Advanced Basic Theoretical Reinforcement Learning Result

By advanced, I mean that one would not find these results in a graduate level RL class, or even popular RL textbooks. The results are mostly found in RL research papers.

By basic, I mean that these results seem general enough that they can be used as building block for more complicated result.

Theory usually progresses one lemma at a time, so I thought it is a good idea to collate these results into one document for easy reference.

For each result, if a cleanly written proof is available, then I link to it. Otherwise, I provide the proof myself.

The theoretical result in each section focuses on one theme, such as model-based RL.

1 Relating performances of a policy in 2 dynamics model to differences between the 2 models

1.1 trajectory-level performance vs. per-state state-action values

Given 2 dynamics model M and \widehat{M} , how can we related the <u>trajectory-level performance</u> of a policy π in the 2 models to the differences in <u>per-state</u> state-action values when states are sampled from either dynamics model? Formal result is in subsection 3.1 [1].

1.2 trajectory-level performance vs. per-state per-action model outputs

Given 2 dynamics model M and \widehat{M} , how can we related the <u>trajectory-level performance</u> of a policy π in the 2 models to the difference in per-state per-action differences in output of the 2 models? Formal result is in subsection 3.2 [1].

2 Relating per-state action distribution and future state visitation distribution

Given two policies π_1 and π_2 , how can we related their per-state difference in action distribution to the differences in their future discounted state visitation distribution? Formal result is in subsection 3.3 [2].

3 Formal Results

3.1 Formal Result for subsection 1.1

M(.|s,a) denotes the distribution on the next state given the current state and action.

 $S_t^{\pi,M}$ denotes the random variable of the states at step t when we execute policy π on the dynamic model starting from S_0 .

 P_X denotes the density function for the random variable X.

 $\rho^{\pi,M}$ denotes the discounted distribution of the states visited by π on M. That is, $\rho^{\pi,M} = (1-\gamma)\sum_{t=0}^{\infty} \gamma^t P_{S_t^{\pi,M}}$.

$$V^{\pi,M} = E_{s_0 \sim P_{S_0}} [V^{\pi,M}(s_0)].$$

The rest of the notation should be standard in the RL world. Please refer to section 2 in the paper.

Let
$$G^{\pi,\widehat{M},M}(s,a) = \underset{\widehat{s}' \sim \widehat{M}(.|s,a)}{E} \left[V^{\pi,\widehat{M}}(\widehat{s}') \right] - \underset{s' \sim M(.|s,a)}{E} \left[V^{\pi,\widehat{M}}(s') \right]$$

For any policy π and dynamical model M, \widehat{M} , we have:

$$V^{\pi,\widehat{M}} - V^{\pi,M} = \frac{\gamma}{1 - \gamma} \mathop{E}_{\substack{S \sim \rho^{\pi,M} \\ A \sim \pi(.|S)}} \left[G^{\pi,\widehat{M},M}(S,A) \right]$$

Proof is available in subsection 4.1

3.2 Formal result for subsection 1.2

The notation is the same as in subsection 3.1. Suppose that $V^{\pi,\widehat{M}}$ is L-Lipschitz, that is:

$$\forall s, s' \in S, |V^{\pi,\widehat{M}}(s) - V^{\pi,\widehat{M}}(s')| \le L \|s - s'\|$$

For any policy π and deterministic dynamical model M, \widehat{M} , we have:

$$|V^{\pi,\widehat{M}} - V^{\pi,M}| \leq \frac{\gamma}{1 - \gamma} L \underset{\substack{S \sim \rho^{\pi,M} \\ A \sim \pi(.|S)}}{E} \left[\left\| \widehat{M}(S,A) - M(S,A) \right\| \right]$$

Proof is available in subsection 4.2

3.3 Formal result for section 2

Let
$$d_{\pi} = (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t d_{\pi}^t$$
.

Given any two policy π_1 and π_2 such that $E_{s \sim d_{\pi_1}} [D_{TV}(\pi_1(\cdot|s), \pi_2(\cdot|s))] \leq \alpha$ then we have:

$$||d_{\pi_1} - d_{\pi_2}||_1 \le \frac{2\alpha}{1 - \gamma}$$

The proof is clearly presented in the appendix of [2].

4 Proofs

4.1 Proof for subsection 3.1

Let $W_j(s)$ be the cumulative reward when we use dynamical model M for j steps and then use \widehat{M} , that is:

$$W_{j}(s) = \underbrace{E}_{\begin{subarray}{l} \forall t \geq 0, A_{t} \sim \pi(.|S_{t}) \\ \forall j \geq t \geq 0, S_{t+1} \sim M(.|S_{t}, A_{t}) \end{subarray}}_{\begin{subarray}{l} \forall t > j, S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}} \left[\sum_{t=0}^{\infty} \gamma^{t} R(S_{t}, A_{t}) \middle| S_{0} = s \right]$$

By definition, $W_{\infty}(s) = V^{\pi,M}(s)$ and $W_0(s) = V^{\pi,\widehat{M}}(s)$, then:

$$V^{\pi,M}(s) - V^{\pi,\widehat{M}}(s) = \sum_{j=0}^{\infty} (W_{j+1}(s) - W_j(s))$$

We now decompose $W_i(s)$ into sum of rewards obtained when sampling under M and \widehat{M} :

$$\begin{split} W_{j}(s) &= \underbrace{E}_{\begin{subarray}{c} \forall t \geq 0, A_{t} \sim \pi(.|S_{t}) \\ \forall t > j, S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}}_{\begin{subarray}{c} \forall t \geq 0, S_{t+1} \sim M(.|S_{t}, A_{t}) \end{subarray}} \left[\sum_{t=0}^{\infty} \gamma^{t} R(S_{t}, A_{t}) \middle| S_{0} = s \right] \\ &= \underbrace{E}_{\begin{subarray}{c} \forall t \geq 0, A_{t} \sim \pi(.|S_{t}) \\ \forall j \geq t \geq 0, S_{t+1} \sim M(.|S_{t}, A_{t}) \end{subarray}}_{\begin{subarray}{c} \forall t \geq j, S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}} \left[\sum_{t=0}^{j} \gamma^{t} R(S_{t}, A_{t}) \middle| S_{0} = s \right] + \underbrace{E}_{\begin{subarray}{c} \forall t \geq 0, A_{t} \sim \pi(.|S_{t}) \\ \forall t > j, S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}}_{\begin{subarray}{c} \forall t \geq 0, A_{t} \sim \pi(.|S_{t}) \\ \forall t \geq j, S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}} \left[\sum_{t=0}^{j} \gamma^{t} R(S_{t}, A_{t}) \middle| S_{0} = s \right] + \underbrace{E}_{\begin{subarray}{c} \forall t \geq 0, A_{t} \sim \pi(.|S_{t}) \\ \forall t \geq j, S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}}_{\begin{subarray}{c} \beta \in S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}} \left[\sum_{t=0}^{j} \gamma^{t} R(S_{t}, A_{t}) \middle| S_{0} = s \right] + \underbrace{E}_{\begin{subarray}{c} S_{j} \sim \widehat{M}(.|S_{0} = s) \\ \widehat{S}_{j+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}}_{\begin{subarray}{c} \beta \in S_{t+1} \sim \widehat{M}(.|S_{t}, A_{t}) \end{subarray}} \left[\gamma^{j+1} V^{\pi,\widehat{M}}(\widehat{S}_{j+1}) \right] \right] \\ A_{j} \sim \pi(.|S_{j}) \end{subarray}}$$

Proof of the last equality is below:

$$\begin{split} & \underset{\substack{y_1 \geq y_2 \geq y_3 \leq y_4 = -M(|S_1,A_2|) \\ y_2 \geq y_2 \geq y_4 = -M(|S_1,A_2|) \\ y_2 \geq y_3 \leq y_4 = -M(|S_1,A_2|) \\ y_3 = y_4 \\ y_4 = y_4 = y_4 = y_4 = y_4 = y_4 = y_4 \\ y_5 = y_4 = y_4 = y_4 = y_4 = y_4 \\ y_5 = y_4 = y_4 = y_4 = y_4 = y_4 \\ y_5 = y_4 = y_4 = y_4 = y_4 \\ y_5 = y_5 \\ y_5 = y_5 \\ y_6 = y_6 \\ y_6 = y_$$

By a similar argument:

$$W_{j+1}(s) = \underbrace{E}_{\substack{\forall t \geq 0, A_t \sim \pi(.|S_t) \\ \forall j \geq t \geq 0, S_{t+1} \sim M(.|S_t, A_t) \\ \forall t > j, S_{t+1} \sim \widehat{M}(.|S_t, A_t)}} \left[\sum_{t=0}^{j} \gamma^t R(S_t, A_t) \Big| S_0 = s \right] + \underbrace{E}_{\substack{S_j \sim \rho_{S_j}^{\pi, M}(.|S_0 = s) \\ A_j \sim \pi(.|S_j)}} \left[\underbrace{E}_{S_{j+1} \sim M(.|S_j, A_j)} \left[\gamma^{j+1} V^{\pi, \widehat{M}}(S_{j+1}) \right] \right]$$

Thus:

$$W_{j+1}(s) - W_{j}(s) = \gamma^{j+1} \underbrace{E}_{\substack{S_{j} \sim \rho_{S_{j}}^{\pi, M}(.|S_{0}=s) \\ A_{j} \sim \pi(.|S_{j})}} \left[\underbrace{E}_{\substack{S_{j+1} \sim M(.|S_{j}, A_{j})}} \left[V^{\pi, \widehat{M}}(S_{j+1}) \right] - \underbrace{E}_{\widehat{S}_{j+1} \sim \widehat{M}(.|S_{j}, A_{j})} \left[V^{\pi, \widehat{M}}(\widehat{S}_{j+1}) \right] \right]$$

$$\begin{split} &\Rightarrow \sum_{j=0}^{\infty} [W_{j+1}(s) - W_{j}(s)] \\ &= \sum_{j=0}^{\infty} \left[\gamma^{j+1} \sum_{\substack{S_{j} \sim \rho_{S_{j}}^{\pi,M}(.|S_{0}=s) \\ A_{j} \sim \pi(.|S_{j})}} \left[\sum_{S_{j+1} \sim M(.|S_{j},A_{j})} \left[V^{\pi,\widehat{M}}(S_{j+1}) \right] - \sum_{\widehat{S}_{j+1} \sim \widehat{M}(.|S_{j},A_{j})} \left[V^{\pi,\widehat{M}}(\widehat{S}_{j+1}) \right] \right] \right] \\ &= \sum_{j=0}^{\infty} \left[\gamma^{j+1} \sum_{s'} \rho_{S_{j}}^{\pi,M}(s'|S_{0}=s) \sum_{A_{j} \sim \pi(.|s')} \left[\sum_{S_{j+1} \sim M(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(S_{j+1}) \right] - \sum_{\widehat{S}_{j+1} \sim \widehat{M}(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(\widehat{S}_{j+1}) \right] \right] \right] \\ &= \frac{\gamma}{1-\gamma} \sum_{j=0}^{\infty} \left[\gamma^{j}(1-\gamma) \sum_{s'} \rho_{S_{j}}^{\pi,M}(s'|S_{0}=s) \sum_{A_{j} \sim \pi(.|s')} \left[\sum_{S_{j+1} \sim M(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(S_{j+1}) \right] - \sum_{\widehat{S}_{j+1} \sim \widehat{M}(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(\widehat{S}_{j+1}) \right] \right] \\ &= \frac{\gamma}{1-\gamma} \sum_{s'} \left[\sum_{j=0}^{\infty} \gamma^{j}(1-\gamma) \rho_{S_{j}}^{\pi,M}(s'|S_{0}=s) \sum_{A_{j} \sim \pi(.|s')} \left[\sum_{S_{j+1} \sim M(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(S_{j+1}) \right] - \sum_{\widehat{S}_{j+1} \sim \widehat{M}(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(\widehat{S}_{j+1}) \right] \right] \right] \\ &= \frac{\gamma}{1-\gamma} \sum_{s'} \left[\rho^{\pi,M}(s'|S_{0}=s) \sum_{A_{j} \sim \pi(.|s')} \left[\sum_{S_{j+1} \sim M(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(S_{j+1}) \right] - \sum_{\widehat{S}_{j+1} \sim \widehat{M}(.|s',A_{j})} \left[V^{\pi,\widehat{M}}(\widehat{S}_{j+1}) \right] \right] \right] \\ &= \frac{\gamma}{1-\gamma} \sum_{S \sim \rho^{\pi,M}(.|S_{0}=s)} \left[\sum_{S_{j+1} \sim M(.|S_{j},A_{j})} \left[V^{\pi,\widehat{M}}(S') \right] - \sum_{\widehat{S}_{j} \sim \widehat{M}(.|S_{j},A_{j})} \left[V^{\pi,\widehat{M}}(\widehat{S}') \right] \right] \\ &= \frac{\gamma}{1-\gamma} \sum_{S \sim \rho^{\pi,M}(.|S_{0}=s)} \left[\sum_{S_{j+1} \sim M(.|S_{j},A_{j})} \left[V^{\pi,\widehat{M}}(S') \right] - \sum_{\widehat{S}_{j} \sim \widehat{M}(.|S_{j},A_{j})} \left[V^{\pi,\widehat{M}}(\widehat{S}') \right] \right] \end{aligned}$$

Thus:

$$\begin{split} V^{\pi,\widehat{M}} - V^{\pi,M} &= \underset{S_0 \sim \rho_{S_0}}{E} \left[V^{\pi,\widehat{M}}(S_0) - V^{\pi,M}(S_0) \right] \\ &= \underset{S_0 \sim \rho_{S_0}}{E} \left[\frac{\gamma}{1 - \gamma} \underset{S \sim \rho^{\pi,M}(.|S_0 = s)}{E} \left[\underset{S' \sim M(.|S,A)}{E} \left[V^{\pi,\widehat{M}}(S') \right] - \underset{\widehat{S}' \sim \widehat{M}(.|S,A)}{E} \left[V^{\pi,\widehat{M}}(\widehat{S}') \right] \right] \\ &= \frac{\gamma}{1 - \gamma} \underset{S_0 \sim \rho_{S_0}}{E} \left[\underset{S \sim \rho^{\pi,M}(.|S_0 = s)}{E} \left[\underset{S' \sim M(.|S,A)}{E} \left[V^{\pi,\widehat{M}}(S') \right] - \underset{\widehat{S}' \sim \widehat{M}(.|S,A)}{E} \left[V^{\pi,\widehat{M}}(\widehat{S}') \right] \right] \right] \\ &= \frac{\gamma}{1 - \gamma} \underset{S \sim \rho^{\pi,M}}{E} \left[\underset{S' \sim M(.|S,A)}{E} \left[V^{\pi,\widehat{M}}(S') \right] - \underset{\widehat{S}' \sim \widehat{M}(.|S,A)}{E} \left[V^{\pi,\widehat{M}}(\widehat{S}') \right] \right] \\ &= \frac{\gamma}{1 - \gamma} \underset{S \sim \rho^{\pi,M}}{E} \left[G^{\pi,\widehat{M},M}(S,A) \right] \\ &= \frac{\gamma}{1 - \gamma} \underset{S \sim \rho^{\pi,M}}{E} \left[G^{\pi,\widehat{M},M}(S,A) \right] \end{split}$$

4.2 Proof for subsection 3.2

From subsection 3.1:

$$\begin{split} V^{\pi,\widehat{M}} - V^{\pi,M} &= \frac{\gamma}{1 - \gamma} \mathop{E}_{\substack{S \sim \rho^{\pi,M} \\ A \sim \pi(.|S)}} \left[\mathop{E}_{\hat{s}' \sim \widehat{M}(.|s,a)} \left[V^{\pi,\widehat{M}}(\hat{s}') \right] - \mathop{E}_{s' \sim M(.|s,a)} \left[V^{\pi,\widehat{M}}(s') \right] \right] \\ &= \frac{\gamma}{1 - \gamma} \mathop{E}_{\substack{S \sim \rho^{\pi,M} \\ A \sim \pi(.|S)}} \left[V^{\pi,\widehat{M}}(\widehat{M}(s,a)) - V^{\pi,\widehat{M}}(M(s,a)) \right] \\ &\Rightarrow |V^{\pi,\widehat{M}} - V^{\pi,M}| \leq \frac{\gamma}{1 - \gamma} \mathop{E}_{\substack{S \sim \rho^{\pi,M} \\ A \sim \pi(.|S)}} \left[|V^{\pi,\widehat{M}}(\widehat{M}(s,a)) - V^{\pi,\widehat{M}}(M(s,a))| \right] \\ &\leq \frac{\gamma}{1 - \gamma} \mathop{E}_{\substack{S \sim \rho^{\pi,M} \\ A \sim \pi(.|S)}} \left[\left\| \widehat{M}(s,a) - M(s,a) \right\| \right] \\ &\leq \frac{\gamma}{1 - \gamma} \mathop{E}_{\substack{S \sim \rho^{\pi,M} \\ A \sim \pi(.|S)}} \left[\left\| \widehat{M}(s,a) - M(s,a) \right\| \right] \end{split}$$

References

- [1] H. Xu, Y. Li, Y. Tian, T. Darrell, and T. Ma, "Algorithmic framework for model-based reinforcement learning with theoretical guarantees," *CoRR*, vol. abs/1807.03858, 2018. [Online]. Available: http://arxiv.org/abs/1807.03858
- [2] W. Sun, G. J. Gordon, B. Boots, and J. A. Bagnell, "Dual policy iteration," CoRR, vol. abs/1805.10755, 2018. [Online]. Available: http://arxiv.org/abs/1805.10755