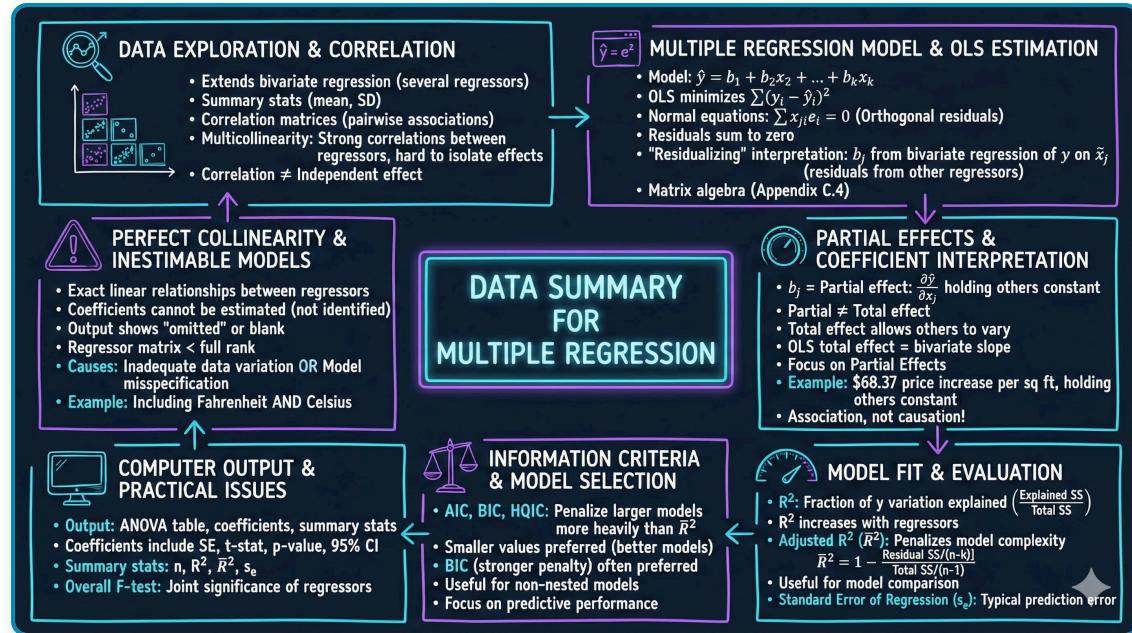


# Chapter 10: Data Summary for Multiple Regression

metricsAI: An Introduction to Econometrics with Python and AI in the Cloud

Carlos Mendez



This notebook provides an interactive introduction to multiple regression analysis. You'll learn how to work with multiple explanatory variables, interpret partial effects, assess model fit, and detect multicollinearity. All code runs directly in Google Colab without any local setup.

Open in Colab

## | Chapter Overview

This chapter extends bivariate regression to the more realistic case where we want to predict an outcome using **multiple explanatory variables** simultaneously. Multiple regression allows us to estimate the partial effect of each variable while controlling for others—a crucial feature for empirical economic analysis.

### Learning Objectives:

By the end of this chapter, you will be able to:

1. Extend bivariate regression concepts to multiple regression with several regressors
2. Interpret pairwise correlations and use them for exploratory data analysis
3. Understand the ordinary least squares (OLS) method for multiple regression
4. Interpret partial effects: how one regressor affects  $y$  while holding others constant
5. Distinguish between partial effects and total effects
6. Evaluate model fit using R-squared and adjusted R-squared
7. Understand information criteria (AIC, BIC) for model selection
8. Recognize when regression coefficients cannot be estimated (perfect collinearity)

**Dataset used:**

- **AED\_HOUSE.DTA:** 29 houses sold in Davis, California (1999) with price, size, bedrooms, bathrooms, lot size, age, and month sold

**Key economic question:** What is the effect of house size on price **after controlling** for other characteristics like bedrooms, bathrooms, and age?

**Chapter outline:**

- 10.1 Example: House Price and Characteristics
- 10.2 Two-Way Scatterplots
- 10.3 Correlation Analysis
- 10.4 Multiple Regression Estimation
- 10.5 Partial Effects — The FWL Theorem
- 10.6 Model Fit Statistics
- 10.7 Model Comparison
- 10.8 Inestimable Models and Multicollinearity
- Key Takeaways
- Practice Exercises
- Case Studies

**Estimated time:** 60-75 minutes

## Setup

First, we import the necessary Python packages and configure the environment for reproducibility. All data will stream directly from GitHub.

In [1]:

```
# Import required packages
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
import statsmodels.api as sm
from statsmodels.formula.api import ols
from scipy import stats
import random
import os

# Set random seeds for reproducibility
RANDOM_SEED = 42
random.seed(RANDOM_SEED)
np.random.seed(RANDOM_SEED)
os.environ['PYTHONHASHSEED'] = str(RANDOM_SEED)

# GitHub data URL
GITHUB_DATA_URL = "https://raw.githubusercontent.com/quarcs-lab/data-open/master/AED/"

# Set plotting style
sns.set_style("whitegrid")
plt.rcParams['figure.figsize'] = (10, 6)

print("Setup complete! Ready to analyze house price data.")
```

Setup complete! Ready to analyze house price data.

## 10.1: Example - House Price and Characteristics

We begin with a real estate dataset from Davis, California. Understanding house prices is a classic economic application because prices reflect both fundamental characteristics (size, bedrooms) and market conditions.

The dataset contains:

- **Price:** Sale price in dollars
- **Size:** House size in square feet
- **Bedrooms:** Number of bedrooms
- **Bathrooms:** Number of bathrooms
- **Lotsize:** Size of lot (1=small, 2=medium, 3=large)
- **Age:** House age in years
- **Monthsold:** Month of year house was sold

**Economic motivation:** A simple regression of price on bedrooms might find a positive relationship, but is this because bedrooms directly add value, or because houses with more bedrooms tend to be larger? Multiple regression helps us disentangle these effects.

In [2]:

```
# Load house price data
data_house = pd.read_stata(GITHUB_DATA_URL + 'AED_HOUSE.DTA')

# Display first few observations
print("First 10 observations:")
print(data_house[['price', 'size', 'bedrooms', 'bathrooms', 'lotsize', 'age',
'monthsold']].head(10))

# Display data summary
print("\nSummary statistics:")
print(data_house.describe())
```

First 10 observations:

	price	size	bedrooms	bathrooms	lotsize	age	monthsold
0	204000	1400	3	2.0	1	31.0	7
1	212000	1600	3	3.0	2	33.0	5
2	213000	1800	3	2.0	2	51.0	4
3	220000	1600	3	2.0	1	49.0	4
4	224500	2100	4	2.5	2	47.0	6
5	229000	1700	4	2.5	2	35.0	3
6	230000	2100	4	2.0	2	34.0	8
7	233000	1700	3	2.0	1	40.0	6
8	235000	1700	4	2.0	2	29.0	7
9	235000	1600	3	2.0	3	35.0	5

Summary statistics:

	price	size	bedrooms	bathrooms	lotsize	age	\
count	29.000000	29.000000	29.000000	29.000000	29.000000	29.000000	
mean	253910.344828	1882.758621	3.793103	2.206897	2.137931	36.413792	
std	37390.710695	398.272130	0.675030	0.341144	0.693034	7.118975	
min	204000.000000	1400.000000	3.000000	2.000000	1.000000	23.000000	
25%	233000.000000	1600.000000	3.000000	2.000000	2.000000	31.000000	
50%	244000.000000	1800.000000	4.000000	2.000000	2.000000	35.000000	
75%	270000.000000	2000.000000	4.000000	2.500000	3.000000	39.000000	
max	375000.000000	3300.000000	6.000000	3.000000	3.000000	51.000000	

	monthsold	list
count	29.000000	29.000000
mean	5.965517	257824.137931
std	1.679344	40860.264099
min	3.000000	199900.000000
25%	5.000000	239000.000000
50%	6.000000	245000.000000
75%	7.000000	269000.000000
max	8.000000	386000.000000

## Bivariate vs. Multiple Regression

Let's compare a simple regression (price on bedrooms only) with a multiple regression (price on bedrooms AND size). This illustrates how controlling for other variables changes coefficient estimates.

**Key insight:** In the bivariate regression, the bedrooms coefficient captures both the direct effect of bedrooms and the indirect effect through correlation with size. In multiple regression, we isolate the **partial effect** of bedrooms holding size constant.

In [3]:

```
# Bivariate regression: price ~ bedrooms
model_bivariate = ols('price ~ bedrooms', data=data_house).fit()

print("=" * 70)
print("BIVARIATE REGRESSION: price ~ bedrooms")
print("=" * 70)
print(model_bivariate.summary())

# Multiple regression: price ~ bedrooms + size
model_multiple = ols('price ~ bedrooms + size', data=data_house).fit()

print("\n" + "=" * 70)
print("MULTIPLE REGRESSION: price ~ bedrooms + size")
print("=" * 70)
print(model_multiple.summary())

# Compare bedrooms coefficient
print("\n" + "=" * 70)
print("COEFFICIENT COMPARISON")
print("=" * 70)
print(f"Bedrooms coefficient (bivariate): ${model_bivariate.params['bedrooms']:.2f}")
print(f"Bedrooms coefficient (multiple): ${model_multiple.params['bedrooms']:.2f}")
print(f"Change: ${model_multiple.params['bedrooms'] - model_bivariate.params['bedrooms']:.2f}")
print("\nThe coefficient drops dramatically because bedrooms was capturing size effects.")
```

```
=====
BIVARIATE REGRESSION: price ~ bedrooms
=====
              OLS Regression Results
=====
Dep. Variable:          price    R-squared:           0.183
Model:                 OLS     Adj. R-squared:       0.152
Method:                Least Squares   F-statistic:        6.030
Date:      Wed, 21 Jan 2026   Prob (F-statistic):  0.0208
Time:          15:16:42    Log-Likelihood:     -343.06
No. Observations:      29     AIC:                  690.1
Df Residuals:          27     BIC:                  692.9
Df Model:                   1
Covariance Type:        nonrobust
=====
```

	coef	std err	t	P> t	[0.025	0.975]
Intercept	1.641e+05	3.71e+04	4.423	0.000	8.8e+04	2.4e+05
bedrooms	2.367e+04	9637.976	2.456	0.021	3891.805	4.34e+04

```
=====
Omnibus:                  23.468   Durbin-Watson:            0.345
Prob(Omnibus):             0.000    Jarque-Bera (JB):        35.285
Skew:                      1.939    Prob(JB):                  2.18e-08
Kurtosis:                  6.764    Cond. No.                  23.8
=====
```

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

```
=====
MULTIPLE REGRESSION: price ~ bedrooms + size
=====
              OLS Regression Results
=====
```

	coef	std err	t	P> t	[0.025	0.975]
Intercept	1.117e+05	2.76e+04	4.048	0.000	5.5e+04	1.68e+05
bedrooms	1553.4580	7846.866	0.198	0.845	-1.46e+04	1.77e+04
size	72.4081	13.300	5.444	0.000	45.070	99.746

```
=====
Omnibus:                  0.516   Durbin-Watson:            1.230
Prob(Omnibus):             0.773    Jarque-Bera (JB):        0.609
Skew:                      -0.086   Prob(JB):                  0.737
Kurtosis:                  2.311    Cond. No.                  1.21e+04
=====
```

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.  
[2] The condition number is large, 1.21e+04. This might indicate that there are strong multicollinearity or other numerical problems.

```
=====
COEFFICIENT COMPARISON
=====
```

```
Bedrooms coefficient (bivariate): $23,667.30
Bedrooms coefficient (multiple): $1,553.46
```

Change: \$-22,113.84

The coefficient drops dramatically because bedrooms was capturing size effects.

### Key Concept: Partial Effects vs. Total Effects in Multiple Regression

In bivariate regression, the bedrooms coefficient (23,667) captures both the direct effect of bedrooms and the indirect effect through the bedrooms coefficient drop to 1,553 — the **partial effect** holding size constant. This dramatic change illustrates why controlling for confounders is essential for isolating individual variable effects.

## | 10.2: Two-Way Scatterplots

Before running multiple regression, it's useful to visualize pairwise relationships between variables. A **scatterplot matrix** shows all two-way scatterplots simultaneously.

### What to look for:

- Strong linear relationships (potential predictors of price)
- Correlation between explanatory variables (potential multicollinearity)
- Outliers or non-linear patterns

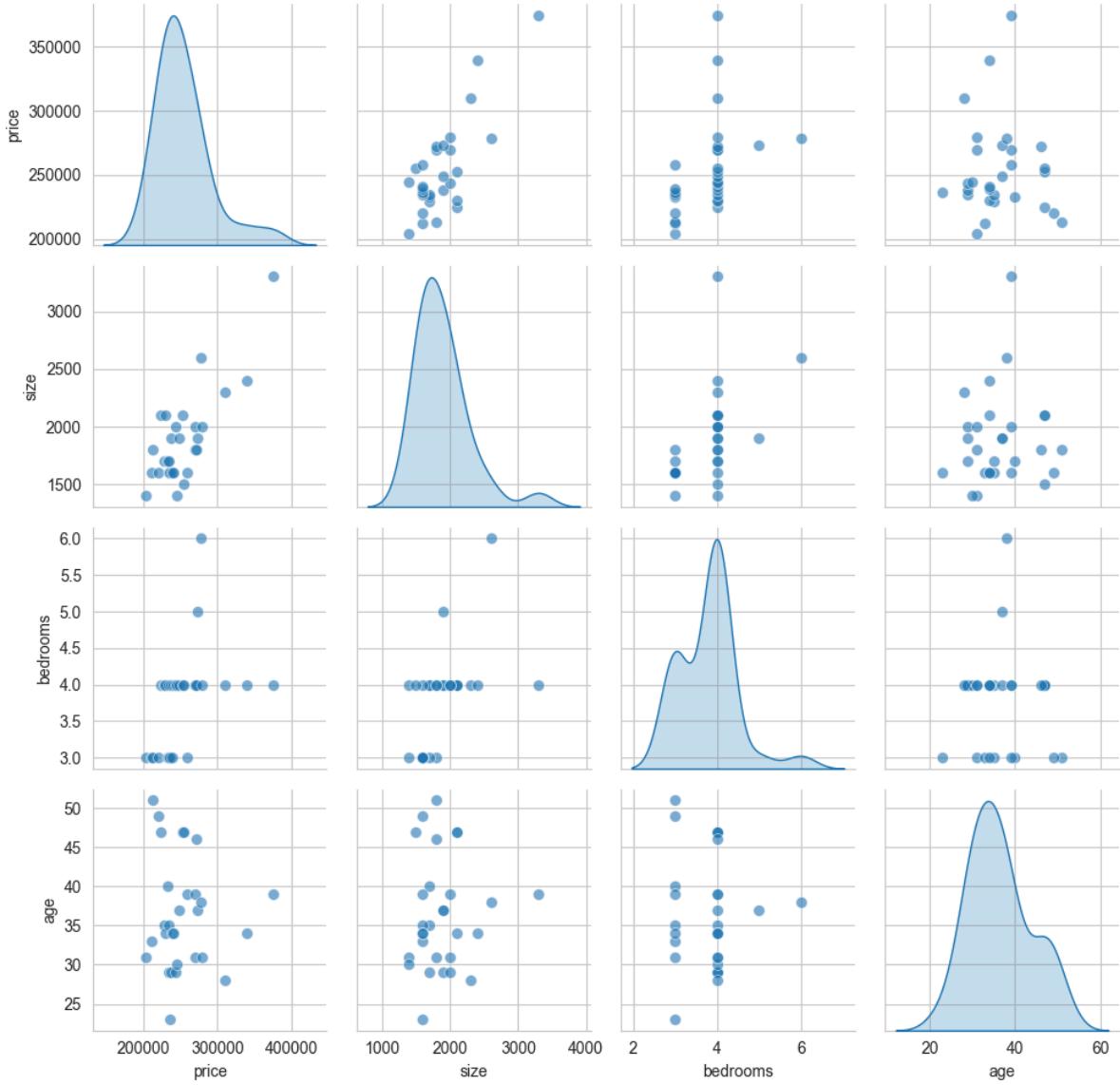
The diagonal shows the distribution of each variable using kernel density estimates (KDE).

In [4]:

```
# Create scatterplot matrix
plot_vars = ['price', 'size', 'bedrooms', 'age']
g = sns.pairplot(data_house[plot_vars], diag_kind='kde', plot_kws={'alpha': 0.6, 's': 50})
g.fig.suptitle('Figure 10.1: Scatterplot Matrix - House Price Data',
               fontsize=14, fontweight='bold', y=1.00)
plt.tight_layout()
plt.show()

print("Scatterplot matrix created.")
print("Notice: Price shows strongest relationship with Size.")
```

**Figure 10.1: Scatterplot Matrix - House Price Data**



Scatterplot matrix created.  
Notice: Price shows strongest relationship with Size.

### Key Concept: Exploratory Data Analysis with Scatterplot Matrices

Pairwise scatterplot matrices display all two-way relationships simultaneously, revealing linear associations, nonlinearities, clusters, and outliers before formal modeling. They also highlight potential multicollinearity: if two regressors are tightly correlated (e.g., size and bedrooms), their individual effects may be hard to separate in a regression.

## 10.3: Correlation Analysis

The **correlation coefficient** measures the strength of linear association between two variables, ranging from -1 (perfect negative) to +1 (perfect positive).

### Correlation matrix insights:

- Price is most correlated with Size ( $r = 0.79$ ), then Bedrooms ( $r = 0.43$ )
- Size and Bedrooms are correlated ( $r = 0.52$ ), which can cause multicollinearity
- Correlation  $\neq$  causation (merely shows association)

A correlation heatmap provides visual representation with color intensity showing strength of correlation.

In [5]:

```
# Calculate correlation matrix
corr_vars = ['price', 'size', 'bedrooms', 'bathrooms', 'lotsize', 'age', 'monthsold']
corr_matrix = data_house[corr_vars].corr()

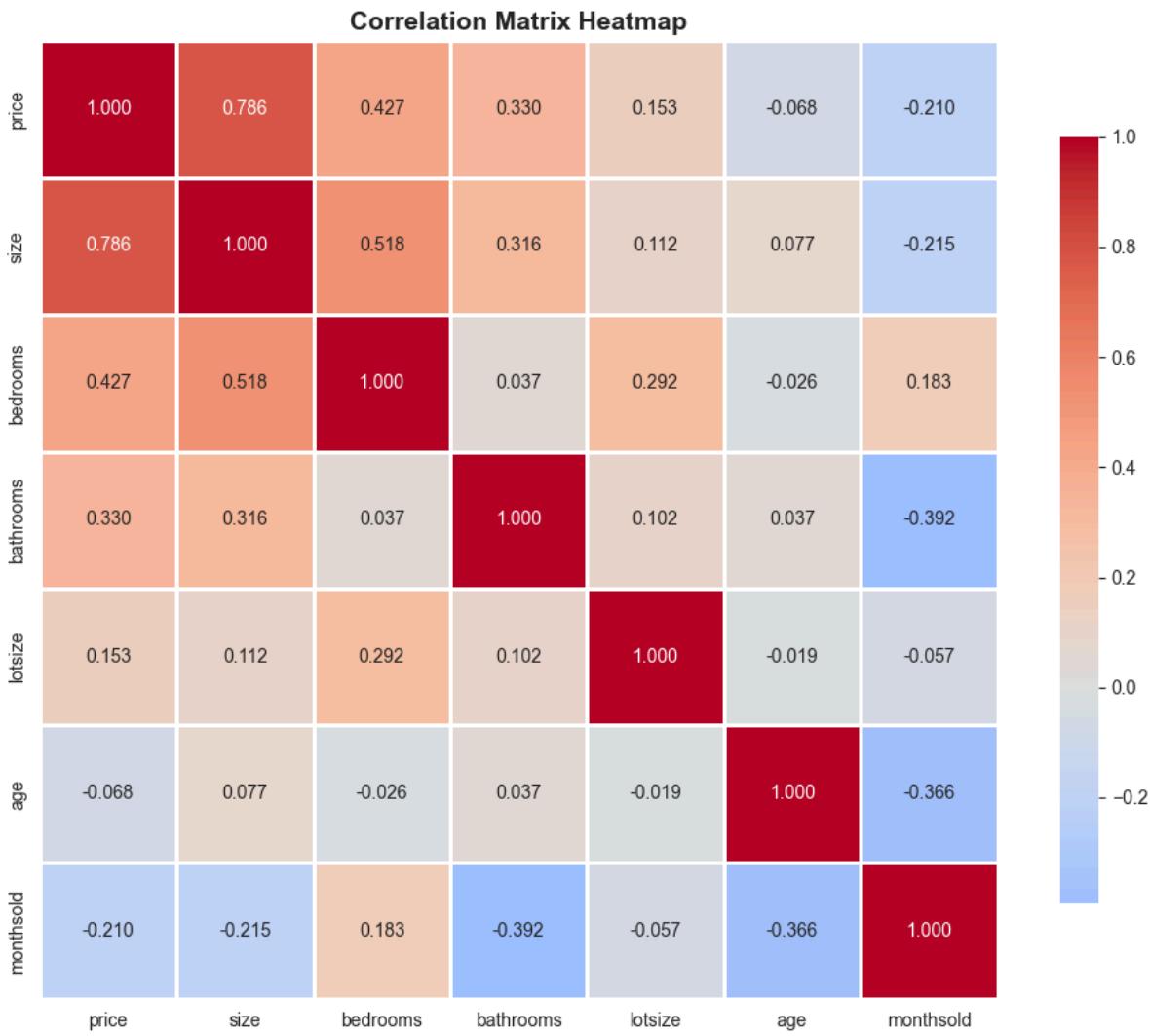
print("Correlation Matrix:")
print(corr_matrix)

# Visualize correlation matrix as heatmap
fig, ax = plt.subplots(figsize=(10, 8))
sns.heatmap(corr_matrix, annot=True, fmt='.3f', cmap='coolwarm', center=0,
            square=True, linewidths=1, cbar_kws={"shrink": 0.8})
ax.set_title('Correlation Matrix Heatmap', fontsize=14, fontweight='bold')
plt.tight_layout()
plt.show()

print("\nKey observations:")
print(f" - Price-Size correlation: {corr_matrix.loc['price', 'size']:.3f} (strongest predictor)")
print(f" - Price-Bedrooms correlation: {corr_matrix.loc['price', 'bedrooms']:.3f}")
print(f" - Size-Bedrooms correlation: {corr_matrix.loc['size', 'bedrooms']:.3f} (multicollinearity concern)")
```

Correlation Matrix:

	price	size	bedrooms	bathrooms	lotsize	age	monthsold
price	1.000000	0.785782	0.427275	0.329793	0.153479	-0.068015	-0.209985
size	0.785782	1.000000	0.517630	0.316338	0.112437	0.076925	-0.214511
bedrooms	0.427275	0.517630	1.000000	0.037435	0.292206	-0.026140	0.182512
bathrooms	0.329793	0.316338	0.037435	1.000000	0.101575	0.037018	-0.392310
lotsize	0.153479	0.112437	0.292206	0.101575	1.000000	-0.019220	-0.057140
age	-0.068015	0.076925	-0.026140	0.037018	-0.019220	1.000000	0.366207
monthsold	-0.209985	-0.214511	0.182512	-0.392310	-0.057140	-0.366207	1.000000



**Key observations:**

- Price-Size correlation: 0.786 (strongest predictor)
- Price-Bedrooms correlation: 0.427
- Size-Bedrooms correlation: 0.518 (multicollinearity concern)

### Key Concept: Correlation vs. Causation in Multivariate Analysis

*High bivariate correlation (e.g., bedrooms-price,  $r = 0.43$ ) may diminish or vanish after controlling for confounders. In our data, bedrooms correlate with price largely because bigger houses have more bedrooms. Multiple regression isolates each variable's partial contribution, revealing that size — not bedrooms — drives most of the price variation.*

Having explored the data visually and through correlations, we now estimate the formal multiple regression model to quantify partial effects.

## 10.4: Multiple Regression Estimation

Now we estimate the **full multiple regression model** with all available predictors. The regression equation is:

$$\widehat{\text{price}} = b_1 + b_2 \times \text{size} + b_3 \times \text{bedrooms} + b_4 \times \text{bathrooms} + b_5 \times \text{lotsize} + b_6 \times \text{age} + b_7 \times \text{monthsold}$$

**Ordinary Least Squares (OLS)** chooses coefficients  $b_1, \dots, b_7$  to minimize the sum of squared residuals:

$$\min \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

where  $y_i$  is the actual price and  $\hat{y}_i$  is the predicted price.

In [6]:

```
# Estimate full multiple regression model
model_full = ols('price ~ size + bedrooms + bathrooms + lotsize + age + monthsold',
                 data=data_house).fit()

print("=" * 70)
print("FULL MULTIPLE REGRESSION MODEL")
print("=" * 70)
print(model_full.summary())
```

```
=====
FULL MULTIPLE REGRESSION MODEL
=====
                OLS Regression Results
=====
Dep. Variable:          price    R-squared:         0.651
Model:                 OLS     Adj. R-squared:      0.555
Method:                Least Squares   F-statistic:       6.826
Date:      Wed, 21 Jan 2026   Prob (F-statistic): 0.000342
Time:          15:16:45   Log-Likelihood:   -330.74
No. Observations:      29     AIC:                  675.5
Df Residuals:          22     BIC:                  685.1
Df Model:                   6
Covariance Type:    nonrobust
=====
```

	coef	std err	t	P> t	[0.025	0.975]
Intercept	1.378e+05	6.15e+04	2.242	0.035	1.03e+04	2.65e+05
size	68.3694	15.389	4.443	0.000	36.454	100.285
bedrooms	2685.3151	9192.526	0.292	0.773	-1.64e+04	2.17e+04
bathrooms	6832.8800	1.57e+04	0.435	0.668	-2.58e+04	3.94e+04
lotsize	2303.2214	7226.535	0.319	0.753	-1.27e+04	1.73e+04
age	-833.0386	719.335	-1.158	0.259	-2324.847	658.770
monthsold	-2088.5036	3520.898	-0.593	0.559	-9390.399	5213.392

```
=====
Omnibus:                      1.317   Durbin-Watson:           1.259
Prob(Omnibus):                0.518   Jarque-Bera (JB):        0.980
Skew:                          0.151   Prob(JB):                  0.612
Kurtosis:                     2.152   Cond. No.            2.59e+04
=====
```

Notes:

- [1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
- [2] The condition number is large, 2.59e+04. This might indicate that there are strong multicollinearity or other numerical problems.

## Coefficient Interpretation with Confidence Intervals

Each regression coefficient represents a **partial effect**: the expected change in price when that variable increases by one unit, **holding all other variables constant**.

### Example interpretation (Size coefficient):

- Coefficient ≈ \$68.37 per square foot
- Interpretation: A one square foot increase in house size is associated with a \$68.37 increase in price, holding bedrooms, bathrooms, lot size, age, and month sold constant.

The 95% confidence interval tells us the range of plausible values for each coefficient.

In [7]:

```
# Display coefficients with 95% confidence intervals
conf_int = model_full.conf_int(alpha=0.05)
coef_table = pd.DataFrame({
    'Coefficient': model_full.params,
    'Std. Error': model_full.bse,
    'CI Lower': conf_int.iloc[:, 0],
    'CI Upper': conf_int.iloc[:, 1],
    't-statistic': model_full.tvalues,
    'p-value': model_full.pvalues
})

print("Coefficients with 95% Confidence Intervals:")
print(coef_table)

print("\nKey result: Size is the only statistically significant predictor (p < 0.05).")
```

Coefficients with 95% Confidence Intervals:				
	Coefficient	Std. Error	CI Lower	CI Upper
Intercept	137791.065699	61464.951869	10320.557398	265261.573999
size	68.369419	15.389472	36.453608	100.285230
bedrooms	2685.315122	9192.525674	-16378.816300	21749.446543
bathrooms	6832.880015	15721.191544	-25770.875723	39436.635753
lotsize	2303.221371	7226.535205	-12683.695364	17290.138107
age	-833.038602	719.334544	-2324.847139	658.769936
monthsold	-2088.503625	3520.897859	-9390.398871	5213.391620

	t-statistic	p-value
Intercept	2.241783	0.035387
size	4.442610	0.000205
bedrooms	0.292119	0.772932
bathrooms	0.434629	0.668065
lotsize	0.318717	0.752947
age	-1.158068	0.259254
monthsold	-0.593174	0.559114

Key result: Size is the only statistically significant predictor (p < 0.05).

### Key Concept: Interpreting Partial Effects in Multiple Regression

Each coefficient  $b_j$  measures the expected change in  $y$  when  $x_j$  increases by one unit, **holding all other regressors constant**. For example, a size coefficient of 68.37 means each additional square foot is associated with a 68.37 price increase, controlling for bedrooms, bathrooms, lot size, age, and month sold. Statistical significance is assessed through confidence intervals and t-tests.

## | 10.5: Partial Effects - The FWL Theorem

The **Frisch-Waugh-Lovell (FWL) Theorem** states that the coefficient on any variable in multiple regression equals the coefficient from a bivariate regression of  $y$  on the

**residualized** version of that variable.

### Demonstration:

1. Regress `size` on all other variables, obtain residuals  $\tilde{\text{size}}$
2. Regress `price` on  $\tilde{\text{size}}$  only
3. The coefficient will exactly match the `size` coefficient from the full multiple regression

**Intuition:** The residual  $\tilde{\text{size}}$  represents the variation in size that is **not explained** by other variables. This is why multiple regression isolates partial effects.

In [8]:

```
# Step 1: Regress size on all other variables
model_size_on_others = ols('size ~ bedrooms + bathrooms + lotsize + age + monthsold',
                           data=data_house).fit()
resid_size = model_size_on_others.resid

# Step 2: Regress price on residualized size
data_house['resid_size'] = resid_size
model_price_on_resid = ols('price ~ resid_size', data=data_house).fit()

# Compare coefficients
print("=" * 70)
print("DEMONSTRATION: FWL THEOREM (Partial Effects)")
print("=" * 70)
print(f"Size coefficient from FULL multiple regression:\n{model_full.params['size']:.10f}")
print(f"Coefficient on residualized size (bivariate):\n{model_price_on_resid.params['resid_size']:.10f}")
print(f"Difference (numerical precision):\n{abs(model_full.params['size'] - model_price_on_resid.params['resid_size']):.15f}")
print("\nThese coefficients are identical! This proves the partial effect interpretation.")

=====
```

```
=====
DEMONSTRATION: FWL THEOREM (Partial Effects)
=====
Size coefficient from FULL multiple regression: 68.3694189767
Coefficient on residualized size (bivariate): 68.3694189767
Difference (numerical precision): 0.00000000000057
```

These coefficients are identical! This proves the partial effect interpretation.

### Key Concept: The Frisch-Waugh-Lovell (FWL) Theorem

The partial effect of  $x_j$  in a multiple regression equals the slope from a bivariate regression of  $y$  on  $\tilde{x}_j$ , where  $\tilde{x}_j$  is the residual from regressing  $x_j$  on all other regressors. Intuitively,  $\tilde{x}_j$  captures the variation in  $x_j$  that is independent of the other variables — this is exactly what multiple regression uses to estimate partial effects.

Now that we understand partial effects and the FWL theorem, let's evaluate how well the overall regression model fits the data.

## 10.6: Model Fit Statistics

Several statistics summarize how well the regression model fits the data:

### R-squared ( $R^2$ ):

- Fraction of variation in  $y$  explained by the regressors
- Formula:  $R^2 = \frac{\text{Explained SS}}{\text{Total SS}} = 1 - \frac{\text{Residual SS}}{\text{Total SS}}$
- Range: 0 to 1 (higher is better fit)
- **Problem:** Always increases when adding variables (even irrelevant ones)

### Adjusted R-squared ( $\bar{R}^2$ ):

- Penalizes model complexity (number of parameters  $k$ )
- Formula:  $\bar{R}^2 = 1 - \frac{\text{RSS}/(n-k)}{\text{TSS}/(n-1)}$
- Can decrease when adding unhelpful variables
- Preferred for model comparison

### Root MSE (Standard error of regression):

- Typical size of prediction error
- Formula:  $s_e = \sqrt{\frac{1}{n-k} \sum (y_i - \hat{y}_i)^2}$
- Same units as  $y$  (dollars in our case)

In [9]:

```
# Calculate and display model fit statistics
n = len(data_house)
k = len(model_full.params) # includes intercept
df = n - k

print("=" * 70)
print("MODEL FIT STATISTICS")
print("=" * 70)
print(f"Sample size (n): {n}")
print(f"Number of parameters (k): {k}")
print(f"Degrees of freedom (n-k): {df}")
print(f"\nR-squared: {model_full.rsquared:.6f}")
print(f"Adjusted R-squared: {model_full.rsquared_adj:.6f}")
print(f"Root MSE: ${np.sqrt(model_full.mse_resid):,.2f}")

# Verify R2 = [Corr(y, ŷ)]2
predicted = model_full.fittedvalues
corr_y_yhat = np.corrcoef(data_house['price'], predicted)[0, 1]
print(f"\nVerification: R2 = [Corr(y, ŷ)]2")
print(f" Correlation(y, ŷ): {corr_y_yhat:.6f}")
print(f" [Correlation(y, ŷ)]2: {corr_y_yhat**2:.6f}")
print(f" R2 from model: {model_full.rsquared:.6f}")
print(f" Match: {np.isclose(corr_y_yhat**2, model_full.rsquared)}")
```

```
=====
MODEL FIT STATISTICS
=====
Sample size (n): 29
Number of parameters (k): 7
Degrees of freedom (n-k): 22

R-squared: 0.650553
Adjusted R-squared: 0.555249
Root MSE: $24,935.73

Verification: R2 = [Corr(y, ŷ)]2
 Correlation(y, ŷ): 0.806569
 [Correlation(y, ŷ)]2: 0.650553
 R2 from model: 0.650553
Match: True
```

## Information Criteria (AIC and BIC)

Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are more sophisticated measures that penalize model complexity:

$$AIC = n \times \ln(\hat{\sigma}_e^2) + n(1 + \ln 2\pi) + 2k$$

$$BIC = n \times \ln(\hat{\sigma}_e^2) + n(1 + \ln 2\pi) + k \times \ln(n)$$

**Key points:**

- Lower values are better (unlike R<sup>2</sup>)
- BIC penalizes complexity more heavily than AIC
- Useful for comparing non-nested models

- Different software packages may use different conventions (scaling by  $n$ )

In [10]:

```
# Calculate information criteria
print("=" * 70)
print("INFORMATION CRITERIA")
print("=" * 70)

# AIC and BIC from statsmodels
print(f"AIC (statsmodels): {model_full.aic:.4f}")
print(f"BIC (statsmodels): {model_full.bic:.4f}")

# Manual calculation (Stata convention)
rss = np.sum(model_full.resid ** 2)
aic_stata = n * np.log(rss/n) + n * (1 + np.log(2*np.pi)) + 2*k
bic_stata = n * np.log(rss/n) + n * (1 + np.log(2*np.pi)) + k*np.log(n)

print(f"\nAIC (Stata convention): {aic_stata:.4f}")
print(f"BIC (Stata convention): {bic_stata:.4f}")
print("\nNote: Different conventions yield different values, but ranking is consistent.")
```

```
=====
INFORMATION CRITERIA
=====
AIC (statsmodels): 675.4824
BIC (statsmodels): 685.0535

AIC (Stata convention): 675.4824
BIC (Stata convention): 685.0535
```

Note: Different conventions yield different values, but ranking is consistent.

### Key Concept: Model Selection with Adjusted R-squared vs. Information Criteria

*Adjusted R<sup>2</sup>* penalizes complexity mildly by dividing sums of squares by degrees of freedom. Information criteria (AIC, BIC) impose stronger penalties:  $BIC = n \ln(\hat{\sigma}_e^2) + k \ln(n)$ . Smaller AIC/BIC values indicate better models. BIC is generally preferred because its penalty grows with sample size, favoring more parsimonious specifications.

## | 10.7: Model Comparison

It's often useful to compare multiple model specifications side-by-side. Here we compare:

- **Simple model:** Price predicted by size only
- **Full model:** Price predicted by all variables

**Key comparison points:**

- How much does  $R^2$  improve?
- Does adjusted  $R^2$  improve (accounting for added complexity)?
- How do coefficient estimates change?

**Economic interpretation:** If adding 5 more variables only modestly improves fit, the simple model may be preferred (parsimony principle).

In [11]:

```
# Estimate simple model (size only)
model_simple = ols('price ~ size', data=data_house).fit()

# Create comparison table
print("=" * 70)
print("MODEL COMPARISON: Simple vs. Full")
print("=" * 70)

comparison_stats = pd.DataFrame({
    'Model': ['Simple (size only)', 'Full (all variables)'],
    'R2': [model_simple.rsquared, model_full.rsquared],
    'Adj R2': [model_simple.rsquared_adj, model_full.rsquared_adj],
    'AIC': [model_simple.aic, model_full.aic],
    'BIC': [model_simple.bic, model_full.bic],
    'N': [n, n]
})

print(comparison_stats.to_string(index=False))

print("\nInterpretation:")
print(" - R2 increases from {:.3f} to {:.3f} (+{:.3f})".format(
    model_full.rsquared - model_simple.rsquared))
print(" - Adj R2 DECREASES from {:.3f} to {:.3f} ({:.3f})".format(
    model_full.rsquared_adj - model_simple.rsquared_adj))
print(" - This suggests the added variables don't improve fit enough to justify complexity.")
print(" - Simple model may be preferred (parsimony principle.)")
```

```
=====
MODEL COMPARISON: Simple vs. Full
=====
      Model      R2  Adj R2      AIC      BIC  N
Simple (size only) 0.617453  0.603285  668.106844  670.841436  29
Full (all variables) 0.650553  0.555249  675.482401  685.053472  29
```

Interpretation:  
 -  $R^2$  increases from 0.617 to 0.651 (+0.033)  
 - Adj  $R^2$  DECREASES from 0.603 to 0.555 (-0.048)  
 - This suggests the added variables don't improve fit enough to justify complexity.  
 - Simple model may be preferred (parsimony principle).

### Key Concept: The Parsimony Principle

*Simpler models are preferred unless additional variables meaningfully improve fit. In our house price example, adding five variables beyond size barely increased  $R^2$  ( $0.618 \rightarrow 0.651$ ) while adjusted  $R^2$  actually fell ( $0.603 \rightarrow 0.555$ ). Check adjusted  $R^2$ , AIC, and BIC — if they don't improve, the simpler model is preferred.*

Having compared models, we now turn to a critical pitfall: when regressors are too closely related to separate their individual effects.

## | 10.8: Inestimable Models and Multicollinearity

**Perfect multicollinearity** occurs when one regressor is an exact linear combination of others. In this case, OLS cannot estimate all coefficients (the model is "inestimable").

**Example:** If we create `size_twice = 2 × size` and include both `size` and `size_twice` as regressors, the model is perfectly collinear.

**Variance Inflation Factor (VIF)** detects multicollinearity:

- VIF measures how much a variable's variance is inflated due to correlation with other regressors
- Formula:  $VIF_j = \frac{1}{1-R_j^2}$  where  $R_j^2$  is from regressing  $x_j$  on all other  $x$ 's
- **Rule of thumb:** VIF > 10 indicates problematic multicollinearity

**Consequences of high multicollinearity:**

- Large standard errors (imprecise estimates)
- Unstable coefficients (small data changes → big estimate changes)
- Difficulty interpreting individual effects

In [12]:

```
# Demonstrate perfect multicollinearity
print("=" * 70)
print("DEMONSTRATION: Perfect Multicollinearity")
print("=" * 70)

# Create a perfectly collinear variable
data_house['size_twice'] = 2 * data_house['size']

print("Creating variable: size_twice = 2 × size")
print("Attempting to estimate: price ~ size + size_twice + bedrooms\n")

try:
    model_collinear = ols('price ~ size + size_twice + bedrooms', data=data_house).fit()
    print("Model estimated (software automatically dropped one variable):")
    print(model_collinear.summary())
except Exception as e:
    print(f"Error: {type(e).__name__}")
    print(f"Message: {str(e)}")
```

```
=====
DEMONSTRATION: Perfect Multicollinearity
=====
Creating variable: size_twice = 2 × size
Attempting to estimate: price ~ size + size_twice + bedrooms

Model estimated (software automatically dropped one variable):
    OLS Regression Results
=====
Dep. Variable:          price    R-squared:       0.618
Model:                 OLS      Adj. R-squared:   0.589
Method:                Least Squares   F-statistic:     21.03
Date:      Wed, 21 Jan 2026   Prob (F-statistic): 3.68e-06
Time:          15:16:45      Log-Likelihood:   -332.03
No. Observations:      29      AIC:             670.1
Df Residuals:          26      BIC:             674.2
Df Model:                  2
Covariance Type:    nonrobust
=====
            coef    std err        t     P>|t|      [0.025    0.975]
-----
Intercept  1.117e+05  2.76e+04     4.048    0.000    5.5e+04  1.68e+05
size       14.4816    2.660      5.444    0.000     9.014   19.949
size_twice  28.9633    5.320      5.444    0.000    18.028   39.898
bedrooms   1553.4580  7846.866     0.198    0.845   -1.46e+04  1.77e+04
-----
Omnibus:           0.516   Durbin-Watson:     1.230
Prob(Omnibus):    0.773   Jarque-Bera (JB):  0.609
Skew:              -0.086   Prob(JB):        0.737
Kurtosis:          2.311   Cond. No.       1.96e+17
-----
Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.
[2] The smallest eigenvalue is 1.39e-26. This might indicate that there are
strong multicollinearity problems or that the design matrix is singular.
```

## | Variance Inflation Factors (VIF)

Now let's calculate VIF for each variable in our full model to check for multicollinearity problems.

In [13]:

```
# Calculate VIF for each variable in the full model
from statsmodels.stats.outliers_influence import variance_inflation_factor

X = data_house[['size', 'bedrooms', 'bathrooms', 'lotsize', 'age', 'monthsold']]
vif_data = pd.DataFrame()
vif_data["Variable"] = X.columns
vif_data["VIF"] = [variance_inflation_factor(X.values, i) for i in range(X.shape[1])]

print("=" * 70)
print("VARIANCE INFLATION FACTORS (VIF)")
print("=" * 70)
print(vif_data.to_string(index=False))
print("\nInterpretation:")
print(" - VIF > 10: Problematic multicollinearity")
print(" - VIF 5-10: Moderate multicollinearity")
print(" - VIF < 5: Low multicollinearity")

max_vif = vif_data['VIF'].max()
if max_vif > 10:
    print(f"\nWarning: Maximum VIF = {max_vif:.2f} indicates multicollinearity issues.")
elif max_vif > 5:
    print(f"\nNote: Maximum VIF = {max_vif:.2f} shows moderate multicollinearity.")
else:
    print(f"\nGood: Maximum VIF = {max_vif:.2f} - no serious multicollinearity detected.")
```

```
=====
VARIANCE INFLATION FACTORS (VIF)
=====
Variable      VIF
size 40.133832
bedrooms 57.819254
bathrooms 34.741906
lotsize 11.969605
age 21.024092
monthsold 12.795557

Interpretation:
 - VIF > 10: Problematic multicollinearity
 - VIF 5-10: Moderate multicollinearity
 - VIF < 5: Low multicollinearity

Warning: Maximum VIF = 57.82 indicates multicollinearity issues.
```

### Key Concept: Detecting Multicollinearity with VIF

The Variance Inflation Factor (VIF) quantifies how much a coefficient's variance is inflated due to correlation with other regressors.  $VIF_j = 1/(1 - R_j^2)$ , where  $R_j^2$  is from regressing  $x_j$  on all other regressors.

A VIF above 5 suggests moderate concern; above 10 indicates severe multicollinearity that inflates standard errors and destabilizes estimates. Perfect collinearity ( $VIF \rightarrow \infty$ ) makes coefficients inestimable.

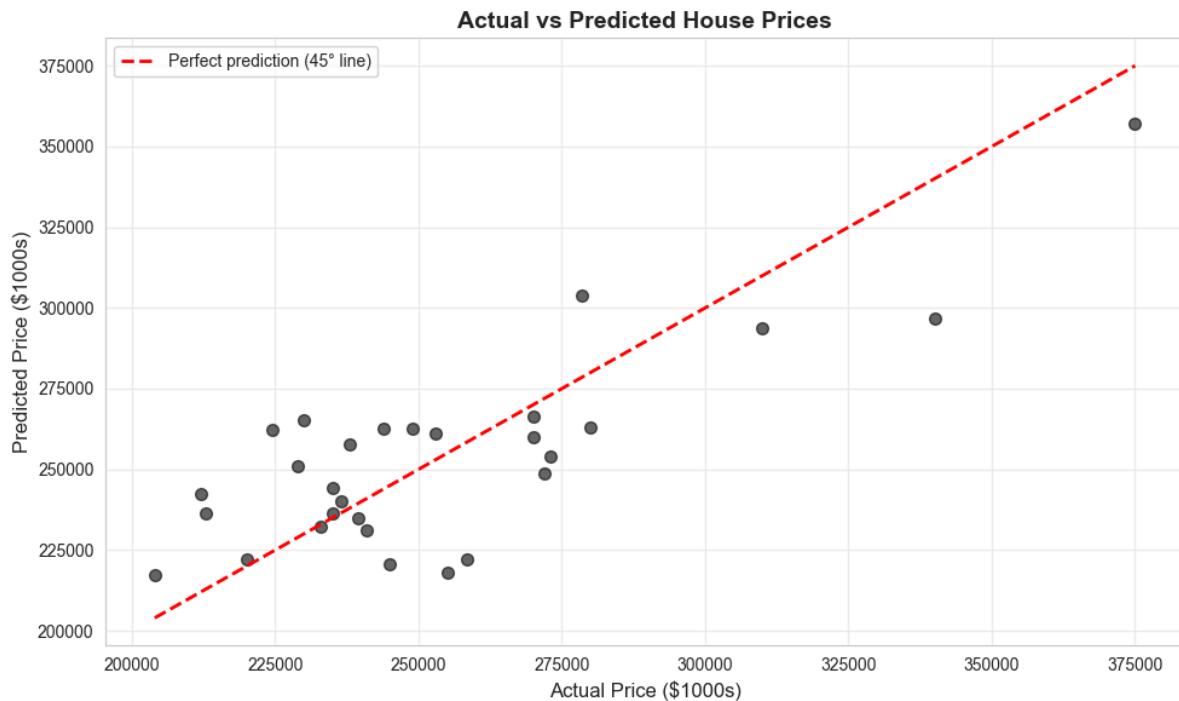
## Visualization: Actual vs. Predicted Values

A plot of actual vs. predicted values helps visualize model fit. Points close to the 45-degree line indicate accurate predictions.

In [14]:

```
# Create actual vs predicted plot
fig, ax = plt.subplots(figsize=(10, 6))
ax.scatter(data_house['price'], model_full.fittedvalues, alpha=0.6, s=50, color='black')
ax.plot([data_house['price'].min(), data_house['price'].max()],
        [data_house['price'].min(), data_house['price'].max()],
        'r--', linewidth=2, label='Perfect prediction (45° line)')
ax.set_xlabel('Actual Price ($1000s)', fontsize=12)
ax.set_ylabel('Predicted Price ($1000s)', fontsize=12)
ax.set_title('Actual vs Predicted House Prices', fontsize=14, fontweight='bold')
ax.legend()
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()

print("Points close to the red line indicate accurate predictions.")
```



Points close to the red line indicate accurate predictions.

## Visualization: Residual Plot

A **residual plot** (residuals vs. fitted values) helps diagnose model problems:

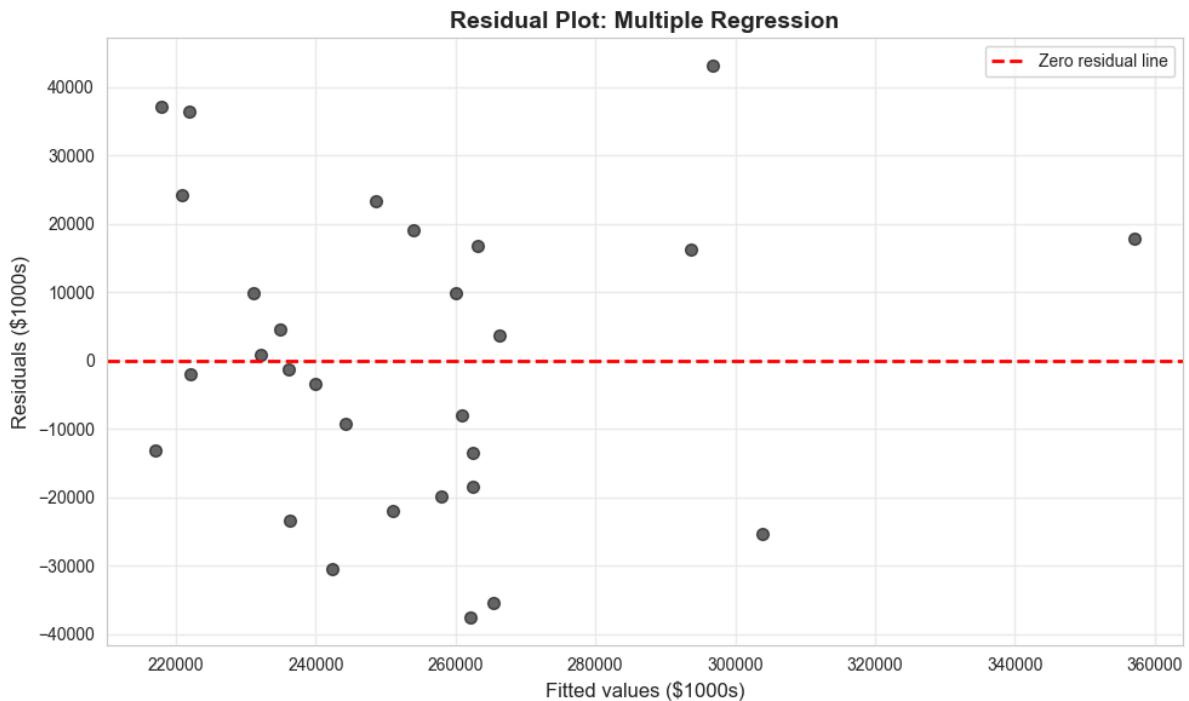
- **Random scatter around zero:** Good (model assumptions satisfied)
- **Patterns (curves, funnels):** Bad (model misspecification or heteroskedasticity)

- **Outliers:** Investigate unusual observations

In [15]:

```
# Create residual plot
fig, ax = plt.subplots(figsize=(10, 6))
ax.scatter(model_full.fittedvalues, model_full.resid, alpha=0.6, s=50, color='black')
ax.axhline(y=0, color='red', linestyle='--', linewidth=2, label='Zero residual line')
ax.set_xlabel('Fitted values ($1000s)', fontsize=12)
ax.set_ylabel('Residuals ($1000s)', fontsize=12)
ax.set_title('Residual Plot: Multiple Regression', fontsize=14, fontweight='bold')
ax.legend()
ax.grid(True, alpha=0.3)
plt.tight_layout()
plt.show()

print("Random scatter around zero suggests model assumptions are reasonable.")
```



Random scatter around zero suggests model assumptions are reasonable.

## Visualization: Coefficient Plot with Confidence Intervals

A **coefficient plot** displays estimated coefficients with their 95% confidence intervals.

This makes it easy to see:

- Which coefficients are significantly different from zero (CI doesn't include zero)
- Relative magnitude of effects
- Precision of estimates (narrow vs. wide CIs)

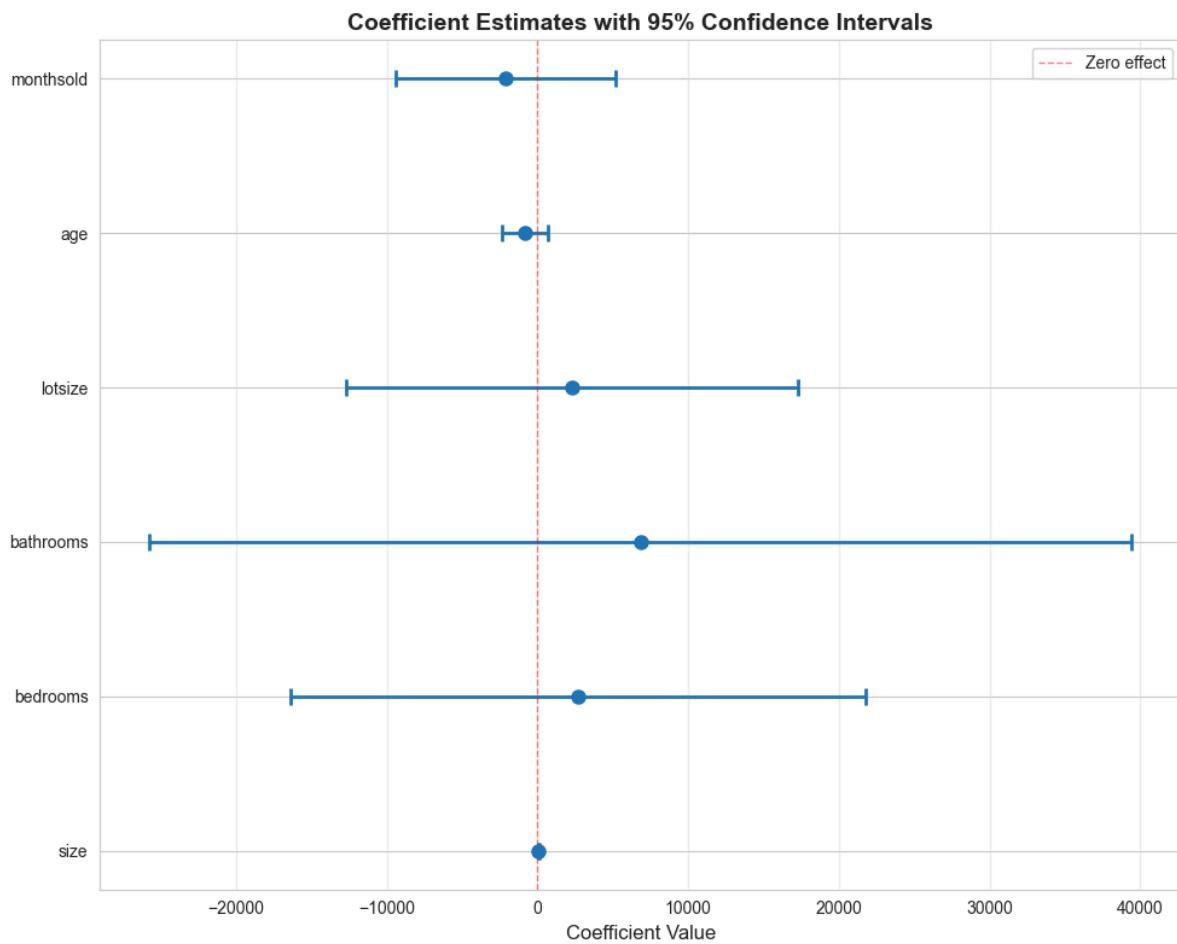
In [16]:

```
# Create coefficient plot with confidence intervals
fig, ax = plt.subplots(figsize=(10, 8))

# Exclude intercept for better visualization
params_no_int = model_full.params[1:]
ci_no_int = conf_int.iloc[1:, :]

y_pos = np.arange(len(params_no_int))
ax.errorbar(params_no_int.values, y_pos,
            xerr=[params_no_int.values - ci_no_int.iloc[:, 0].values,
                   ci_no_int.iloc[:, 1].values - params_no_int.values],
            fmt='o', markersize=8, capsize=5, capthick=2, linewidth=2)
ax.set_yticks(y_pos)
ax.set_yticklabels(params_no_int.index)
ax.axvline(x=0, color='red', linestyle='--', linewidth=1, alpha=0.5, label='Zero effect')
ax.set_xlabel('Coefficient Value', fontsize=12)
ax.set_title('Coefficient Estimates with 95% Confidence Intervals',
             fontsize=14, fontweight='bold')
ax.legend()
ax.grid(True, alpha=0.3, axis='x')
plt.tight_layout()
plt.show()

print("Coefficients whose CI crosses zero are not statistically significant.")
```



Coefficients whose CI crosses zero are not statistically significant.

# Key Takeaways

## Data Exploration and Correlation:

- Multiple regression extends bivariate regression to include several regressors simultaneously
- Summary statistics and two-way scatterplots provide initial overview of variable relationships
- Correlation matrices reveal pairwise associations (e.g., price-size correlation of 0.79)
- Correlation can be misleading — bedrooms correlate with price, but may simply reflect house size

## OLS Estimation:

- Multiple regression model:  $\hat{y} = b_1 + b_2x_2 + b_3x_3 + \dots + b_kx_k$
- OLS minimizes the sum of squared residuals:  $\min \sum (y_i - \hat{y}_i)^2$
- The coefficient  $b_j$  can be computed by bivariate regression of  $y$  on  $\tilde{x}_j$  (residualized  $x_j$ )

## Partial Effects:

- Each coefficient  $b_j$  measures the partial effect: the change in  $\hat{y}$  when  $x_j$  changes by one unit, **holding all other regressors constant**
- Partial effects differ from total effects (which allow other regressors to vary)
- OLS measures association, not causation — use careful language ("associated with")

## Model Fit:

- $R^2$  measures the fraction of variation in  $y$  explained by all regressors (0 to 1)
- Adjusted  $R^2$  penalizes model complexity: can decrease when adding weak regressors
- Standard error of regression  $s_e$  measures typical prediction error in units of  $y$
- Example: Adding 5 regressors increased  $R^2$  from 0.618 to 0.651 but decreased  $\bar{R}^2$  from 0.603 to 0.555

## Information Criteria:

- AIC, BIC, and HQIC penalize larger models more heavily than adjusted  $R^2$
- Smaller values indicate better models

- BIC is generally preferred (stronger complexity penalty than AIC)

### Multicollinearity:

- Perfect collinearity makes some coefficients inestimable (computer shows "omitted")
- VIF detects multicollinearity:  $VIF > 10$  indicates problematic,  $VIF > 5$  moderate concern
- High multicollinearity inflates standard errors and makes estimates unstable

**Python tools used:** `statsmodels` (OLS, VIF), `seaborn` (pairplot, heatmap), `pandas` (DataFrames), `matplotlib` (coefficient plots, diagnostics)

**Next steps:** Chapter 11 covers **statistical inference** for multiple regression — hypothesis tests, confidence intervals, and overall F-tests for model significance.

Congratulations on completing Chapter 10! You now have the tools to estimate and evaluate multiple regression models, a foundation for the rest of the course.

## I Practice Exercises

Test your understanding of multiple regression concepts with these exercises.

---

### Exercise 1: Residuals and Fitted Values

A regression leads to the fitted equation  $\hat{y} = 2 + 3x_2 + 4x_3$ .

- What is the predicted value for observation  $(x_2, x_3) = (2, 1)$ ?
  - If the actual value is  $y = 9$ , what is the residual?
  - Is this observation over-predicted or under-predicted?
- 

### Exercise 2: Partial vs. Total Effects

Suppose OLS regression on the same dataset leads to:

- Bivariate:  $\hat{y} = 6 + 5x_2$
  - Multiple:  $\hat{y} = 2 + 3x_2 + 4x_3$
- Why does the coefficient on  $x_2$  change from 5 to 3?
  - Which coefficient (5 or 3) represents the partial effect of  $x_2$ ? Explain.

- c) Under what condition would the bivariate and multiple regression coefficients on  $x_2$  be equal?
- 

### Exercise 3: R-squared and Adjusted R-squared

OLS regression of  $y$  on  $x$  for a sample of size  $n = 53$  leads to Residual SS = 20 and Total SS = 50.

- Compute  $R^2$ .
  - Compute the standard error of the regression  $s_e$ .
  - Compute the correlation between  $y$  and  $\hat{y}$ .
  - If we add 3 more regressors and the Residual SS drops to 18, compute the new  $R^2$  and adjusted  $R^2$ . Does the model improve?
- 

### Exercise 4: Information Criteria

Consider two models estimated on  $n = 29$  observations:

- Model A (2 parameters): AIC = 668.1, BIC = 670.8
  - Model B (7 parameters): AIC = 675.5, BIC = 685.1
- Which model is preferred by AIC? By BIC?
  - Why does BIC penalize Model B more heavily than AIC?
  - What economic reasoning supports the simpler model in this case?
- 

### Exercise 5: Multicollinearity Detection

A multiple regression of house prices yields the following VIF values:

Variable	VIF
Size	40.1
Bedrooms	57.8
Bathrooms	34.7
Age	21.0

- Which variables show the most severe multicollinearity?
- What consequences does high VIF have for coefficient estimates and inference?

c) Suggest a strategy to reduce multicollinearity in this model.

---

### Exercise 6: Model Building

You have data on employee wages with variables: education (years), experience (years), age, tenure (years at current firm), and gender.

- a) Why might including both age and experience create multicollinearity?
- b) Propose two alternative model specifications and explain the trade-offs.
- c) How would you use adjusted  $R^2$  and BIC to select the preferred specification?

## | Case Studies

### Case Study: Multiple Regression for Cross-Country Productivity

In this case study, you will apply multiple regression techniques to investigate how human capital and physical capital jointly explain cross-country differences in labor productivity. Using data from 108 countries over 1990-2014, you will explore correlations, estimate multiple regression models, interpret partial effects, and compare model specifications.

**Dataset:** Mendez Convergence Clubs Data

- **Source:** Mendez (2020), 108 countries, 1990-2014
- **Key variables:**
  - `lp` — Labor productivity (GDP per worker)
  - `rk` — Physical capital per worker
  - `hc` — Human capital index
  - `rgdppc` — Real GDP per capita
  - `region` — Geographic region

**Research question:** How do human capital and physical capital jointly explain cross-country productivity differences?

```

# Load the Mendez convergence clubs dataset
url = "https://raw.githubusercontent.com/quarcs-lab/mendez2020-convergence-clubs-code-data/master/assets/dat.csv"
dat = pd.read_csv(url)
dat_2014 = dat[dat['year'] == 2014].dropna(subset=['lp', 'rk', 'hc']).copy()
print(f"Cross-section sample: {len(dat_2014)} countries (year 2014)")
dat_2014.head()

```

## Task 1: Explore Productivity Data (Guided)

Explore the cross-country productivity dataset.

```

# Step 1: Create log-transformed variables
dat_2014['ln_lp'] = np.log(dat_2014['lp'])
dat_2014['ln_rk'] = np.log(dat_2014['rk'])

# Step 2: Summary statistics for key variables
print(dat_2014[['lp', 'rk', 'hc', 'ln_lp', 'ln_rk']].describe())

# Step 3: Create a scatterplot matrix
sns.pairplot(dat_2014[['ln_lp', 'ln_rk', 'hc']], diag_kind='kde')
plt.suptitle('Scatterplot Matrix: Productivity Variables', y=1.02)
plt.show()

```

### Questions:

- Which variable shows the strongest visual association with  $\ln(\text{lp})$ ?
- Do you see any nonlinear patterns or outliers?

## Task 2: Correlation Analysis (Guided)

Compute and visualize the correlation matrix.

```

# Step 1: Compute correlation matrix
corr = dat_2014[['ln_lp', 'ln_rk', 'hc']].corr()
print(corr)

# Step 2: Visualize as heatmap
sns.heatmap(corr, annot=True, fmt='.3f', cmap='coolwarm', center=0, square=True)
plt.title('Correlation Matrix: Productivity Variables')
plt.show()

```

### Questions:

- Which predictor is most strongly correlated with  $\ln(\text{lp})$ ?
- Are  $\ln(\text{rk})$  and  $\text{hc}$  correlated with each other? What implications does this have for multiple regression?

### Key Concept: Functional Form and Cross-Country Comparisons

When comparing countries with vastly different income levels, logarithmic transformations are essential. Using  $\ln(lp)$  and  $\ln(rk)$  compresses the scale so that both Luxembourg and Malawi can be meaningfully compared. Coefficients on log-transformed variables have elasticity interpretations, while coefficients on level variables (like  $hc$ ) represent semi-elasticities.

## Task 3: Multiple Regression Estimation (Semi-guided)

Estimate bivariate and multiple regression models for labor productivity.

```
# Model 1: ln(lp) ~ ln(rk) only
model1 = ols('ln_lp ~ ln_rk', data=dat_2014).fit()
print(model1.summary())

# Model 2: ln(lp) ~ hc only
model2 = ols('ln_lp ~ hc', data=dat_2014).fit()
print(model2.summary())

# Model 3: ln(lp) ~ ln(rk) + hc (multiple regression)
model3 = ols('ln_lp ~ ln_rk + hc', data=dat_2014).fit()
print(model3.summary())
```

#### Questions:

- How do the coefficients on  $\ln(rk)$  change between Model 1 and Model 3?
- Does adding human capital improve the model? Compare  $R^2$  and adjusted  $R^2$ .

## Task 4: Interpret Partial Effects (Semi-guided)

Interpret the coefficients from the multiple regression model.

```
# Display coefficients with confidence intervals
print("Model 3 Coefficients:")
print(model3.params)
print("\n95% Confidence Intervals:")
print(model3.conf_int())
```

#### Questions:

- Interpret the coefficient on  $\ln(rk)$  in Model 3. What does it mean in economic terms?
- Interpret the coefficient on  $hc$ . How does a one-unit increase in human capital relate to productivity?
- Are both coefficients statistically significant? How do you know?

## Task 5: Model Selection (Independent)

Compare models using fit statistics and information criteria.

### Your tasks:

1. Create a comparison table with  $R^2$ , adjusted  $R^2$ , AIC, and BIC for all three models
2. Which model is preferred by adjusted  $R^2$ ? By AIC? By BIC?
3. Does the parsimony principle favor one model over another?
4. Calculate VIF for Model 3 — is multicollinearity a concern?

*Hint: Use `model.rsquared`, `model.rsquared_adj`, `model.aic`, `model.bic` and `variance_inflation_factor()` from statsmodels.*

## Task 6: Development Policy Brief (Independent)

Write a 200-300 word policy brief summarizing your findings.

### Your brief should address:

1. What are the key determinants of cross-country labor productivity?
2. What is the relative importance of physical capital vs. human capital?
3. How do partial effects differ from bivariate associations?
4. What policy implications follow from your analysis?
5. What limitations should policymakers keep in mind (association vs. causation)?

### Key Concept: Multiple Regression in Development Economics

*Cross-country productivity regressions are central to development economics. By controlling for both physical capital ( $\ln(rk)$ ) and human capital ( $hc$ ), we can assess each factor's **partial contribution** to productivity. This addresses a key policy question: should developing countries invest in machines or education? Multiple regression helps disentangle these effects, though causal claims require careful identification strategies.*

## What You've Learned

In this case study, you applied the full multiple regression toolkit to cross-country productivity data:

- Explored data with scatterplot matrices and correlation analysis
- Estimated bivariate and multiple regression models, observing how coefficients change
- Interpreted partial effects of physical and human capital on productivity
- Compared model specifications using adjusted  $R^2$ , AIC, and BIC
- Connected statistical findings to development policy questions

These skills form the foundation for more advanced empirical analysis in the chapters ahead.