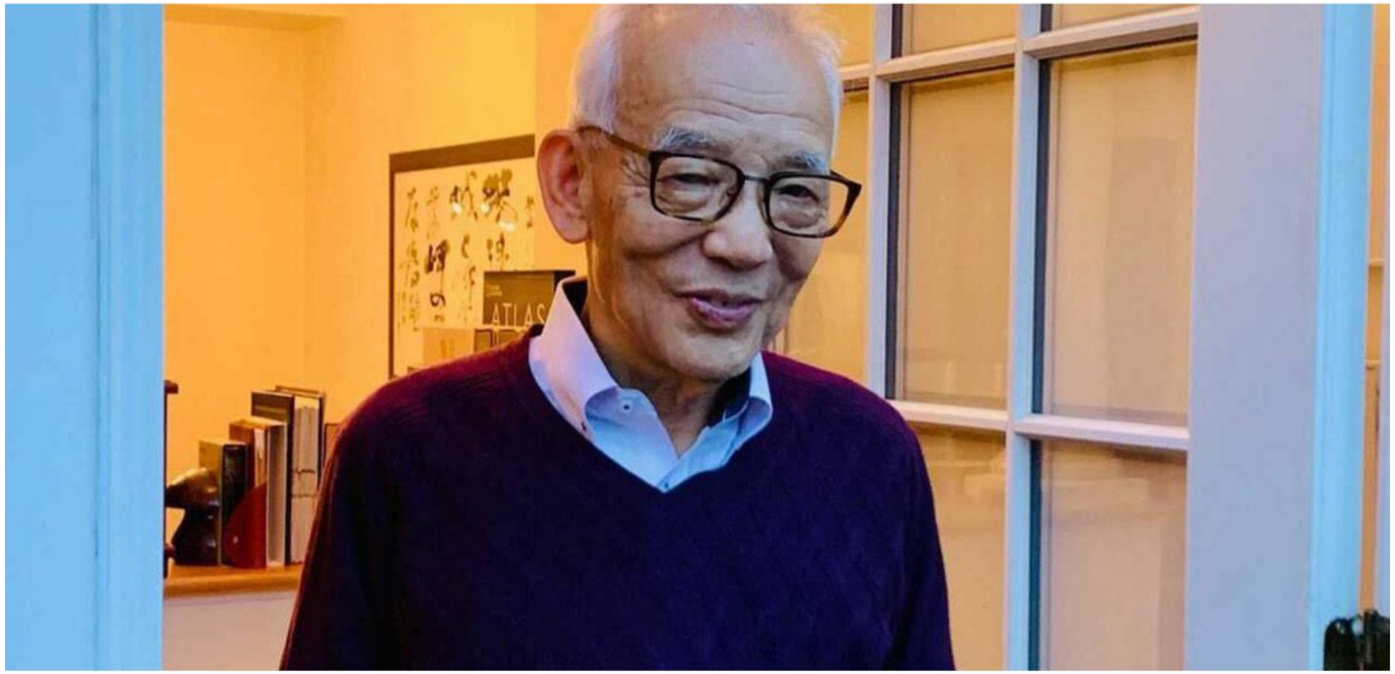


Princeton's Syukuro Manabe receives Nobel Prize in physics

Manabe is a senior meteorologist, Princeton AOS and GFDL

https://phys.org/news/2021-10-nobel-prize-physics-awarded-scientists.html?fbclid=IwAR2m-UphoXJHAJLdb_JWFRUw4tRt1yD5KaN_-iU7A37N2jWUv20FzvIC4FU

https://www.nature.com/articles/d41586-021-02703-3

- In the late 1960's, Syukuro "Suki" Manabe and Kirk Bryan began to develop a general circulation model of the coupled atmosphere-ocean-land system, which eventually became a very powerful tool for the simulation of Global warming.
- "*One complex system of vital importance to humankind is Earth's climate. Syukuro Manabe demonstrated how increased levels of carbon dioxide in the atmosphere lead to increased temperatures at the surface of the Earth,*" the Royal Swedish Academy of Sciences.
- In 1968, Princeton University created the precursor to today's Program in Atmospheric and Oceanic Sciences (AOS), dedicated to understanding key mechanisms driving global climate systems.

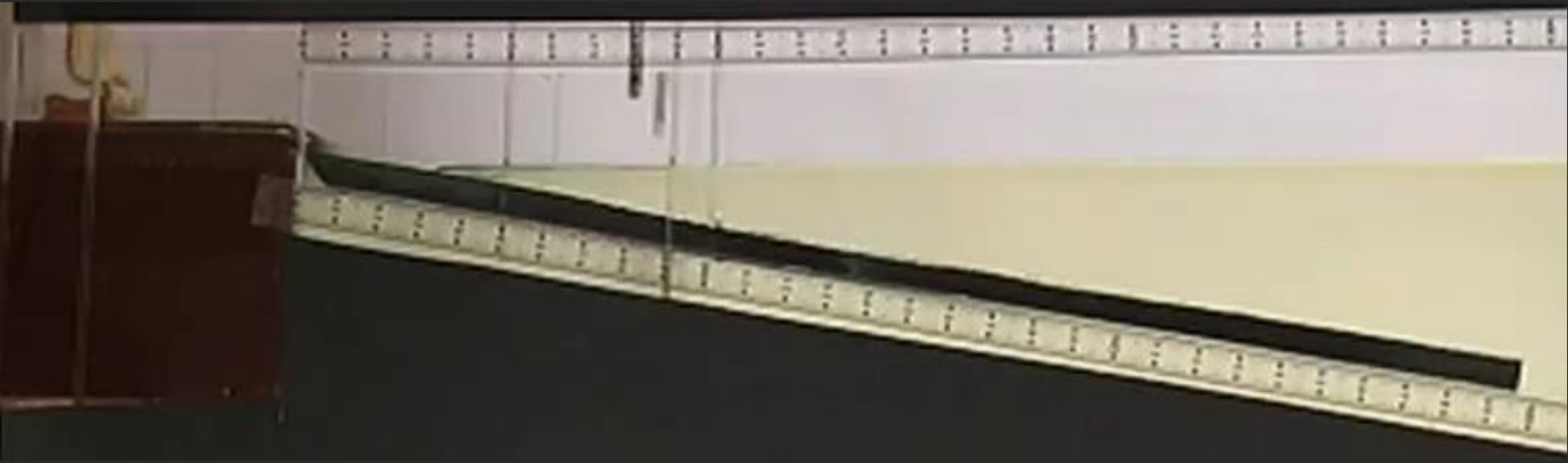
Application to Ice dynamics

(Non-Newtonian viscous gravity current)

Antarctic ice sheet

Ice shelf intuition

Rosalyn et al, JFM (2010)



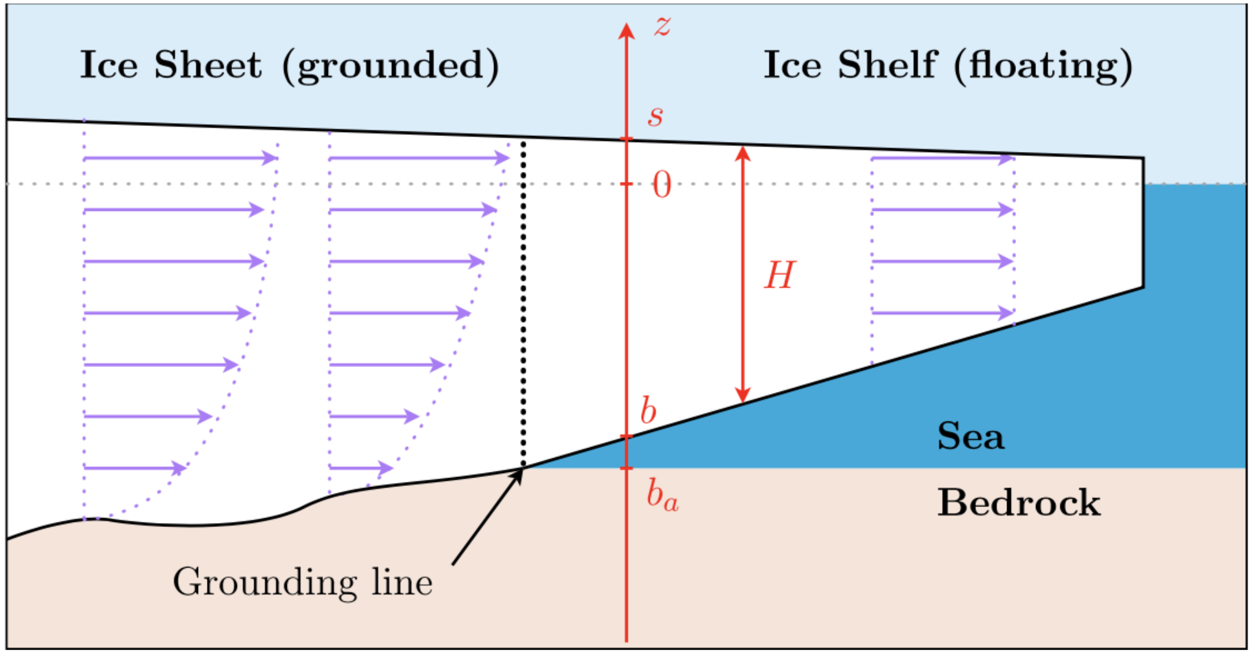
H: Thickness
u: Velocities
a: Accumulation rate
ν: Viscosity
B: **Ice Hardness**

E.g., 1D Ice-shelf

- Modelled by time independent, incompressible Navier-Stokes Equations

- The Shallow Shelf Approximation (SSA) – width >> height
→ depth averaged equation

- Non-dimensionalize the equations



H: Thickness
u: Velocities
a: Accumulation rate
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B: **Ice Hardness**

E.g., 1D Ice-shelf

Conservation of momentum

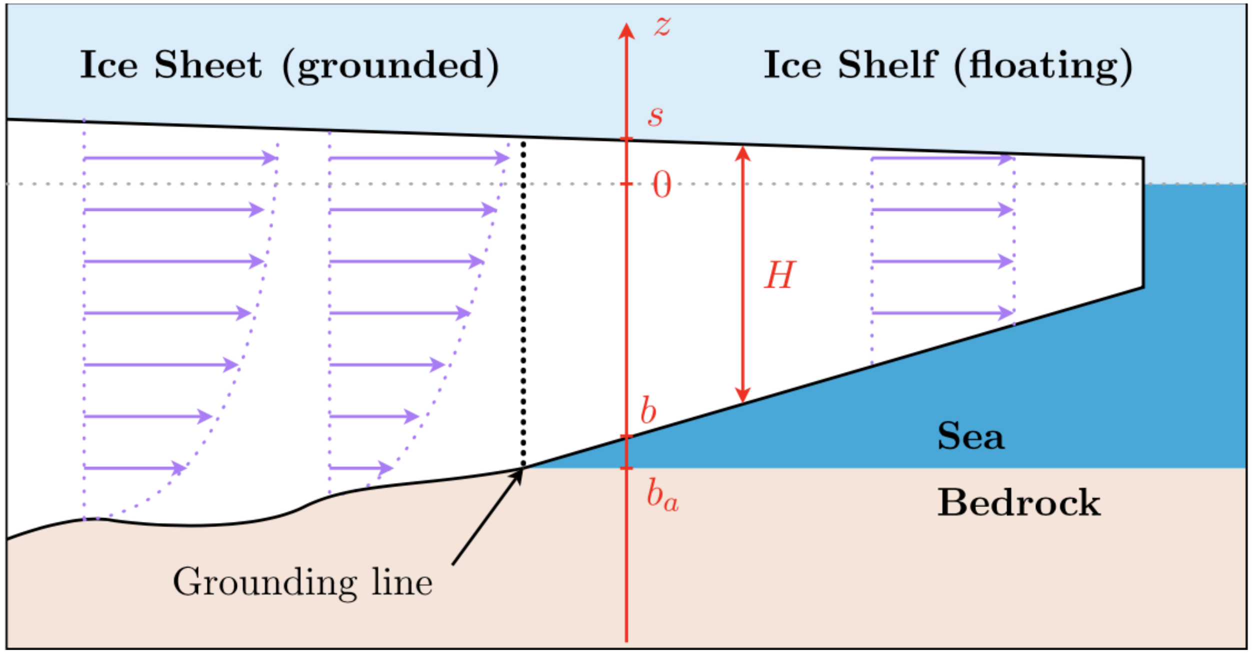
$$\underbrace{\frac{\partial}{\partial x}\left(4\overline{\nu}H\frac{\partial u}{\partial x}\right)}_{\text{viscous effects}} = \underbrace{\rho_i(1-\frac{\rho_i}{\rho_w})gH\frac{\partial H}{\partial x}}_{\text{gravitational effects}}$$

Conservation of mass

$$\underbrace{\frac{\partial(uH)}{\partial x}}_{\text{flux divergence}} = \underbrace{a}_{\text{snow accrumulation}}$$

Non – newtonian rheology

$$\underbrace{\overline{\nu}}_{\text{effective viscosity}} = \frac{\overline{B}}{2}\left|\frac{\partial u}{\partial x}\right|^{1/n-1}, n = 3,\ \overline{B} = \frac{1}{H}\int_{z_b}^{z_s} B\,\mathrm{d}z$$



H: Thickness
u: Velocities
a: Accumulation rate
ν: Viscosity
B: **Ice Hardness**

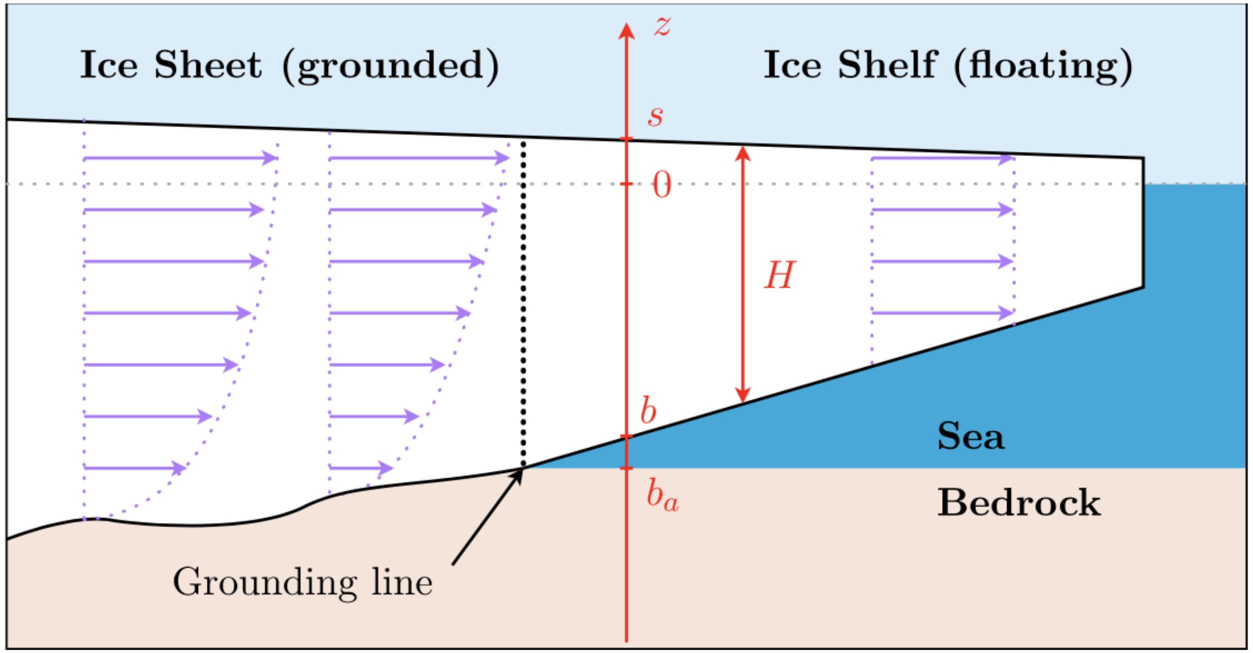
E.g., 1D Ice-shelf (dimensionless)

Conservation of momentum

$$\underbrace{\nu^* \frac{\partial}{\partial \tilde{x}} \left(\tilde{B}\tilde{H} \left| \frac{\partial \tilde{u}}{\partial \tilde{x}} \right|^{\frac{1}{n}-1} \frac{\partial \tilde{u}}{\partial \tilde{x}} \right)}_{\text{viscous effects}} = \underbrace{\tilde{H}\frac{\partial \tilde{H}}{\partial \tilde{x}}}_{\text{gravitational effects}}$$

Conservation of mass

$$\underbrace{\frac{\partial(\tilde{u}\tilde{H})}{\partial \tilde{x}}}_{\text{flux divergence}} = \underbrace{A_0}_{\text{snow accrumulation}}$$



Non – dimensionalize our parameters!

$$u = U_0\tilde{u},\ x = L_x\tilde{x},\ H = Z_0\tilde{H},\ \overline{B} = B_0\tilde{B},\ A_0 = \frac{aL_x}{U_0 Z_0},\ \nu^* = \frac{4B_0 U_0^{1/n}}{2\rho_i g\delta Z_0 L_x^{1/n}} \left(\frac{\text{viscous effects}}{\text{gravitational effects}} \right)$$

H: Thickness
u: Velocities
a: Accumulation rate
ν: Viscosity
B: **Ice Hardness**

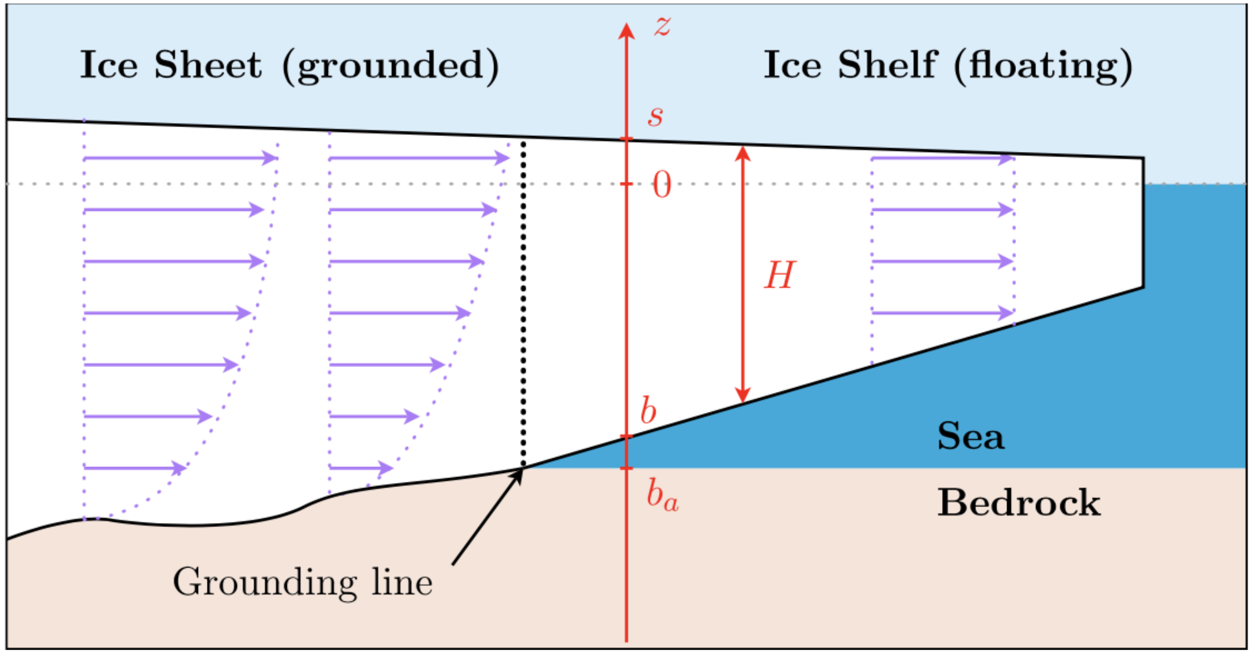
E.g., 1D Ice-shelf (dimensionless)

Conservation of momentum

$$\underbrace{2\nu^* \tilde{B} \left|\frac{\partial u}{\partial x}\right|^{1/n-1} \frac{\partial \tilde{u}}{\partial \tilde{x}}}_{\text{viscous effects}} = \underbrace{\tilde{H}}_{\text{gravitational effects}}$$

Conservation of mass

$$\underbrace{\frac{\partial(\tilde{u}\tilde{H})}{\partial \tilde{x}}}_{\text{flux divergence}} = \underbrace{A_0}_{\text{snow accrumulation}}$$



Non – dimensionalize our parameters!

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Forward vs inverse problems

H: Thickness
u: Velocities
a: Accumulation rate
ν: Viscosity
B: **Ice Hardness**

Conservation of momentum

$$\underbrace{2\nu^* \tilde{B} \left| \frac{\partial u}{\partial x} \right|^{1/n-1} \frac{\partial \tilde{u}}{\partial \tilde{x}}}_{\text{viscous effects}} = \underbrace{\tilde{H}}_{\text{gravitational effects}}$$

Conservation of mass

$$\underbrace{\frac{\partial(\tilde{u}\tilde{H})}{\partial \tilde{x}}}_{\text{flux divergence}} = \underbrace{A_0}_{\text{snow accrumulation}}$$

- **Forward problem**

Given: eqn, $\nu^*, A_0, \tilde{B}(\tilde{x})$, BC

Solve for: $\tilde{u}(\tilde{x}), \tilde{H}(\tilde{x})$

- **Inverse problem**

Given: eqn, $\nu^*, A_0, \tilde{u}(\tilde{x}), \tilde{H}(\tilde{x})$

Invert for $\tilde{B}(\tilde{x})$

Non − dimensionalize our parameters!

$$u = U_0\tilde{u},\ x = L_x\tilde{x},\ H = Z_0\tilde{H},\ \overline{B} = B_0\tilde{B},\ A_0 = \frac{aL_x}{U_0Z_0},\ \nu^* = \frac{4B_0U_0^{1/n}}{2\rho_i g\delta Z_0 L_x^{1/n}} \left(\frac{\text{viscous effects}}{\text{gravitational effects}}\right)$$

Optimally weighted loss functions for solving PDEs with Neural Networks

Remco van der Meer,[1,2] Cornelis Oosterlee,[1,2] and Anastasia Borovykh[1,3]

Loss function

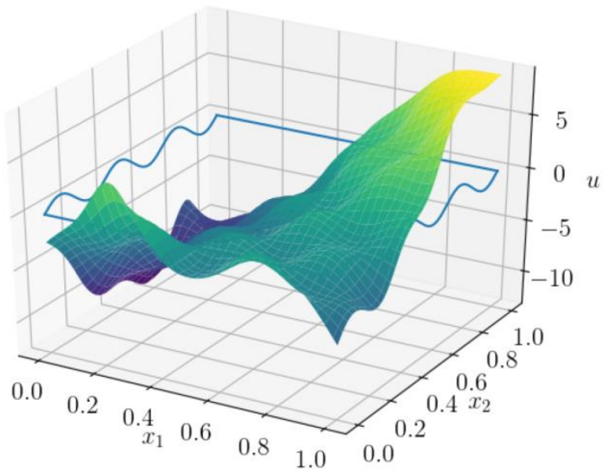
Proposed by
van der Meer et al (2021)

$$\boxed{MSE = (1-\gamma)MSE_{data} + \gamma\ MSE_{eqn}}$$

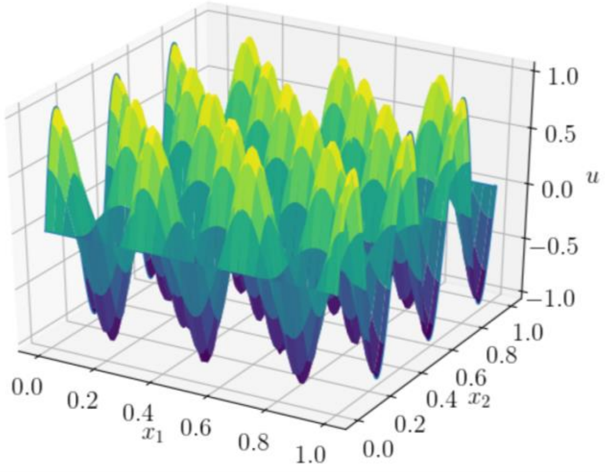
γ is a hyper-parameters that determines the importance of minimizing MSE_{data} vs MSE_{eqn} to obtain the correct prediction

the Poisson equation can be defined by

$$\begin{cases} \nabla^2 u(x,y) = F(x,y) & \text{in } \Omega, \\ u(x,y) = G(x,y) & \text{on } \partial\Omega. \end{cases}$$



$\gamma = 1/2$



$\gamma = 10^{-5}$

Why is minimizing BC loss more important than minimizing eqn loss?

E.g., 1D ice shelf- forward

Problem statement

Eqns:
$$\begin{cases} 2\nu^* \tilde{B} \left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}}\right)^{1/n} = \tilde{H} & , \tilde{x} \in [0,1] \\ \frac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 \end{cases}$$

BCs: $\tilde{u}(0) = 1$

$\tilde{H}(0) = h_0$

Known parameters:
$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$

NN output: $\tilde{u}$, $\tilde{H}$

NN architecture: 2 hidden layers 50 neurons per layer

Data points: 1 pt at x=0

Collocation points: N pts within the domain

Training data (from ground truth):

$\{\tilde{x}_d^i, \tilde{u}_d^i, \tilde{H}_d^i\}_{i=1}^m$, $m = 1$

Collocation points:

$\{\tilde{x}_f^i\}_{i=1}^N$, $N = 201$

Physics equations:

$$f \equiv 2\nu^* \tilde{B}(\tilde{u}_{\tilde{x}})^{1/n} - \tilde{H}$$
$$g \equiv (\tilde{u}\tilde{H})_{\tilde{x}} - A_0$$

Loss function:

$$MSE = (1-\gamma)\frac{1}{m}\Sigma_{i=1}^m(|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \tilde{H}_d^i|^2)$$

Data loss

$$+\gamma\frac{1}{N}\Sigma_{i=1}^N(|f(\tilde{x}_f^i)|^2 + |g(\tilde{x}_f^i)|^2)$$

Equation loss

E.g., 1D ice shelf- <u>forward</u>

Problem statement

Eqns:
$$\begin{cases} 2\nu^* \tilde{B} \left(\dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}}\right)^{1/n} = \tilde{H} & , \tilde{x} \in [0,1] \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 & \end{cases}$$

BCs: $\tilde{u}(0) = 1$

$\tilde{H}(0) = h_0$

Known parameters:
$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

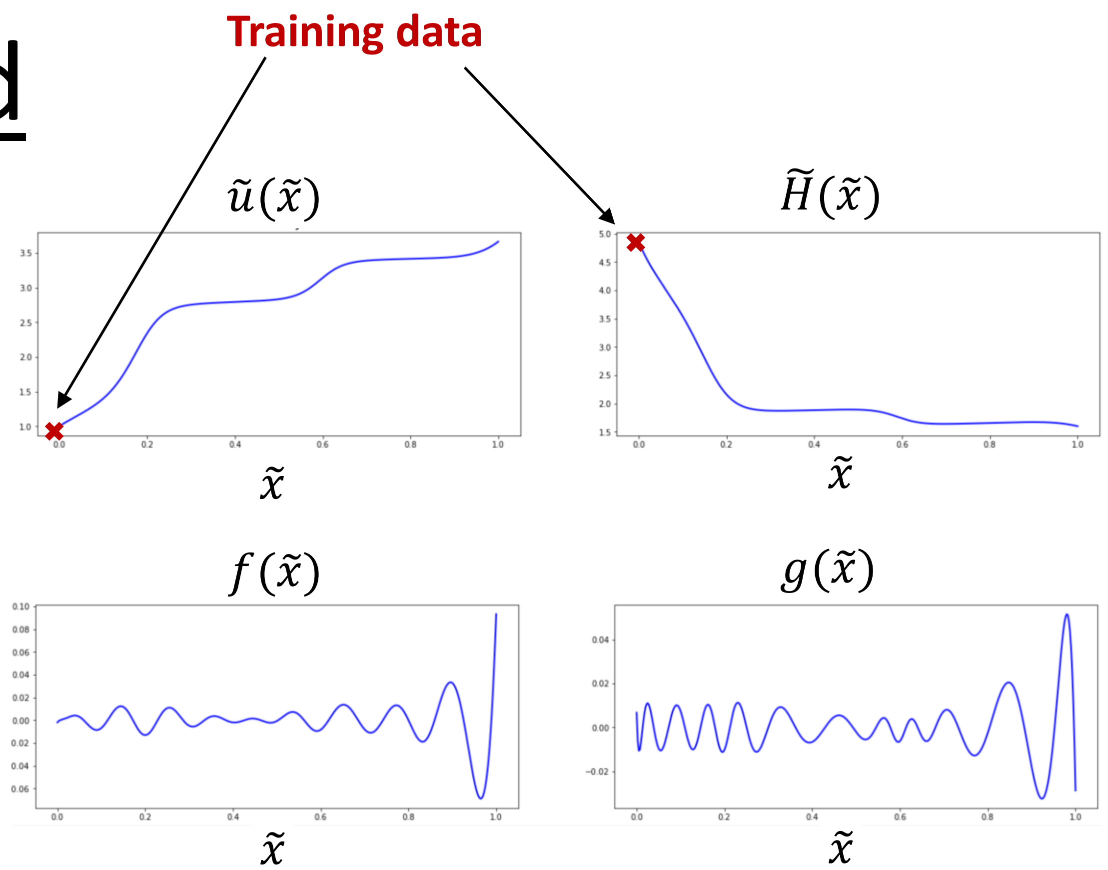
NN input: $\tilde{x}$

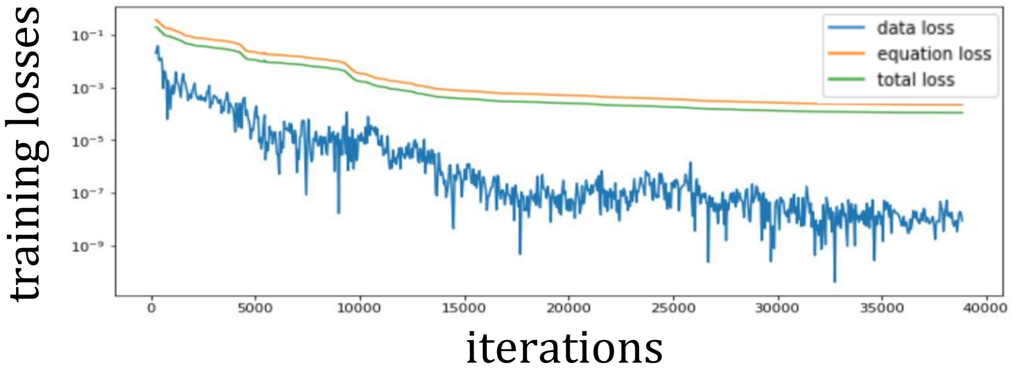
NN output: $\tilde{u}$, $\tilde{H}$

NN architecture: 2 hidden layers 50 neurons per layer

Data points: 1 pt at x=0

Collocation points: N pts within the domain





E.g., 1D ice shelf- <u>inverse</u>

Given training data of $\tilde{u}(\tilde{x}), \widetilde{H}(\tilde{x})$, find $\tilde{B}(\tilde{x})$ without $\tilde{B}$ training data!

Problem statement

Eqns:
$$\begin{cases} 2\nu^* \tilde{B}\left(\dfrac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{x}}\right)^{1/n} = \tilde{H} \qquad , \tilde{x} \in [0,1] \\ \cancel{\dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0} \end{cases}$$

BCs: $\cancel{\tilde{u}(0) = 1}$

$\cancel{\tilde{H}(0) = h_0}$

Known parameters:
$$\begin{cases} \cancel{\tilde{B} = \cos(5\pi\tilde{x}) + 2} \\ \cancel{A_0 = 1} \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$
NN output: $\tilde{u}, \widetilde{H}, \tilde{B}$
NN architecture: 4 hidden layers 100 neurons per layer
Data points: 1 pt at x=0
Collocation points: N pts within the domain

Training data (from ground truth):

$\{\tilde{x}_d^i, \tilde{u}_d^i, \widetilde{H}_d^i\}_{i=1}^m, \qquad m = 80,401$

Collocation points:

$\left\{\tilde{x}_f^i\right\}_{i=1}^N, \qquad N = 201$

Physics equations:

$$f \equiv 2\nu^* \tilde{B}(\tilde{u}_{\tilde{x}})^{1/n} - \widetilde{H}$$

Loss function:

$$MSE = (1-\gamma)\frac{1}{m}\sum_{i=1}^m \left(|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \widetilde{H}_d^i|^2\right)$$

Data loss

$$+\gamma \frac{1}{N}\sum_{i=1}^N \left(|f(\tilde{x}_f^i)|^2\right)$$

Equation loss

E.g., 1D ice shelf- <u>inverse</u>

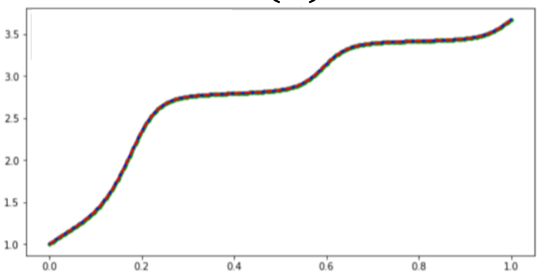
Given training data of $\tilde{u}(\tilde{x}), \tilde{H}(\tilde{x})$, find $\tilde{B}(\tilde{x})$ without $\tilde{B}$ training data!

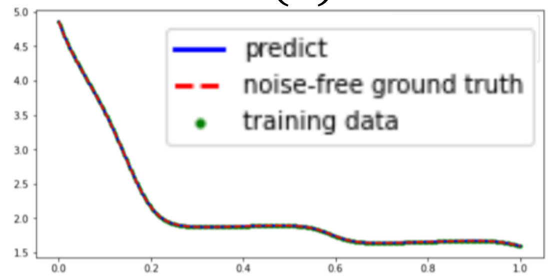
Training data (from ground truth):

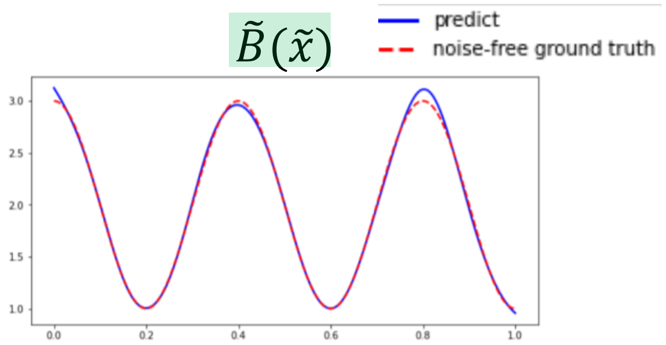
Inverted $\tilde{B}$:

$\tilde{u}(\tilde{x})$ $\tilde{H}(\tilde{x})$ $\tilde{B}(\tilde{x})$

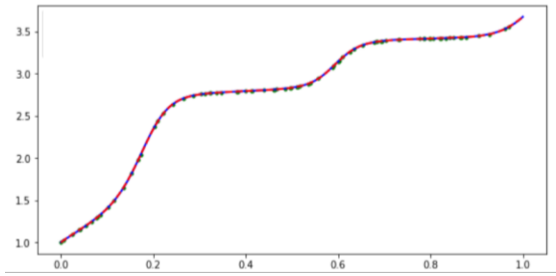
(1) $m = 401$, No noise
$\gamma = 0.1$

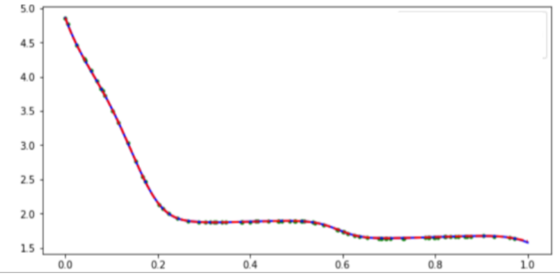


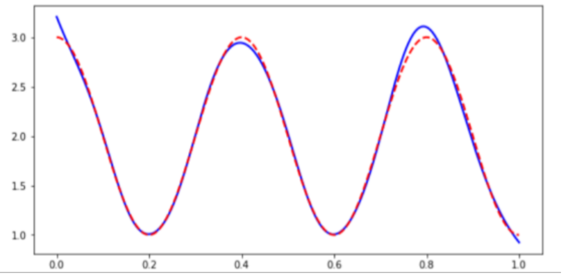




(2) $m = 80$, No noise
$\gamma = 0.1$

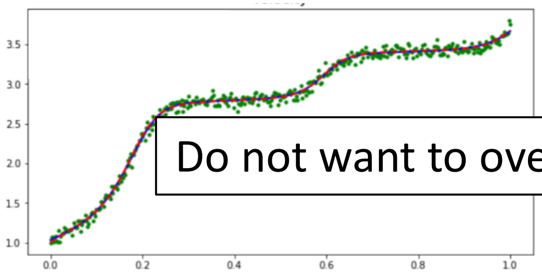


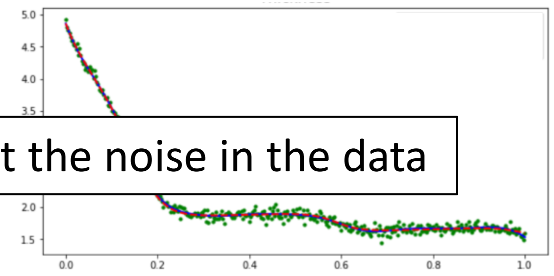




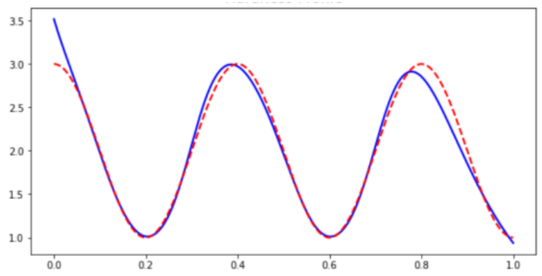
(3) $m = 401$, 5% noise
$\gamma = 0.9$

Should you increase/decrease γ?





Do not want to overfit the noise in the data



$\tilde{x}$ $\tilde{x}$ $\tilde{x}$

E.g., 1D ice shelf- forward (2nd order)

Problem statement

Eqns:
$$\begin{cases} \nu^* \dfrac{\partial}{\partial \tilde{x}} \left(\tilde{B}\tilde{H} \left| \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right|^{\frac{1}{n}-1} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right) = \tilde{H} \dfrac{\partial \tilde{H}}{\partial \tilde{x}} \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 \end{cases} \quad , \tilde{x} \in [0, 1]$$

BCs:
$$\tilde{u}(0) = 1$$
$$\tilde{H}(0) = h_0$$
$$\nu^* \tilde{B}\tilde{H}\tilde{u}_{\tilde{x}}^{\frac{1}{n}} = \tilde{H}^2/2, \quad \tilde{x} = 1$$

Known parameters:
$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$
NN output: $\tilde{u}$, $\tilde{H}$

Training data (from ground truth):
$$\{\tilde{x}_d^i, \tilde{u}_d^i, \tilde{H}_d^i\}_{i=1}^m, \{\tilde{x}_d^i, \tilde{u}_{\tilde{x}d}^{\ i}\}_{i=1}^m, \qquad m = 1$$

Collocation points:
$$\{\tilde{x}_f^i\}_{i=1}^N, \qquad N = 201$$

Physics equations:
$$f \equiv \nu^* \left(\tilde{B}\tilde{H}\tilde{u}_{\tilde{x}} |\tilde{u}_{\tilde{x}}|^{\frac{1}{n}-1} \right)_{\tilde{x}} - \tilde{H}\tilde{H}_{\tilde{x}}$$
$$g \equiv \left(\tilde{u}\tilde{H}\right)_{\tilde{x}} - A_0$$

Loss function:

Data loss

$$MSE = (1-\gamma)\frac{1}{m}\sum_{i=1}^m (|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \tilde{H}_d^i|^2$$

$$+\gamma \frac{1}{N}\sum_{i=1}^N (|f(\tilde{x}_f^i)|^2 + |g(\tilde{x}_f^i)|^2)$$ Equation loss

E.g., 1D ice shelf- forward (2^{nd} order)

Problem statement

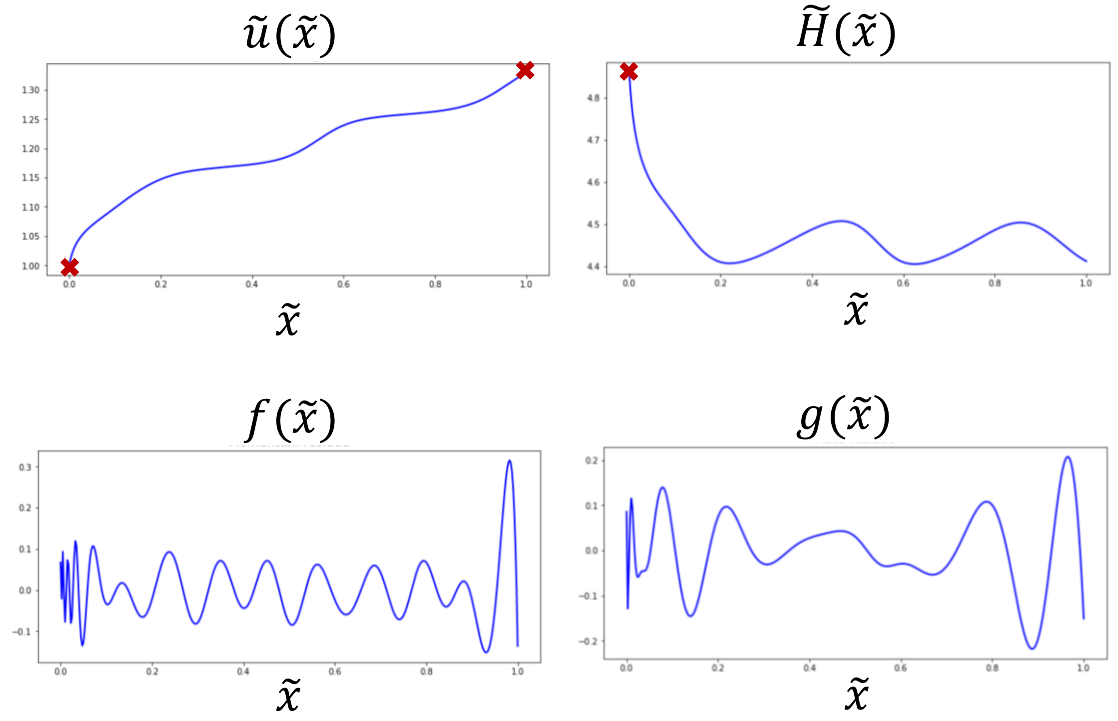
Eqns: $$\begin{cases} \nu^* \dfrac{\partial}{\partial \tilde{x}} \left(\tilde{B}\tilde{H} \left| \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right|^{\frac{1}{n}-1} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right) = \tilde{H} \dfrac{\partial \tilde{H}}{\partial \tilde{x}} \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 \end{cases} \quad , \tilde{x} \in [0, 1]$$

BCs: $$\tilde{u}(0) = 1$$
$$\tilde{H}(0) = h_0$$
$$\nu^* \tilde{B}\tilde{H}\tilde{u}_{\tilde{x}}^{\frac{1}{n}} = \tilde{H}^2/2, \quad \tilde{x} = 1$$

Known parameters: $$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$

NN output: $\tilde{u}$, $\tilde{H}$



Pause and Ponder

Problem statement

Eqns:
$$\begin{cases} \nu^* \dfrac{\partial}{\partial \tilde{x}} \left(\tilde{B}\tilde{H} \left| \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right|^{\frac{1}{n}-1} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right) = \tilde{H} \dfrac{\partial \tilde{H}}{\partial \tilde{x}} \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 \end{cases} \quad , \tilde{x} \in [0,1]$$

BCs:
$$\tilde{u}(0) = 1$$
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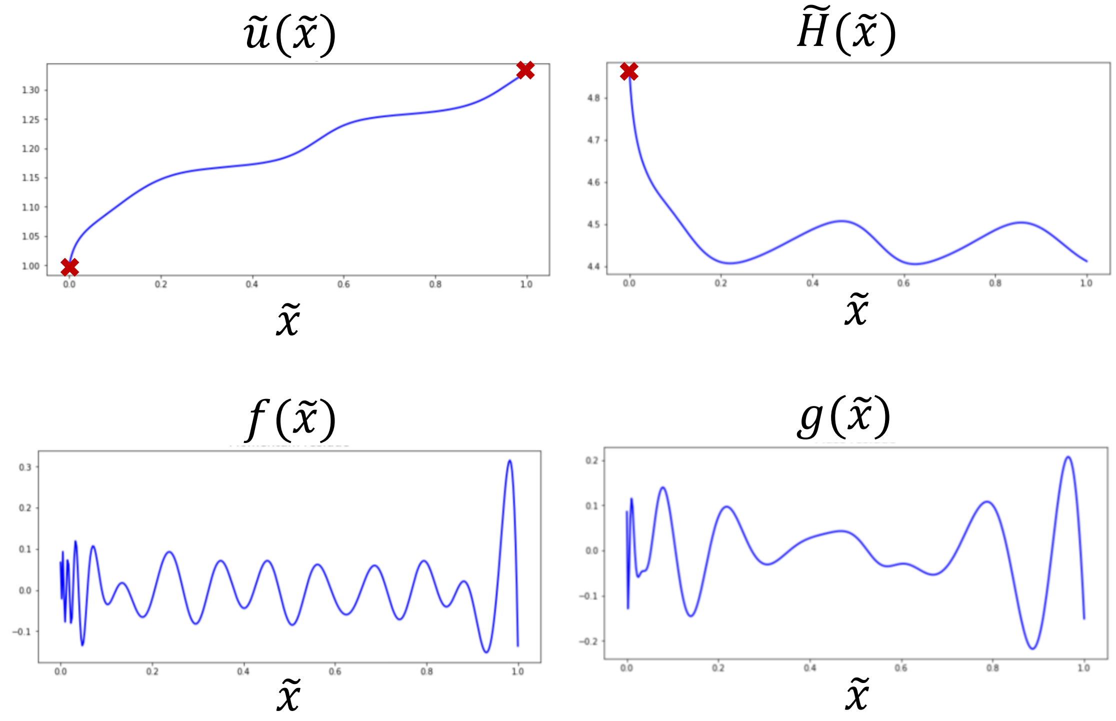
$$\nu^* \tilde{B}\tilde{H}\tilde{u}_{\tilde{x}}^{\frac{1}{n}} = \tilde{H}^2/2, \quad \tilde{x} = 1$$

Known parameters:
$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$
NN output: $\tilde{u}$, $\tilde{H}$



- It looks like the NN is having troubles making precise predictions. What could be the problems?

Pause and Ponder

Problem statement

Eqns:

$$\begin{cases} \nu^* \dfrac{\partial}{\partial \tilde{x}} \left(\tilde{B}\tilde{H} \left| \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right|^{\frac{1}{n}-1} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right) = \tilde{H} \dfrac{\partial \tilde{H}}{\partial \tilde{x}} & , \tilde{x} \in [0,1] \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 & \end{cases}$$

BCs: $\tilde{u}(0) = 1$

$\tilde{H}(0) = h_0$

$\nu^* \tilde{B}\tilde{H}\tilde{u}_{\tilde{x}}^{\frac{1}{n}} = \tilde{H}^2/2, \quad \tilde{x} = 1$

Known parameters:

$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$

NN output: $\tilde{u}$, $\tilde{H}$

$\tilde{u}_{\tilde{x}} > 0, \; n > 0$

Eqn contains derivative of $(\tilde{u}_{\tilde{x}})^{1/n}$

$\tilde{u}_{\tilde{x}} \sim \tilde{u}(0) + c\tilde{x}$ near $\tilde{x} = 0$

$\dfrac{d(\tilde{x}^{1/3})}{d\tilde{x}} = \dfrac{1}{3}\tilde{x}^{-\frac{2}{3}}$ blows up at $\tilde{x} = 0$

- **Derivative of a term with fractional power 1/n** is potentially causing the problem!
- Can you think of a way to resolve this?

E.g., 1D ice shelf- forward (2nd order)

Problem statement

Eqns:
$$\begin{cases} \nu^* \dfrac{\partial}{\partial \tilde{x}} \left(\tilde{B}\tilde{H} \left| \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right|^{\frac{1}{n}-1} \dfrac{\partial \tilde{u}}{\partial \tilde{x}} \right) = \tilde{H} \dfrac{\partial \tilde{H}}{\partial \tilde{x}} \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 \end{cases} \quad , \tilde{x} \in [0,1]$$

BCs:
$$\tilde{u}(0) = 1$$
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$$\nu^* \tilde{B}\tilde{H}\tilde{u}_{\tilde{x}}^{\frac{1}{n}} = \tilde{H}^2/2, \quad \tilde{x} = 1$$

Known parameters:
$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$
NN output: $\tilde{u}$, $\tilde{H}$

Training data (from ground truth):

$$\{\tilde{x}_d^i, \tilde{u}_d^i, \tilde{H}_d^i\}_{i=1}^m, \{\tilde{x}_d^i, \tilde{u}_{\tilde{x}d}^{\ i}\}_{i=1}^m, \qquad m = 1$$

Collocation points:

$$\{\tilde{x}_f^i\}_{i=1}^N, \qquad N = 201$$

Physics equations:

$$f \equiv \nu^* \left(\tilde{B}\tilde{H}\tilde{u}_{\tilde{x}} |\tilde{u}_{\tilde{x}}|^{\frac{1}{n}-1} \right)_{\tilde{x}} - \tilde{H}\tilde{H}_{\tilde{x}}$$
$$g \equiv (\tilde{u}\tilde{H})_{\tilde{x}} - A_0$$

Loss function:

Data loss

$$MSE = (1-\gamma)\frac{1}{m}\sum_{i=1}^m (|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \tilde{H}_d^i|^2 + |\tilde{u}_{\tilde{x}}(\tilde{x}_d^i) - \tilde{u}_{\tilde{x}d}^{\ i}|^2)$$

$$+\gamma \frac{1}{N}\sum_{i=1}^N (|f(\tilde{x}_f^i)|^2 + |g(\tilde{x}_f^i)|^2)$$ Equation loss

E.g., 1D ice shelf- forward (2nd order)

Problem statement

Eqns:
$$\begin{cases} \tilde{v}^n \equiv \dfrac{d\tilde{u}}{d\tilde{x}} \\ \nu^* \dfrac{d}{d\tilde{x}}\left(\tilde{B}\tilde{H}\tilde{v}\right) = \tilde{H}\dfrac{d\tilde{H}}{d\tilde{x}} & , \tilde{x} \in [0,1] \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 \end{cases}$$

Change of variable

BCs:
$$\tilde{u}(0) = 1$$
$$\tilde{H}(0) = h_0$$
$$\tilde{v}(1) \equiv \frac{\tilde{H}(1)}{2\nu^* \tilde{B}(1)}$$

Known parameters:
$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$

NN output: $\tilde{u}, \tilde{H}, \tilde{v}$

Training data (from ground truth):

$$\{\tilde{x}_d^i, \tilde{u}_d^i, \tilde{H}_d^i\}_{i=1}^m, \{\tilde{x}_d^i, \tilde{v}_d^i\}_{i=1}^m, \qquad m = 1$$

Collocation points:

$$\{\tilde{x}_f^i\}_{i=1}^N, \qquad N = 201$$

Physics equations:

$$e \equiv \tilde{v}^n - \tilde{u}_{\tilde{x}}$$
$$f \equiv \nu^*\left(\tilde{B}\tilde{H}\tilde{v}\right)_{\tilde{x}} - \tilde{H}\tilde{H}_{\tilde{x}}$$
$$g \equiv \left(\tilde{u}\tilde{H}\right)_{\tilde{x}} - A_0$$

Loss function:

Data loss

$$MSE = (1-\gamma)\frac{1}{m}\sum_{i=1}^m (|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \tilde{H}_d^i|^2 + |\tilde{v}(\tilde{x}_d^i) - \tilde{v}_d^i|^2)$$

$$+\gamma \frac{1}{N}\sum_{i=1}^N (|f(\tilde{x}_f^i)|^2 + |g(\tilde{x}_f^i)|^2$$

Equation loss

E.g., 1D ice shelf- <u>forward</u> (2^{nd} order)

Problem statement

Eqns:
$$\begin{cases} \tilde{v}^n \equiv \dfrac{d\tilde{u}}{d\tilde{x}} \\ \nu^* \dfrac{d}{d\tilde{x}}\left(\tilde{B}\tilde{H}\tilde{v}\right) = \tilde{H}\dfrac{d\tilde{H}}{d\tilde{x}} \qquad , \tilde{x} \in [0,1] \\ \dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0 \end{cases}$$

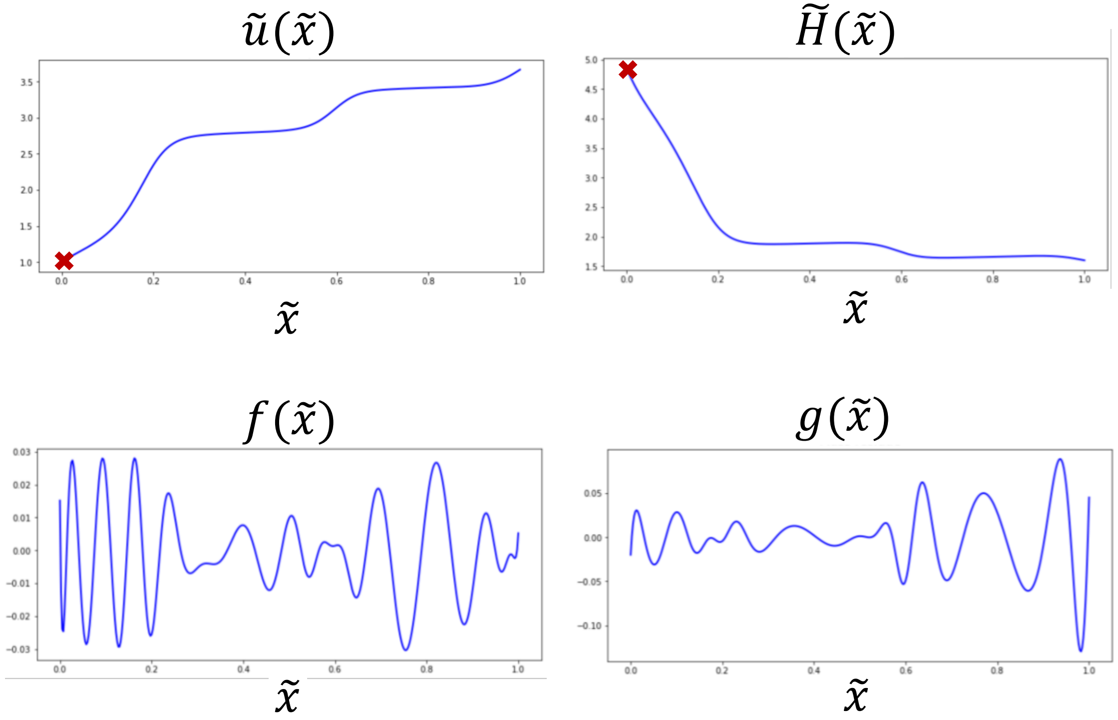
Change of variable

BCs:
$$\tilde{u}(0) = 1$$
$$\tilde{H}(0) = h_0$$
$$\tilde{v}(1) \equiv \frac{\tilde{H}(1)}{2\nu^* \tilde{B}(1)}$$

Known parameters:
$$\begin{cases} \tilde{B} = \cos(5\pi\tilde{x}) + 2 \\ A_0 = 1 \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$

NN output: $\tilde{u}$, $\tilde{H}$, $\tilde{v}$



Good prediction!

$$O(f), O(g) \sim 10^{-2} \ll O(\tilde{u}), O(\tilde{H}) \sim 1$$

E.g., 1D ice shelf- inverse (2^{nd} order)

Problem statement

Eqns:

$$\begin{cases} \tilde{v}^n \equiv \dfrac{d\tilde{u}}{d\tilde{x}} \\ \nu^* \dfrac{d}{d\tilde{x}}\left(\tilde{B}\tilde{H}\tilde{v}\right) = \tilde{H}\dfrac{d\tilde{H}}{d\tilde{x}} \qquad , \tilde{x} \in [0,1] \\ \cancel{\dfrac{\mathrm{d}(\tilde{u}\tilde{H})}{\mathrm{d}\tilde{x}} = A_0} \end{cases}$$

BCs:
$$\cancel{\tilde{u}(0) = 1}$$
$$\cancel{\tilde{H}(0) = h_0}$$
$$\cancel{\tilde{v}(1) = \frac{\tilde{H}(1)}{2\nu^*\tilde{B}(1)}}$$

Known parameters:
$$\begin{cases} \cancel{\tilde{B} = \cos(5\pi\tilde{x}) + 2} \\ \cancel{A_0 = 1} \\ \nu^* = 1/2 \end{cases}$$

NN input: $\tilde{x}$

NN output: $\tilde{u}, \tilde{v}, \tilde{H}, \tilde{B}$

Training data (from ground truth):

$$\{\tilde{x}_d^i, \tilde{u}_d^i, \tilde{H}_d^i, \tilde{v}_d^i\}_{i=1}^{m}, \qquad m = 401$$

Collocation points:

$$\{\tilde{x}_f^i\}_{i=1}^{N}, \qquad N = 201$$

Physics equations:

$$e \equiv \tilde{v}^n - \tilde{u}_{\tilde{x}}$$
$$f \equiv \nu^*\left(\tilde{B}\tilde{H}\tilde{v}\right)_{\tilde{x}} - \tilde{H}\tilde{H}_{\tilde{x}}$$
$$g \equiv \left(\tilde{u}\tilde{H}\right)_{\tilde{x}} - A_0$$

Loss function:

Data loss

$$MSE = (1-\gamma)\frac{1}{m}\sum_{i=1}^{m}\left(|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \tilde{H}_d^i|^2 + |\tilde{v}(\tilde{x}_d^i) - \tilde{v}_d^i|^2\right)$$

$$+\gamma\frac{1}{N}\sum_{i=1}^{N}\left(|f(\tilde{x}_f^i)|^2 + |e(\tilde{x}_f^i)|^2\right)$$

Equation loss

Pause and Ponder

Given training data of $\tilde{u}(\tilde{x}), \tilde{H}(\tilde{x})$, find $\tilde{B}(\tilde{x})$ without $\tilde{B}$ training data?

Problem statement

Eqns:
$$\begin{cases} \tilde{v}^n \equiv \frac{d\tilde{u}}{d\tilde{x}} \\ \nu^* \frac{d}{d\tilde{x}}\left(\tilde{B}\tilde{H}\tilde{v}\right) = \tilde{H}\frac{d\tilde{H}}{d\tilde{x}} \end{cases}, \tilde{x} \in [0, 1]$$

Known parameters: $\nu^* = 1/2$

- Can $\tilde{B}(\tilde{x})$ be **uniquely** determined?

NN input: $\tilde{x}$
NN output: $\tilde{u}, \tilde{v}, \tilde{H}, \tilde{B}$

Training data (from ground truth):

$$\left\{\tilde{x}_d^i, \tilde{u}_d^i, \tilde{H}_d^i, \tilde{v}_d^i\right\}_{i=1}^m, \qquad m = 401$$

Collocation points:

$$\left\{\tilde{x}_f^i\right\}_{i=1}^N, \qquad N = 201$$

Physics equations:

$$e \equiv \tilde{v}^n - \tilde{u}_{\tilde{x}}$$
$$f \equiv \nu^*\left(\tilde{B}\tilde{H}\tilde{v}\right)_{\tilde{x}} - \tilde{H}\tilde{H}_{\tilde{x}}$$
$$g \equiv \left(\tilde{u}\tilde{H}\right)_{\tilde{x}} - A_0$$

Loss function:

Data loss

$$MSE = (1-\gamma)\frac{1}{m}\sum_{i=1}^m\left(|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \tilde{H}_d^i|^2 + |\tilde{v}(\tilde{x}_d^i) - \tilde{v}_d^i|^2\right)$$

$$+\gamma\frac{1}{N}\sum_{i=1}^N\left(|f(\tilde{x}_f^i)|^2 + |e(\tilde{x}_f^i)|^2\right)$$ Equation loss

E.g., 1D ice shelf- inverse (2nd order)

Problem statement

Eqns: $\begin{cases} \tilde{v}^n \equiv \dfrac{d\tilde{u}}{d\tilde{x}} \\ \nu^* \dfrac{d}{d\tilde{x}}\left(\tilde{B}\tilde{H}\tilde{v}\right) = \tilde{H}\dfrac{d\tilde{H}}{d\tilde{x}} \end{cases}$ $, \tilde{x} \in [0, 1]$

Known parameters: $\nu^* = 1/2$

BCs: $\tilde{B}(0) = 3$
or $\tilde{B}(1) = \cos(5\pi) + 2$

- We need **1 BC for B** to **uniquely** determine $\tilde{B}(\tilde{x})$!

NN input: $\tilde{x}$
NN output: $\tilde{u}, \tilde{v}, \tilde{H}, \tilde{B}$

Training data (from ground truth):

$\{\tilde{x}_d^i, \tilde{u}_d^i, \tilde{H}_d^i, \tilde{v}_d^i\}_{i=1}^m$, $m = 401$

$\{\tilde{x}_d^i, \tilde{B}_d^i\}_{i=1}^m$, $m = 1$

Collocation points:

$\{\tilde{x}_f^i\}_{i=1}^N$, $N = 201$

Physics equations:

$e \equiv \tilde{v}^n - \tilde{u}_{\tilde{x}}$
$f \equiv \nu^*\left(\tilde{B}\tilde{H}\tilde{v}\right)_{\tilde{x}} - \tilde{H}\tilde{H}_{\tilde{x}}$
$g \equiv \left(\tilde{u}\tilde{H}\right)_{\tilde{x}} - A_0$

Loss function:

Data loss

$$MSE = (1-\gamma)\frac{1}{m}\sum_{i=1}^m\left(|\tilde{u}(\tilde{x}_d^i) - \tilde{u}_d^i|^2 + |H(\tilde{x}_d^i) - \tilde{H}_d^i|^2 + |\tilde{v}(\tilde{x}_d^i) - \tilde{v}_d^i|^2\right) + |\tilde{B}(\tilde{x}_d^i) - \tilde{B}_d^i|^2$$

$$+\gamma\frac{1}{N}\sum_{i=1}^N\left(|f(\tilde{x}_f^i)|^2 + |e(\tilde{x}_f^i)|^2\right)$$ Equation loss

Pause and Ponder

Problem statement

Eqns: $\begin{cases} \tilde{v}^n \equiv \dfrac{d\tilde{u}}{d\tilde{x}} \\ \nu^* \dfrac{d}{d\tilde{x}}\left(\tilde{B}\tilde{H}\tilde{v}\right) = \tilde{H}\dfrac{d\tilde{H}}{d\tilde{x}} \end{cases}$ $,\ \tilde{x} \in [0, 1]$

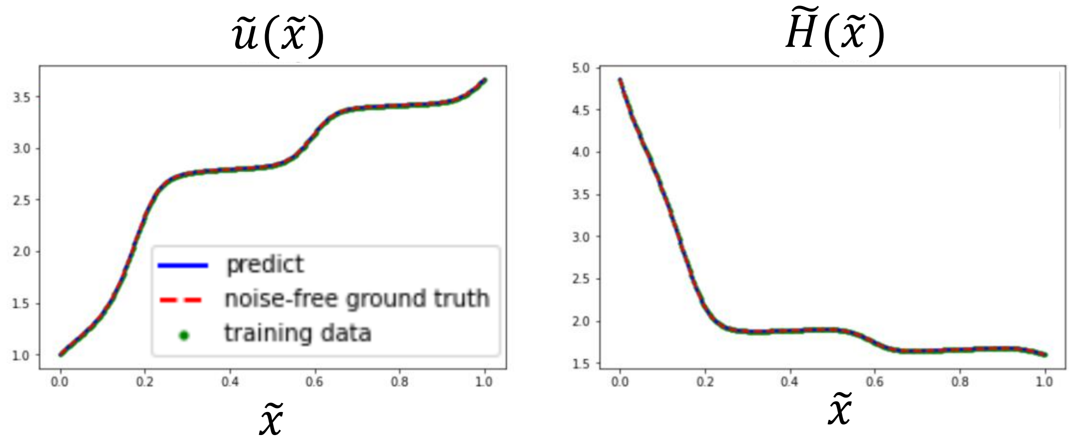
Known parameters: $\nu^* = 1/2$

BCs: $\tilde{B}(0) = 3$
or $\tilde{B}(1) = \cos(5\pi) + 2$

- Should we use $\tilde{B}(0)$ or $\tilde{B}(1)$ as BC?

NN input: $\tilde{x}$
NN output: $\tilde{u}$, $\tilde{v}$, $\tilde{H}$, $\tilde{B}$

Training data (from ground truth):



Inverted $\tilde{B}$:

(1) Exp 1

(2) Exp 2