ECOG 315 / ECON 181, Summer 2025
Advanced Research Methods and Statistical Programming
Week 6 Lecture Slides

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July 2, 2025

Introduction to Dynamic Models

- Want to model decisions and outcomes of **agents** over (discrete) time
- Formal name: microeconomic dynamic stochastic optimization problems
- Outcomes probably subject to risk: random shocks from some distribution
- Standard assumptions: agents know their model and observe their current state, choose optimal action according to their preferences
- Almost always assume that agents have **time consistent preferences**: expectations of preferences in future align with actual preferences in future
- Upshot: want to maximize discounted sum of utility flows, geometric discounting

Ingredients of a MicroDSOP

What do we need to put into a MicroDSOP?

- Who are the agents? Who is this model about?
- How fast does time move? How long is a "period"?
- What do they want? What are their preferences?
- What do the agents observe about their situation when they act?
- What actions can agents take? What constraints do they face?
- What sources of risk/uncertainty to the agents face (shocks)?
- How do actions and shocks generate next period's state?

Jumping into the Deep End: Consumption-Saving Models

Let's sketch out a (very) basic consumption-saving model:

- Who is this model about? Person who earns income, consumes, and saves
- How long is a "period"? Usually a year or a quarter, depends
- What do they want? They like to consume; CRRA preferences (ρ)
- What do agents observe? They know their market resources M_t or cash-on-hand
- What actions can agents take? Divide M_t between consumption C_t and assets A_t
- What constraints do they face? Can't borrow assets: $A_t \geq 0$.
- What are the risks? Labor income Y_t is drawn iid from distribution F
- How does the situation change? Assets earn interest at risk-free factor $\mathsf{R} = (1 + r)$

Translating the Model Into Math

$$\text{consumer's problem:} \quad \max_{C_t} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t U(C_t) \;\middle|\; M_0\right] \quad \text{s.t.}$$

$$\begin{aligned}
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
A_t &= M_t - C_t, \\
A_t &\geq 0, \\
K_{t+1} &= A_t, \\
M_{t+1} &= \mathsf{R}K_{t+1}, + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}$$

Translating the Model Into Math

$$\text{consumer's (shorter) problem:} \quad \max_{C_t} \mathbb{E}\left[\sum_{t=0}^{T} \beta^t \frac{C_t^{1-\rho}}{1-\rho} \;\middle|\; M_0\right] \quad \text{s.t.}$$

$$C_t \leq M_t,$$

$$M_{t+1} = \mathsf{R}(M_t - C_t) + Y_{t+1},$$

$$Y_{t+1} \sim F.$$

Taking it One Period at a Time: Bellman Representation

- The model has $T + 1$ periods, and T could be big
- T random income shocks will realize along the way
- Bellman insight: in period t, assume that you will act optimally in all later periods; now decision is only between current utility flow and future value (of resources)
- Bellman value function $V_t(X_t)$ maps from current state X_t to $\mathbb{R}$, representing expected PDV of utility flows from taking optimal actions from t onward
- Upshot: Can solve the model by backward induction, starting from $t = T$

Putting the Model into Bellman Form

$$
\begin{aligned}
V_0(M_0) &= \max_{C_t} \ \mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t U(C_t) \ \middle| \ M_0 \right] \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}
$$

Putting the Model into Bellman Form

$$
\begin{aligned}
V_0(M_0) &= \max_{C_t} \; \mathbb{E}_0 \left[U(C_0) + \sum_{t=1}^{T} \beta^t U(C_t) \;\middle|\; M_0 \right] \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}
$$

Putting the Model into Bellman Form

$$
\begin{aligned}
V_0(M_0) &= \max_{C_t} \left\{ U(C_0) + \mathbb{E}_0 \left[\sum_{t=1}^{T} \beta^t U(C_t) \;\middle|\; M_0 \right] \right\} \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}
$$

Putting the Model into Bellman Form

$$
\begin{aligned}
V_0(M_0) &= \max_{C_t} \left\{ U(C_0) + \beta \mathbb{E}_0 \left[\sum_{t=1}^{T} \beta^{t-1} U(C_t) \;\middle|\; M_0 \right] \right\} \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}
$$

Putting the Model into Bellman Form

$$
\begin{aligned}
V_0(M_0) &= \max_{C_t} \left\{ U(C_0) + \beta \mathbb{E}_0 \mathbb{E}_1 \left[\sum_{t=1}^{T} \beta^{t-1} U(C_t) \;\middle|\; M_1 \right] \right\} \text{ s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}
$$

Putting the Model into Bellman Form

$$V_0(M_0) = \max_{C_t} \left\{ U(C_0) + \beta \mathbb{E}_0 \overbrace{\mathbb{E}_1 \left[\sum_{t=1}^{T} \beta^{t-1} U(C_t) \;\middle|\; M_1 \right]}^{\equiv V_1(M_1)} \right\} \text{ s.t.}$$

$$\begin{aligned}
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}$$

Putting the Model into Bellman Form

$$
\begin{aligned}
V_0(M_0) &= \max_{C_0} \ \{U(C_0) + \beta \mathbb{E}_0 V_1(M_1)\} \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
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Putting the Model into Bellman Form

$$
\begin{aligned}
V_t(M_t) &= \max_{C_t} \; \{U(C_t) + \beta \mathbb{E}_t V_{t+1}(M_{t+1})\} \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}
$$

Solving By Backward Induction

- Bellman: the **value function** is all you need to know about the future
- Value function: expected PDV of making optimal choices from t onward (conditional on state)
- Start from the end of the problem, work backward
- Solving terminal period $t = T$ is trivial: there's no future, consume it all!
- Terminal period value function is then just $V_T(M_T) = U(M_T)$
- Solving non-terminal periods is where it gets more interesting

Solving the Terminal period

$$
\begin{aligned}
V_T(M_T) &= \max_{C_T} \ \{U(C_T) + \beta \mathbb{E}_T V_{T+1}(M_{T+1})\} \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\leq M_t, \\
M_{t+1} &= \mathsf{R}(M_t - C_t) + Y_{t+1}, \\
Y_{t+1} &\sim F.
\end{aligned}
$$

Solving the Terminal period

$$V_T(M_T) = \max_{C_T} \left\{ U(C_T) + \beta \mathbb{E}_T \overbrace{V_{T+1}(M_{T+1})}^{=0,\text{ no future}} \right\} \quad \text{s.t.}$$

$$U(C_t) = \frac{C_t^{1-\rho}}{1-\rho},$$

$$C_t \leq M_t,$$

$$M_{t+1} = \mathsf{R}(M_t - C_t) + Y_{t+1},$$

$$Y_{t+1} \sim F.$$

Solving the Terminal period

$$
\begin{aligned}
V_T(M_T) &= \max_{C_T} \; \{U(C_T) + \beta \cdot 0\} \quad \text{s.t.} \\
U(C_t) &= \frac{C_t^{1-\rho}}{1-\rho}, \\
C_t &\le M_t.
\end{aligned}
$$

Solving the Terminal period

$$V_T(M_T) = \max_{C_T} \left\{ \frac{C_T^{1-\rho}}{1-\rho} + \right\} \quad \text{s.t.}$$

$$C_T \leq M_T.$$

Solving the Terminal period

$$V_T(M_T) = \frac{M_T^{1-\rho}}{1-\rho}.$$

Solving Non-Terminal Periods

- There is no closed form solution to non-terminal periods
- Problem can only be solved **numerically** on a computer
- Numeric solution: approximate representation of true solution to chosen accuracy
- Solve optimization problem on a finite set of state space gridpoints
- Represent value function and policy function (consumption function) with approximating interpolations
- But how do we actually do that? Several ways to go about this

Method 1: Explicit Value Maximization

- Bellman representation says to maximize sum of current period utility and expected discounted future value, so... let's do that
- Choose a set of gridpoints for M_t, starting at/near zero
- For each M_t, search for the C_t that maximizes utility plus expected future value
- Construct value function by interpolating on (M_t, V_t) pairs: connect the dots
- Construct consumption function by interpolating on (M_t, C_t) pairs

Method 2: First Order Conditions

- When we maximize $U(C_t) + \beta\mathbb{E}V_{t+1}(M_{t+1})$, we are implicitly finding a solution to the first order conditions for optimality
- Derivative of maximand w.r.t C_t: $U'(C_t) - \mathrm{R}\beta\mathbb{E}V'_{t+1}(M_{t+1})$
- FOC for interior solution: that should be zero
- FOC for constrained solution: it should be positive, but $C_t = M_t$
- To do this, we need the **marginal value function** $V'_t(M_t)$
- Envelope condition: $V'_t(M_t) = U'(C_t)$ (for optimal C_t)
- Don't need to compute the value function at all! Consumption function is sufficient statistic for marginal value function!

Logic of the Envelope Condition

$$
\begin{aligned}
V_t(M_t) &= \max_{C_t} \; \{U(C_t) + \beta \mathbb{E}_t V_{t+1}(M_{t+1})\}, \\
\mathbf{C}_t(M_t) &\equiv \arg\max_{C_t} \; \{U(C_t) + \beta \mathbb{E}_t V_{t+1}(M_{t+1})\}, \\
U'(C't) &= \mathsf{R}\beta \mathbb{E}_t V'_{t+1}(M_{t+1}) \text{ (first order condition)}
\end{aligned}
$$

Logic of the Envelope Condition

$$
\begin{aligned}
V_t(M_t) &= U(\mathbf{C}_t(M_t)) + \beta \mathbb{E}_t V_{t+1}(M_{t+1}) \Longrightarrow \\
V_t(M_t) &= U(\mathbf{C}_t(M_t)) + \beta \mathbb{E}_t V_{t+1}(\mathsf{R}(M_t - \mathbf{C}_t(M_t)) + Y_{t+1}) \Longrightarrow \\
V_t'(M_t) &= \mathbf{C}_t'(M_t) U'(\mathbf{C}_t(M_t)) + (1 - \mathbf{C}_t'(M_t)) \mathsf{R} \beta \mathbb{E}_t V_{t+1}'(\mathsf{R}(M_t - \mathbf{C}_t(M_t)) + Y_{t+1}) \Longrightarrow \\
V_t'(M_t) &= \mathbf{C}_t'(M_t) U'(C_t) + (1 - \mathbf{C}_t'(M_t)) \mathsf{R} \beta \mathbb{E}_t V_{t+1}'(M_{t+1}) \Longrightarrow \\
V_t'(M_t) &= \mathbf{C}_t'(M_t) U'(C_t) + (1 - \mathbf{C}_t'(M_t)) U'(C_t) \Longrightarrow \\
V_t'(M_t) &= (\mathbf{C}_t'(M_t) + 1 - \mathbf{C}_t'(M_t)) U'(C_t) \Longrightarrow \\
V_t'(M_t) &= U'(C_t).
\end{aligned}
$$