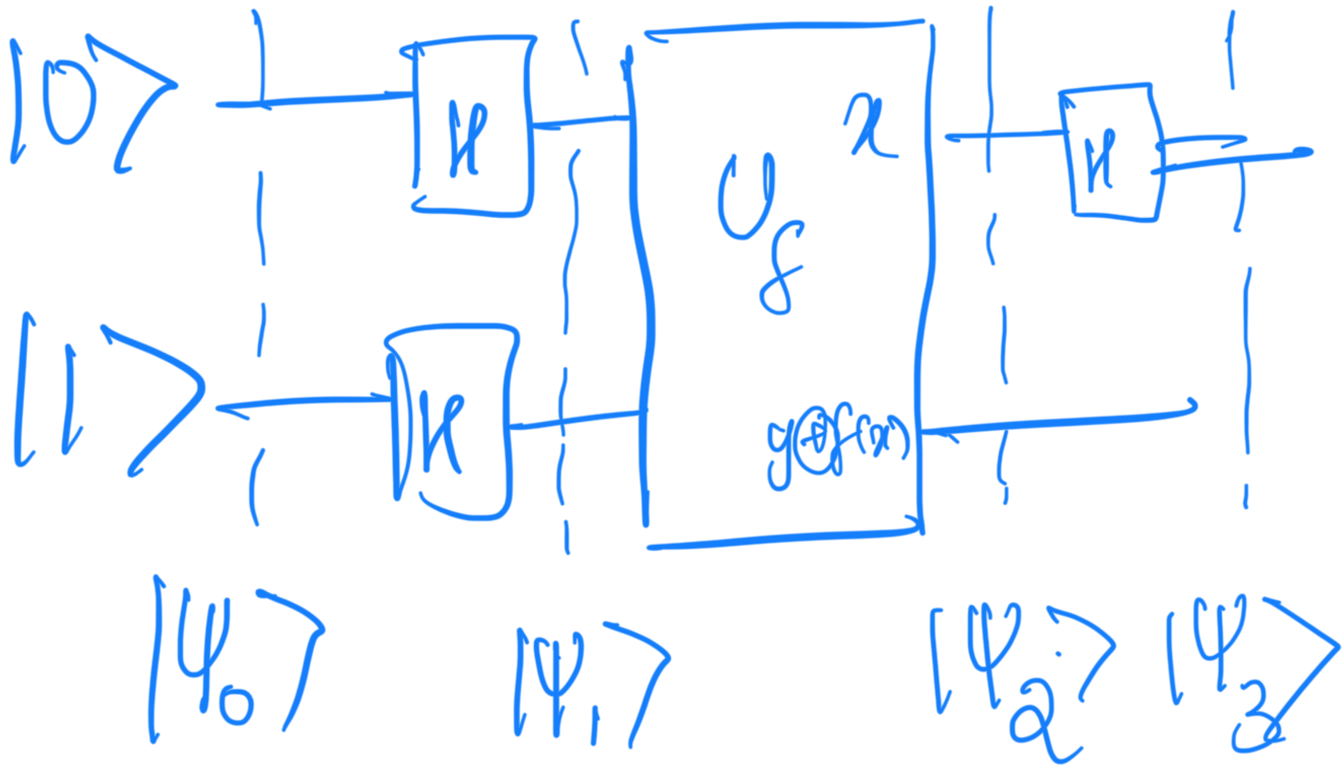
Deutsch's algorithm



$$|\psi_0\rangle = |0\rangle|1\rangle$$

$|\psi_1\rangle$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

$\sqrt{2}\left[\qquad\right]$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] \otimes \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle]$$

$$= \frac{1}{2}[|00\rangle - |01\rangle + |10\rangle - |11\rangle]$$

Oracle Application (U_f)

$$\boxed{U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle}$$

$$|\psi_2\rangle = U_f\left(\underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|+\rangle} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= |+\rangle\left[\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) \oplus f(x)\right]$$

Case 1: $f(x) = 0$

Then,

$$|\psi_2\rangle = |+\rangle\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$$

Case 2 : $f(x) = 1$

$$|\psi_2\rangle = |+\rangle\left[\frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)\right]$$

Generalize to say,

$$|\psi_2\rangle = (-1)^{f(x)} \underbrace{\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]}_{|x\rangle} \underbrace{\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]}_{|y\rangle}$$

NOTICE: Second Qubit isn't changed after oracle application!!!

Expand further:

$$|\psi_2\rangle = \frac{1}{2}\left[(-1)^{f(x)}\left(|00\rangle - |01\rangle + |10\rangle + |11\rangle\right)\right]$$

$$= \frac{1}{2}\left((-1)^{f(x)}|0\rangle(|0\rangle - |1\rangle) + (-1)^{f(x)}|1\rangle(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}\left[\left((-1)^{f(0)}|0\rangle(|0\rangle - |1\rangle)\right) + (-1)^{f(1)}|1\rangle(|0\rangle - |1\rangle)\right]$$

There are 2 cases here,

$$f(0) = f(1)$$

or

$$f(0) \neq f(1)$$

Case 1

$$f(0) = f(1)$$

- $f(0) = f(1) = 0$.

Then $|\psi_2\rangle = \frac{1}{2}\Big((-1)^0 |0\rangle(|0\rangle - |1\rangle) + (-1)^0 |1\rangle(|0\rangle - |1\rangle)\Big)$

$$= \frac{1}{2}\Big[\big(|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle - |1\rangle)\big)\Big]$$

$$= \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

- $f(0) = f(1) = 1$

Then $|\psi_2\rangle = \frac{1}{2}\Big((-1)^1 |0\rangle(|0\rangle - |1\rangle) + (-1)^1 |1\rangle(|0\rangle - |1\rangle)\Big)$

$$= \frac{1}{2}\Big(-(|0\rangle)(|0\rangle - |1\rangle) - |1\rangle(|0\rangle - |1\rangle) \Big)$$

$$= -\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

In Essence,

$$|\psi_2\rangle = \begin{cases} \pm\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right], \\ \qquad \text{given } f(0) = f(1) \end{cases}$$

Second case,

$$f(0) \neq f(1)$$

$$|\psi_2\rangle = \pm\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

So in essence;

$$|\psi_2\rangle = \begin{cases} \pm\left[\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)=f(1) \\ \pm\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right]\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)\neq f(1) \end{cases}$$

Lastly,

let's move onto $|\psi_3\rangle$

Apply Hadamard onto first qubit:

$$|\psi_3\rangle = \begin{cases} \pm|0\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)=f(1) \\ \pm|1\rangle\left[\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right], & f(0)\neq f(1) \end{cases}$$

So what's the conclusion???

If $f(0) = f(1) \rightarrow$ Constant function

then output after measuring 1st qubit:

$$|0\rangle$$

If $f(0) \neq f(1) \rightarrow$ balanced function

then output after measuring 1st qubit

$$|1\rangle$$