$$U_{12} = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2) \; ; \; \gamma_1 = c_1^\dagger + c_1 \; ; \; \gamma_2 = i(c_1^\dagger - c_1)$$

$$\gamma_1\gamma_2|00\rangle = \gamma_1(\gamma_2|00\rangle) = \gamma_1(i(c_1^\dagger|00\rangle - c_1|00\rangle))$$
$$= \gamma_1(i(|10\rangle - 0)) = i(c_1^\dagger|10\rangle + c_1|10\rangle)$$
$$= i(0 + |00\rangle) = i|00\rangle$$

$$U_{12}|00\rangle = \frac{1}{\sqrt{2}}(1+i)|00\rangle$$

$$\gamma_1\gamma_2|01\rangle = \gamma_1(i(c_1^\dagger|01\rangle - c_1|01\rangle)) = \gamma_1(i(|11\rangle - 0))$$
$$= i(c_1^\dagger|11\rangle + c_1|11\rangle) = i(0 + |01\rangle) = i|01\rangle$$

$$U_{12}|01\rangle = \frac{1}{\sqrt{2}}(1+i)|01\rangle$$

$$\gamma_1\gamma_2|10\rangle = \gamma_1(i(c_1^\dagger|10\rangle - c_1|10\rangle)) = \gamma_1(i(0 - |00\rangle))$$
$$= -i(c_1^\dagger|00\rangle + c_1|00\rangle) = -i|10\rangle$$

$$U_{12}|10\rangle = \frac{1}{\sqrt{2}}(1-i)|10\rangle$$

$$\gamma_1\gamma_2|11\rangle = \gamma_1(i(c_1^\dagger|11\rangle - c_1|11\rangle)) = \gamma_1(i(0 - |01\rangle))$$
$$= -i(c_1^\dagger|01\rangle - c_1|01\rangle) = -i|11\rangle$$

$$U_{12}|11\rangle = \frac{1}{\sqrt{2}}(1-i)|11\rangle$$

$$\Rightarrow U_{12} = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} e^{i\frac{\pi}{4}} & & & \\ & e^{i\frac{\pi}{4}} & & \\ & & e^{-i\frac{\pi}{4}} & \\ & & & e^{-i\frac{\pi}{4}} \end{bmatrix} \end{array}$$

$$U_{23} = \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_3) \; ; \quad \gamma_2 = i(c_1^+ - c_1) \; ; \quad \gamma_3 = c_2^+ + c_2$$

$$\gamma_2\gamma_3|00\rangle = \gamma_2(c_2^+|00\rangle + c_2|00\rangle) = \gamma_2(|01\rangle + 0)$$

$$= i(c_1^+|01\rangle - c_1|01\rangle) = i(|11\rangle - 0) = i|11\rangle$$

$$U_{23}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$$

$$\gamma_2\gamma_3|01\rangle = \gamma_2(c_2^+|01\rangle + c_2|01\rangle) = \gamma_2(0 + |00\rangle)$$

$$= i(c_1^+|00\rangle - c_1|00\rangle) = i|10\rangle$$

$$U_{23}|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle)$$

$$\gamma_2\gamma_3|10\rangle = \gamma_2(c_2^+|10\rangle + c_2|10\rangle) = \gamma_2(|11\rangle + 0)$$

$$= i(c_1^+|11\rangle - c_1|11\rangle) = -i|01\rangle$$

$$U_{23}|10\rangle = \frac{1}{\sqrt{2}}(|10\rangle - i|01\rangle)$$

$$\gamma_2\gamma_3|11\rangle = \gamma_2(c_2^+|11\rangle + c_2|11\rangle) = \gamma_2(0 + |10\rangle)$$

$$= i(c_1^+|10\rangle - c_1|10\rangle) = -i|00\rangle$$

$$U_{23}|11\rangle = \frac{1}{\sqrt{2}}(|11\rangle - i|00\rangle)$$

$$\Rightarrow \quad U_{23} = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$U_{34} = \frac{1}{\sqrt{2}}(1+\gamma_3\gamma_4); \quad \gamma_3 = c_2^\dagger + c_2; \quad \gamma_4 = i(c_2^\dagger - c_2)$$

$$\gamma_3\gamma_4|00\rangle = \gamma_3\left(i\left(c_2^\dagger|00\rangle - c_2|00\rangle\right)\right) = \gamma_3\left(i\left(|01\rangle - 0\right)\right)$$

$$= i\left(c_2^\dagger|01\rangle + c_2|01\rangle\right) = i|00\rangle$$

$$U_{34}|00\rangle = \frac{1}{\sqrt{2}}(1+i)|00\rangle$$

$$\gamma_3\gamma_4|01\rangle = \gamma_3\left(i\left(c_2^\dagger|01\rangle - c_2|01\rangle\right)\right) = \gamma_3\left(i\left(0 - |00\rangle\right)\right)$$

$$= -i\left(c_2^\dagger|00\rangle + c_2|00\rangle\right) = -i|01\rangle$$

$$U_{34}|01\rangle = \frac{1}{\sqrt{2}}(1-i)|01\rangle$$

$$\gamma_3\gamma_4|10\rangle = \gamma_3\left(i\left(c_2^\dagger|10\rangle - c_2|10\rangle\right)\right) = \gamma_3\left(i\left(|11\rangle - 0\right)\right)$$

$$= i\left(c_2^\dagger|11\rangle + c_2|11\rangle\right) = i|10\rangle$$

$$U_{23}|10\rangle = \frac{1}{\sqrt{2}}(1+i)|10\rangle$$

$$\gamma_3\gamma_4|11\rangle = \gamma_3\left(i\left(c_2^\dagger|11\rangle - c_2|11\rangle\right)\right) = \gamma_3\left(i\left(0 - |10\rangle\right)\right)$$

$$= -i\left(c_1^\dagger|10\rangle + c_1|10\rangle\right) = -i|11\rangle$$

$$U_{23}|11\rangle = \frac{1}{\sqrt{2}}(1-i)|11\rangle$$

$$\Rightarrow U_{34} = \begin{bmatrix} e^{i\frac{\pi}{4}} & & & \\ & e^{-i\frac{\pi}{4}} & & \\ & & e^{i\frac{\pi}{4}} & \\ & & & e^{-i\frac{\pi}{4}} \end{bmatrix}$$