Notes for the reading club

Reading Club

March 6, 2023

Abstract

The Yellow Book Notes. It is good to write notes!

Contents

1 Preliminary

1.1 Conventions

Metric tensor and Coordinate.– The metric tensor in Minkowski and Eclidean space-time is defined as

$$\eta = \begin{Bmatrix} +1 & & \\ & -1 & \\ & & ... \end{Bmatrix} \tag{1}$$

and

$$\eta = \begin{Bmatrix} +1 & & \\ & +1 & \\ & & ... \end{Bmatrix} \tag{2}$$

respectively, where the first index is the time. In the Yellow Book, without specifications, we are working in Eclidean space. The coordinate is defined as $x^\mu = \{t, \vec{x}\}$. So that the norm of a vector in Minkowski space-time is $x^\mu x_\mu = t^2 - r^2$.

γ *matrices.*– The γ matrices follow the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}. \tag{3}$$

In Minkowski space time, the γ matrices can be chosen as

$$\begin{aligned} \gamma^0 &= \sigma^x \\ \gamma^1 &= -i\sigma^y, \end{aligned} \tag{4}$$

while in Elidean space, they can be chosen as

$$\begin{aligned} \gamma^0 &= \sigma^x \\ \gamma^1 &= \sigma^y. \end{aligned} \tag{5}$$

1.2 Free fermions

In Minkowski space time, the Lagrange density for the free fermion reads

$$\mathcal{L} = \frac{g}{2}\left(\psi^1 \, i(\partial t + \partial x)\, \psi^1 + \psi^2 \, i(\partial t - \partial x)\, \psi^2\right). \tag{6}$$

In terms of $\psi = (\psi^1, \psi^2)$, one can write the theory as

$$\begin{aligned} \mathcal{L} &= \frac{g}{2}\left(\psi^\dagger i\partial_t \psi + \psi^\dagger\, \sigma^z\, i\partial_x \psi \right) \\ &= \frac{g}{2}\left(\psi^\dagger \sigma^x \sigma^x i\partial_t\psi + \psi^\dagger\, -i\sigma^x\sigma^y\, i\partial_x\psi \right) \\ &= \frac{g}{2}\psi^\dagger \sigma^x (\sigma^x i\partial_t - i\sigma^y i\partial_x)\psi \\ &= \frac{g}{2}\psi^\dagger \gamma^0 i\gamma^\mu \partial_\mu \psi \end{aligned} \tag{7}$$

where we used

$$\gamma^0 = \sigma^x \quad \gamma^1 = -i\sigma^y \tag{8}$$

1.2.1 Wick rotation

It is usually more convenient to work in Euclidean space rather than Minkowski space time. Upon doing the Wick rotation, the action changes as

$$i\, S_M \rightarrow -\, S_E. \tag{9}$$

Specifically,

$$
\begin{aligned}
i\, S[\psi] &= i \int dx dt\, \frac{g}{2} \psi^\dagger \gamma^0 i \gamma^\mu \partial_\mu \psi \\
&= i^2 \int dx dt\, \frac{g}{2} \psi^\dagger \partial_t \psi + i^2 \int dx dt\, \frac{g}{2} \psi^\dagger \sigma^x (-i) \sigma^y \partial_x \psi \\
&= -\int dx d\tau\, \frac{g}{2} \psi^\dagger \partial_\tau \psi - \int dx d(-it)\, \frac{g}{2} \psi^\dagger \sigma^x \sigma^y \partial_x \psi \\
&= -\int dx d\tau\, \frac{g}{2} \psi^\dagger \sigma^x \sigma^x \partial_\tau \psi - \int dx d\tau\, \frac{g}{2} \psi^\dagger \sigma^x \sigma^y \partial_x \psi \\
&= -\int dx d\tau\, \frac{g}{2} \psi^\dagger \gamma_E^0 \gamma_E^\mu \partial_\mu \psi
\end{aligned} \tag{10}
$$

where $\tau = -it$. The Eulidean space action can be written as

$$S_E = \int d^2 x\, \frac{g}{2}\, \psi^\dagger \gamma_E^0 \gamma_E^\mu \partial_\mu \psi \tag{11}$$

1.2.2 1+1d free fermions: Legendre transformation: from $\mathcal{L}$ to H

A lattice version free fermion theory Eq. 2.38 reads

$$\mathcal{L} = \frac{i}{2} \sum_n \left(\psi_n \dot\psi_n + \psi_n \psi_{n+1} \right). \tag{12}$$

The canonical momentum corresponding to ψ_n is

$$\pi_n = \frac{\partial \mathcal{L}}{\partial \dot\psi_n} = -\frac{i}{2} \psi_n. \tag{13}$$

So that the Hamiltonian is

$$
\begin{aligned}
\mathcal{H} &= \sum_n \pi_n \dot\psi_n - \mathcal{L} \\
&= -\frac{i}{2} \sum_n \psi_n \dot\psi_n - \frac{i}{2} \sum_n \left(\psi_n \dot\psi_n + \psi_n \psi_{n+1} \right) \\
&= -i \sum_n \psi_n \dot\psi_n - \frac{i}{2} \sum_n \psi_n \psi_{n+1}.
\end{aligned} \tag{14}
$$

While it shoud be

$$\mathcal{H} = -\frac{i}{2} \sum_n \psi_n \psi_{n+1}. \tag{15}$$

If we'd like to keep defining the derivative of Grassmann number according to the order of left-to-right, we need to define the Hamiltonian as

$$\mathcal{H} = \sum_n \dot\psi_n \pi_n - \mathcal{L} = -\frac{i}{2} \sum_n \psi_n \psi_{n+1} \tag{16}$$

3

1.3 Free boson

The action for the free boson in the Minkoswki space time reads

$$S = \frac{1}{2}g \int dx\, dt\ \partial_\mu \phi \partial^\mu \phi, \tag{17}$$

where ϕ is a real scalar field. After Wick rotation $\tau = it$, it becomes

$$
\begin{aligned}
i\,S &= \frac{i}{2}g \int dx\, dt\ \partial_t \phi \partial_t \phi - \frac{i}{2}g \int dx\, dt\ \partial_x \phi \partial_x \phi \\
&= -\frac{1}{2}g \int dx\, d\tau\ \partial_\tau \phi \partial_\tau \phi - \frac{1}{2}g \int dx\, dit\ \partial_x \phi \partial_x \phi \\
&= -\frac{1}{2}g \int dx\, d\tau\ \partial_\mu \phi \partial^\mu \phi
\end{aligned}
\tag{18}
$$

The Euclidean action reads

$$S_E = \frac{1}{2}g \int d^2x\ \partial_\mu \phi \partial^\mu \phi. \tag{19}$$

The two point correlation up to a constant term is

$$\langle \phi(x)\phi(y) \rangle = -\frac{1}{2\pi g}\ln(\rho). \tag{20}$$

1.4 Symmetries at the classical level

The action becomes different after a coordinate transformation. We say it has a symmetry if it remains unchanged and a Noether current can be derived from the symmetry. The coordinate transformation is denoted as

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \tag{21}$$

and the field changes according to

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta F}{\delta \omega_a}(x) \tag{22}$$

where ω_a is a constant and small parameter.

By definition, the change of the action δS disappears for a symmetric transformation. We can get nothing new from this. If we allow ω_a to be arbitrary, the leading contribution to δS becomes

$$\delta S = -\int d^2x\, j^\mu \partial_\mu \omega_a, \tag{23}$$

where we introduced the the current j^μ. We assume it decreases fast when approaching infinite. So that one obtains

$$\delta S = \int d^2x\ \partial_\mu j^\mu\ \omega_a. \tag{24}$$

This equations holds for all the field configurations. If we require the field configuration to be the one obeying the equation, the action should be invariant for arbitrary coordinate transformation and one finds the conservation of j^μ

$$\partial_\mu j^\mu = 0. \tag{25}$$

1.4.1 Energy-momentum tensor

The canonical energy-momentum tensor is defined to be the Noether current of the translation transformation

$$x'^\mu = x^\mu + \epsilon^\nu \delta^\mu_\nu \tag{26}$$

$$T^{\mu\nu} = -\eta^{\mu\nu} L + \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\nu \phi. \tag{27}$$

This definition of $T^{\mu\nu}$ is not guaranteed to be symmetric between the two indices (The requirement of a symmetric $T^{\mu\nu}$ will be clear later).

Another definition that makes the energy-momentum tensor symmetric follows. In the coordinate transformation, if we also consider the variance of the metric tensor

$$\delta g_{\mu\nu} = -\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu \tag{28}$$

the action remains invariant since this is nothing but a reparametrization of the theory. So that one finds

$$\delta S = 0 = -\frac{1}{2} \int d^d x \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \right) \left(T^{\mu\nu} + 2 \frac{\delta S}{\delta g_{\mu\nu}} \right). \tag{29}$$

So that one can define the energy-momentum tensor as

$$T^{\mu\nu} = -2 \frac{\delta S}{\delta g_{\mu\nu}} \tag{30}$$

up to a surface term.

Another way to make the energy-momentum tensor symmetric is add a surface term to the canonical one. One can show that with rotation symmetry, such a term can be constructed to make $T^{\mu\nu}$ symmetric.

1.5 Symmetry at the quantum level

All the field configurations contribute to the quantum theory, so that one has no Noether current in general. Still the symmetry has constraints to the quantum theory. For the $n-$point correlation functions, one has

$$\langle \phi(x'_1)...\phi(x'_n) \rangle = \frac{1}{Z} \int [D\phi] \, \phi(x'_1)...\phi(x'_n) \, e^{-S[\phi]} \tag{31}$$

$$= \frac{1}{Z} \int [D\phi'] \, \phi'(x'_1)...\phi'(x'_n) \, e^{-S'[\phi']} \tag{32}$$

$$= \frac{1}{Z} \int [D\phi] \, F(\phi(x_1))...F(\phi(x_n)) \, e^{-S[\phi]} \tag{33}$$

$$= \langle \, F(\phi(x_1))...F(\phi(x_n)) \, \rangle \tag{34}$$

in which we assumed the functional integral measure does not change and the coordinate transformation is a rigid one (ω_a is a constant).

1.5.1 Ward identity

As stated above there is no conserved current at the quantum level. The infinitesimal coordinate transformation at the quantum level results in the so-called Ward identity.

We denote the change of fields as

$$\phi'(x) = \phi(x) - i\omega_a \, G_a \, \phi(x). \tag{35}$$

The infinitesimal coordinate transformation (ω_a now is arbitrary) changes the correlation as (We only consider the first order perturbation contribution)

$$
\begin{aligned}
\langle \phi'(x_1)...\phi'(x_n) \rangle &= \langle \phi(x_1)...\phi(x_n) \rangle &\text{(36)} \\
&= \frac{1}{Z} \int [D\phi'](X + \delta X) e^{-S[\phi] - \int d^d x \partial_\mu j^\mu \omega_a} &\text{(37)} \\
&= \frac{1}{Z} \int [D\phi](X + \delta X) e^{-S[\phi] - \int d^d x \partial_\mu j^\mu \omega_a} &\text{(38)} \\
&= \langle X \rangle - \int [D\phi] \int d^d x \, X \partial_\mu j^\mu \omega_a e^{-S[\phi]} - \int [D\phi] \delta X e^{-S[\phi]} &\text{(39)}
\end{aligned}
$$

so that one finds

$$\langle \delta X \rangle = \int d^d x \, \partial_\mu \langle j^\mu X \rangle \omega_a(x). \tag{40}$$

As

$$
\begin{aligned}
\delta X &= -i \sum_i \phi(x_1)...G_a \phi(x_i)...\phi(x_n) \omega_a(x_i) &\text{(41)} \\
&= -i \int d^d x \sum_i \phi(x_1)...G_a \phi(x_i)...\phi(x_n) \delta(x - x_i) \omega_a(x) &\text{(42)}
\end{aligned}
$$

Since ω_a is arbitrary, one obtains the Ward identity

$$\partial_\mu \langle j^\mu X \rangle = -i \sum_i \delta(x - x_i) \, \langle \phi(x_1)...G_a \phi(x_i)...\phi(x_n) \rangle. \tag{43}$$

So that for each symmetry, there exists a Ward identity, i.e., a constraint to the correlation function. With enough symmetries, one can get all the information of the correlation functions.

1.6   Renormalization group

1.6.1   Dimensional analysis and renormalizability of QFT

Let's start with the canonical dimension of fields and couplings in the action,

$$S = \int d^d x \, \mathcal{L}(\phi, \lambda). \tag{44}$$

Since the action is dimensionless, every term in $\mathcal{L}$ has an energy scaling dimension of

$$\Delta(\mathcal{L}) = [\mathcal{L}] = \omega^d \tag{45}$$

which determines the canonical dimension fields and couplings. The renormalizability of a QFT is directly obtained from the energy dimension of Feynman diagrams,

$$\mathcal{D} = d - E_\phi \Delta(\phi) - \Delta(\lambda_i) \tag{46}$$

where E_ϕ is the number of external fields and λ_i the couplings in the theory. A nice discussion about renormalizability can be found online (https://web2.ph.utexas.edu/ vadim/ Classes/2022f/notes.html).

Super-renormalizable theories have only couplings with positive dimensions. For such theories, there are finite Feynman diagrams become divergent in the perturbation calculation. Renormalizable theories have couplings with non-negative dimensions, in which a finite number of couplings have zero dimensions. There exists infinite number of divergent Feynman diagrams, but the number of divergent amplitudes is finite. If there is at least one coupling with a negative dimension, the theory is non-renormalizable.

1.6.2 Wilson-Kadanoff renormalization scheme

The renormalization group builds up the modern understanding of QFT, which is regarded as an effective field theory. Here we briefly recall the Wilson-Kadanoff scheme. In this scheme, a momentum cutoff $k < \Lambda$ is introduced.

One first divides modes into fast $\Lambda/s < k < \Lambda$ and slow $k < \Lambda/s$ parts. The fast modes are integrated to result in a new theory

$$e^{-S'(\phi)_{\Lambda/s}} = \int D\phi_{\Lambda/s < k < \Lambda} \, e^{-S_{\Lambda}(\phi)} \tag{47}$$

with a smaller cutoff Λ/s. This theory can not be compared with the original theory, since they have different cutoffs. Another rescaling step

$$k' = ks \tag{48}$$

is required to restore the

2 Conformal group in d=2

In $d = 2$ we have infinitely many *local* conformal transformations. The 6 parameter subgroup of conformal transformations that are everywhere well defined is the *global* conformal group $SL(2, \mathbb{C})/\mathbb{Z}_2$.

3 Normal ordering

For free fields the OPE of the field with itself contains only one term with a constant prefactor. It can be regularized by normal ordering the fields, or equivalently, subtracting its expectation value. Using the former prescription for $T(z)T(w)$ only kills the $\propto c$ term. So clearly we need a more elaborate definition of normal ordering. We shall define proper normal ordering for general fields as subtracting all the singular terms from the OPE. We will write this normal ordering as

$$(AB)(z). \tag{49}$$

Concretely, given the OPE

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{\{AB\}_n(w)}{(z-w)^n} \tag{50}$$

we have that

$$(AB)(w) = \{AB\}_0(w). \tag{51}$$

Equivalently, we can compute the normal ordering of fields using contour integral methods:

$$(AB)(w) = \frac{1}{2\pi i} \oint \frac{dz}{z-w} A(z)B(w). \tag{52}$$

The contraction of fields contains only the singular terms of the OPE:

$$\overbrace{A(z)B}(w) = \sum_{n=1}^{N} \frac{\{AB\}_n(w)}{(z-w)^n}. \tag{53}$$

We now want to express the modes of the normal ordered field in terms of the modes of the input fields. Given fields A and B and points $|z| > |x| > |w|$ we write

$$A(z) = \sum_n (z-x)^{-n-h_A} A_n(x) \tag{54a}$$

$$B(w) = \sum_n (w-x)^{-n-h_B} B_n(x). \tag{54b}$$

Contour integrating ultimately results in:

$$(AB)_m = \sum_{n \le -h_A} A_n B_{m-n} + \sum_{n > -h_A} B_{m-n} A_n, \tag{55}$$

where we defined the modes of (AB) as:

$$(AB)(z) = \sum_n z^{-n-h_A-h_B}(AB)_n. \tag{56}$$

Some warnings are in place:

1. Normal ordering is not commutative: $(AB)(z) \neq (BA)(z)$.

2. Normal ordering is not associative: $((AB)C)(z) \neq (A(BC))(z)$.

3. With this definition of normal ordering, Wick's theorem needs to be revisited. This is done in Appendix 6.B of the Book.

4   Conformal families and Operator algebra

> There's nothing stronger than family.
>
> D. T.

The goal of this section is to introduce the notion of conformal blocks and associated to this the method of conformal bootstrapping as a way to solve CFTs, ie. compute the correlation functions, explicitly. Before that we revisit the notion of descendant fields and conformal families.

In the following we mostly only care about the holomorphic part of fields.

4.1   Descendant fields

A descendant is generated from a primary by acting with the Virasoro operators:

$$\phi^{(-n)}(w) = (L_{-n}\phi)(w) = \frac{1}{2\pi i} \oint_w dz \frac{1}{(z-w)^{n-1}} T(z)\phi(w), \tag{57}$$

in particular:

$$\phi^{(0)}(w) = h\phi(w), \qquad \phi^{(-1)} = \partial\phi(w). \tag{58}$$

Consider following correlation function of states that are part of the same family:

$$\langle (L_{-n}\phi)(w)X \rangle, \tag{59}$$

where X denotes a string of primary fields: $X = \phi_1(w_1)...\phi_N(w_N)$. After a computation one finds:

$$\langle (L_{-n}\phi)(w)X \rangle = \mathcal{L}_{-n}\langle \phi(w)X \rangle \qquad (n \geq 1). \tag{60}$$

With the differential operator

$$\mathcal{L}_{-n} = \sum_i \left\{ \frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i} \right\}. \tag{61}$$

In other words, knowing all the correlation functions between primaries, $\langle \phi(w)X \rangle$, is sufficient to compute the correlation functions that involve descendants by applying the differential operators $\mathcal{L}_{-n}$. More generally, for descendants of the form

$$\phi^{(-k,-n)}(w) = (L_{-k}L_{-n}\phi)(w), \tag{62}$$

and so on, we find in a similar way that

$$\left\langle (L_{-k_1}...L_{-k_n}\phi)(w)X \right\rangle = \mathcal{L}_{-k_1}...\mathcal{L}_{-k_n}\langle \phi(w)X \rangle \qquad (n \geq 1) \tag{63}$$

4.2 Conformal families

A *conformal family* is a set of states that transform according to a representation of the conformal group. A family contains a primary and its descendants. We will denote the conformal family associated with the primary ϕ by $[\phi]$. First descendants of a primary are sometimes called *secondary fields*. Another way to say that a conformal family transforms under itself is to say that the OPE of $T(z)$ with any member of the family will be composed solely of other members within the same family. Concretely:

$$T(z)\phi^{(-n)}(w) = \frac{cn(n^2-1)/12}{(z-w)^{n+2}}\phi(w) + \sum_{k=1}^n \frac{n+k}{(z-w)^{k+2}}\phi^{(k-n)}(w) + \sum_{k\geq 0}(z-w)^{k-2}\phi^{(-k,-n)}(w) \tag{64}$$

4.3 The operator algebra

The two - and three point functions of a CFT are fixed by conformal invariance. However, we need additional dynamical information to compute the three-point fusion coefficients C_{ijk} (for example using a conformal bootstrap approach). This information is contained in the *operator algebra*. The OPE which also includes the regular terms of all primary fields with each other. Using the operator algebra we can reduce all correlation functions to two-point correlation functions.

First we choose a basis of fields such that $C_{\alpha\beta} = \delta_{\alpha\beta}$ in

$$\left\langle \phi_\alpha(w,\overline{w})\phi_\beta(z,\overline{z}) \right\rangle = \frac{C_{\alpha\beta}}{(w-z)^{2h}(\overline{w}-\overline{z})^{2\overline{h}}}. \tag{65}$$

This implies that states belonging to different conformal families are always orthogonal. From scale invariance it follows that:

$$\phi_1(z,\overline{z})\phi_2(0,0) = \sum_p \sum_{\{k,\overline{k}\}} C_{12}^{p\{k,\overline{k}\}} z^{h_p-h_1-h_2+K}\overline{z}^{\overline{h}_p-\overline{h}_1-\overline{h}_2+\overline{K}} \phi_p^{(k,\overline{k})}(0,0). \tag{66}$$

We introduced the notation $K = \sum_i k_i$.

Writing

$$C_{12}^{p\{0,0\}} \equiv C_{12}^p = C_{p12}, \tag{67}$$

we find that

$$C_{12}^{p\{k,\bar{k}\}} = C_{12}^p \beta_{12}^{p\{k\}} \overline{\beta}_{12}^{p\{\bar{k}\}}. \tag{68}$$

This means that descendants fields are correlated to a given third field only if the primary is correlated. An the holomorphic and antiholomorphic parts factorize.

An example is given in the Book. Even in a relatively simple case, finding the three point function is not straightforward!

In conclusion, given the central charge, the conformal dimensions and the three-point coefficients C_{pnm}, one can - in principle - determine the operator algebra. Using the operator algebra, all the n-point correlation functions can be computed and the entire theory is solved.

4.4 Conformal blocks

Let us illustrate how the four-point functions can be reduced to three-point functions using the machinery introduced in the previous sections.

We consider the four-point function

$$\left\langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \phi_3(z_3, \bar{z}_3) \phi_4(z_4, \bar{z}_4) \right\rangle. \tag{69}$$

For sake of simplicity, we shall carry out a global conformal transformation to put $z_4 = 0$, $z_1 = \infty$, $z_2 = 1$, $z_3 = x$. We define:

$$G_{34}^{21}(x, \bar{x}) = \left\langle h_1, \bar{h}_1 \middle| \phi_2(1, 1) \phi_3(x, \bar{x}) \middle| h_4, \bar{h}_4 \right\rangle. \tag{70}$$

Note the order of the indices!

Using operator algebra techniques, we can write this function as:

$$G_{34}^{21}(x, \bar{x}) = \sum_p C_{34}^p C_{12}^p A_{34}^{21}(p|x, \bar{x}). \tag{71}$$

The sum over p is a sum over intermediate conformal families that play the role of mediating channels in the scattering from fields from $(0, x)$ towards $(1, \infty)$. These functions $A_{34}^{21}(p|x, \bar{x})$ are called *partial waves*. They can be depicted by:

The partial wave factorizes in a holomorphic and antiholomorphic part, according to:

$$A_{34}^{21}(p|x, \bar{x}) = \mathcal{F}_{34}^{21}(p|x) \overline{\mathcal{F}}_{34}^{21}(p|\bar{x}), \tag{72}$$

we call these functions $\mathcal{F}$, the *conformal blocks*. There is a recipe to compute these conformal blocks, even though it is a pain to compute these in practice. Physically speaking, these conformal blocks are the part in the four-point function that is fixed by conformal invariance. They depend on the anharmonic ratios via a series expansion. The remaining elements are the three-point coefficients, which are *not* fixed by conformal invariance.

$$\sum_p C_{nm}^p C_{lk}^p \quad\text{[diagram]}\quad = \quad \sum_q C_{nl}^q C_{mk}^q \quad\text{[diagram]}$$

4.5 Crossing symmetry and the conformal bootstrap

What happens if we would choose instead of $z_4 = 0$, $z_1 = \infty$, $z_2 = 1$, $z_3 = x$ a different order of the fields? Following identities can be obtained relatively easy:

$$G_{34}^{21}(x, \overline{x}) = G_{32}^{41}(1-x, 1-\overline{x}), \tag{73}$$

and

$$G_{34}^{21}(x, \overline{x}) = \frac{1}{x^{2h_3}\overline{x}^{2\overline{h}_3}} G_{31}^{24}(1/x, 1/\overline{x}). \tag{74}$$

These identities are specific instances of the *crossing symmetry* of the functions G. Explicitly we can write the first identity as

$$\sum_p C_{21}^p C_{34}^p \mathcal{F}_{34}^{21}(p|x)\overline{\mathcal{F}}_{34}^{21}(p|\overline{x}) = \sum_q C_{41}^q C_{32}^q \mathcal{F}_{32}^{41}(p|1-x)\overline{\mathcal{F}}_{32}^{41}(p|1-\overline{x}), \tag{75}$$

which has a aesthetically pleasing pictorial interpretation: Crossing symmetry imposes N^4 constrains on the $N^3 + N$ parameters C_{mn}^p and h_p. Exploiting crossing symmetry to compute these parameters is called *conformal* bootstrap.

5 Modular invariance

Up until now we considered only CFTs on the infinite plane/Riemann sphere and at the exact fixed point. In that case the Hilbert space is a tensor product of a holomorphic and (completely decoupled) anti-holomorphic part. These parts don't interact and can this in principle describe completely different theories. In this section we will consider CFTs on the torus, still at the fixed point, in which case the two parts will need to be coupled in such a way that *modular invariance* is preserved. This might seem somewhat ad hoc to define theories on a higher genus surface, but from a string theory point of view this is very natural. CFTs on higher genus surfaces describe higher order closed string scattering amplitudes. From a more stat mech point of view, a theory on a torus is simply a theory with periodic boundary conditions in both directions.

5.1 Conformal field theory on the torus

We will define the *periods* of the lattice as linearly independent complex numbers ω_1, ω_2. The only dependence of the partition function can be on the *modular* parameter $\tau = \omega_1/\omega_2$. As this is independent of the surface area and the absolute orientation of the periods.

5.1.1 The partition function

In the operator formalism the partition function Z and its dependence on τ stands central. We shall define the time and space direction to be the imaginary and real axis respectively. In that case:

$$Z = \text{Tr}\left(\exp(-\{H\,\text{Im}\,\omega_2 - iP\,\text{Re}\,\omega_2\})\right), \tag{76}$$

using

$$H = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \tag{77}$$

$$P = \frac{2\pi}{L}\left(L_0 - \bar{L}_0\right), \tag{78}$$

[1] (the factor $c/12$ ensures the energy is zero in the infinite length case), we find:

$$Z = \text{Tr}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\right), \tag{79}$$

where

$$q = \exp(2\pi i \tau) \tag{80}$$

$$\bar{q} = \exp(-2\pi i \bar{\tau}). \tag{81}$$

We will see that the partition function will be a bilinear combination of characters.

5.1.2 Modular invarariance

Suppose ω_i' define the same lattice, then necessarily:

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{82}$$

Where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. The matrix has an integer inverse, and describes basis transformations that leave the surface area of the torus invariant.

Under a $SL(2,\mathbb{Z})$ transformation:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1. \tag{83}$$

This leads to the modular group $SL(2,\mathbb{Z})/\mathbb{Z}_2 = PSL(2,\mathbb{Z})$. Where the $\mathbb{Z}_2$ subgroup that is modded out corresponds to the symmetry $a, b, c, d \mapsto -a, -b, -c, -d$.

5.1.3 Generators and the fundamental domain

The modular group is generated by:

$$T : \tau \mapsto \tau + 1, \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{84}$$

$$S : \tau \mapsto -\frac{1}{\tau}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{85}$$

And you can check $(ST)^3 = S^2 = -1.$[2]

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[1]Remark: there is a misprint in the book: P shouldn't have an i.
[2]Remark: there is a misprint in the book: S is missing a minus sign and $(ST)^3 = S^2 = 1$ should be -1.

Geometrically, the modular group is generated by *Dehn twists*: T amounts to cutting the torus at fixed time and turning one of the ends 2π before gluing it back together. $U = TST$ does the same for fixed space. These are operators which are not smoothly connected to the identity.

A *fundamental domain F* is a part of the upper half complex plane that can be mapped to every other point of the complex plane by modular transformations. A convention for F_0 is:

$$z \in F_0 \quad \text{if} \quad \begin{cases} \operatorname{Im} z > 0, -\frac{1}{2} \leq \operatorname{Re} z \leq 0, & |z| \geq 1 \\ \operatorname{Im} z > 0, 0 < \operatorname{Re} z < \frac{1}{2}, & |z| \geq 1 \end{cases} \tag{86}$$

5.2   The free boson on the torus

Recall Dedekind's η function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \tag{87}$$

it can be shown that under modular transformations:

$$\eta(\tau + 1) = e^{i\pi/12} \eta(\tau) \tag{88}$$

$$\eta(-1/\tau) = \sqrt{-i\tau} \, \eta(\tau). \tag{89}$$

Without zero mode the boson partition function can be written as:

$$Z_{\text{bos}}(\tau) = \frac{1}{(\operatorname{Im} \tau)^{1/2} |\eta(\tau)|^2}. \tag{90}$$

This is a modular invariant combination of τ and can be obtained in the path-integral formalism by expanding the boson field in the eigenfunctions of the Laplacian and using ζ-regularization.

5.3   Free fermions on the Torus

Fermion fields can be periodic or anti-periodic in two directions on the lattice:

$$\psi(z + \omega_1) = e^{2\pi i v} \psi(z) \tag{91}$$

$$\psi(z + \omega_2) = e^{2\pi i u} \psi(z), \tag{92}$$

$v, u = 0, \frac{1}{2}$, and we shall call the periodic boundary condition *Ramond* and the anti-periodic boundary condition *Neveu-Schwarz*. This leads to four different sectors. A set of boundary conditions is called a *spin structure* for the fermion on the torus. Because ψ and $\bar{\psi}$ decouple we can write:

$$Z_{v,u} = |d_{v,u}|^2. \tag{93}$$

A tedious computation leads to

$$d_{0,0} = \frac{1}{\sqrt{2}} \operatorname{Tr}(-1)^F q^{L_0 - 1/48} = \frac{1}{\sqrt{2}} \operatorname{Tr}(-1)^F q^{\sum_k k b_{-k} b_k + 1/24}, \tag{94}$$

$$d_{0,\frac{1}{2}} = \frac{1}{\sqrt{2}} \operatorname{Tr} q^{L_0 - 1/48} = \frac{1}{\sqrt{2}} \operatorname{Tr} q^{\sum_k k b_{-k} b_k + 1/24}, \tag{95}$$

$$d_{\frac{1}{2},0} = \operatorname{Tr}(-1)^F q^{L_0 - 1/48} = \operatorname{Tr}(-1)^F q^{\sum_k k b_{-k} b_k - 1/48}, \tag{96}$$

$$d_{\frac{1}{2},\frac{1}{2}} = \operatorname{Tr} q^{L_0 - 1/48} = \operatorname{Tr} q^{\sum_k k b_{-k} b_k - 1/48}, \tag{97}$$

or

$$d_{0,0} = 0, \tag{98}$$

$$d_{0,\frac{1}{2}} = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}, \tag{99}$$

$$d_{\frac{1}{2},0} = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}, \tag{100}$$

$$d_{\frac{1}{2},\frac{1}{2}} = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}. \tag{101}$$

Remember the characters

$$\chi_{(c,h)}(\tau) = \operatorname{Tr} q^{L_0 - c/24}. \tag{102}$$

And the expressions for L_0:

$$L_0 = \sum_{k>0} k b_{-k} b_k \qquad \left(k \in \mathbb{Z} + \frac{1}{2}\right), \tag{103}$$

$$L_0 = \sum_{k>0} k b_{-k} b_k + \frac{1}{16} \qquad (k \in \mathbb{Z}), \tag{104}$$

in the NS and R sectors.

Consider e.g. the NS sector. Since L_0 takes both half-integer and integer values, this character is the sum of at least two simple Virasoro characters. From the fact that $c = 1/2$ we find that

$$\chi_{1,1} = q^{-1/48} \frac{1}{2} \operatorname{Tr}\left(1 + (-1)^F\right) q^{L_0} \tag{105}$$

$$\chi_{2,1} = q^{-1/48} \frac{1}{2} \operatorname{Tr}\left(1 - (-1)^F\right) q^{L_0}. \tag{106}$$

Comparing with the partition function:

$$\chi_{1,1} = \frac{1}{2}\left(d_{\frac{1}{2},\frac{1}{2}} + d_{\frac{1}{2},0}\right), \tag{107}$$

$$\chi_{2,1} = \frac{1}{2}\left(d_{\frac{1}{2},\frac{1}{2}} - d_{\frac{1}{2},0}\right). \tag{108}$$

Similarly for the R sector:

$$\chi_{1,2} = \frac{1}{\sqrt{2}} d_{0,\frac{1}{2}}. \tag{109}$$

From the modular transformation properties of the d's, it follows there are two modular invariant partition functions: the one which only contains (R,R) and is zero and the one that combines the other sectors as:

$$Z = Z_{\frac{1}{2},\frac{1}{2}} + Z_{0,\frac{1}{2}} + Z_{\frac{1}{2},0} \tag{110}$$

$$= \left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| \tag{111}$$

$$= 2\left(|\chi_{1,1}|^2 + |\chi_{2,1}|^2 + |\chi_{1,2}|^2\right). \tag{112}$$

This is twice the partition function of the Ising model on the torus!

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A   Central extensions of Lie algebras

In this section $\mathfrak{g}, \mathfrak{h}, \ldots$ denote (possibly infinite) Lie algebras over some field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. This section is mainly based on Wikipedia and [1].

A.1   Extensions

Definition: A *Lie algebra extension* is a short exact sequence of Lie algebras:

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g}. \tag{113}$$

One calls $\mathfrak{e}$ an extension of $\mathfrak{g}$ by $\mathfrak{h}$. By exactness of the sequence one has $\mathfrak{g} \cong \mathfrak{e}/\operatorname{Im}\iota$.

Definition: A *central extension* is an extension $\mathfrak{e}$ of $\mathfrak{g}$ by $\mathfrak{h}$, such that $\operatorname{Im}\iota$ is contained in the center of $\mathfrak{e}$, $\iota(\mathfrak{h}) \subseteq Z(\mathfrak{e})$.

Notice that for a central extension $\mathfrak{h}$ is necessarily abelian. We now introduce a notion of trivial central extensions as follows:

Definition: A Lie algebra extension

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \tag{114}$$

splits if there exists a Lie algebra morphism $\beta : \mathfrak{g} \mapsto \mathfrak{e}$ such that $\pi \circ \beta = \operatorname{id}_{\mathfrak{e}}$. β is called a splitting map.

A central extension

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g}. \tag{115}$$

that splits is trivial in the sense that it is equivalent[3] to one where $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{h}$.

Let us now consider a central extension and a map (not necesserily a Lie algebra homomorphism) $\beta : \mathfrak{g} \to \mathfrak{e}$ such that $\pi \circ \beta = \operatorname{id}_{\mathfrak{e}}$. From this map construct $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ as follows:

$$\Theta(x, y) := [\Theta(x), \Theta(y)] - \Theta([x, y]). \tag{116}$$

This map is:

1. Antisymmetric.

2. Bilinear.

3. Satisfies $\Theta(x, [y, z]) + \Theta(y, [z, x]) + \Theta(z, [x, y]) = 0$.

Given Θ one can now show that there is an isomorphism between the vector spaces $\mathfrak{e} \cong \mathfrak{g} \oplus \mathfrak{h}$ that is given by:

$$\Psi : \mathfrak{g} \oplus \mathfrak{h} \mapsto \mathfrak{e} : (x, y) \mapsto \beta(x) + y. \tag{117}$$

A Lie bracket on $\mathfrak{g} \oplus \mathfrak{h}$ is given by:

$$[x \oplus z, y \oplus z']_{\mathfrak{e}} := [x, y]_{\mathfrak{g}} + \Theta(x, y). \tag{118}$$

Lemma: In the above construction β is a splitting map if and only if

$$\Theta(x, y) = \mu([x, y]), \tag{119}$$

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[3]To do: introduce the notion of equivalent extensions.

for some $\mu \in \text{Hom}(\mathfrak{g}, \mathfrak{h})$.

Now comes the classification of central extensions of Lie algebras:

Theorem: Every central extension comes from a map Θ that satisfies the above properties (1-3). Conversely, every central extension gives rise to a map Θ that satisfies the above properties (1-3).

A.2   Lie algebra cohomology

The classification of Lie algebra extensions is very satisfying. It smells a lot like a cohomological classification. Indeed, the extensions are classified by functions depending on two variables satisfying the condition (3) that is exactly the one needed to fulfill the Jacobi identity of the central extension. Moreover, the central extension is trivial if the *2-cocycle* Θ is trivial in the following sense: $\Theta(x, y) = \mu([x, y])$. This is reminiscent of considering 2-cocycles to be trivial if they are equal to a coboundary. Let us put this on a bit more rigorous footing.

Definitions:

1. $Z^2(\mathfrak{g}, \mathfrak{h}) = \left\{ \Theta \in \Lambda^2(\mathfrak{g}, \mathfrak{h}) | \Theta : (3) \right\}$.

2. $B^2(\mathfrak{g}, \mathfrak{h}) = \{ \Theta : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{h} | \exists \mu \in \text{Hom}(\mathfrak{g}, \mathfrak{h}) : \Theta(-, -) = \mu([-, -]) \}$.

3. $H^2(\mathfrak{g}, \mathfrak{h}) := Z^2(\mathfrak{g}, \mathfrak{h})/B^2(\mathfrak{g}, \mathfrak{h})$.

H^2 is of course called the second cohomology group. We thus obtain the following reformulation of the classification of central extensions:

Theorem: The equivalence classes of central extensions

$$\mathfrak{h} \xrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \tag{120}$$

are in one-to-one correspondence with the elements of $H^2(\mathfrak{g}, \mathfrak{h})$.

For completeness, let us introduce a notion of cochain complexes for Lie algebras. A cochain f is a alternating multilinear map f:

$$f : \Lambda^n \mathfrak{g} \mapsto \mathfrak{h}. \tag{121}$$

Here, $\mathfrak{h}$ is considered a $\mathfrak{g}$-module or -representation.

The differential of an n-cochain is given by

$$(df)(x_1, \ldots, x_{n+1}) = \sum_i (-1)^{i+1} x_i f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) +$$
$$\sum_{i<j} (-1)^{i+j} f\left([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}\right), \tag{122}$$

so for example, with trivial action we obtain

$$(df)(x_1, x_2) = f([x_1, x_2]), \tag{123}$$

and

$$(df)(x_1, x_2, x_3) = -f([x_1, x_2], x_3) + f([x_1, x_3], x_2) - f([x_2, x_3], x_1)$$
$$= -f([x_1, x_2], x_3) - f([x_3, x_1], x_2) - f([x_2, x_3], x_1) \tag{124}$$
$$= f(x_3, [x_1, x_2]) + f(x_2, [x_3, x_1]) + f(x_1, [x_2, x_3]).$$

So clearly, $Z^2(\mathfrak{g}, \mathfrak{h})$ defined above is the group of 2-cocycles satisfying $d\Theta = 0$ and $B^2(\mathfrak{g}, \mathfrak{h})$ the set of coboundaries: $\Theta = d\mu$.

References

[1] M. Schottenloher, *Central extensions of lie algebras and Bargmann's theorem*, In *A Mathematical Introduction to Conformal Field Theory*, pp. 63–73. Springer (2008).