ZQCD$_2$

1 Lattice setup

1.1 Action

Using standard Wilson discretization, the lattice action corresponding to our effective theory reads

$$S = S_{W_3} + S_{\mathcal{Z}} + V(\Sigma, \Pi) \tag{1.1}$$

where

$$S_{W_3} = \beta \sum_{n,i<j} 1 - \frac{1}{2} \operatorname{tr} U_{ij} \tag{1.2}$$

$$S_Z = 2\left(\frac{4}{\beta}\right) \sum_{n,i} \left[\Sigma^2(n) - \Sigma(n)\Sigma(n+\hat{i})\right] \tag{1.3}$$

$$+ \left(\frac{4}{\beta}\right) \sum_{n,i} \operatorname{tr}\left[\Pi(n)^2 - \Pi(n)U_i(n)\Pi(n+\hat{i})U_i^\dagger(n)\right] \tag{1.4}$$

$$V(\Sigma, \Pi) = \left(\frac{4}{\beta}\right) \sum_n \left[\hat{b}_1\Sigma^2 + \hat{b}_2\Pi_a^2 + \hat{c}_1\Sigma^4 + \hat{c}_2(\Pi_a^2)^2 + \hat{c}_3\Sigma^2\Pi_a^2\right] \tag{1.5}$$

where β is given by

$$\beta = \frac{4}{ag_3^2} \tag{1.6}$$

The fields $\Sigma(n)$ and Π_n^a are components (scalar fields) of a $SU(2)$ adjoint fields $\mathcal{Z}$

$$\mathcal{Z}(n) = \Sigma(n)\mathbb{1} + i\Pi_n^a \cdot \sigma_a = \begin{pmatrix} \Sigma + i\Pi_3 & \Pi_2 + i\Pi_1 \\ -\Pi_2 + i\Pi_1 & \Sigma - i\Pi_3 \end{pmatrix} \tag{1.7}$$

!Disclaimer! This is a 3D theory, a toy model. Up to this moment, it has *nothing* to do with QCD, Yang-Mills or my grandma. Namely, β, $\hat{c}_i$ and $\hat{b}_i$, they have no counterpart in the real world: treat them as parameters in your toy model.

Also, this theory has its own **continuum limit**, i. e. I have to take $\beta \to \infty$ while keeping fixed some observables. Fixed to which value? It's a toy model, we have no physical reference, we just choose arbitrarily a value. *How many observables do I need to take the continuum limit? (1 for each coupling?)*

Where is the catch? The theory with completely generic coefficients is a toy model, but dimensional reduction tells us that *in a specific region of the couplings space* this toy model has the same physics as the real world.

1.1.1 Coefficients

In the continuum, this toy model has arbitrary b_i and c_i coefficients, but we are interested in a subregion of parameter space corresponding to those values coming from the matching of this tool theory to Yang-Mills$_2$ at high T. This region is given in terms only of the following 3 parameters: g_3 (the coupling of our 3d gauge fields), g (the coupling of the 4d theory) and r (the ratio between the heavy mass of the field that is integrated out in the matching and the temperature).

$$b_1 = -\frac{1}{4}r^2T^2 \tag{1.8}$$

$$b_2 = -\frac{1}{4}r^2T^2 + 0.441841g^2T^2 \tag{1.9}$$

$$c_1 = 0.0311994r^2 + 0.0135415g^2 \tag{1.10}$$

$$c_2 = 0.0311994r^2 + 0.008443432g^2 \tag{1.11}$$

$$c_3 = 0.0623987r^2 \tag{1.12}$$

and

$$g_3^2 = g^2T \tag{1.13}$$

The coefficients on the lattice ($\hat{b}_i$, $\hat{c}_i$) are related to the continuum ones (b_i, c_i) using lattice perturbation theory and are given by

$$\hat{c}_i = c_i \tag{1.14}$$

$$\hat{b}_1 = \frac{b_1}{g_3^4} - \frac{2.38193365}{4\pi}(2\hat{c}_1 + \hat{c}_3)\beta + \\ + \frac{1}{16\pi^2}\left[(48\hat{c}_1^2 + 12\hat{c}_3^2 - 12\hat{c}_3)(\log 1.5\beta + 0.08849) - 6.9537\hat{c}_3\right] + \mathcal{O}\left(\beta^{-1}\right) \tag{1.15}$$

$$\hat{b}_2 = \frac{b_2}{g_3^4} - \frac{0.7939779}{4\pi}(10\hat{c}_2 + \hat{c}_3 + 2)\beta + \\ + \frac{1}{16\pi^2}\left[(80\hat{c}_2^2 + 4\hat{c}_3^2 - 40\hat{c}_2)(\log 1.5\beta + 0.08849) - 23.17895\hat{c}_2 - 8.66687\right] + \mathcal{O}\left(\beta^{-1}\right) \tag{1.16}$$

Note Note that although the continuum coefficients b_1 and b_2 are dimensionful (they contain the temperature T), their lattice version $\hat{b}_i$ are dimensionless, as expected.

1.2 HMC Forces

In the following, we use the notation $\Sigma_n \equiv \Sigma(n)$ and $\Pi_n^a \equiv \Pi_a(n)$. In order to take derivatives of functions of matrices $M \in SU(2)$, we use the following rule[1]

$$\frac{\mathrm{d}f[M]}{\mathrm{d}M} = \frac{\mathrm{d}}{\mathrm{d}\omega_a} f\left[M \to e^{i\omega_a t_a} M\right]\Big|_{\omega=0} \tag{1.17}$$

The generators of $SU(2)$ are $t_a \equiv i\sigma_a$.

$$F_U(n)_i^a = F_{\text{wilson}}(n)_i^a - \frac{4i}{\beta} \operatorname{tr}\left[\Pi_n(t_a U_i(n) - U_i(n)t_a)\Pi_{n+\hat{i}} U_i^\dagger(n)\right] \tag{1.18}$$

$$F_\Sigma(n) = \frac{8}{\beta}\left[\left(6 + \hat{b}_1 + \hat{c}_3(\Pi_n^a)^2\right)\Sigma_n + 2\hat{c}_1\Sigma_n^3 - \sum_i \left(\Sigma_{n+\hat{i}} + \Sigma_{n-\hat{i}}\right)\right] \tag{1.19}$$

$$\begin{aligned} F_\Pi(n)^a = &\frac{4}{\beta}\left[2\hat{b}_2\Pi_n^a + 2\hat{c}_3\Sigma_n(\Pi_n^a)^2 + 4\hat{c}_3\left((\Pi_n^a)^3 + \Pi_n^a\sum_{b\neq a}(\Pi_n^b)^2\right)\right] + \\ &+ \frac{4}{\beta}\operatorname{tr}\left[3i(t_a\Pi_n + \Pi_n t_a)\Pi_n - it_a\Pi_n W(n)\right] \end{aligned} \tag{1.20}$$

In the last equation,

$$W(n) = \sum_i \left[U_i(n)\Pi_{n+\hat{i}}U_i^\dagger(n) + U_i^\dagger(n-\hat{i})\Pi_{n-\hat{i}}U_i(n-\hat{i})\right] \tag{1.21}$$

[1]In the case of $f = \text{tr}$, expand $e^{i\omega}$, take out the ω_a from the trace, derive, set to zero every ω left.