

$$U_{12} = \frac{1}{\sqrt{2}}(1 + \sigma_1 \sigma_2) ; \sigma_1 = C_1^\dagger + C_1 ; \sigma_2 = i(C_1^\dagger - C_1)$$

$$\begin{aligned} \sigma_1 \sigma_2 |00\rangle &= \sigma_1 (\sigma_2 |00\rangle) = \sigma_1 (i(C_1^\dagger |00\rangle - C_1 |00\rangle)) \\ &= \sigma_1 (i(|10\rangle - 0)) = i(C_1^\dagger |10\rangle + C_1 |10\rangle) \\ &= i(0 + |00\rangle) = i|00\rangle \end{aligned}$$

$$U_{12} |00\rangle = \frac{1}{\sqrt{2}}(1 + i)|00\rangle$$

$$\begin{aligned} \sigma_1 \sigma_2 |01\rangle &= \sigma_1 (i(C_1^\dagger |01\rangle - C_1 |01\rangle)) = \sigma_1 (i(|11\rangle - 0)) \\ &= i(C_1^\dagger |11\rangle + C_1 |11\rangle) = i(0 + |01\rangle) = i|01\rangle \end{aligned}$$

$$U_{12} |01\rangle = \frac{1}{\sqrt{2}}(1 + i)|01\rangle$$

$$\begin{aligned} \sigma_1 \sigma_2 |10\rangle &= \sigma_1 (i(C_1^\dagger |10\rangle - C_1 |10\rangle)) = \sigma_1 (i(0 - |00\rangle)) \\ &= -i(C_1^\dagger |00\rangle + C_1 |00\rangle) = -i|10\rangle \end{aligned}$$

$$U_{12} |10\rangle = \frac{1}{\sqrt{2}}(1 - i)|10\rangle$$

$$\begin{aligned} \sigma_1 \sigma_2 |11\rangle &= \sigma_1 (i(C_1^\dagger |11\rangle - C_1 |11\rangle)) = \sigma_1 (i(0 - |01\rangle)) \\ &= -i(C_1^\dagger |01\rangle - C_1 |01\rangle) = -i|11\rangle \end{aligned}$$

$$U_{12} |11\rangle = \frac{1}{\sqrt{2}}(1 - i)|11\rangle$$

$$\Rightarrow U_{12} = \begin{bmatrix} e^{-i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$$



$$U_{23} = \frac{1}{\sqrt{2}} (1 + i\gamma_2 \gamma_3); \quad \gamma_2 = i(C_1^\dagger - C_1); \quad \gamma_3 = C_2^\dagger + C_2$$

$$\begin{aligned} \gamma_2 \gamma_3 |00\rangle &= \gamma_2 (C_2^\dagger |00\rangle + C_2 |00\rangle) = \gamma_2 (|01\rangle + 0) \\ &= i(C_1^\dagger |01\rangle - C_1 |01\rangle) = i(|11\rangle - 0) = i|11\rangle \end{aligned}$$

$$U_{23}|00\rangle = \frac{1}{\sqrt{2}} (|00\rangle + i|11\rangle)$$

$$\begin{aligned} \gamma_2 \gamma_3 |01\rangle &= \gamma_2 (C_2^\dagger |01\rangle + C_2 |01\rangle) = \gamma_2 (0 + |00\rangle) \\ &= i(C_1^\dagger |00\rangle - C_1 |00\rangle) = i|10\rangle \end{aligned}$$

$$U_{23}|01\rangle = \frac{1}{\sqrt{2}} (|01\rangle + i|10\rangle)$$

$$\begin{aligned} \gamma_2 \gamma_3 |10\rangle &= \gamma_2 (C_2^\dagger |10\rangle + C_2 |10\rangle) = \gamma_2 (|11\rangle + 0) \\ &= i(C_1^\dagger |11\rangle - C_1 |11\rangle) = -i|01\rangle \end{aligned}$$

$$U_{23}|10\rangle = \frac{1}{\sqrt{2}} (|10\rangle - i|01\rangle)$$

$$\begin{aligned} \gamma_2 \gamma_3 |11\rangle &= \gamma_2 (C_2^\dagger |11\rangle + C_2 |11\rangle) = \gamma_2 (0 + |10\rangle) \\ &= i(C_1^\dagger |10\rangle - C_1 |10\rangle) = -i|00\rangle \end{aligned}$$

$$U_{23}|11\rangle = \frac{1}{\sqrt{2}} (|11\rangle - i|00\rangle)$$

$$\Rightarrow U_{23} = \begin{array}{c|c|c|c} & 00 & 01 & 10 & 11 \\ \hline \begin{bmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \end{array}$$



$$U_{34} = \frac{1}{\sqrt{2}}(1 + i\sigma_3\sigma_4); \quad \sigma_3 = c_2^\dagger + c_2; \quad \sigma_4 = i(c_2^\dagger - c_2)$$

$$\begin{aligned} \sigma_3\sigma_4|00\rangle &= \sigma_3(i(c_2^\dagger|00\rangle - c_2|00\rangle)) = \sigma_3(i(|01\rangle - 0)) \\ &= i(c_2^\dagger|01\rangle + c_2|01\rangle) = i|00\rangle \end{aligned}$$

$$U_{34}|00\rangle = \frac{1}{\sqrt{2}}(1+i)|00\rangle$$

$$\begin{aligned} \sigma_3\sigma_4|01\rangle &= \sigma_3(i(c_2^\dagger|01\rangle - c_2|01\rangle)) = \sigma_3(i(0 - |00\rangle)) \\ &= -i(c_2^\dagger|00\rangle + c_2|00\rangle) = -i|01\rangle \end{aligned}$$

$$U_{34}|01\rangle = \frac{1}{\sqrt{2}}(1-i)|01\rangle$$

$$\begin{aligned} \sigma_3\sigma_4|10\rangle &= \sigma_3(i(c_2^\dagger|10\rangle - c_2|10\rangle)) = \sigma_3(i(|11\rangle - 0)) \\ &= i(c_2^\dagger|11\rangle + c_2|11\rangle) = i|10\rangle \end{aligned}$$

$$U_{34}|10\rangle = \frac{1}{\sqrt{2}}(1+i)|10\rangle$$

$$\begin{aligned} \sigma_3\sigma_4|11\rangle &= \sigma_3(i(c_2^\dagger|11\rangle - c_2|11\rangle)) = \sigma_3(i(0 - |10\rangle)) \\ &= -i(c_2^\dagger|10\rangle + c_2|10\rangle) = -i|11\rangle \end{aligned}$$

$$U_{34}|11\rangle = \frac{1}{\sqrt{2}}(1-i)|11\rangle$$

$$\Rightarrow U_{34} = \begin{bmatrix} e^{-i\frac{\pi}{4}} & & & \\ & e^{i\frac{\pi}{4}} & & \\ & & e^{-i\frac{\pi}{4}} & \\ & & & e^{i\frac{\pi}{4}} \end{bmatrix}$$