider the four-point function

$$
\left\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\phi_4(z_4,\overline{z}_4)\right\rangle.
$$
(228)

For sake of simplicity, we shall carry out a global conformal transformation to put $z_4 = 0$, $z_1 = \infty$, $z_2 = 1$, $z_3 = x$. We define:

$$
G_{34}^{21}(x,\overline{x}) = \left\langle h_1,\overline{h}_1 \middle| \phi_2(1,1)\phi_3(x,\overline{x}) \middle| h_4,\overline{h}_4\right\rangle.
$$
(229)

Note the order of the indices!

Using operator algebra techniques, we can write this function as:

$$G_{34}^{21}(x, \overline{x}) = \sum_p C_{34}^p C_{12}^p A_{34}^{21}(p|x, \overline{x}). \tag{230}$$

The sum over p is a sum over intermediate conformal families that play the role of mediating channels in the scattering from fields from $(0, x)$ towards $(1, \infty)$. These functions $A_{34}^{21}(p|x, \overline{x})$ are called *partial waves*. They can be depicted by:

$$A_{kl}^{ji}(p|x, \overline{x}) =$$

The partial wave factorizes in a holomorphic and antiholomorphic part, according to:

$$A_{34}^{21}(p|x, \overline{x}) = \mathcal{F}_{34}^{21}(p|x)\overline{\mathcal{F}}_{34}^{21}(p|\overline{x}), \tag{231}$$

we call these functions $\mathcal{F}$, the *conformal blocks*. There is a recipe to compute these conformal blocks, even though it is a pain to compute these in practice. Physically speaking, these conformal blocks are the part in the four-point function that is fixed by conformal invariance. They depend on the anharmonic ratios via a series expansion. The remaining elements are the three-point coefficients, which are *not* fixed by conformal invariance.

Crossing symmetry and the conformal bootstrap
What happens if we would choose instead of $z_4 = 0$, $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ a different order of the fields? Following identities can be obtained relatively easy:

$$G_{34}^{21}(x, \overline{x}) = G_{32}^{41}(1 - x, 1 - \overline{x}), \tag{232}$$

and

$$G_{34}^{21}(x, \overline{x}) = \frac{1}{x^{2h_3}\overline{x}^{2\overline{h}_3}} G_{31}^{24}(1/x, 1/\overline{x}). \tag{233}$$

These identities are specific instances of the *crossing symmetry* of the functions G. Explicitly we can write the first identity as

$$\sum_p C_{21}^p C_{34}^p \mathcal{F}_{34}^{21}(p|x)\overline{\mathcal{F}}_{34}^{21}(p|\overline{x}) = \sum_q C_{41}^q C_{32}^q \mathcal{F}_{32}^{41}(p|1 - x)\overline{\mathcal{F}}_{32}^{41}(p|1 - \overline{x}), \tag{234}$$

which has a aesthetically pleasing pictorial interpretation: Crossing symmetry imposes N^4 constrains on the $N^3 + N$ parameters C_{mn}^p and h_p. Exploiting crossing symmetry to compute these parameters is called *conformal* bootstrap.

$$\sum_p C_{nm}^p C_{lk}^p \quad \diagram \quad = \quad \sum_q C_{nl}^q C_{mk}^q \quad \diagram$$

4 Modular invariance

Up until now we considered only CFTs on the infinite plane/Riemann sphere and at the exact fixed point. In that case the Hilbert space is a tensor product of a holomorphic and (completely decoupled) anti-holomorphic part. These parts don't interact and can this in principle describe completely different theories. In this section we will consider CFTs on the torus, still at the fixed point, in which case the two parts will need to be coupled in such a way that *modular invariance* is preserved. This might seem somewhat ad hoc to define theories on a higher genus surface, but from a string theory point of view this is very natural. CFTs on higher genus surfaces describe higher order closed string scattering amplitudes. From a more stat mech point of view, a theory on a torus is simply a theory with periodic boundary conditions in both directions.

4.1 Conformal field theory on the torus

We will define the *periods* of the lattice as linearly independent complex numbers ω_1, ω_2. The only dependence of the partition function can be on the *modular* parameter $\tau = \omega_1/\omega_2$. As this is independent of the surface area and the absolute orientation of the periods. **The partition function** In the operator formalism the partition function Z and its dependence on τ stands central. We shall define the time and space direction to be the imaginary and real axis respectively. In that case:

$$Z = \text{Tr}\left(\exp(-\{H \,\text{Im}\, \omega_2 - iP \,\text{Re}\, \omega_2\})\right), \tag{235}$$

using

$$H = \frac{2\pi}{L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right) \tag{236}$$

$$P = \frac{2\pi}{L}\left(L_0 - \overline{L}_0\right), \tag{237}$$

[2] (the factor $c/12$ ensures the energy is zero in the infinite length case), we find:

$$Z = \text{Tr}\left(q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - c/24}\right), \tag{238}$$

where

$$q = \exp(2\pi i \tau) \tag{239}$$

$$\overline{q} = \exp(-2\pi i \overline{\tau}). \tag{240}$$

We will see that the partition function will be a bilinear combination of characters.

Modular invarariance Suppose ω_i' define the same lattice, then necessarily:

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{241}$$

Where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. The matrix has an integer inverse, and describes basis transformations that leave the surface area of the torus invariant.

Under a $SL(2, \mathbb{Z})$ transformation:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1. \tag{242}$$

---

[2]Remark: there is a misprint in the book: P shouldn't have an i.

This leads to the modular group $SL(2,\mathbb{Z})/\mathbb{Z}_2 = PSL(2,\mathbb{Z})$. Where the $\mathbb{Z}_2$ subgroup that is modded out corresponds to the symmetry $a,b,c,d \mapsto -a,-b,-c,-d$.

Generators and the fundamental domain The modular group is generated by:

$$T : \tau \mapsto \tau + 1, \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{243}$$

$$S : \tau \mapsto -\frac{1}{\tau}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{244}$$

And you can check $(ST)^3 = S^2 = -1.$[3]

Geometrically, the modular group is generated by *Dehn twists*: T amounts to cutting the torus at fixed time and turning one of the ends 2π before gluing it back together. $U = TST$ does the same for fixed space. These are operators which are not smoothly connected to the identity.

A *fundamental domain* F is a part of the upper half complex plane that can be mapped to every other point of the complex plane by modular transformations. A convention for F_0 is:

$$z \in F_0 \quad \text{if} \quad \begin{cases} \operatorname{Im} z > 0, -\frac{1}{2} \le \operatorname{Re} z \le 0, & |z| \ge 1 \\ \operatorname{Im} z > 0, 0 < \operatorname{Re} z < \frac{1}{2}, & |z| \ge 1 \end{cases} \tag{245}$$

4.2 The free boson on the torus

Recall Dedekind's η function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{246}$$

it can be shown that under modular transformations:

$$\eta(\tau + 1) = e^{i\pi/12} \eta(\tau) \tag{247}$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau). \tag{248}$$

Without zero mode the boson partition function can be written as:

$$Z_{\text{bos}}(\tau) = \frac{1}{(\operatorname{Im}\tau)^{1/2} |\eta(\tau)|^2}. \tag{249}$$

This is a modular invariant combination of τ and can be obtained in the path-integral formalism by expanding the boson field in the eigenfunctions of the Laplacian and using ζ-regularization.

4.3 Free fermions on the Torus

Fermion fields can be periodic or anti-periodic in two directions on the lattice:

$$\psi(z + \omega_1) = e^{2\pi i v} \psi(z) \tag{250}$$

$$\psi(z + \omega_2) = e^{2\pi i u} \psi(z), \tag{251}$$

$v, u = 0, \frac{1}{2}$, and we shall call the periodic boundary condition *Ramond* and the anti-periodic boundary condition *Neveu-Schwarz*. This leads to four different sectors. A set of boundary

---

[3]Remark: there is a misprint in the book: S is missing a minus sign and $(ST)^3 = S^2 = 1$ should be -1.

conditions is called a *spin structure* for the fermion on the torus. Because ψ and $\overline{\psi}$ decouple we can write:

$$Z_{\nu,u} = |d_{\nu,u}|^2. \tag{252}$$

A tedious computation leads to

$$d_{0,0} = \frac{1}{\sqrt{2}} \mathrm{Tr}\,(-1)^F q^{L_0 - 1/48} = \frac{1}{\sqrt{2}} \mathrm{Tr}\,(-1)^F q^{\sum_k k b_{-k} b_k + 1/24}, \tag{253}$$

$$d_{0,\frac{1}{2}} = \frac{1}{\sqrt{2}} \mathrm{Tr}\, q^{L_0 - 1/48} = \frac{1}{\sqrt{2}} \mathrm{Tr}\, q^{\sum_k k b_{-k} b_k + 1/24}, \tag{254}$$

$$d_{\frac{1}{2},0} = \mathrm{Tr}\,(-1)^F q^{L_0 - 1/48} = \mathrm{Tr}\,(-1)^F q^{\sum_k k b_{-k} b_k - 1/48}, \tag{255}$$

$$d_{\frac{1}{2},\frac{1}{2}} = \mathrm{Tr}\, q^{L_0 - 1/48} = \mathrm{Tr}\, q^{\sum_k k b_{-k} b_k - 1/48}, \tag{256}$$

or

$$d_{0,0} = 0, \tag{257}$$

$$d_{0,\frac{1}{2}} = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}, \tag{258}$$

$$d_{\frac{1}{2},0} = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}, \tag{259}$$

$$d_{\frac{1}{2},\frac{1}{2}} = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}. \tag{260}$$

Remember the characters

$$\chi_{(c,h)}(\tau) = \mathrm{Tr}\, q^{L_0 - c/24}. \tag{261}$$

And the expressions for L_0:

$$L_0 = \sum_{k>0} k b_{-k} b_k \qquad \left(k \in \mathbb{Z} + \frac{1}{2}\right), \tag{262}$$

$$L_0 = \sum_{k>0} k b_{-k} b_k + \frac{1}{16} \qquad (k \in \mathbb{Z}), \tag{263}$$

in the NS and R sectors.

Consider e.g. the NS sector. Since L_0 takes both half-integer and integer values, this character is the sum of at least two simple Virasoro characters. From the fact that $c = 1/2$ we find that

$$\chi_{1,1} = q^{-1/48} \frac{1}{2} \mathrm{Tr}\left(1 + (-1)^F\right) q^{L_0} \tag{264}$$

$$\chi_{2,1} = q^{-1/48} \frac{1}{2} \mathrm{Tr}\left(1 - (-1)^F\right) q^{L_0}. \tag{265}$$

Comparing with the partition function:

$$\chi_{1,1} = \frac{1}{2}\left(d_{\frac{1}{2},\frac{1}{2}} + d_{\frac{1}{2},0}\right), \tag{266}$$

$$\chi_{2,1} = \frac{1}{2}\left(d_{\frac{1}{2},\frac{1}{2}} - d_{\frac{1}{2},0}\right). \tag{267}$$

Similarly for the R sector:

$$\chi_{1,2} = \frac{1}{\sqrt{2}} d_{0,\frac{1}{2}}. \tag{268}$$

From the modular transformation properties of the d's, it follows there are two modular invariant partition functions: the one which only contains (R,R) and is zero and the one that combines the other sectors as:

$$Z = Z_{\frac{1}{2},\frac{1}{2}} + Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},0} \tag{269}$$

$$= \left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| \tag{270}$$

$$= 2\left(|\chi_{1,1}|^2 + |\chi_{2,1}|^2 + |\chi_{1,2}|^2\right). \tag{271}$$

This is twice the partition function of the Ising model on the torus!

5 FRS/Categorical construction of RCFT

The mordern lanuage (probaly not the final lanuage of course) for RCFTs is the so-called category theory. The chiral symmetry of a RCFT is described by a (semisimple) modular tensor category $\mathcal{C}$. A full CFT requires the knowledge of how to combine the chiral and anti-chiral parts. Historically modular invariance has been utilized to construct a full CFT. However there exist non-physical modular invariant theories. The required part is a semisimple module category $\mathcal{M}$ over $\mathcal{C}$. Physically $\mathcal{M}$ contain information of how bulk fields interact with boundaries or more generally defects. Category theory provides a unified and beautiful lanuage of how bulk, boudary and defect fields interact, which compose a full CFT.

5.1 Chiral symmetry in RCFT

5.2 Verlinde lines in diagonal RCFT

5.3 Global symmetry and orbifold RCFT with discrete torsion

With a given CFT $\mathcal{T}$ defined on $\mathcal{M}$ with a discrete global group symmetry G, it is possible to define a different CFT $\tilde{T} \sim T/G$ by gauging the global symmetry G [3–5]. The orbifold theory $\tilde{T}$ then has a fusion category symmetry $\text{Rep}(G)$. A symmetry operator or simple object in $\text{Rep}(G)$ is labeled by G-representation. One-dimensional symmetry operators are invitable, while higher dimensional representations correspond to non-invitable symmetries.

We first review the global symmetry G in RCFT. Physically each symmetry operator g can be represented by a line operator $\mathcal{L}_g$. We further require that $\mathcal{L}_g$ commutes with the chiral vertext algebra in RCFT, hence $\mathcal{L}_g$ is topological, which are denoted as topological defect lines (TDLs) [4]. Two symmetry operators can be fused into a new one, in which the fusion rule follows the multiplication of group elements. When fusing three operators, there are two different ways. These two fusion orders are related by a F-move, characterized by $\alpha(g, h, k)$: 3-cocycles $G^3 \to U(1)$ up to coboundaries. A non-trivial 3-cocycle means the global symmetry G is anomalous in which the fusion of $\mathcal{L}_g$ not associative.

When G is non-anomalous, we can construct orbifold theories. We start by inserting a symmetry projector $P := \frac{1}{|G|} \sum_{g \in G} \mathcal{L}_g$ in the partition function

$$Z = \text{Tr}\left(P \, q^{L_0} \overline{q}^{\overline{L}_0} \right). \tag{272}$$

However, this theory is usually not modular invariant, since a modular transformation can modifies the symmetry action in the time direction and generate different terms. One needs to consider the so-called twisted sectors in the Hilbert space $\mathcal{H}_h$ to construct a modular invariant theory. This twisted sector $\mathcal{H}_h$ is difind according to

$$\phi(e^{2\pi i}z, e^{-2\pi i}\overline{z}) = \mathcal{L}_h \circ \phi(z, \overline{z}). \tag{273}$$

Now after symmetrilization, one writes down the orbifold theory

$$Z(q, \overline{q}) = \frac{1}{|G|} \sum_{[g,h]=0} \text{Tr}_{\mathcal{H}_h}\left(\mathcal{L}_g \, q^{L_0} \overline{q}^{\overline{L}_0} \right). \tag{274}$$

For a given symmetry group G, there may exist different orbifold ways. It turns out one can use the second cohomology group $H^2(G, U(1))$ to classify orbifolds. This is known as discrete torsion in the literature. Specifically one can use $\epsilon(g, h)$ to characterize each orbifold theory

$$Z(q, \overline{q}) = \frac{1}{|G|} \sum_{[g,h]=0} \epsilon(g, h) \text{Tr}_{\mathcal{H}_h}\left(\mathcal{L}_g \, q^{L_0} \overline{q}^{\overline{L}_0} \right), \tag{275}$$

---

[4]There may exist conformal line operators which are not topological.

where $\epsilon(g,h) := \nu(g,h)/\nu(h,g)$ with $\nu(g_1, g_2)$ being two cocycles. The symmetry projector becomes $P_h := \frac{1}{|G|} \sum_g \epsilon(g,h) \mathcal{L}_g$.

5.4 Duality TDL and Self-dual orbifold

5.5 Topological defects in Ising CFT

Besides the closed manifolds, CFTs can also be defined on manifols with boundaries, or defects in more general setting. A boundary can be understood a special defect between a CFT and a trivial CFT with a central charge $c = 0$. On the other hand a defect between CFT1 and CFT2 can be regarded as a boundary of stacking the two CFTs using the folding trick. It is natural to require the conformal symmetry still holds at boundaries and defects. For a boundary state $|a\rangle$ this means

$$\left(L_n - \overline{L}_{-n}\right)|a\rangle = 0. \tag{276}$$

For a defect O_α it requires

$$\left[L_n - \overline{L}_{-n}, O_\alpha\right] = 0. \tag{277}$$

Usually we require stronger constraint for the conformal defects

$$\begin{aligned} [L_n, O_\alpha] &= 0 \\ \left[\overline{L}_n, O_\alpha\right] &= 0 \end{aligned} \tag{278}$$

which are now called topological defects. They correspond to the topological aspects of CFT and serve the role of intertwiners between conformal boundaries. Since the topological defects commute with Virasoro generators, they can be viewed as genearlized symmetries.

Let's start with a simple example — the Ising CFT $\mathcal{M}(4,3)$. There are three conformal boundaries/topological defects in Ising CFT which are labeled by primaries. Two defects can be fused into other defects when put close to each other. As a result their fusion forms a fusion algebra. In Ising CFT, the fusion algebra of defects is the same as that of primaries. This is true for all A-type minimal models.

5.6 Boundary states

6 Other topics

6.1 Boundary conformal field theory

CFT can also be defined in a manifold with boundaries, in which the nice local properties are still applied from the CFT defined on a plane. A boundary conformal field theory (BCFT) is a CFT defined on a two dimension manifold with one dimension boundaries. It is important theoretical tool in the study of surface critical behavior, defects and open string theories. Note that one can also consider extended symmetry in a CFT, so that the Virasoro algebra is a subalgebra of the extended symmetry. The corresponding boundary condition (resp. boudary state) in the open (resp. closed) sector should also respect the extended algebra. It was shown than there are also symmetry-breaking boundary conditions which only respect the conformal symmetry. Here we only discuss conformal symmetry.

BCFT in the open sector We define a BCFT on a cylinder with two boundaries B_a and B_b. The complex coordinate is denoted by $\omega = \tau + i\sigma$, where we identify the time τ ($\tau \in [0, 2\pi t]$) direction to be periodic and space σ ($\sigma \in [0, \pi]$) to be open. When the time direction range is unbounded $t \to \infty$, the cylinder becomes a strip. Since the Hamiltonian H_{ab} generating time evolution is defined at a fixed time with two boundaries, the whole Hilbert space is constrained by the boundary conditions at B_a and B_b. To apply the powerful conformal symmetry to the strip, one can apply the conformal mapping $z = e^{\omega} = e^{\tau} e^{i\sigma}$ to relate the strip to the upper half-plane. The energy-momentum tensor has to be real at the boundary, so that

$$T(z) - \overline{T}(\bar{z})|_{\sigma=0,\pi} = 0 \tag{279}$$

or in terms of Laurent series

$$\sum_n L_n z^{-n-2} - \sum_n \overline{L}_n \bar{z}^{-n-2}|_{\sigma=0,\pi} = 0. \tag{280}$$

One then finds the two Virasoro algebra now reduces to a single algebra under the constraint $L_n = \overline{L}_n$. This is naturally satisfied if one identifies the antiholomorphic coordinate $\bar{z}$ in the upper half-plane to be the holomorphic one z^* in the lower half-plane. As a result, a BCFT defined on a strip is mapped to a chiral CFT defined on a plane.

With this identification, one can express operators defined on a cylinder in terms of those defined on a plane. The Hamiltonian operator is

$$H_{ab} = L_0 - \frac{c}{24}. \tag{281}$$

The partition function can also be written down

$$Z_{ab} = \text{Tr}\left(e^{-2\pi t H_{ab}}\right) = \text{Tr}\left(q^{L_0 - c/24}\right), \tag{282}$$

where $q = e^{\pi i \tau_{open}}$ and the modular parameter $\tau_{open} = it$. As stated above, the Hilbert space is determined by the boundaries $B_{a/b}$. A conformal family $[\phi_h(z)]$ appearing in the chiral theory should be consistent with the boundary conditions. The partition function can be written as

$$Z_{ab} = \sum_h n_{ab}^h \chi_h(q) \tag{283}$$

where the non-negative integer n_{ab}^h selects the conformal families appearing in the theory and the character $\chi_h(q) = \text{Tr}_h\left(q^{L_0 - c/24}\right)$.

BCFT in the closed sector In the open sector the time direction is chosen to be periodic. Meanwhile one has the freedom to define the space σ ($\sigma \in [0, 2\pi]$) direction to be periodic and time τ to be open ($\tau \in [0, 2\pi l]$), which is the closed sector. The calculation of partition function becomes a time evolution

$$Z_{ab} = \langle a|e^{-2\pi l H}|b\rangle = \langle a|e^{-2\pi l(L_0 + \overline{L}_0 - c/12)}|b\rangle, \tag{284}$$

where the time evolution operator H defined on a cylinder is the dilation operator on the plane $H = L_0 + \overline{L}_0 - c/12$.

The physical constraint Eq. 279 in the closed sector now becomes

$$z^2 T(z) - \overline{z}^2 \overline{T}(\overline{z})|_{\tau=0,2\pi l} = 0. \tag{285}$$

Note that due to the choice of time direction, the boundary condition is a constraint to the boundary state $|a\rangle$ and $|b\rangle$. In terms of Laurent modes, this is

$$\left(L_n - \overline{L}_{-n}\right)|a\rangle = 0. \tag{286}$$

The basis state of the solution is the so-called Ishibashi state

$$|h\rangle\rangle = \sum_n |h, n\rangle \otimes U\overline{|h, n\rangle}, \tag{287}$$

where U is an anti-unitary operator which commutes with Virasoro generators, h denotes a conformal family ϕ_h. If there are a finite number representations of the Virasoro algebra in the theory, one can construct physical boundary states, the Cardy states, in terms of the Ishibashi states

$$|a\rangle = \sum_h c_a^h |h\rangle\rangle. \tag{288}$$

The partition function Z_{ab} then reads

$$Z_{ab} = \sum_h \left(c_a^h\right)^* c_b^h \chi_h(\tilde{q}) \tag{289}$$

where $\tilde{q} = e^{2\pi i \tau_{closed}}$ with the modular parameter $\tau_{closed} = 2i\,l = i/t$.

Open-closed BCFT duality The open and closed sector theories are essentially describing the same Z_{ab}. They are related through a modular transformation of the modular parameter $\tau \to -1/\tau$ which exchange the role of space and time. The characters are related by a S matrix

$$\chi_j(\tilde{q}) = \sum_i S_j^i \chi_i(q). \tag{290}$$

This gives the Cardy conditions to the BCFT

$$n_{ab}^h = \sum_i \left(c_a^i\right)^* c_b^i S_i^h$$
$$\left(c_a^h\right)^* c_b^h = \sum_i n_{ab}^i S_i^h. \tag{291}$$

For minimal models the Cardy states can be constructed by requiring

$$c_a^h = S_a^h / \sqrt{S_0^h}$$
$$n_{ab}^h = \sum_i \frac{\left(S_a^i\right)^* S_b^i S_h^i}{S_0^i}. \tag{292}$$

Comparing with the Verlinde formula, one can identify the coefficient n_{ab}^h is the same as the fusion multiplicities N_{ab}^h. It is now clear that the conformal families appearing in the open sector theory is determined by the fusion of the primaries representing the Cardy states in the closed sector.

6.2   Bosonization

In the discussion of free compact boson theory, we have seen the vertex operator $V_1 = \,:e^{i\phi(z,\bar{z})}:$ has the same scaling dimension as a fermion operator and more importantly the free complex fermion and and the free compact boson (with a radius $R = 1$) share the same partiton function. These facts indicate that the fermion and boson theories are dual to each other.

6.2.1   Free compact boson

Here we review the free compact boson theory in (1+1) d

$$S = \frac{1}{4\pi K} \int \partial\varphi \overline{\partial}\varphi, \tag{293}$$

where we used the convention in the note. Under this convention, the two-point correlation of free boson $\langle\varphi(x)\varphi(y)\rangle = -2K\log(r)$. However, it is clear that K is directly related to the compactified radius R of φ, meaning that one has the freedom to fix either K or R to a constant without causing any trouble. We introduce $\tilde{R} := R/\sqrt{K}$ for convenience. Then the scalar is compactified as $\varphi \sim \varphi + 2\pi\tilde{R}\sqrt{K}$.

With the chiral mass center introduced above the chiral part of the compact boson field $\varphi(z,\bar{z}) = \phi(z) + \overline{\phi}(\bar{z})$ can be expanded in terms of current operators. Recall that the two-point correlation of free boson $\langle\varphi(x)\varphi(y)\rangle = -2K\log(r)$ [5]. The current operator can be defind as

$$j(z) = i\partial\varphi/\sqrt{K}, \tag{294}$$

which ensures the the current operator normalized as $\langle j(z)j(z')\rangle = 1/z^2$.

The chiral/anti-chiral part of the free compact boson can be written as (z is defined on the plane)

$$\phi(z)/\sqrt{K} = \phi_0 - i j_0 \ln z + i \sum_{n\neq 0} \frac{1}{n} j_n z^{-n}$$
$$\overline{\phi}(\bar{z})/\sqrt{K} = \overline{\phi}_0 - i \overline{j}_0 \ln \bar{z} + i \sum_{n\neq 0} \frac{1}{n} \overline{j}_n \bar{z}^{-n}. \tag{295}$$

Note here the complex coordinate denotes $z = e^{\omega} = e^{\tau+i\sigma}$. ω is the coordinate defined on the cylinder. Whenever we use a quantum theory, we should consult to the radial quantization. In terms of the chiral term, we can write down the free scalar $\varphi = \phi + \overline{\phi}$ and its dual field $\theta = \phi - \overline{\phi}$ as

$$\varphi/\sqrt{K} = \varphi_0 - i\tau\left(j_0 + \overline{j}_0\right) + \sigma\left(j_0 - \overline{j}_0\right) + i \sum_{n\neq 0} \frac{1}{n}\left(j_n z^{-n} + \overline{j}_n \bar{z}^{-n}\right)$$
$$\theta/\sqrt{K} = \theta_0 - i\tau\left(j_0 - \overline{j}_0\right) + \sigma\left(j_0 + \overline{j}_0\right) + i \sum_{n\neq 0} \frac{1}{n}\left(j_n z^{-n} - \overline{j}_n \bar{z}^{-n}\right). \tag{296}$$

Let's derive the quantum version (radial quantization) of the free compact boson in terms of the current operator. Recall the Hamiltonian for the free compact boson reads

$$H = \int dx \left(2\pi K \Pi^2 + \frac{1}{8\pi K}(\partial_x\phi)^2\right), \tag{297}$$

---

[5]In Ref. [1], K is set to be $K = 1$.

where the canonical momentum $\Pi_\varphi = \frac{1}{4\pi K}\dot{\phi}$, which satisfies the standard canonical commutation (equal-time)

$$\left[\varphi(x),\,\Pi_\varphi(y)\right] = i\,\delta(x-y) \tag{298}$$

The whole theory can be actuall reproduced using the current algebra j_n and $\bar{j}_n$. One can write down (note the quantization is defined on the plane, or the so-called radial quantization)

$$\begin{aligned}
\partial_x \varphi &= i z\,\partial\varphi - i\bar{z}\,\overline{\partial}\varphi = j_n/z^n - \bar{j}_n/\bar{z}^n \\
\partial_t \varphi &= i z\,\partial\varphi + i\bar{z}\,\overline{\partial}\varphi = j_n/z^n + \bar{j}_n/\bar{z}^n \\
\partial_x \theta &= i z\,\partial\theta - i\bar{z}\,\overline{\partial}\theta = j_n/z^n + \bar{j}_n/\bar{z}^n = \partial_t \varphi \\
\partial_t \theta &= i z\,\partial\theta + i\bar{z}\,\overline{\partial}\theta = j_n/z^n - \bar{j}_n/\bar{z}^n = \partial_x \varphi
\end{aligned} \tag{299}$$

One can verify the canonical communicator for the scalar φ through the current algebra

$$\begin{aligned}
\left[\phi(x),\,\phi(y)\right] &= [\phi_0, -i\,j_0\log(\omega)] - [\phi_0, -i\,j_0\log(z)] - \sum_{n\neq 0}\frac{1}{nm}[j_n, j_m]\frac{1}{z^n\omega^m} \\
&= \log(\omega/z) + \sum_{n\neq 0}\frac{1}{n}\frac{\omega^n}{z^n} \\
&= i(y-x) - \sum_{n\neq 0}\frac{1}{n}e^{in(x-y)} \\
&= -2\pi i\,H(x-y)
\end{aligned} \tag{300}$$

and

$$\left[\bar{\phi}(x),\,\bar{\phi}(y)\right] = 2\pi i\,H(x-y) \tag{301}$$

where $H(x)$ is the step function. Now one can easily find that

$$\begin{aligned}
\left[\phi(x),\,\partial_y\phi(y)\right] &= i\sum_n e^{in(x-y)} = -2\pi i\delta(x-y) \\
\left[\bar{\phi}(x),\,\partial_y\bar{\phi}(y)\right] &= -i\sum_n e^{in(x-y)} = -2\pi i\delta(x-y)
\end{aligned} \tag{302}$$

and

$$\begin{aligned}
[\varphi(x),\,\varphi(y)] &= 0 \\
[\theta(x),\,\theta(y)] &= 0 \\
[\varphi(x),\,\theta(y)] &= -4\pi i\,H(x-y) \\
\left[\varphi(x),\,\Pi_\varphi(y)\right] &= \left[\varphi(x),\,\partial_y\theta(y)/4\pi\right] = i\delta(x-y) \\
\left[\theta(x),\,\Pi_\theta(y)\right] &= \left[\theta(x),\,\partial_y\varphi(y)/4\pi\right] = i\delta(x-y)
\end{aligned} \tag{303}$$

Thus the Hamiltonian can be written down in terms of current operators

$$\begin{aligned}
HK &= \frac{1}{4\pi}\int_0^{2\pi} d\sigma \sum_{n,m}\left(\frac{j_n j_m}{z^{n+m}} + \frac{\bar{j}_n \bar{j}_m}{\bar{z}^{n+m}}\right) \\
&= \frac{1}{2}j_0 j_0 + \frac{1}{2}\bar{j}_0\bar{j}_0 + \sum_{n>0} j_{-n}j_n + \bar{j}_{-n}\bar{j}_n.
\end{aligned} \tag{304}$$

We thus identify the Virasoro generator

$$\begin{aligned}
L_0 &= \frac{1}{2K}j_0 j_0 + \frac{1}{K}\sum_{n>0} j_{-n}j_n \\
\overline{L}_0 &= \frac{1}{2K}\bar{j}_0\bar{j}_0 + \frac{1}{K}\sum_{n>0}\bar{j}_{-n}\bar{j}_n.
\end{aligned} \tag{305}$$

It is clear that j_0 and $\bar{j}_0$ are converved quantuties. In the following discussions, we will use the notation j_0 interchangably to represent both the operator and its charge. Now let's derive their charges in the quanttum theory. This relies on the modular invariance of the torus partition function. Thanks to the current operator algebra, we can construct the Hilbert space as

$$|j_0; \, n_1, n_2, n_3, \dots\rangle = j_{-1}^{n_1} j_{-2}^{n_2} j_{-3}^{n_3} \dots |j_0\rangle, \tag{306}$$

where j_0 is a conserved quantity. The torus partition function thus reads

$$Z_{cir}(q, \overline{q}) = q^{j_0^2/2K} \, \overline{q}^{\bar{j}_0^2/2K} \, Z'_{boson}(q, \overline{q}), \tag{307}$$

where Z'_{boson} denotes the part without zero modes. A modular T transformation $\tau \to \tau+1, \overline{\tau} \to \overline{\tau}+1$ induces $q \to e^{2\pi i}q$ and $\overline{q} \to e^{-2\pi i}\overline{q}$. Note that the constant K in the Hamiltonian does not change the energy spectrum and Hilbert space at all, which can be absorbed into the definition of the modular parameter q and $\overline{q}$.

The partition function reads

$$\begin{aligned} Z_{cir}(e^{2\pi i}q, e^{-2\pi i}\overline{q}) &= e^{2\pi i\left(j_0^2-\bar{j}_0^2\right)/2} \, q^{j_0^2/2} \, \overline{q}^{\bar{j}_0^2/2} \, Z'_{boson}(q, \overline{q}) \\ &= e^{2\pi i\left(j_0-\bar{j}_0\right)\left(j_0+\bar{j}_0\right)/2} \, q^{j_0^2/2} \, \overline{q}^{\bar{j}_0^2/2} \, Z'_{boson}(q, \overline{q}) \\ &= e^{2\pi i m\left(j_0+\bar{j}_0\right)/2\tilde{R}} \, q^{j_0^2/2} \, \overline{q}^{\bar{j}_0^2/2} \, Z'_{boson}(q, \overline{q}) \end{aligned} \tag{308}$$

A modular invariant partition function thus requires

$$\left(j_0 + \bar{j}_0\right)/2\tilde{R} = e, \tag{309}$$

where the electronic charge e is also an integer[6]. One then finds the zero mode of the current operator is quantitized to be

$$\begin{aligned} j_0 &= e\frac{1}{\tilde{R}} + m\frac{\tilde{R}}{2} \\ \bar{j}_0 &= e\frac{1}{\tilde{R}} - m\frac{\tilde{R}}{2}. \end{aligned} \tag{310}$$

From the quantization of zero modes j_0, it is also clear that the dual scalar $\theta(x)$ is compactified as

$$\theta \sim \theta + 4\pi/\tilde{R} \sim \theta + 2\pi\frac{2}{\tilde{R}} \sim \theta + 2\pi\tilde{R}_\theta \tag{311}$$

or in other words the duality transformation $(\varphi \to \theta)$ changes the radius $\tilde{R} \to 2/\tilde{R}$.

It is clear now K and R are not independent. The theory is actuall determined by the ratio $\tilde{R} := R/\sqrt{K}$. A parameter K introduced in the action is equivalently to say the free scalar has a compactified radius $R = \sqrt{K}$. We can either choose $K = 1$

$$\begin{aligned} \varphi &\sim \varphi + 2\pi R \\ \theta &\sim \theta + 2\pi/R_\theta \sim \theta + 4\pi/R \\ V_{e,m} &\sim e^{ie\varphi/R} \, e^{im\theta R/2} \end{aligned} \tag{312}$$

where the vertex operator has a dimension $\Delta_{e,m} = \frac{1}{2}\left(\left(\frac{e}{R} + \frac{mR}{2}\right)^2, \left(\frac{e}{R} - \frac{mR}{2}\right)^2\right)$, or $R = 1$

$$\begin{aligned} \varphi &\sim \varphi + 2\pi \\ \theta &\sim \theta + 4\pi K \\ V_{e,m} &\sim e^{ie\varphi} \, e^{im\theta/2K} \end{aligned} \tag{313}$$

---

[6]See a more rigorous argument in the Yellow Book [6]

where $\Delta_{e,m} = \frac{1}{2}\left(\left(e\sqrt{K} + \frac{m}{2\sqrt{K}}\right)^2, \left(e\sqrt{K} - \frac{m}{2\sqrt{K}}\right)^2\right)$. In the later case, one can also redefine $\theta \to \theta/K$ to simplify the notation. In this note we choose the first convention and set $K = 1$.

Another way to understand the quantization of $j_0 + \bar{j}_0$

The above derivation for the quantization of $j_0 + \bar{j}_0$ uses the modular invarariance. Another simple way is to look at the conjugate momentum of the scalar φ

$$\Pi_0 = \int dx\, \Pi_\varphi = \int dx\, \frac{1}{4\pi}\left(j_0 + \bar{j}_0\right) + \sum_{n \neq 0} \ldots = \frac{j_0 + \bar{j}_0}{2} \tag{314}$$

The vertex operator can be interpreted as the displace operator $e^{i\Pi_0\varphi}$, which means $e^{i\frac{1}{2}(j_0+\bar{j}_0)2\pi R}$ is single valued. This gives rise to the quantization condition

$$j_0 + \bar{j}_0 \sim \frac{2e}{R} \tag{315}$$

which means the total momentum has to be integers.

6.2.2 Symmetries in free compact boson

In this section we discuss the continuoius symmetry (internal) in the free compact boson. Note that we used $z = \tau + i\sigma$ in the convention, which means

$$\phi(z) \tag{316}$$

It is evident that there is a $U(1)_\varphi$ symmetry which acts as $\varphi \to \varphi + $constant. The corresponding Noether's current reads

$$j_\varphi^\mu = \partial^\mu\varphi. \tag{317}$$

The componenets are (Note the upper index here)

$$\begin{aligned} j_\varphi^0 &\sim \partial^t\varphi = i\left(\partial\varphi + \bar{\partial}\varphi\right) \\ j_\varphi^1 &\sim \partial^x\varphi = -i\left(\partial\varphi - \bar{\partial}\varphi\right) \end{aligned} \tag{318}$$

The charge Q_φ (electric) is

$$Q_\varphi = \int dx\, j_\varphi^0 = 2\pi\left(j_0 + \bar{j}_0\right) = e\frac{4\pi}{R}. \tag{319}$$

It indicates that Q_φ is nothing but the winding number of the dual scalar θ. Note that we can see this diractly by expressing the charge density in terms of the dual scalar θ through

$$j_\varphi^0 = i\left(\partial\varphi + \bar{\partial}\varphi\right) = i\partial_\tau\varphi = \partial_x\theta. \tag{320}$$

It is clear that the electric charge e is indeed the winding number of θ

$$Q_\varphi = \int dx\, \partial_x\theta = e\frac{4\pi}{R} \tag{321}$$

The charge e of φ is the winding number of its dual scalar θ.

The massless free scalar theory also enjoys another $U(1)_\theta$ symmetry acting as $\theta \to \theta + $constant. From the current operator expression for the Hamiltonian, one can write down the action in terms of the θ scalar

$$S = \frac{1}{8\pi}\int(\partial\theta)^2 \tag{322}$$

which gives a Noether's current

$$j_\theta^\mu \sim \partial^\mu \theta = \epsilon^{\mu\nu} j_{\varphi\,\nu} \tag{323}$$

whose componenets

$$
\begin{aligned}
j_\theta^0 &\sim \partial^t \theta = i\left(\partial\theta + \overline{\partial}\theta\right) = -\partial^x \varphi = -j_\varphi^1 \\
j_\theta^1 &\sim \partial^x \theta = -i\left(\partial\theta - \overline{\partial}\theta\right) = -\partial^t \varphi = -j_\varphi^0
\end{aligned}
\tag{324}
$$

The charge for the θ scalar, the magnetic charge, reads

$$Q_\theta = \int dx\, j_\theta^0 = \int dx\, \partial_x \varphi = m\, 2\pi R. \tag{325}$$

The θ scalar charge m is nothing but the winding number of the scalar field φ.

In summary, in the massless free compact boson theory, one can show that the charge of a field is the winding number of its dual field. These calculations show fascinating examples of particle-vertex duality.

6.2.3 Bosonization dictonary for the free fermion

Sine the scalar field is dimensionless in free compact boson, the physical quantites should be expressed in terms of the so-called vertex operator. [7] Due to the quantization of the zero modes for the current operator, the vertex operator should have the structure

$$V_{e,m} := e^{i\varphi e/R} e^{i\theta mR/2} = e^{i\phi(e/R+mR/2)} e^{i\overline{\phi}(e/R-mR/2)}, \tag{326}$$

where e and m denote the electric and magnetic charge respectively.

The simplest way for bosonization of fermions is by matching the conformal dimension of the fermion operator to the vertext operator. Take the left mover ψ_+, a holomorphic field $(1/2, 0)$, as an example

$$
\begin{aligned}
|e/R + mR/2| &= 1 \\
e/R - mR/2 &= 0.
\end{aligned}
\tag{327}
$$

One can easily find two solutions, either $V_{\frac{1}{2},1}$ with $R = 1$ or $V_{1,\frac{1}{2}}$ with $R = 2$. These two solutions are equivalent to each other since they are related to each other through a duality transformation $\varphi \to \theta$.

Here we take the first one ($R = 1$) to build up the bosonization dictionary for the free fermion. One can see that the fermion operator corresponds to a half charge boson operator, which means that the fermion degrees of freedom are always in pairs when using the bosonization to study a fermion theory.

Note that one has the freedom to set the creation or annihalation operator to be the vertex operator. For example

$$
\begin{aligned}
\psi_+^\dagger &\sim V_{(\frac{1}{2},1)} \sim e^{i\phi} \\
\psi_-^\dagger &\sim V_{(\frac{1}{2},-1)} \sim e^{i\overline{\phi}}.
\end{aligned}
\tag{328}
$$

However, it is better to choose a different convention

$$
\begin{aligned}
\psi_+^\dagger &\sim V_{(\frac{1}{2},1)} \sim e^{i\phi} \\
\psi_-^\dagger &\sim V_{(\frac{1}{2},-1)}^\dagger \sim e^{-i\overline{\phi}}.
\end{aligned}
\tag{329}
$$

In this convention the fermion charge is the winding number of the φ scalar or the momentum charge for the dual scalar θ.

---

[7] In this note, we assume all physical fields are normalized, even if not explicitly stated.

This convention ensures that the density fluctuation can be written down as the spatial derivative of the scalare field φ, rather than the spatial derivative of its dual field θ, which is consistent with the fact that we already choose φ to write down the Hamiltonian density.

Construction approach of abelian bosonization dictonary

The above bosonization convention is not natural to understand. One may use a hydrodynamic viewpoint to construct the bosonization. First it is convenient to identify the totoal fermin charge density as

$$\rho(x) \sim \partial_x \varphi, \tag{330}$$

which can be seen as the definition of the scalar φ. This also means we identify the fermion charge density to the momentum of θ scalar field.

A fermion creation or annihalation operator changes the fermion charge, hence the scalar $\varphi(x)$. The momentum operator satisfies such a requirement. One can choose the scalar operator $e^{i(j_0 - \bar{j}_0)\theta(x)/2} \sim e^{i\theta(x)/2}$ to construct the fermion operator. The last missing part is the Fermi-Dirac anticommutation relation. One can use the Jordan-Wigner transformation to realize the Fermi-Dirac condtion. In conclusion the fermion operator must take the form $\psi \sim e^{i\theta/2} e^{i\varphi/2}$.

Bosonization dictonary

In the following we are going to show how to express physical quantities of free fermion theory in terms of free compact boson. Using the convention discussed above, the fermion density reads

$$\rho(x) = \bar{\psi}\gamma^0 \psi \sim i\left(\partial \varphi - \bar{\partial}\varphi\right) \sim \partial_x \varphi \sim j_1^\varphi \sim j_0^\theta \tag{331}$$

It is clear that the fermion charge corresponds to magnetic charge for the dual scalar field θ or the winding number of the scalar field φ.

The axial current density maps to

$$\rho_a = -i\bar{\psi}\sigma^y \psi \sim \left(\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-\right) \sim i\left(\partial \varphi + \bar{\partial}\varphi\right) \sim j_1^\theta \sim j_0^\varphi \tag{332}$$

which is the electronic charge of the scalar φ or the vertex topological charge of the dual scalar θ.

A mass term is mapped to be

$$\bar{\psi}\psi \sim \psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+ \sim e^{i\varphi} + h.c. \sim \cos(\varphi) \tag{333}$$

whose dimensional is $(1/2, 1/2)$.

6.2.4 Bosonization on the lattice

On the lattice, the slow modes are those close to the Fermi point $\pm k_f$. Assuming the energy dispersion

$$E_k \sim -\cos(k), \tag{334}$$

one can identify fermions around $\mp k_f$ as the left/right mover ψ_+/ψ_-. Note there is another subtle point that we have normalized the correlation of fermion fields to $\langle \psi_+ \psi_+^\dagger \rangle \sim \frac{1}{z}$, while the canonical anticommutation relation requires the two-point correlation $\frac{1}{2\pi}\frac{1}{z-w}$. This can be shown explicitly via (here we showed the expectation of the anticommutator)

$$\psi_+ \psi_+^\dagger \sim \frac{1}{z-w} = \lim_{\epsilon \to 0} \frac{1}{\epsilon + i(x-y)} = -iP\left(\frac{1}{x-y}\right) + \pi\delta(x-y)$$

$$\psi_+^\dagger \psi_+ \sim \frac{1}{w-z} = \lim_{\epsilon \to 0} \frac{1}{\epsilon + i(y-x)} = +iP\left(\frac{1}{x-y}\right) + \pi\delta(x-y) \tag{335}$$

$$\{\psi_+, \psi_+^\dagger\} \sim \psi_+ \psi_+^\dagger + \psi_+^\dagger \psi_+ \sim 2\pi\delta(x-y)$$

Thus one identifies that the slow modes are close to $\pm k_f$ ($k_f > 0$),

$$c(t,x) \sim e^{-ik_f(t+x)}\psi_+/\sqrt{2\pi} + e^{-ik_f(t-x)}\psi_-/\sqrt{2\pi}. \tag{336}$$

Q: Should we use 4π here? The holomorphic and antiholomorphic both contributes a factor $\delta(x-y)$, which gives rise a wrong anticommutator for the lattice fermion c. Is this the fermion doubling problem? A probaly better understanding is to treat the two branches independently

$$\begin{aligned}
c_+(t,x) &\sim \frac{1}{\sqrt{2\pi}}e^{-ik_f(t+x)}\psi_+ \sim \frac{1}{\sqrt{2\pi}}e^{-ik_f(t+x)}e^{-i\phi} \\
c_-(t,x) &\sim \frac{1}{\sqrt{2\pi}}e^{-ik_f(t-x)}\psi_- \sim \frac{1}{\sqrt{2\pi}}e^{-ik_f(t-x)}e^{i\bar\phi}.
\end{aligned} \tag{337}$$

The lattice fermon density is

$$\begin{aligned}
2\pi\rho &:= 2\pi c^\dagger c \\
&\sim 2\pi\rho_0 + \psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_- + \left(e^{2ik_f x}\psi_+^\dagger\psi_- + h.c.\right) \\
&= 2\pi\rho_0 + i\left(\partial\varphi - \bar\partial\varphi\right) + 2\cos\left(\varphi + 2k_f x\right) \\
&= 2\pi\rho_0 + \partial_x\varphi + 2\cos\left(\varphi + 2k_f x\right)
\end{aligned} \tag{338}$$

where ρ_0 is added since the lattice operator is not normalized. The total charge is given by

$$\begin{aligned}
Q &\sim \int dx\,\rho \\
&= \int dx\,\rho_0 + \int dx\left(j_0 - \bar{j}_0\right) \\
&= Q_0 + m/R
\end{aligned} \tag{339}$$

This is consistent with the convention we chosed for the bosonization dictonary.

Symmetries on the lattice
The fermions on the lattice has a global $U(1)$ symmetry and translation invarariance T, defined as

$$\begin{aligned}
c_n &\to e^{i\alpha}c_n \\
c_n &\to c_{n+1}
\end{aligned} \tag{340}$$

while for the field theory they are realized very differently

$$\begin{aligned}
\psi &\to e^{i\alpha}\psi \\
\psi &\to e^{-ik_f\sigma^z}\psi.
\end{aligned} \tag{341}$$

They becomes the internal symmetry for the field theory. Interestingly, they are related to the chiral symmetry. It is known as an quantum anomaly, which prohibits a symmetric gapped ground state.

Renormalization effect from the interaction
As analysed above, the density-density interactions become free in the bonson language. However, it should be noted that, this renormalization or mapping procedure should be done in the ***Hamiltonian*** language since the density-density interactions are equal-time process. Note that chaning the radius is not something simply multipling the scalar φ with a factor. One should remember that this is a quantum theoy and in the whole process we heavily rely on the current operators. For example, the spatial derivative $\partial_x\varphi$ is related to the magnetic charge,

which gets a factor R, while the time derivative $\partial_0 \varphi$ is related the electric charge, which obtains a factor $1/R$. Or in a different viewpoint, in a quantum theory ($t = 0$), the time derivative part $\partial_0 \varphi$ should be understood as the corresponding canonical momentum Π. Once we build up the mapping, the calculation of free compact boson can be carried out in the Lagrangian framework, which is simply a computation technique. One can find the two density-density interactions discussed above change the radius of the free compact boson from 1 to R.

Another point is how to understand the fermion operator in an interacting theory. The correct logic is that the bosonization build up the relation between the fermion operator and the scalar field on the algebra level. The true ground state after the interaction effect is the one of the renormalized free scalar theory. To calculate, for example the correlation function with respect to the true ground state (after the renormalization of the interaction), one should write down the fermon operator in terms of the new scalar field

$$\psi_+^\dagger := V_{(\frac{1}{2},1)} \sim e^{i\phi_0} \sim e^{i\frac{\varphi_0}{2}} e^{i\frac{\theta_0}{2}} \sim e^{i\frac{\varphi}{2R}} e^{i\frac{\theta R}{2}} \sim e^{i\left(\frac{1}{2R}+\frac{R}{2}\right)\phi} e^{i\left(\frac{1}{2R}-\frac{R}{2}\right)\bar{\phi}} \tag{342}$$

where φ_0 and θ_0 denote the bare scalar fields.

6.2.5 Bosonization for quantum spin chains

We use bosonization study the spin-half xxz chain

$$H = \frac{1}{2} \sum_n \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + g S_z^n S_z^{n+1} \right) = \sum_n \left(S_n^\dagger S_{n+1}^- + h.c. + g S_z^n S_z^{n+1} \right) \tag{343}$$

where $S^\dagger := S^x + i S^y$. Indentifying the up spin state to the empty fermion state

$$\begin{aligned} |\uparrow\rangle &\longleftrightarrow |0\rangle \\ |\downarrow\rangle &\longleftrightarrow |1\rangle \end{aligned} \tag{344}$$

one can use the transformation

$$\begin{aligned} S_n^z &= \frac{1}{2} - c_n^\dagger c_n \\ S_n^\dagger &= (-1)^n c_n \prod_{i<n} e^{i\pi c_i^\dagger c_i} \\ &= (-1)^n \left(e^{-i k_f x} e^{i\phi} + e^{i\bar{\phi}} \right) e^{i\frac{\theta}{2}} \\ &= (-1)^n e^{i\varphi/2} + e^{i\varphi/2} e^{i\theta} \end{aligned} \tag{345}$$

to rewrite the spin model as

$$H = -\sum_n \left(c_n^\dagger c_{n+1} + h.c. \right) = -2 \sum_k \cos(k) c_k^\dagger c_k. \tag{346}$$

From the dispersion, one can find that the fermi point $k_f = \pi/2$.

In terms of scalar fields, close to the free fermion point ($g = 0$) spin operators are expressed as

$$\begin{aligned} S^z(x) &\sim -\frac{1}{2\pi} \partial_x \varphi - \frac{1}{\pi} \cos\left(\varphi + 2 k_f x \right) \\ S^\dagger(n) &\sim (-1)^n c_n \prod_{i<n} e^{i\pi c_i^\dagger c_i} \\ &\sim (-1)^n \left(e^{-i k_f x} e^{-i\phi} + e^{i k_f x} e^{i\bar{\phi}} \right) e^{i\pi \int \left(\frac{1}{2\pi} \partial_x \varphi + \rho_0 \right)} \\ &\sim (-1)^n \left(e^{-i\phi} + e^{2i k_f x} e^{i\bar{\phi}} \right) e^{i\frac{1}{2}\varphi} \\ &\sim (-1)^n e^{-i\theta/2} + e^{i(2k_f + \pi)x} e^{-i\theta/2} e^{i\varphi} \end{aligned} \tag{347}$$

where we used $k_f = \pi \rho_0$. Note that according to the Luttinger theorem in 1D, the volume encircled by the fermi point can not be changed by the interactions, which gives rise to the value of $k_f = \pi/2$ at the free fermion point $g = 0$.

Turning on interactions $|g| < 1$, the dictonary becomes

$$S^z(x) \sim -\frac{1}{2\pi R}\partial_x \varphi - \frac{1}{\pi}\cos\left(\varphi/R + 2k_f x\right)$$
$$S^\dagger(n) \sim (-1)^n e^{-i\theta R/2} + e^{i(2k_f + \pi)x} e^{-i\theta R/2} e^{i\varphi/R} \tag{348}$$

where the radius of the scalar fields are renormalized by the interactions.

Correlation functions
Using the mapping above, one can calculate the equal-time spin-spin operator correlation functions

$$\langle S_n^z S_0^z \rangle \sim c_1 \langle \partial_n \varphi \, \partial_0 \varphi \rangle + c_2 (-1)^n \left\langle e^{i\varphi(n)} e^{-i\varphi(0)} \right\rangle + h.c.$$
$$\sim \frac{c_1}{r^2} + (-1)^n \frac{c_2}{r^{2/R^2}} \tag{349}$$
$$\langle S_n^\dagger S_0^- \rangle \sim (-1)^n \frac{1}{r^{R^2/2}} + \frac{1}{r^{2/R^2 + R^2/2}}.$$

where R is the raidus of the free compact boson

$$R = \sqrt{2 - \frac{2}{\pi}\mathrm{acos}(g)}. \tag{350}$$

At the free fermion ($g = 0$) point, the radisu $R = 1$ as discussed previously. At the $SU(2)$ symmetric point (g=1), one has $R = \sqrt{2}$, which is the self-dual point between the scalar fields φ and θ (R = 2/R).

Possible relevant perturbations and boudary states
As analysed in the last section for the free fermions, the $U(1)$ symmetry and translation T are anomalous that they prevent the existence of a symmetric gapped ground state. The mixed anomaly can also be the $U_\phi(1)$ and $U_\theta(1)$. Here we consider perturbations breaking at least part of the symmetries.

(1) Uniform magnetic field in the xy plane. Recal that the spin operator in the boson field language reads

$$S^\dagger \sim (-1)^n e^{-i\theta/2} + e^{-i\theta/2} e^{i\varphi}. \tag{351}$$

The first term has strong phase fluctuation. The contribution to the uniform magnetic field thus becomes

$$e^{-i\theta(n)/2} - e^{-i\theta(n+1)/2} \sim \partial_x \theta \sim i\partial_\tau \varphi \tag{352}$$

which is noting but the $U_\varphi(1)$ charge density. It can not be renormalized in the RG process. The second term is the vertex operator $V_{(1,1)}$ with a scaling dimension $\Delta_{(1,1)} = \frac{1}{R^2} + \frac{R^2}{4}$. It is relevant in the Luttinger liquid phase described the free compact boson. It breaks both $U(1)$ symmetries and results in an ordered state.

(2) Spin-Peierls (dimerization or VBS)

$$D(n) := (-1)^n \left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y\right) \sim \sin(\varphi) \tag{353}$$

This term pinns down the φ field, hence breaks the $U_\varphi(1)$ symmetry. It also breaks the translation symmetry, which is realized by $\theta \to \theta + 2\pi$ and $\varphi \to \varphi + \pi$. Hence, it also breaks both $U(1)$ symmetries and results in an ordered state.

(3) Uniform magnetic field in the z direction. One finds

$$S^z \sim -\frac{1}{2\pi}\partial_x \varphi - \frac{1}{\pi}\cos\!\left(\varphi + 2k_f x\right). \tag{354}$$

The first term is a current operator. The second one has strong fluctuation and also becomes a current operator. There is emergent $U_\varphi(1) \otimes U_\theta(1)$ symmetry at the thermodynamic limit.

7 Appendix

7.1 Central extensions of Lie algebras

In this section $\mathfrak{g}, \mathfrak{h}, \cdots$ denote (possibly infinite) Lie algebras over some field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. This section is mainly based on Wikipedia and [7].

Extensions

Definition: A *Lie algebra extension* is a short exact sequence of Lie algebras:

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g}. \tag{355}$$

One calls $\mathfrak{e}$ an extension of $\mathfrak{g}$ by $\mathfrak{h}$. By exactness of the sequence one has $\mathfrak{g} \cong \mathfrak{e} / \operatorname{Im} \iota$.

Definition: A *central extension* is an extension $\mathfrak{e}$ of $\mathfrak{g}$ by $\mathfrak{h}$, such that $\operatorname{Im} \iota$ is contained in the center of $\mathfrak{e}$, $\iota(\mathfrak{h}) \subseteq Z(\mathfrak{e})$.

Notice that for a central extension $\mathfrak{h}$ is necessarily abelian. We now introduce a notion of trivial central extensions as follows:

Definition: A Lie algebra extension

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \tag{356}$$

splits if there exists a Lie algebra morphism $\beta : \mathfrak{g} \mapsto \mathfrak{e}$ such that $\pi \circ \beta = \operatorname{id}_{\mathfrak{e}}$. β is called a splitting map.

A central extension

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g}. \tag{357}$$

that splits is trivial in the sense that it is equivalent[8] to one where $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{h}$.

Let us now consider a central extension and a map (not necesserily a Lie algebra homomorphism) $\beta : \mathfrak{g} \to \mathfrak{e}$ such that $\pi \circ \beta = \operatorname{id}_{\mathfrak{e}}$. From this map construct $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ as follows:

$$\Theta(x, y) := [\Theta(x), \Theta(y)] - \Theta([x, y]). \tag{358}$$

This map is:

1. Antisymmetric.

2. Bilinear.

3. Satisfies $\Theta(x, [y, z]) + \Theta(y, [z, x]) + \Theta(z, [x, y]) = 0$.

Given Θ one can now show that there is an isomorphism between the vector spaces $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{h}$ that is given by:

$$\Psi : \mathfrak{g} \oplus \mathfrak{h} \mapsto \mathfrak{e} : (x, y) \mapsto \beta(x) + y. \tag{359}$$

A Lie bracket on $\mathfrak{g} \oplus \mathfrak{h}$ is given by:

$$[x \oplus z, y \oplus z']_{\mathfrak{e}} := [x, y]_{\mathfrak{g}} + \Theta(x, y). \tag{360}$$

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[8]To do: introduce the notion of equivalent extensions.

Lemma: In the above construction β is a splitting map if and only if

$$\Theta(x, y) = \mu([x, y]), \tag{361}$$

for some $\mu \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$.

Now comes the classification of central extensions of Lie algebras:

Theorem: Every central extension comes from a map Θ that satisfies the above properties (1-3). Conversely, every central extension gives rise to a map Θ that satisfies the above properties (1-3).

Lie algebra cohomology

The classification of Lie algebra extensions is very satisfying. It smells a lot like a cohomological classification. Indeed, the extensions are classified by functions depending on two variables satisfying the condition (3) that is exactly the one needed to fulfill the Jacobi identity of the central extension. Moreover, the central extension is trivial if the *2-cocycle* Θ is trivial in the following sense: $\Theta(x, y) = \mu([x, y])$. This is reminiscent of considering 2-cocycles to be trivial if they are equal to a coboundary. Let us put this on a bit more rigorous footing.

Definitions:

1. $Z^2(\mathfrak{g}, \mathfrak{h}) = \left\{ \Theta \in \Lambda^2(\mathfrak{g}, \mathfrak{h}) | \Theta : (3) \right\}$.

2. $B^2(\mathfrak{g}, \mathfrak{h}) = \{ \Theta : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{h} | \exists \mu \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}) : \Theta(-, -) = \mu([-, -]) \}$.

3. $H^2(\mathfrak{g}, \mathfrak{h}) := Z^2(\mathfrak{g}, \mathfrak{h})/B^2(\mathfrak{g}, \mathfrak{h})$.

H^2 is of course called the second cohomology group. We thus obtain the following reformulation of the classification of central extensions:

Theorem: The equivalence classes of central extensions

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \tag{362}$$

are in one-to-one correspondence with the elements of $H^2(\mathfrak{g}, \mathfrak{h})$.

For completeness, let us introduce a notion of cochain complexes for Lie algebras. A cochain f is a alternating multilinear map f:

$$f : \Lambda^n \mathfrak{g} \mapsto \mathfrak{h}. \tag{363}$$

Here, $\mathfrak{h}$ is considered a $\mathfrak{g}$-module or -representation.

The differential of an n-cochain is given by

$$(df)(x_1, \ldots, x_{n+1}) = \sum_i (-1)^{i+1} x_i f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) + \\ \sum_{i<j} (-1)^{i+j} f\left([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}\right), \tag{364}$$

so for example, with trivial action we obtain

$$(df)(x_1, x_2) = f([x_1, x_2]), \tag{365}$$

and

$$
\begin{aligned}
(df)(x_1, x_2, x_3) &= -f([x_1, x_2], x_3) + f([x_1, x_3], x_2) - f([x_2, x_3], x_1) \\
&= -f([x_1, x_2], x_3) - f([x_3, x_1], x_2) - f([x_2, x_3], x_1) \quad (366) \\
&= f(x_3, [x_1, x_2]) + f(x_2, [x_3, x_1]) + f(x_1, [x_2, x_3]).
\end{aligned}
$$

So clearly, $Z^2(\mathfrak{g}, \mathfrak{h})$ defined above is the group of 2-cocycles satisfying $d\Theta = 0$ and $B^2(\mathfrak{g}, \mathfrak{h})$ the set of coboundaries: $\Theta = d\mu$.

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