Chapter 1

Global Conformal Invariance

1.1 The Conformal Group

We define a conformal transformation as a invertible mapping from $x \to x'$, which leaves the metric tensor $g_{\mu\nu}$ invariant up to a scale.

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu} \tag{1.1}$$

The set of conformal transformations forms a group, which we call the conformal group. Conformal transformations preserves angles between two arbitrary curves.

Consider an infinitesimal transformation $x^\mu \to x'^\mu + \epsilon^\mu(\mathbf{x})$, then the metric tensor changes up to first order in ϵ as

$$g_{\mu\nu} \to g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right). \tag{1.2}$$

The requirement that the metric tensor stays invariant up to a scale corresponds to

$$\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) = f(\mathbf{x}) g_{\mu\nu}. \tag{1.3}$$

After some working out we get

$$(d-1)\partial^2 f = 0 \tag{1.4}$$

Equation 1.4 has the form of a homogeneous Laplace equation. f correspond to the kernel of the Laplace operator in d-dimensions and is the main object of study in Harmonic Analysis. For $d = 1$ equation 1.4 is trivially satisfied and any smooth transformation is conformal. For now we'll consider $d \geq 3$.

Equation 1.4 implies that ϵ is at most quadratic in the coordinates and can in general be written as

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{1.5}$$

We can look at every term and determine what kind of transformation they represent.

$$\text{Translation} : \epsilon^\mu = a^\mu \tag{1.6}$$

$$\text{Dilation} : \epsilon^\mu = \alpha x^\mu \tag{1.7}$$

$$\text{Rotation} : \epsilon^\mu = M^\mu_\nu x^\nu \tag{1.8}$$

$$\text{Special Conformal Transformation} : \epsilon^\mu = 2(\mathbf{x} \cdot \mathbf{b}) x^\mu - b^\mu \mathbf{x}^2 \tag{1.9}$$

In the equations above α is a scalar, M is an antisymmetric tensor and $b_\mu = \frac{c^\sigma_{\sigma\mu}}{d}$. We can also introduce the generators of the infinitesimal transformations.

$$\text{Translation} : P^\mu = -i\partial_\mu \tag{1.10}$$

$$\text{Dilation} : D = -i x^\mu \partial_\mu \tag{1.11}$$

$$\text{Rotation} : L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{1.12}$$

$$\text{Special Conformal Transformation} : K_\mu = -i(2 x_\mu x^\nu \partial_\nu - \mathbf{x}^2 \partial_\mu) \tag{1.13}$$

The conformal group implies a specific structure for functions of N points that are invariant under conformal transformations. Translation and rotation invariance implies that invariants can only depend on the distance $\left|\mathbf{x_i} - \mathbf{x_j}\right|$ between pair of points. Scale invariance implies that only ratios of distances are important, not the distances themselves. Invariance under SCT implies that only invariants of at least 4 points can be constructed. Simplest forms are called anharmonic ratios or cross-ratios. In equation 1.14 is an example of a cross-ratio for 4 points given.

$$\text{cross-ratio} = \frac{|\mathbf{x}_1 - \mathbf{x}_2||\mathbf{x}_3 - \mathbf{x}_4|}{|\mathbf{x}_1 - \mathbf{x}_3||\mathbf{x}_2 - \mathbf{x}_4|} \tag{1.14}$$

1.2 Conformal Invariance of Classical Fields

To know how a classical field $\Phi(x)$ transforms under conformal transformations, it is enough to determine the action of the generators on the field. The following actions can be derived

$$P_\mu \Phi(x) = -i\partial_\mu \Phi(x) \tag{1.15}$$
$$L_{\mu\nu}\Phi(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x) \tag{1.16}$$
$$D\Phi(x) = (-ix^\nu \partial_\nu + \tilde{\Delta})\Phi(x) \tag{1.17}$$
$$K_\mu \Phi(x) = \left[\kappa_\mu + 2x_\mu \tilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\right]\Phi(x) \tag{1.18}$$

Invoking Schur's lemma, we find that $\tilde{\Delta}$ must be a multiple of the identity with $\tilde{\Delta} = -i\Delta\mathbf{1}$ with Δ the scaling dimension of Φ. Notice that $\tilde{\Delta}$ is not Hermitian and representations of the dilation group on classical fields are not unitary.

For a finite conformal transformation of a classical spinless field $\phi(x)$ we have

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x). \tag{1.19}$$

A field transforming according to equation 1.19 is called a quasi-primary field.

1.2.1 Energy-Momentum Tensor

Under a arbitrary transformation $x^\mu \to x^\mu + \epsilon^\mu$, the action changes as

$$\delta S = \int d^d x T^{\mu\nu} \partial_\mu \epsilon_\nu \tag{1.20}$$
$$= \frac{1}{2}\int d^d x T^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu), \tag{1.21}$$

where we assumed T to be symmetric. If the transformation is conformal then

$$\delta S = \frac{1}{d}\int d^d x T^\mu_\mu \partial_\rho \epsilon^\rho, \tag{1.22}$$

which is zero if T is traceless. If the energy-momentum tensor can be made traceless (and symmetric), then conformal invariance follows. Note that the converse is not true. Under certain conditions T can be made traceless if the theory has scale invariance and can be made symmetric if the theory has rotational invariance. If these conditions are satisfied conformal invariance follows from Poincaré invariance and scale invariance.

1.3 Conformal Invariance of Quantum Fields

1.3.1 Correlation Functions

Conformal invariance has some important consequences on correlation functions of quasi-primary fields. For example consider the two point correlation function, which due to equation 1.19 and conformal invariance, has the following form

$$< \phi_1(x_1)\phi_2(x_2) >= \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_2/d} < \phi_1(x_1')\phi_2(x_2') > . \tag{1.23}$$

Translation and rotation invariance require that

$$< \phi_1(x_1)\phi_2(x_2) >= f(|x_1 - x_2|) \tag{1.24}$$

Scale invariance implies that $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$ and thus

$$< \phi_1(x_1)\phi_2(x_2) >= \frac{C_12}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \tag{1.25}$$

Invariance under the special conformal transformation requires

$$< \phi_1(x_1)\phi_2(x_2) >= \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & \Delta_1 = \Delta_2 \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{1.26}$$

Similarly we find for the 3-point correlation function

$$< \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) >= \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_1+\Delta_3-\Delta_2}} . \tag{1.27}$$

Starting from the 4-point correlation function, the form isn't entirely fixed up to a multiplicative constant. The correlation function can gain an extra functional dependence on the cross-ratios.

1.3.2 Ward Identities

There exists a Ward identity corresponding to every symmetry. In case of conformal symmetry we have the following Ward identities. For translation invariance

$$\partial_\mu < T^\mu_\nu X >= -\sum_i \delta(x - x_i)\frac{\partial}{\partial x_i^\nu} < X > . \tag{1.28}$$

For rotation invariance we find

$$< (T^{\rho\nu} - T^{\nu\rho} X >= -i\sum_i \delta(x - x_i) S_i^{\nu\rho} < X > . \tag{1.29}$$

For scale invariance we get

$$< T^\mu_\mu X >= -\sum_i \delta(x - x_i)\Delta_i < X > . \tag{1.30}$$