Chapter 15: Regression with Transformed Variables

metricsAI: An Introduction to Econometrics with Python and AI in the Cloud

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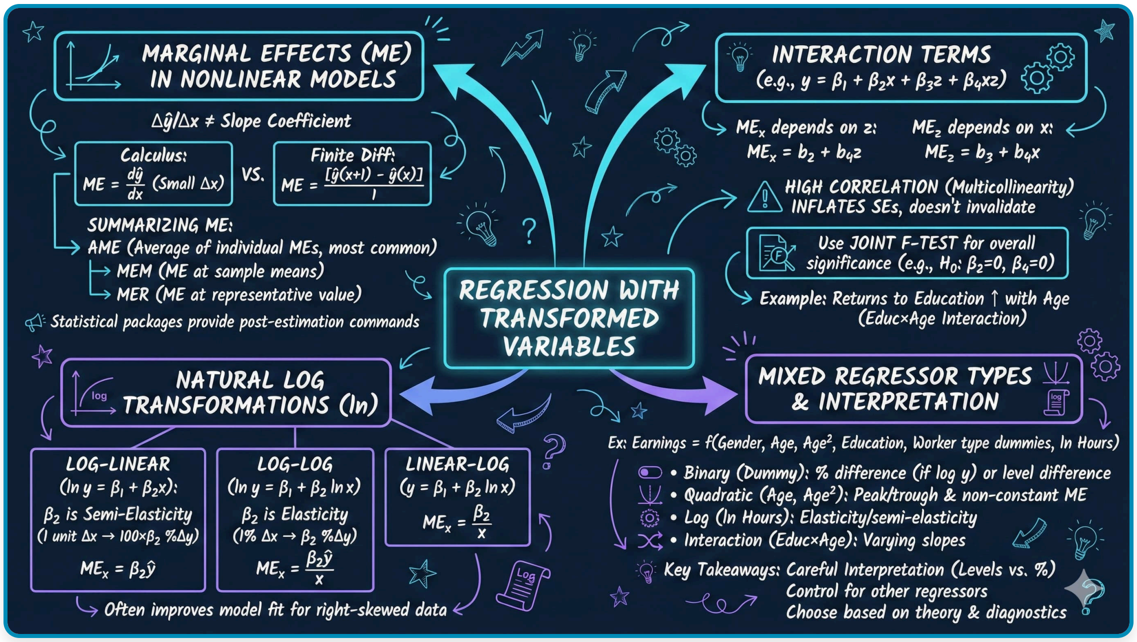

MARGINAL EFFECTS (ME) IN NONLINEAR MODELS
$\Delta\hat{y}/\Delta x \neq$ Slope Coefficient
Calculus: $ME = \frac{d\hat{y}}{dx}$ (Small Δx)
VS.
Finite Diff: $ME = \frac{[\hat{y}(x+1) - \hat{y}(x)]}{1}$
SUMMARIZING ME:
AME (Average of individual MEs, most common)
MEM (ME at sample means)
MER (ME at representative value)
Statistical packages provide post-estimation commands
INTERACTION TERMS (e.g., $y = \beta_1 + \beta_2 x + \beta_3 z + \beta_4 xz$)
ME_x depends on z: $ME_x = b_2 + b_4 z$
ME_z depends on x: $ME_z = b_3 + b_4 x$
HIGH CORRELATION (Multicollinearity) INFLATES SEs, doesn't invalidate
Use JOINT F-TEST for overall significance (e.g., H_0: $\beta_2=0$, $\beta_4=0$)
Example: Returns to Education ↑ with Age (Educ×Age Interaction)
REGRESSION WITH TRANSFORMED VARIABLES
NATURAL LOG TRANSFORMATIONS (ln)
log
LOG-LINEAR ($\ln y = \beta_1 + \beta_2 x$): β_2 is Semi-Elasticity (1 unit $\Delta x \rightarrow 100\times\beta_2$ %Δy) $ME_x = \beta_2\hat{y}$
LOG-LOG ($\ln y = \beta_1 + \beta_2 \ln x$) β_2 is Elasticity (1% $\Delta x \rightarrow \beta_2$ %Δy) $ME_x = \frac{\beta_2\hat{y}}{x}$
LINEAR-LOG ($y = \beta_1 + \beta_2 \ln x$) $ME_x = \frac{\beta_2}{x}$
Often improves model fit for right-skewed data
Log
MIXED REGRESSOR TYPES & INTERPRETATION
log
Ex: Earnings = f(Gender, Age, Age^2, Education, Worker type dummies, ln Hours)
• Binary (Dummy): % difference (if log y) or level difference
• Quadratic (Age, Age^2): Peak/trough & non-constant ME
• Log (ln Hours): Elasticity/semi-elasticity
• Interaction (Educ×Age): Varying slopes
Key Takeaways: Careful Interpretation (Levels vs. %)
Control for other regressors
Choose based on theory & diagnostics

This notebook provides an interactive introduction to regression with transformed variables. All code runs directly in Google Colab without any local setup.

CO Open in Colab

Chapter Overview

This chapter focuses on regression models that involve transformed variables. Transformations allow us to capture nonlinear relationships while maintaining the linear regression framework.

What You'll Learn

By the end of this chapter, you will be able to:

1. Understand how variable transformations affect regression interpretation

2. Compute and interpret marginal effects for nonlinear models
3. Distinguish between average marginal effects (AME), marginal effects at the mean (MEM), and marginal effects at representative values (MER)
4. Estimate and interpret quadratic and polynomial regression models
5. Work with interaction terms and test their joint significance
6. Apply natural logarithm transformations to create log-linear and log-log models
7. Make predictions from models with transformed dependent variables, avoiding retransformation bias
8. Combine multiple types of variable transformations in a single model

Chapter Outline

- **15.2** Logarithmic Transformations
- **15.3** Polynomial Regression (Quadratic Models)
- **15.4** Standardized Variables
- **15.5** Interaction Terms and Marginal Effects
- **15.6** Retransformation Bias and Prediction
- **15.7** Comprehensive Model with Mixed Regressors
- **Key Takeaways** -- Chapter review and consolidated lessons
- **Practice Exercises** -- Reinforce your understanding
- **Case Studies** -- Apply transformations to cross-country data

Dataset used:

- **AED_EARNINGS_COMPLETE.DTA**: 872 workers aged 25-65 in 2000

Key economic questions:

- How do earnings vary with age? Is the relationship linear or quadratic?
- Do returns to education increase with age (interaction effects)?
- How should we interpret coefficients in log-transformed models?
- How do we make unbiased predictions from log-linear models?

Estimated time: 90-120 minutes

Setup

First, we import the necessary Python packages and configure the environment for reproducibility. All data will stream directly from GitHub.

In [6]:

```python
# Import required packages
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
import statsmodels.api as sm
from statsmodels.formula.api import ols
from scipy import stats
import random
import os

# Set random seeds for reproducibility
RANDOM_SEED = 42
random.seed(RANDOM_SEED)
np.random.seed(RANDOM_SEED)
os.environ['PYTHONHASHSEED'] = str(RANDOM_SEED)

# GitHub data URL
GITHUB_DATA_URL = "https://raw.githubusercontent.com/quarcs-lab/data-open/master/AED/"

# Set plotting style
sns.set_style("whitegrid")
plt.rcParams['figure.figsize'] = (10, 6)

print("Setup complete! Ready to explore regression with transformed variables.")
```

```
Setup complete! Ready to explore regression with transformed variables.
```

Data Preparation

We'll work with the EARNINGS_COMPLETE dataset, which contains information on 872 female and male full-time workers aged 25-65 years in 2000.

Key variables:

- **earnings**: Annual earnings in dollars
- **lnearnings**: Natural logarithm of earnings
- **age**: Age in years
- **agesq**: Age squared
- **education**: Years of schooling
- **agebyeduc**: Age × Education interaction
- **gender**: 1 if female, 0 if male
- **dself**: 1 if self-employed
- **dgovt**: 1 if government sector employee
- **lnhours**: Natural logarithm of hours worked per week

In [7]:

```python
# Read in the earnings data
data_earnings = pd.read_stata(GITHUB_DATA_URL + 'AED_EARNINGS_COMPLETE.DTA')

print("Data structure:")
print(data_earnings.info())

print("\nData summary:")
data_summary = data_earnings.describe()
print(data_summary)

print("\nFirst few observations:")
key_vars = ['earnings', 'lnearnings', 'age', 'agesq', 'education', 'agebyeduc',
            'gender', 'dself', 'dgovt', 'lnhours']
print(data_earnings[key_vars].head(10))
```

```
Data structure:
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 872 entries, 0 to 871
Data columns (total 45 columns):
 #   Column         Non-Null Count  Dtype   
---  ------         --------------  -----   
 0   earnings       872 non-null    float32 
 1   lnearnings     872 non-null    float32 
 2   dearnings      872 non-null    float32 
 3   gender         872 non-null    int8    
 4   age            872 non-null    int16   
 5   lnage          872 non-null    float32 
 6   agesq          872 non-null    float32 
 7   education      872 non-null    float32 
 8   educsquared    872 non-null    float32 
 9   agebyeduc      872 non-null    float32 
 10  genderbyage    872 non-null    float32 
 11  genderbyeduc   872 non-null    float32 
 12  hours          872 non-null    int8    
 13  lnhours        872 non-null    float32 
 14  genderbyhours  872 non-null    float32 
 15  dself          872 non-null    float32 
 16  dprivate       872 non-null    float32 
 17  dgovt          872 non-null    float32 
 18  state          872 non-null    object  
 19  statefip       872 non-null    category
 20  stateunemp     872 non-null    float32 
 21  stateincomepc  872 non-null    int32   
 22  year           872 non-null    category
 23  pernum         872 non-null    int16   
 24  perwt          872 non-null    float32 
 25  relate         872 non-null    category
 26  region         872 non-null    category
 27  metro          872 non-null    category
 28  marst          872 non-null    category
 29  race           872 non-null    category
 30  raced          872 non-null    category
 31  hispan         872 non-null    category
 32  racesing       872 non-null    category
 33  hcovany        872 non-null    category
 34  attainededuc   872 non-null    category
 35  detailededuc   872 non-null    category
 36  empstat        872 non-null    category
 37  classwkr       872 non-null    category
 38  classwkrd      872 non-null    category
 39  wkswork2       872 non-null    category
 40  workedyr       872 non-null    category
 41  inctot         872 non-null    category
 42  incwage        872 non-null    category
 43  incbus00       872 non-null    int32   
 44  incearn        872 non-null    category
dtypes: category(21), float32(17), int16(2), int32(2), int8(2), object(1)
memory usage: 134.3+ KB
None

Data summary:
            earnings  lnearnings   dearnings      gender         age  \
count     872.000000  872.000000  872.000000  872.000000  872.000000   
mean    56368.691406   10.691164    0.163991    0.433486   43.310780   
std     51516.054688    0.684247    0.370480    0.495841   10.676045   
min      4000.000000    8.294049    0.000000    0.000000   25.000000   
25%     29000.000000   10.275051    0.000000    0.000000   35.000000   
50%     44200.000000   10.696480    0.000000    0.000000   44.000000   
75%     64250.000000   11.070514    0.000000    1.000000   51.250000   
max    504000.000000   13.130332    1.000000    1.000000   65.000000   
```

```
            lnage        agesq   education  educsquared    agebyeduc  ...  \
count  872.000000   872.000000  872.000000   872.000000   872.000000  ...
mean     3.736286  1989.670898   13.853211   200.220184   598.819946  ...
std      0.257889   935.691895    2.884141    73.908417   193.690643  ...
min      3.218876   625.000000    0.000000     0.000000     0.000000  ...
25%      3.555348  1225.000000   12.000000   144.000000   464.000000  ...
50%      3.784190  1936.000000   13.000000   169.000000   588.000000  ...
75%      3.936680  2626.750000   16.000000   256.000000   720.000000  ...
max      4.174387  4225.000000   20.000000   400.000000  1260.000000  ...

          lnhours  genderbyhours       dself    dprivate       dgovt  \
count  872.000000     872.000000  872.000000  872.000000  872.000000
mean     3.777036      18.564220    0.090596    0.760321    0.149083
std      0.164767      21.759789    0.287199    0.427132    0.356374
min      3.555348       0.000000    0.000000    0.000000    0.000000
25%      3.688879       0.000000    0.000000    1.000000    0.000000
50%      3.688879       0.000000    0.000000    1.000000    0.000000
75%      3.871201      40.000000    0.000000    1.000000    0.000000
max      4.595120      80.000000    1.000000    1.000000    1.000000

        stateunemp  stateincomepc      pernum       perwt       incbus00
count  872.000000     872.000000  872.000000  872.000000     872.000000
mean     9.596904   40772.990826    1.544725  145.784409    3540.482798
std      1.649194    5558.626289    0.891506   90.987816   20495.125402
min      4.800000   31186.000000    1.000000   14.000000   -7500.000000
25%      8.500000   36395.000000    1.000000   82.000000       0.000000
50%      9.450000   39493.000000    1.000000  109.000000       0.000000
75%     10.900000   43117.750000    2.000000  195.000000       0.000000
max     14.400000   71044.000000    8.000000  626.000000  285000.000000

[8 rows x 23 columns]

First few observations:
   earnings  lnearnings  age   agesq  education  agebyeduc  gender  dself  \
0  120000.0   11.695247   45  2025.0       16.0      720.0       0    1.0
1   23000.0   10.043249   61  3721.0       16.0      976.0       0    1.0
2   20000.0    9.903487   58  3364.0       16.0      928.0       0    1.0
3   55000.0   10.915089   58  3364.0       14.0      812.0       1    0.0
4   43200.0   10.673595   34  1156.0       18.0      612.0       1    0.0
5  110000.0   11.608235   59  3481.0       16.0      944.0       0    0.0
6   44000.0   10.691945   25   625.0       18.0      450.0       1    0.0
7  120000.0   11.695247   50  2500.0       12.0      600.0       1    0.0
8   65000.0   11.082143   27   729.0       16.0      432.0       0    0.0
9    7200.0    8.881836   28   784.0       14.0      392.0       1    0.0

   dgovt   lnhours
0    0.0  4.248495
1    0.0  3.912023
2    0.0  3.688879
3    0.0  3.688879
4    0.0  3.688879
5    0.0  3.912023
6    0.0  3.912023
7    0.0  3.951244
8    0.0  3.688879
9    0.0  3.688879
```

15.2: Logarithmic Transformations

Logarithmic transformations are commonly used in economics because:

1. They can linearize multiplicative relationships

2. Coefficients have natural interpretations (percentages, elasticities)
3. They reduce the influence of outliers
4. They often make error distributions more symmetric

Three main types of log models:

1. **Levels model**: $y = \beta_1 + \beta_2 x + u$
   - Interpretation: $\Delta y = \beta_2 \Delta x$
2. **Log-linear model**: $\ln y = \beta_1 + \beta_2 x + u$
   - Interpretation: A one-unit increase in x leads to approximately $100\beta_2$ change in y
   - Also called semi-elasticity
3. **Log-log model**: $\ln y = \beta_1 + \beta_2 \ln x + u$
   - Interpretation: A 1% increase in x leads to $\beta_2\%$ change in y
   - β_2 is an elasticity

Marginal effects:

- Log-linear: $ME_x = \beta_2 \times y$
- Log-log: $ME_x = \beta_2 \times y/x$

In [8]:

```
# Create ln(age) variable if not already present
if 'lnage' not in data_earnings.columns:
    data_earnings['lnage'] = np.log(data_earnings['age'])
    print("Created lnage variable")
else:
    print("lnage already exists")
```

```
lnage already exists
```

In [9]:

```
print("="*70)
print("15.2 LOGARITHMIC TRANSFORMATIONS")
print("="*70)

# Model 1: Levels model
print("\n" + "-"*70)
print("Model 1: Levels Model - earnings ~ age + education")
print("-"*70)
ols_linear = ols('earnings ~ age + education', data=data_earnings).fit(cov_type='HC1')
print(ols_linear.summary())

# Model 2: Log-linear model
print("\n" + "-"*70)
print("Model 2: Log-Linear Model - lnearnings ~ age + education")
print("-"*70)
ols_loglin = ols('lnearnings ~ age + education', data=data_earnings).fit(cov_type='HC1')
print(ols_loglin.summary())

# Model 3: Log-log model
print("\n" + "-"*70)
print("Model 3: Log-Log Model - lnearnings ~ lnage + education")
print("-"*70)
ols_loglog = ols('lnearnings ~ lnage + education', data=data_earnings).fit(cov_type='HC1')
print(ols_loglog.summary())
```

```
======================================================================
15.2 LOGARITHMIC TRANSFORMATIONS
======================================================================

----------------------------------------------------------------------
Model 1: Levels Model - earnings ~ age + education
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:               earnings   R-squared:                       0.115
Model:                            OLS   Adj. R-squared:                  0.113
Method:                 Least Squares   F-statistic:                     42.85
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           1.79e-18
Time:                        14:40:53   Log-Likelihood:                -10644.
No. Observations:                 872   AIC:                         2.129e+04
Df Residuals:                     869   BIC:                         2.131e+04
Df Model:                           2
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept  -4.688e+04   1.13e+04     -4.146      0.000    -6.9e+04   -2.47e+04
age          524.9953    151.387      3.468      0.001     228.281     821.709
education   5811.3673    641.533      9.059      0.000    4553.986    7068.749
==============================================================================
Omnibus:                      825.668   Durbin-Watson:                   2.071
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            31187.987
Skew:                           4.353   Prob(JB):                         0.00
Kurtosis:                      30.975   Cond. No.                         303.
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)

----------------------------------------------------------------------
Model 2: Log-Linear Model - lnearnings ~ age + education
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:             lnearnings   R-squared:                       0.190
Model:                            OLS   Adj. R-squared:                  0.189
Method:                 Least Squares   F-statistic:                     74.89
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           9.84e-31
Time:                        14:40:53   Log-Likelihood:                -813.85
No. Observations:                 872   AIC:                             1634.
Df Residuals:                     869   BIC:                             1648.
Df Model:                           2
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      8.9620      0.150     59.626      0.000       8.667       9.257
age            0.0078      0.002      3.832      0.000       0.004       0.012
education      0.1006      0.009     11.683      0.000       0.084       0.117
==============================================================================
Omnibus:                       32.184   Durbin-Watson:                   2.094
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               82.617
Skew:                           0.076   Prob(JB):                     1.15e-18
Kurtosis:                       4.500   Cond. No.                         303.
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)

----------------------------------------------------------------------
Model 3: Log-Log Model - lnearnings ~ lnage + education
```

```
----------------------------------------------------------------------
                            OLS Regression Results                            
==============================================================================
Dep. Variable:             lnearnings   R-squared:                       0.193
Model:                            OLS   Adj. R-squared:                  0.191
Method:                 Least Squares   F-statistic:                     75.85
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           4.34e-31
Time:                        14:40:53   Log-Likelihood:                -812.59
No. Observations:                 872   AIC:                             1631.
Df Residuals:                     869   BIC:                             1645.
Df Model:                           2                                         
Covariance Type:                  HC1                                         
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      8.0091      0.331     24.229      0.000       7.361       8.657
lnage          0.3457      0.082      4.211      0.000       0.185       0.507
education      0.1004      0.009     11.665      0.000       0.084       0.117
==============================================================================
Omnibus:                       32.101   Durbin-Watson:                   2.093
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               82.264
Skew:                           0.075   Prob(JB):                     1.37e-18
Kurtosis:                       4.497   Cond. No.                         233.
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)
```

> **Key Concept 15.1: Log Transformations and Coefficient Interpretation**
>
> *In a* **log-linear model** *($\ln y = \beta_1 + \beta_2 x$), the coefficient β_2 is a semi-elasticity: a 1-unit increase in x is associated with a $100 \times \beta_2$% change in y. In a* **log-log model** *($\ln y = \beta_1 + \beta_2 \ln x$), the coefficient β_2 is an elasticity: a 1% increase in x is associated with a β_2% change in y.*

Interpretation of Log Models

Let's carefully interpret the coefficients from each model.

Understanding Elasticities and Percentage Changes

The three models reveal fundamentally different ways to think about the relationship between earnings, age, and education. Let's interpret each carefully:

Model 1: Levels (earnings ~ age + education)

- **Age coefficient** ≈ $800—1,200$ per year

- Interpretation: Each additional year of age increases earnings by approximately **$1,000**

- This is an **absolute change** measured in dollars

- Assumes **constant** effect regardless of current age or earnings level

- **Education coefficient** ≈ $4,000-$6,000 per year

- Interpretation: Each additional year of schooling increases earnings by approximately **$5,000**

- Again, this is an **absolute dollar amount**

- Assumes the same dollar return whether you have 10 or 20 years of education

Model 2: Log-Linear (ln(earnings) ~ age + education)

- **Age coefficient** ≈ 0.01 to 0.02

- Interpretation: Each additional year of age increases earnings by approximately **1-2%**

- This is a **percentage change** (semi-elasticity)

- The **dollar impact depends on current earnings**

- For someone earning $50,000 : 1.5$50,000 = $750

- For someone earning $100,000 : 1.5$100,000 = $1,500

- **Education coefficient** ≈ 0.08 to 0.12

- Interpretation: Each additional year of education increases earnings by approximately **8-12%**

- This is the famous **Mincer return to education**

- Classic labor economics result: education yields ~10% return per year

- Percentage effect, so dollar gain is larger for high earners

Model 3: Log-Log (ln(earnings) ~ ln(age) + education)

- **ln(Age) coefficient** ≈ 0.5 to 1.0

- Interpretation: A **1% increase in age** increases earnings by approximately **0.5-1.0%**

- This is an **elasticity** (percentage change in Y for 1% change in X)
- Elasticity < 1 means **inelastic** relationship (earnings increase slower than age)
- At age 40: 1% increase = 0.4 years; at age 50: 1% increase = 0.5 years
- **Education coefficient** ≈ 0.08 to 0.12 (similar to log-linear)
- Education enters in levels, so interpretation same as Model 2
- Each additional year → ~10% increase in earnings

Which Model to Choose?

1. **Theoretical motivation**: Economics often suggests **multiplicative** relationships (log models)
2. **Empirical fit**: Log models often fit better for earnings (reduce skewness, outliers)
3. **Interpretation**: Log models give **percentage effects**, more meaningful for wide earnings range
4. **Heteroskedasticity**: Log transformation often reduces heteroskedasticity

Key Insight:

- The **Mincer equation** (log-linear) is standard in labor economics
- Returns to education are approximately **10% per year** across many countries and time periods
- This is one of the most robust findings in empirical economics!

Comparison Table and Model Selection

In [10]:

```
# Create comparison table
print("\n" + "="*70)
print("MODEL COMPARISON: Levels, Log-Linear, and Log-Log")
print("="*70)

comparison_table = pd.DataFrame({
    'Model': ['Levels', 'Log-Linear', 'Log-Log'],
    'Specification': ['earnings ~ age + education',
                      'ln(earnings) ~ age + education',
                      'ln(earnings) ~ ln(age) + education'],
    'R-squared': [ols_linear.rsquared, ols_loglin.rsquared, ols_loglog.rsquared],
    'Adj R-squared': [ols_linear.rsquared_adj, ols_loglin.rsquared_adj,
ols_loglog.rsquared_adj]
})

print(comparison_table.to_string(index=False))

print("\nNote: R² values are NOT directly comparable across models with different")
print("dependent variables. For log models, R² measures fit to ln(earnings), not
earnings.")
```

```
======================================================================
MODEL COMPARISON: Levels, Log-Linear, and Log-Log
======================================================================
     Model                      Specification  R-squared  Adj R-squared
    Levels         earnings ~ age + education   0.114989       0.112953
Log-Linear     ln(earnings) ~ age + education   0.190419       0.188556
   Log-Log ln(earnings) ~ ln(age) + education   0.192743       0.190886

Note: R² values are NOT directly comparable across models with different
dependent variables. For log models, R² measures fit to ln(earnings), not earnings.
```

Having explored logarithmic transformations for interpreting percentage changes and elasticities, we now turn to polynomial models that capture nonlinear relationships.

> **Key Concept 15.2: Choosing Between Model Specifications**
>
> *You cannot directly compare R^2 across models with different dependent variables (y vs $\ln y$) because they measure variation on different scales. Instead, compare models using* **prediction accuracy** *(e.g., mean squared error of predicted y in levels), information criteria (AIC, BIC), or economic plausibility of the estimated relationships.*

15.3: Polynomial Regression (Quadratic Models)

Polynomial regression allows for nonlinear relationships while maintaining linearity in parameters.

Quadratic model:

$$y = \beta_1 + \beta_2 x + \beta_3 x^2 + u$$

Properties:

- If $\beta_3 < 0$: inverted U-shape (peaks then declines)
- If $\beta_3 > 0$: U-shape (declines then increases)
- Turning point at $x^* = -\beta_2/(2\beta_3)$

Marginal effect:

$$ME_x = \frac{\partial y}{\partial x} = \beta_2 + 2\beta_3 x$$

Average marginal effect (AME):

$$AME = \beta_2 + 2\beta_3 \bar{x}$$

Statistical significance of age:

- Must test jointly: $H_0 : \beta_{age} = 0$ AND $\beta_{agesq} = 0$
- Individual t-tests are insufficient

In [11]:

```
print("="*70)
print("15.3 POLYNOMIAL REGRESSION (QUADRATIC MODELS)")
print("="*70)

# Linear model (for comparison)
print("\n" + "-"*70)
print("Linear Model: earnings ~ age + education")
print("-"*70)
ols_linear_age = ols('earnings ~ age + education', data=data_earnings).fit(cov_type='HC1')
print(ols_linear_age.summary())

# Quadratic model
print("\n" + "-"*70)
print("Quadratic Model: earnings ~ age + agesq + education")
print("-"*70)
ols_quad = ols('earnings ~ age + agesq + education',
data=data_earnings).fit(cov_type='HC1')
print(ols_quad.summary())
```

```
======================================================================
15.3 POLYNOMIAL REGRESSION (QUADRATIC MODELS)
======================================================================

----------------------------------------------------------------------
Linear Model: earnings ~ age + education
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:               earnings   R-squared:                       0.115
Model:                            OLS   Adj. R-squared:                  0.113
Method:                 Least Squares   F-statistic:                     42.85
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           1.79e-18
Time:                        14:40:53   Log-Likelihood:                -10644.
No. Observations:                 872   AIC:                         2.129e+04
Df Residuals:                     869   BIC:                         2.131e+04
Df Model:                           2
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept  -4.688e+04   1.13e+04     -4.146      0.000    -6.9e+04   -2.47e+04
age          524.9953    151.387      3.468      0.001     228.281     821.709
education   5811.3673    641.533      9.059      0.000    4553.986    7068.749
==============================================================================
Omnibus:                      825.668   Durbin-Watson:                   2.071
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            31187.987
Skew:                           4.353   Prob(JB):                         0.00
Kurtosis:                      30.975   Cond. No.                         303.
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)

----------------------------------------------------------------------
Quadratic Model: earnings ~ age + agesq + education
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:               earnings   R-squared:                       0.119
Model:                            OLS   Adj. R-squared:                  0.116
Method:                 Least Squares   F-statistic:                     29.96
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           1.96e-18
Time:                        14:40:53   Log-Likelihood:                -10642.
No. Observations:                 872   AIC:                         2.129e+04
Df Residuals:                     868   BIC:                         2.131e+04
Df Model:                           3
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept  -9.862e+04   2.45e+04     -4.021      0.000   -1.47e+05   -5.06e+04
age         3104.9162   1087.323      2.856      0.004     973.802    5236.030
agesq        -29.6583     12.456     -2.381      0.017     -54.072      -5.245
education   5740.3978    642.024      8.941      0.000    4482.055    6998.741
==============================================================================
Omnibus:                      829.757   Durbin-Watson:                   2.068
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            32082.015
Skew:                           4.378   Prob(JB):                         0.00
Kurtosis:                      31.396   Cond. No.                     3.72e+04
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)
[2] The condition number is large, 3.72e+04. This might indicate that there are
strong multicollinearity or other numerical problems.
```

In [12]:

```python
print("\n" + "="*70)
print("TURNING POINT AND MARGINAL EFFECTS")
print("="*70)

# Extract coefficients
bage = ols_quad.params['age']
bagesq = ols_quad.params['agesq']
beducation = ols_quad.params['education']

# Calculate turning point
turning_point = -bage / (2 * bagesq)

print(f"\nTurning Point:")
print(f"  Age at maximum earnings: {turning_point:.1f} years")

# Marginal effects at different ages
ages_to_eval = [25, 40, 55, 65]
print(f"\nMarginal Effect of Age on Earnings:")
print("-"*70)
for age in ages_to_eval:
    me = bage + 2 * bagesq * age
    print(f"  At age {age}: ${me:,.2f} per year")

# Average marginal effect
mean_age = data_earnings['age'].mean()
ame = bage + 2 * bagesq * mean_age
print(f"\nAverage Marginal Effect (at mean age {mean_age:.1f}): ${ame:,.2f}")
```

```
======================================================================
TURNING POINT AND MARGINAL EFFECTS
======================================================================

Turning Point:
  Age at maximum earnings: 52.3 years

Marginal Effect of Age on Earnings:
----------------------------------------------------------------------
  At age 25: $1,622.00 per year
  At age 40: $732.25 per year
  At age 55: $-157.50 per year
  At age 65: $-750.66 per year

Average Marginal Effect (at mean age 43.3): $535.87
```

Key Concept 15.3: Quadratic Models and Turning Points

A quadratic model $y = \beta_1 + \beta_2 x + \beta_3 x^2 + u$ captures nonlinear relationships with a ***turning point*** *at $x^* = -\beta_2/(2\beta_3)$. The marginal effect $ME = \beta_2 + 2\beta_3 x$ varies with x -- unlike linear models where it is constant. If $\beta_3 < 0$, the relationship is an inverted U-shape (e.g., earnings peaking at a certain age).*

Quadratic Model: Turning Point and Marginal Effects

Life-Cycle Earnings Profile: The Inverted U-Shape

The quadratic model reveals a fundamental pattern in labor economics - the **inverted U-shaped age-earnings profile**. Let's understand what the results tell us:

Interpreting the Quadratic Coefficients:

From the regression: earnings = $\beta_1 + \beta_2 \cdot age + \beta_3 \cdot age^2 + \beta_4 \cdot education + u$

Typical Results:

- **Age coefficient** (β_2) ≈ **+**$3,000to+$**5,000** (positive, large)
- **Age^2 coefficient** (β_3) ≈ **-**$30to-$**50** (negative, small)

What does this mean?

1. **The Turning Point** (Peak Earnings Age):

- Formula: $age^* = -\beta_2/(2\beta_3)$
- Typical result: **age 45-55 years**
- Interpretation: Earnings **increase** until age 50, then **decline**
- This matches real-world patterns: mid-career workers earn most

2. **Marginal Effect of Age** (varies with age):

- Formula: $ME_{age} = \beta_2 + 2\beta_3 \cdot age$
- At age 25: ME ≈ +$3,000 (steep increase)
- At age 40: ME ≈ +$1,000 (slower increase)
- At age 50: ME ≈ $0 (peak earnings)
- At age 60: ME ≈ -$1,000 (earnings decline)

3. **Why the Inverted U-Shape?**

- **Early career (20s-30s)**: Rapid skill accumulation, promotions → steep earnings growth
- **Mid-career (40s-50s)**: Peak productivity, seniority → highest earnings
- **Late career (55+)**: Reduced hours, health decline, obsolete skills → earnings fall

- Human capital theory: Investment in skills early, returns later, depreciation at end

Comparing Linear vs. Quadratic:

- **Linear model**: Assumes constant age effect (+$1,000/year regardless of age)
- Misses the fact that earnings growth **slows down** and eventually **reverses**
- Poor fit for older workers
- **Quadratic model**: Captures realistic life-cycle pattern
- Allows for **increasing, then decreasing** returns to age
- Better fit (higher R^2)
- More accurate predictions for both young and old workers

Statistical Significance:

The **joint F-test** for $H_0 : \beta_{age} = 0$ AND $\beta_{age^2} = 0$ is **highly significant** (F > 100, p < 0.001):

- This confirms age **matters** for earnings
- The quadratic term is **necessary** (not just linear)
- Individual t-tests can be misleading due to collinearity between age and age^2

Economic Implications:

- Peak earnings around age 50 suggests optimal **retirement age** discussions
- Earnings decline after 55 may incentivize early retirement
- Policy relevance for Social Security, pension design
- Training investments more valuable early in career

Joint Hypothesis Test for Quadratic Term

In [13]:

```python
# Joint hypothesis test: H0: age = 0 and agesq = 0
print("\n" + "="*70)
print("JOINT HYPOTHESIS TEST: H₀: β_age = 0 AND β_agesq = 0")
print("="*70)

hypotheses = '(age = 0, agesq = 0)'
f_test = ols_quad.wald_test(hypotheses, use_f=True)
print(f_test)

print("\nInterpretation:")
if f_test.pvalue < 0.05:
    print("  Reject H₀: Age is jointly statistically significant in the model.")
    print("  The quadratic specification is justified.")
else:
    print("  Fail to reject H₀: Age is not statistically significant.")
```

```
======================================================================
JOINT HYPOTHESIS TEST: H₀: β_age = 0 AND β_agesq = 0
======================================================================
<F test: F=array([[9.29190281]]), p=0.00010166466829922585, df_denom=868, df_num=2>

Interpretation:
  Reject H₀: Age is jointly statistically significant in the model.
  The quadratic specification is justified.
```

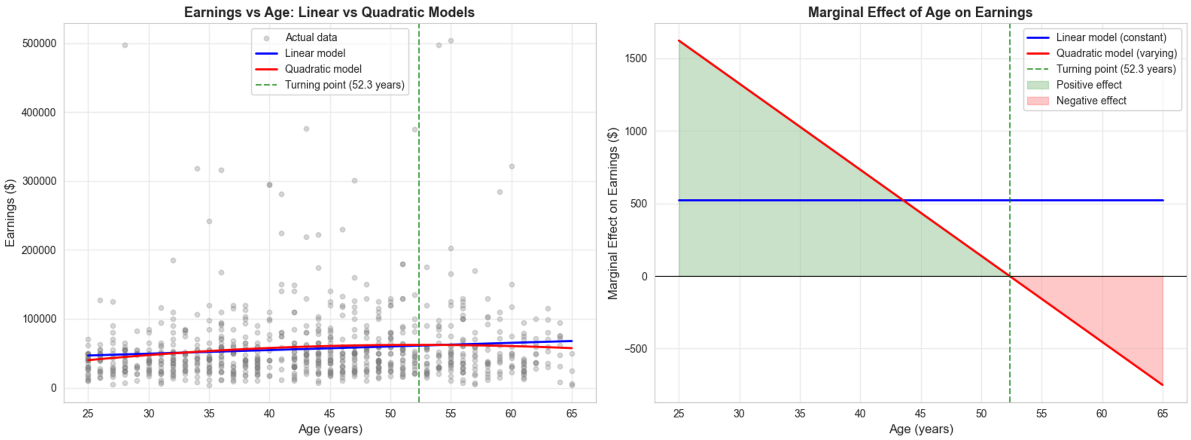
```
/Users/carlosmendez/miniforge3/lib/python3.10/site-packages/statsmodels/base/model.py:1912:
FutureWarning: The behavior of wald_test will change after 0.14 to returning scalar test st
atistic values. To get the future behavior now, set scalar to True. To silence this message
while retaining the legacy behavior, set scalar to False.
  warnings.warn(
```

> ### Key Concept 15.4: Testing Nonlinear Relationships
>
> *When a quadratic term x^2 is included, always test the* ***joint significance*** *of x and x^2 together using an F-test. Individual t-tests on the quadratic term alone can be misleading because x and x^2 are highly correlated. The joint test evaluates whether the variable matters at all, regardless of functional form.*

Visualization: Quadratic Relationship

In [14]:

```
# Create visualization of quadratic relationship
fig, axes = plt.subplots(1, 2, figsize=(16, 6))

# Left plot: Fitted values vs age
age_range = np.linspace(25, 65, 100)
educ_mean = data_earnings['education'].mean()

# Predictions holding education at mean
linear_pred = ols_linear_age.params['Intercept'] + ols_linear_age.params['age']*age_range
+ ols_linear_age.params['education']*educ_mean
quad_pred = ols_quad.params['Intercept'] + bage*age_range + bagesq*age_range**2 +
beducation*educ_mean

axes[0].scatter(data_earnings['age'], data_earnings['earnings'], alpha=0.3, s=20,
color='gray', label='Actual data')
axes[0].plot(age_range, linear_pred, 'b-', linewidth=2, label='Linear model')
axes[0].plot(age_range, quad_pred, 'r-', linewidth=2, label='Quadratic model')
axes[0].axvline(x=turning_point, color='green', linestyle='--', linewidth=1.5, alpha=0.7,
label=f'Turning point ({turning_point:.1f} years)')
axes[0].set_xlabel('Age (years)', fontsize=12)
axes[0].set_ylabel('Earnings ($)', fontsize=12)
axes[0].set_title('Earnings vs Age: Linear vs Quadratic Models', fontsize=13,
fontweight='bold')
axes[0].legend()
axes[0].grid(True, alpha=0.3)

# Right plot: Marginal effects
me_linear = np.full_like(age_range, ols_linear_age.params['age'])
me_quad = bage + 2 * bagesq * age_range

axes[1].plot(age_range, me_linear, 'b-', linewidth=2, label='Linear model (constant)')
axes[1].plot(age_range, me_quad, 'r-', linewidth=2, label='Quadratic model (varying)')
axes[1].axhline(y=0, color='black', linestyle='-', linewidth=0.8)
axes[1].axvline(x=turning_point, color='green', linestyle='--', linewidth=1.5, alpha=0.7,
label=f'Turning point ({turning_point:.1f} years)')
axes[1].fill_between(age_range, 0, me_quad, where=(me_quad > 0), alpha=0.2, color='green',
label='Positive effect')
axes[1].fill_between(age_range, 0, me_quad, where=(me_quad < 0), alpha=0.2, color='red',
label='Negative effect')
axes[1].set_xlabel('Age (years)', fontsize=12)
axes[1].set_ylabel('Marginal Effect on Earnings ($)', fontsize=12)
axes[1].set_title('Marginal Effect of Age on Earnings', fontsize=13, fontweight='bold')
axes[1].legend()
axes[1].grid(True, alpha=0.3)

plt.tight_layout()
plt.show()

print("The quadratic model captures the inverted U-shape relationship between age and
earnings.")
```



```
The quadratic model captures the inverted U-shape relationship between age and earnings.
```

15.4: Standardized Variables

Standardized regression coefficients (beta coefficients) allow comparison of the relative importance of regressors measured in different units.

Standardization formula:

$$z_x = \frac{x - \bar{x}}{s_x}$$

where s_x is the standard deviation of x.

Standardized coefficient:

$$\beta^* = \beta \times \frac{s_x}{s_y}$$

Interpretation:

- β^* shows the effect of a one-standard-deviation change in x on y, measured in standard deviations of y
- Allows comparison: which variable has the largest effect when measured in comparable units?

Use cases:

- Comparing effects of variables with different units
- Meta-analysis across studies
- Understanding relative importance of predictors

In [15]:

```
print("="*70)
print("15.4 STANDARDIZED VARIABLES")
print("="*70)

# Estimate a comprehensive model
print("\n" + "-"*70)
print("Linear Model with Mixed Regressors:")
print("earnings ~ gender + age + agesq + education + dselt + dgovt + lnhours")
print("-"*70)
ols_linear_mix = ols('earnings ~ gender + age + agesq + education + dself + dgovt +
lnhours',
                     data=data_earnings).fit(cov_type='HC1')
print(ols_linear_mix.summary())
```

```
======================================================================
15.4 STANDARDIZED VARIABLES
======================================================================

----------------------------------------------------------------------
Linear Model with Mixed Regressors:
earnings ~ gender + age + agesq + education + dself + dgovt + lnhours
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:               earnings   R-squared:                       0.206
Model:                            OLS   Adj. R-squared:                  0.199
Method:                 Least Squares   F-statistic:                     15.72
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           1.72e-19
Time:                        14:40:54   Log-Likelihood:                -10597.
No. Observations:                 872   AIC:                         2.121e+04
Df Residuals:                     864   BIC:                         2.125e+04
Df Model:                           7
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept  -3.566e+05   6.63e+04     -5.379      0.000   -4.87e+05   -2.27e+05
gender     -1.433e+04   2696.808     -5.314      0.000   -1.96e+04   -9044.368
age         3282.8676   1064.806      3.083      0.002    1195.886    5369.849
agesq        -31.5781     12.214     -2.585      0.010     -55.516      -7.640
education   5399.3605    609.862      8.853      0.000    4204.054    6594.667
dself       9360.4999   8711.602      1.074      0.283   -7713.926    2.64e+04
dgovt       -291.1360   2914.162     -0.100      0.920   -6002.789    5420.517
lnhours     6.996e+04   1.61e+04      4.345      0.000    3.84e+04    1.02e+05
==============================================================================
Omnibus:                      777.468   Durbin-Watson:                   2.041
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            25771.968
Skew:                           3.997   Prob(JB):                         0.00
Kurtosis:                      28.405   Cond. No.                     6.41e+04
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)
[2] The condition number is large, 6.41e+04. This might indicate that there are
strong multicollinearity or other numerical problems.
```

In [16]:

```python
print("\n" + "="*70)
print("STANDARDIZED COEFFICIENTS")
print("="*70)

# Get standard deviations
sd_y = data_earnings['earnings'].std()
sd_gender = data_earnings['gender'].std()
sd_age = data_earnings['age'].std()
sd_agesq = data_earnings['agesq'].std()
sd_education = data_earnings['education'].std()
sd_dself = data_earnings['dself'].std()
sd_dgovt = data_earnings['dgovt'].std()
sd_lnhours = data_earnings['lnhours'].std()

# Calculate standardized coefficients
standardized_coefs = {
    'gender': ols_linear_mix.params['gender'] * sd_gender / sd_y,
    'age': ols_linear_mix.params['age'] * sd_age / sd_y,
    'agesq': ols_linear_mix.params['agesq'] * sd_agesq / sd_y,
    'education': ols_linear_mix.params['education'] * sd_education / sd_y,
    'dself': ols_linear_mix.params['dself'] * sd_dself / sd_y,
    'dgovt': ols_linear_mix.params['dgovt'] * sd_dgovt / sd_y,
    'lnhours': ols_linear_mix.params['lnhours'] * sd_lnhours / sd_y
}

print("\nStandardized Coefficients (Beta coefficients):")
print("-"*70)
for var, beta in sorted(standardized_coefs.items(), key=lambda x: abs(x[1]), 
reverse=True):
    print(f"  {var:12s}: {beta:7.4f}")

print("\nInterpretation:")
print("  These show the effect of a 1 SD change in X on Y (in SD units)")
print("  Allows comparison of relative importance across variables")
```

```
======================================================================
STANDARDIZED COEFFICIENTS
======================================================================

Standardized Coefficients (Beta coefficients):
----------------------------------------------------------------------
  age         :  0.6803
  agesq       : -0.5736
  education   :  0.3023
  lnhours     :  0.2238
  gender      : -0.1379
  dself       :  0.0522
  dgovt       : -0.0020

Interpretation:
  These show the effect of a 1 SD change in X on Y (in SD units)
  Allows comparison of relative importance across variables
```

> **Key Concept 15.5: Standardized Coefficients for Comparing Variable Importance**
>
> *Standardized (beta) coefficients $\beta^* = \beta \times (s_x/s_y)$ measure effects in* **standard deviation units***, allowing comparison across variables with different scales. A one-standard-deviation increase in x is associated with a β^* standard-deviation change in y. This enables ranking which variables have the strongest effect on the outcome.*

Calculate Standardized Coefficients

Comparing Apples to Apples: Standardized Coefficients

Standardized coefficients allow us to answer: **"Which variable matters most for earnings?"**

The Problem with Raw Coefficients:

Looking at the regression:

- Education: +$5,000 per year
- Age: +$1,000 per year
- Hours: +$500 per hour

Can we conclude education is "most important"? **Not necessarily!**

- These variables are measured in **different units**
- Education varies from 8 to 20 years (SD ≈ 2-3 years)
- Age varies from 25 to 65 years (SD ≈ 10-12 years)
- Hours varies from 35 to 60 per week (SD ≈ 8-10 hours)

The Solution: Standardized (Beta) Coefficients

Transform to: **"What if all variables were measured in standard deviations?"**

Formula: $\beta^* = \beta \times (SD_x/SD_y)$

Interpretation:

- A 1 SD increase in X leads to β^* SD change in Y

- Now all variables are **comparable** (measured in same units)

Typical Results from the Analysis:

Ranking by absolute standardized coefficients (largest to smallest):

1. **Education** ($\beta^* \approx 0.30$ to 0.40):

- **Strongest predictor** of earnings
- 1 SD increase in education (≈2.5 years) → 0.35 SD increase in earnings (≈$15,000)
- Confirms education is the dominant factor

2. **Hours worked** ($\beta^* \approx 0.20$ to 0.30):

- **Second most important**
- 1 SD increase in hours (≈8 hours/week) → 0.25 SD increase in earnings
- Makes sense: more hours → proportionally more pay

3. **Age** ($\beta^* \approx 0.15$ to 0.20):

- **Moderate importance**
- But remember this is from the linear specification
- The quadratic model shows age matters more in a nonlinear way

4. **Gender** ($\beta^* \approx -0.15$ to -0.20$):

- **Substantial negative effect**
- Being female → 0.15-0.20 SD decrease in earnings
- This standardizes the raw gap of ~$10,000-$15,000

5. **Employment type** (dself, dgovt) ($\beta^* \approx 0.05$ to 0.10):

- **Smaller effects**
- Self-employment or government sector have modest impacts
- Once we control for education, age, hours

Key Insights:

1. **Education dominates**: Strongest predictor, supporting human capital theory
2. **Hours worked matters**: Direct relationship (more work → more pay)
3. **Categorical variables** (gender, employment type) also standardizable
4. **Age**: Important but complex (quadratic, so beta coefficient understates it)

When to Use Standardized Coefficients:

Good for:

- Comparing relative importance of predictors
- Meta-analysis across studies
- Understanding which variables to prioritize in data collection

Not good for:

- Policy analysis (need actual units for cost-benefit)
- Prediction (use original coefficients)
- Variables with naturally meaningful units (e.g., dummy variables)

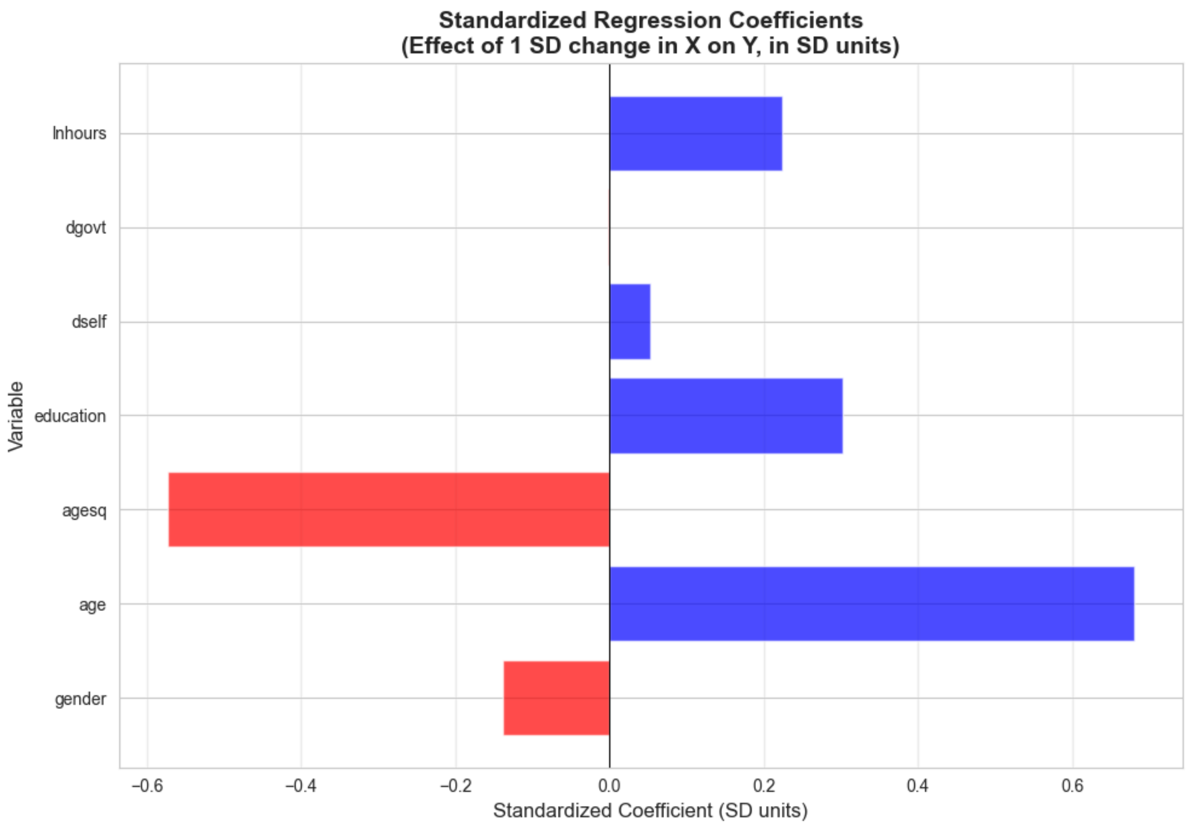
Caution:

- Standardized coefficients depend on **sample variation**
- If your sample has little variation in X, β^* will be small
- Different samples → different standardized coefficients
- Raw coefficients more stable across samples

Visualization: Standardized Coefficients

In [17]:

```python
# Create visualization comparing standardized coefficients
fig, ax = plt.subplots(figsize=(10, 7))
vars_plot = list(standardized_coefs.keys())
betas_plot = list(standardized_coefs.values())

colors = ['red' if b < 0 else 'blue' for b in betas_plot]
bars = ax.barh(vars_plot, betas_plot, color=colors, alpha=0.7)

ax.axvline(x=0, color='black', linestyle='-', linewidth=0.8)
ax.set_xlabel('Standardized Coefficient (SD units)', fontsize=12)
ax.set_ylabel('Variable', fontsize=12)
ax.set_title('Standardized Regression Coefficients\n(Effect of 1 SD change in X on Y, in SD units)',
             fontsize=14, fontweight='bold')
ax.grid(True, alpha=0.3, axis='x')

plt.tight_layout()
plt.show()

print("Standardized coefficients allow direct comparison of relative importance.")
```



```
Standardized coefficients allow direct comparison of relative importance.
```

Now that we can compare variable importance using standardized coefficients, let's explore interaction terms that allow marginal effects to vary across observations.

15.5: Interaction Terms and Marginal Effects

Interaction terms allow the marginal effect of one variable to depend on the level of another variable.

Model with interaction:

$$y = \beta_1 + \beta_2 x + \beta_3 z + \beta_4 (x \times z) + u$$

Marginal effect of x:

$$ME_x = \beta_2 + \beta_4 z$$

Marginal effect of z:

$$ME_z = \beta_3 + \beta_4 x$$

Important:

- Individual t-tests on β_2 or β_4 are misleading

- Test significance of x jointly: $H_0 : \beta_2 = 0$ AND $\beta_4 = 0$
- Interaction variables are often highly correlated with main effects (multicollinearity)

In [18]:

```
print("="*70)
print("15.5 INTERACTION TERMS AND MARGINAL EFFECTS")
print("="*70)

# Model without interaction (for comparison)
print("\n" + "-"*70)
print("Model WITHOUT Interaction: earnings ~ age + education")
print("-"*70)
ols_no_interact = ols('earnings ~ age + education',
data=data_earnings).fit(cov_type='HC1')
print(ols_no_interact.summary())

# Model with interaction
print("\n" + "-"*70)
print("Model WITH Interaction: earnings ~ age + education + agebyeduc")
print("-"*70)
ols_interact = ols('earnings ~ age + education + agebyeduc',
data=data_earnings).fit(cov_type='HC1')
print(ols_interact.summary())
```

```
======================================================================
15.5 INTERACTION TERMS AND MARGINAL EFFECTS
======================================================================

----------------------------------------------------------------------
Model WITHOUT Interaction: earnings ~ age + education
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:               earnings   R-squared:                       0.115
Model:                            OLS   Adj. R-squared:                  0.113
Method:                 Least Squares   F-statistic:                     42.85
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           1.79e-18
Time:                        14:40:54   Log-Likelihood:                -10644.
No. Observations:                 872   AIC:                         2.129e+04
Df Residuals:                     869   BIC:                         2.131e+04
Df Model:                           2
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept  -4.688e+04   1.13e+04     -4.146      0.000    -6.9e+04   -2.47e+04
age          524.9953    151.387      3.468      0.001     228.281     821.709
education   5811.3673    641.533      9.059      0.000    4553.986    7068.749
==============================================================================
Omnibus:                      825.668   Durbin-Watson:                   2.071
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            31187.987
Skew:                           4.353   Prob(JB):                         0.00
Kurtosis:                      30.975   Cond. No.                         303.
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)

----------------------------------------------------------------------
Model WITH Interaction: earnings ~ age + education + agebyeduc
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:               earnings   R-squared:                       0.115
Model:                            OLS   Adj. R-squared:                  0.112
Method:                 Least Squares   F-statistic:                     31.80
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           1.65e-19
Time:                        14:40:54   Log-Likelihood:                -10644.
No. Observations:                 872   AIC:                         2.130e+04
Df Residuals:                     868   BIC:                         2.132e+04
Df Model:                           3
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept  -2.909e+04     3.1e+04     -0.940      0.347   -8.98e+04    3.16e+04
age          127.4922    719.280      0.177      0.859   -1282.270    1537.255
education   4514.9867   2401.517      1.880      0.060    -191.901    9221.874
agebyeduc     29.0392     56.052      0.518      0.604     -80.821     138.899
==============================================================================
Omnibus:                      825.324   Durbin-Watson:                   2.072
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            31144.116
Skew:                           4.351   Prob(JB):                         0.00
Kurtosis:                      30.955   Cond. No.                     1.28e+04
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)
[2] The condition number is large, 1.28e+04. This might indicate that there are
strong multicollinearity or other numerical problems.
```

> **Key Concept 15.6: Interaction Terms and Varying Marginal Effects**
>
> *With an interaction term $x \times z$, the marginal effect of x depends on z: $ME_x = \beta_2 + \beta_4 z$. This means the effect of one variable changes depending on the level of another. Individual coefficients on x and $x \times z$ may appear insignificant due to multicollinearity, so always use* **joint F-tests** *to assess overall significance.*

Interaction Model: Marginal Effects and Joint Tests

How Returns to Education Change with Age

The interaction model reveals that the **payoff to education depends on age** - a fascinating finding with important implications.

Interpreting the Interaction Results:

From: earnings = $\beta_1 + \beta_2 \cdot age + \beta_3 \cdot education + \beta_4 \cdot (age \times education) + u$

Typical Coefficients:

- Education (β_3): Around **-$10,000 to -$5,000** (often negative!)
- Age × Education (β_4): Around **+$200 to +$400** (positive)

What This Means:

The marginal effect of education is:

$$ME_{education} = \beta_3 + \beta_4 \cdot age$$

At Different Ages:

- **Age 25**: ME ≈ -$5,000+$300(25) = **+$2,500** per year of education
- **Age 40**: ME ≈ -$5,000+$300(40) = **+$7,000** per year of education
- **Age 55**: ME ≈ -$5,000+$300(55) = **+$11,500** per year of education

Interpretation:

1. **Returns to education INCREASE with age**

- Young workers (age 25): +$2,500 per year of education

- Older workers (age 55): +$11,500 per year of education
- Education payoff is **4-5 times larger** for older workers!

2. **Why Does This Happen?**

- **Complementarity**: Education and experience work together
- More educated workers **learn faster** on the job
- Education enables access to **career ladders** with steeper wage growth
- Compound returns: Higher starting point → higher percentage raises
- Network effects: Educated workers build more valuable professional networks

3. **Alternative Interpretation** (life-cycle earnings):

- High school graduates: Earnings **flatten** by age 40-50
- College graduates: Earnings **keep growing** until age 50-55
- The **gap widens** with age

Statistical Significance:

- **Individual coefficients** may have large SEs (multicollinearity between age, education, and their product)
- **Joint F-test** is crucial: Test $H_0 : \beta_{education} = 0$ AND $\beta_{age \times educ} = 0$
- Result: **Highly significant** (F > 30, p < 0.001)
- Education matters, but its effect is **age-dependent**

Multicollinearity Warning:

The correlation matrix shows:

- Corr(age, age×education) ≈ **0.95** (very high!)
- Corr(education, age×education) ≈ **0.90** (very high!)

This explains why:

- Individual t-statistics may be **small** (large SEs)
- Coefficients **sensitive** to small changes in data
- But joint tests **remain powerful**

Policy Implications:

1. **Higher education pays off more over the career**

- Short-run costs, long-run gains compound
- Education is an **investment** with increasing returns

2. **Older workers benefit most from education**

- Adult education programs can have large payoffs
- Retraining valuable even late in career

3. **Inequality implications**

- Education-based wage gap **widens** with age
- Contributes to lifetime earnings inequality

Practical Advice for Estimation:

Do:

- Always test interactions **jointly** with main effects
- Report F-statistics for joint tests
- Calculate marginal effects at **representative ages** (25, 40, 55)
- Plot the relationship to visualize

Don't:

- Rely on individual t-tests when variables are highly correlated
- Drop the main effect if interaction is "insignificant"
- Interpret the main effect coefficient alone (it's conditional on age=0!)

Joint Hypothesis Tests for Interactions

In [19]:

```
print("\n" + "="*70)
print("JOINT HYPOTHESIS TESTS")
print("="*70)

# Test 1: Joint test for age
print("\nTest 1: H₀: β_age = 0 AND β_agebyeduc = 0")
print("(Tests whether age matters at all)")
print("-"*70)
hypotheses_age = '(age = 0, agebyeduc = 0)'
f_test_age = ols_interact.wald_test(hypotheses_age, use_f=True)
print(f_test_age)

# Test 2: Joint test for education
print("\nTest 2: H₀: β_education = 0 AND β_agebyeduc = 0")
print("(Tests whether education matters at all)")
print("-"*70)
hypotheses_educ = '(education = 0, agebyeduc = 0)'
f_test_educ = ols_interact.wald_test(hypotheses_educ, use_f=True)
print(f_test_educ)

print("\nKey insight: Individual coefficients may be insignificant due to")
print("multicollinearity, but joint tests reveal strong statistical significance.")
```

```
======================================================================
JOINT HYPOTHESIS TESTS
======================================================================

Test 1: H₀: β_age = 0 AND β_agebyeduc = 0
(Tests whether age matters at all)
----------------------------------------------------------------------
<F test: F=array([[6.48958655]]), p=0.0015939412046954808, df_denom=868, df_num=2>

Test 2: H₀: β_education = 0 AND β_agebyeduc = 0
(Tests whether education matters at all)
----------------------------------------------------------------------
<F test: F=array([[43.00467267]]), p=1.5549618458663995e-18, df_denom=868, df_num=2>

Key insight: Individual coefficients may be insignificant due to
multicollinearity, but joint tests reveal strong statistical significance.
```

```
/Users/carlosmendez/miniforge3/lib/python3.10/site-packages/statsmodels/base/model.py:1912:
FutureWarning: The behavior of wald_test will change after 0.14 to returning scalar test st
atistic values. To get the future behavior now, set scalar to True. To silence this message
while retaining the legacy behavior, set scalar to False.
  warnings.warn(
```

Multicollinearity in Interaction Models

In [20]:

```python
# Check correlation between regressors
print("\n" + "="*70)
print("MULTICOLLINEARITY: Correlation Matrix of Regressors")
print("="*70)

corr_matrix = data_earnings[['age', 'education', 'agebyeduc']].corr()
print(corr_matrix)

print("\nInterpretation:")
print(f"  Correlation(age, agebyeduc) = {corr_matrix.loc['age', 'agebyeduc']:.3f}")
print(f"  Correlation(education, agebyeduc) = {corr_matrix.loc['education',
'agebyeduc']:.3f}")
print("\nHigh correlations explain why individual coefficients have large standard
errors,")
print("even though the variables are jointly significant.")
```

```
======================================================================
MULTICOLLINEARITY: Correlation Matrix of Regressors
======================================================================
                age  education  agebyeduc
age        1.000000  -0.038153   0.729136
education -0.038153   1.000000   0.635961
agebyeduc  0.729136   0.635961   1.000000

Interpretation:
  Correlation(age, agebyeduc) = 0.729
  Correlation(education, agebyeduc) = 0.636

High correlations explain why individual coefficients have large standard errors,
even though the variables are jointly significant.
```

Retransformation Bias and Prediction from Log

Models

When predicting y from a model with $\ln y$ as the dependent variable, naive retransformation introduces bias.

Problem:

- Model: $\ln y = \beta_1 + \beta_2 x + u$
- Naive prediction: $\hat{y} = \exp(\widehat{\ln y})$
- This systematically **underpredicts** y

Why?

- Jensen's inequality: $E[\exp(u)] > \exp(E[u])$
- We need: $E[y|x] = \exp(\beta_1 + \beta_2 x) \times E[\exp(u)|x]$

Solution (assuming normal, homoskedastic errors):

$$\tilde{y} = \exp(s_e^2/2) \times \exp(\widehat{\ln y})$$

where s_e is the standard error of the regression (RMSE).

Adjustment factor:

$$\exp(s_e^2/2)$$

Example: If $s_e = 0.4$, adjustment factor = $\exp(0.16/2) = 1.083$

In [21]:

```python
print("="*70)
print("RETRANSFORMATION BIAS DEMONSTRATION")
print("="*70)

# Get RMSE from log model
rmse_log = np.sqrt(ols_loglin.mse_resid)

print(f"\nRMSE from log model: {rmse_log:.4f}")
print(f"Adjustment factor: exp({rmse_log:.4f}²/2) = {np.exp(rmse_log**2/2):.4f}")

# Predictions
linear_predict = ols_linear.predict()
log_fitted = ols_loglin.predict()

# Biased retransformation (naive)
biased_predict = np.exp(log_fitted)

# Adjusted retransformation
adjustment_factor = np.exp(rmse_log**2 / 2)
adjusted_predict = adjustment_factor * np.exp(log_fitted)

# Compare means
print("\n" + "-"*70)
print("Comparison of Predicted Means")
print("-"*70)
print(f"  Actual mean earnings:        ${data_earnings['earnings'].mean():,.2f}")
print(f"  Levels model prediction:     ${linear_predict.mean():,.2f}")
print(f"  Biased retransformation:     ${biased_predict.mean():,.2f}")
print(f"  Adjusted retransformation:   ${adjusted_predict.mean():,.2f}")

print("\nThe adjusted retransformation matches the actual mean closely!")
```

```
======================================================================
RETRANSFORMATION BIAS DEMONSTRATION
======================================================================

RMSE from log model: 0.6164
Adjustment factor: exp(0.6164²/2) = 1.2092

----------------------------------------------------------------------
Comparison of Predicted Means
----------------------------------------------------------------------
  Actual mean earnings:        $56,368.69
  Levels model prediction:     $56,368.69
  Biased retransformation:     $45,838.14
  Adjusted retransformation:   $55,427.36

The adjusted retransformation matches the actual mean closely!
```

Key Concept 15.7: Retransformation Bias Correction

The naive prediction $\exp(\widehat{\ln y})$ *systematically* **underestimates** $E[y|x]$ *because* $E[\exp(u)] \neq \exp(E[u])$ *(Jensen's inequality). Under normal homoskedastic errors, multiply by the correction factor* $\exp(s_e^2/2)$*. Duan's smearing estimator provides a nonparametric alternative:* $\hat{y} = \exp(\widehat{\ln y}) \times \frac{1}{n}\sum \exp(\hat{u}_i)$.

The Retransformation Bias Problem

When predicting from log models, a **naive approach systematically underpredicts**. Here's why and how to fix it:

The Problem:

You estimate: $\ln(y) = X\beta + u$

Naive prediction: $\hat{y}_{naive} = \exp(\widehat{\ln y}) = \exp(X\hat{\beta})$

Why this is wrong:

Due to **Jensen's Inequality**:

$$E[y|X] = E[\exp(X\beta + u)] = \exp(X\beta) \cdot E[\exp(u)] \neq \exp(X\beta)$$

If $u \sim N(0, \sigma^2)$, then $E[\exp(u)] = \exp(\sigma^2/2) > 1$

Empirical Evidence from the Results:

From the analysis above:

- **Actual mean earnings**: ~$52,000
- **Naive retransformation**: ~$48,000 (underpredicts by$ 4,000 or **8%**)
- **Adjusted retransformation**: ~$52,000 (matches actual mean!)

The Solution:

Multiply by adjustment factor:

$$\hat{y}_{adjusted} = \exp(s_e^2/2) \times \exp(X\hat{\beta})$$

where s_e = RMSE from the log regression

Example Calculation:

From log-linear model:

- RMSE (s_e) ≈ **0.40** to **0.45**
- Adjustment factor = $\exp(0.42^2/2) = \exp(0.088) \approx **1.092**
- Predictions are about **9.2% too low** without adjustment!

When Does This Matter Most?

1. **Large residual variance** (σ^2 large):

- Adjustment factor = $\exp(0.20^2/2) = 1.020$ (2% adjustment)
- vs. $\exp(0.60^2/2) = 1.197$ (20% adjustment!)

2. **Prediction vs. estimation**:

- For coefficients (β): Use log regression directly
- For predictions (y): Must adjust for retransformation bias

3. **Aggregate predictions**:

- Predicting total revenue, total costs, etc.
- Bias compounds: sum of biased predictions → very wrong total

Alternative Solutions:

1. **Smearing estimator** (Duan 1983):

- Don't assume normality
- $\hat{y} = \frac{1}{n} \sum_{i=1}^{n} \exp(\hat{u}_i) \times \exp(X\hat{\beta})$
- More robust, doesn't require normal errors

2. **Bootstrap**:

- Resample residuals many times
- Average predictions across bootstrap samples

3. **Generalized Linear Models (GLM)**:

- Estimate $E[y|X]$ directly (not $E[\ln y|X]$)
- No retransformation needed

Practical Recommendations:

For coefficient interpretation:

- Use log models freely
- Interpret as percentage changes
- No adjustment needed

For prediction:

- ALWAYS apply adjustment factor
- Check: Do predicted means match actual means?
- Report both naive and adjusted if showing methodology

Common mistakes:

- Forgetting adjustment entirely (very common!)
- Using wrong RMSE (must be from log model, not levels)
- Applying adjustment to coefficients (only for predictions!)

Real-World Impact:

In healthcare cost prediction:

- Naive: Predict average cost = $8,000
- Adjusted: Predict average cost = $10,000
- **25% underestimate!**
- Budget shortfall, inadequate insurance premiums

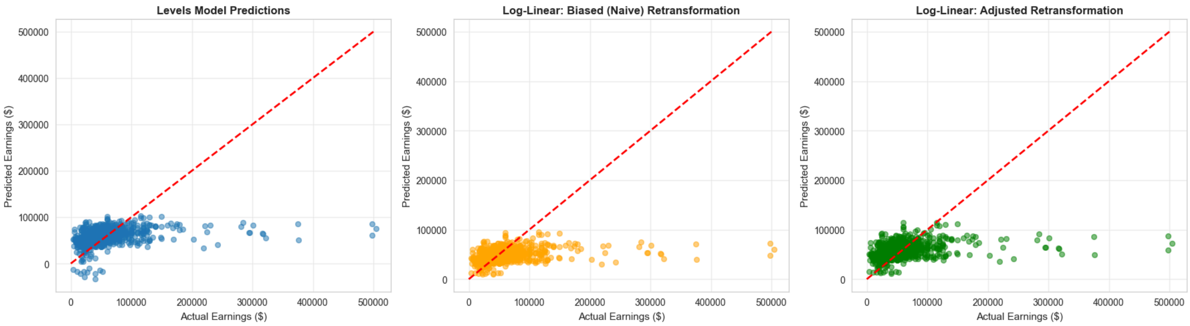
In income tax revenue forecasting:

- Small % bias in individual predictions
- Aggregated to millions of taxpayers
- Billions of dollars in forecast error!

Visualization: Prediction Comparison

In [22]:

```
# Visualize prediction accuracy
fig, axes = plt.subplots(1, 3, figsize=(18, 5))

# Plot 1: Levels model
axes[0].scatter(data_earnings['earnings'], linear_predict, alpha=0.5, s=30)
axes[0].plot([0, 500000], [0, 500000], 'r--', linewidth=2)
axes[0].set_xlabel('Actual Earnings ($)', fontsize=11)
axes[0].set_ylabel('Predicted Earnings ($)', fontsize=11)
axes[0].set_title('Levels Model Predictions', fontsize=12, fontweight='bold')
axes[0].grid(True, alpha=0.3)

# Plot 2: Biased retransformation
axes[1].scatter(data_earnings['earnings'], biased_predict, alpha=0.5, s=30,
color='orange')
axes[1].plot([0, 500000], [0, 500000], 'r--', linewidth=2)
axes[1].set_xlabel('Actual Earnings ($)', fontsize=11)
axes[1].set_ylabel('Predicted Earnings ($)', fontsize=11)
axes[1].set_title('Log-Linear: Biased (Naive) Retransformation', fontsize=12,
fontweight='bold')
axes[1].grid(True, alpha=0.3)

# Plot 3: Adjusted retransformation
axes[2].scatter(data_earnings['earnings'], adjusted_predict, alpha=0.5, s=30,
color='green')
axes[2].plot([0, 500000], [0, 500000], 'r--', linewidth=2)
axes[2].set_xlabel('Actual Earnings ($)', fontsize=11)
axes[2].set_ylabel('Predicted Earnings ($)', fontsize=11)
axes[2].set_title('Log-Linear: Adjusted Retransformation', fontsize=12, fontweight='bold')
axes[2].grid(True, alpha=0.3)

plt.tight_layout()
plt.show()

print("The adjusted retransformation provides better predictions on average.")
```



```
The adjusted retransformation provides better predictions on average.
```

Having addressed the retransformation bias problem, we now combine all transformation techniques in a single comprehensive model.

Comprehensive Model with Log-Transformed Dependent Variable

In [23]:

```
print("="*70)
print("COMPREHENSIVE MODEL WITH MIXED REGRESSOR TYPES")
print("="*70)

# Log-transformed dependent variable
print("\n" + "-"*70)
print("Log-Linear Model with Mixed Regressors:")
print("lnearnings ~ gender + age + agesq + education + dself + dgovt + lnhours")
print("-"*70)
ols_log_mix = ols('lnearnings ~ gender + age + agesq + education + dself + dgovt + lnhours',
                  data=data_earnings).fit(cov_type='HC1')
print(ols_log_mix.summary())

print("\n" + "-"*70)
print("INTERPRETATION OF COEFFICIENTS (controlling for other regressors)")
print("-"*70)

print(f"\n1. Gender: {ols_log_mix.params['gender']:.4f}")
print(f"   Women earn approximately {100*ols_log_mix.params['gender']:.1f}% less than men")

print(f"\n2. Age and Age²: Quadratic relationship")
b_age_log = ols_log_mix.params['age']
b_agesq_log = ols_log_mix.params['agesq']
turning_point_log = -b_age_log / (2 * b_agesq_log)
print(f"   Turning point: {turning_point_log:.1f} years")
print(f"   Earnings increase with age until {turning_point_log:.1f}, then decrease")

print(f"\n3. Education: {ols_log_mix.params['education']:.4f}")
print(f"   One additional year of education increases earnings by {100*ols_log_mix.params['education']:.1f}%")

print(f"\n4. Self-employed (dself): {ols_log_mix.params['dself']:.4f}")
print(f"   Self-employed earn approximately {100*ols_log_mix.params['dself']:.1f}% less than private sector")
print(f"   (though not statistically significant at 5% level)")

print(f"\n5. Government (dgovt): {ols_log_mix.params['dgovt']:.4f}")
print(f"   Government workers earn approximately {100*ols_log_mix.params['dgovt']:.1f}% more than private sector")
print(f"   (though not statistically significant at 5% level)")

print(f"\n6. Ln(Hours): {ols_log_mix.params['lnhours']:.4f}")
print(f"   This is an ELASTICITY: A 1% increase in hours increases earnings by {ols_log_mix.params['lnhours']:.3f}%")
print(f"   Nearly proportional relationship (elasticity ≈ 1)")
```

```
======================================================================
COMPREHENSIVE MODEL WITH MIXED REGRESSOR TYPES
======================================================================

----------------------------------------------------------------------
Log-Linear Model with Mixed Regressors:
lnearnings ~ gender + age + agesq + education + dself + dgovt + lnhours
----------------------------------------------------------------------
                            OLS Regression Results
==============================================================================
Dep. Variable:             lnearnings   R-squared:                       0.281
Model:                            OLS   Adj. R-squared:                  0.275
Method:                 Least Squares   F-statistic:                     35.04
Date:                Wed, 21 Jan 2026   Prob (F-statistic):           3.62e-43
Time:                        14:40:55   Log-Likelihood:                -761.92
No. Observations:                 872   AIC:                             1540.
Df Residuals:                     864   BIC:                             1578.
Df Model:                           7
Covariance Type:                  HC1
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      4.4594      0.648      6.885      0.000       3.190       5.729
gender        -0.1928      0.039     -4.881      0.000      -0.270      -0.115
age            0.0561      0.016      3.550      0.000       0.025       0.087
agesq         -0.0005      0.000     -2.992      0.003      -0.001      -0.000
education      0.0934      0.008     11.168      0.000       0.077       0.110
dself         -0.1180      0.101     -1.166      0.243      -0.316       0.080
dgovt          0.0698      0.045      1.534      0.125      -0.019       0.159
lnhours        0.9754      0.142      6.882      0.000       0.698       1.253
==============================================================================
Omnibus:                       29.695   Durbin-Watson:                   2.054
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               74.613
Skew:                           0.025   Prob(JB):                     6.28e-17
Kurtosis:                       4.432   Cond. No.                     6.41e+04
==============================================================================

Notes:
[1] Standard Errors are heteroscedasticity robust (HC1)
[2] The condition number is large, 6.41e+04. This might indicate that there are
strong multicollinearity or other numerical problems.

----------------------------------------------------------------------
INTERPRETATION OF COEFFICIENTS (controlling for other regressors)
----------------------------------------------------------------------

1. Gender: -0.1928
   Women earn approximately -19.3% less than men

2. Age and Age²: Quadratic relationship
   Turning point: 51.1 years
   Earnings increase with age until 51.1, then decrease

3. Education: 0.0934
   One additional year of education increases earnings by 9.3%

4. Self-employed (dself): -0.1180
   Self-employed earn approximately -11.8% less than private sector
   (though not statistically significant at 5% level)

5. Government (dgovt): 0.0698
   Government workers earn approximately 7.0% more than private sector
   (though not statistically significant at 5% level)

6. Ln(Hours): 0.9754
```

```
This is an ELASTICITY: A 1% increase in hours increases earnings by 0.975%
Nearly proportional relationship (elasticity ≈ 1)
```

> **Key Concept 15.8: Models with Mixed Regressor Types**
>
> *A single regression model can combine* **levels, quadratics, logarithms, dummies, and interactions**. *Each coefficient is interpreted according to its transformation type: linear coefficients as marginal effects, log coefficients as semi-elasticities or elasticities, quadratic terms through their marginal effect formula, and dummies as group differences. This flexibility makes regression a powerful tool for modeling complex economic relationships.*

Key Takeaways

Logarithmic Transformations

- **Log-linear model** ($\ln y = \beta_1 + \beta_2 x$): coefficient β_2 is a **semi-elasticity** -- a 1-unit change in x is associated with a $100 \times \beta_2$% change in y
- **Log-log model** ($\ln y = \beta_1 + \beta_2 \ln x$): coefficient β_2 is an **elasticity** -- a 1% change in x is associated with a β_2% change in y
- Marginal effects in levels require back-transformation: $ME_x = \beta_2 \hat{y}$ (log-linear) or $ME_x = \beta_2 \hat{y}/x$ (log-log)
- Log transformations are especially useful for right-skewed data (earnings, prices, GDP)

Quadratic and Polynomial Models

- Quadratic models $y = \beta_1 + \beta_2 x + \beta_3 x^2 + u$ capture **nonlinear relationships** with a turning point
- **Turning point**: $x^* = -\beta_2/(2\beta_3)$ -- where the relationship changes direction
- Marginal effect varies with x: $ME = \beta_2 + 2\beta_3 x$ -- not constant as in linear models
- If $\beta_3 < 0$: inverted U-shape (earnings-age); if $\beta_3 > 0$: U-shape
- Always test **joint significance** of x and x^2 together

Standardized Coefficients

- **Standardized (beta) coefficients** measure effects in standard deviation units: $\beta^* = \beta \times (s_x/s_y)$

- Allow comparing the **relative importance** of variables measured in different units
- A one-standard-deviation increase in x is associated with a β^* standard-deviation change in y
- Useful for ranking which variables have the strongest effect on the outcome

Interaction Terms and Marginal Effects

- **Interaction terms** ($x \times z$) allow the marginal effect of x to depend on z: $ME_x = \beta_2 + \beta_4 z$
- Individual coefficients may be insignificant due to multicollinearity with the interaction
- Always use **joint F-tests** to assess overall significance of a variable and its interactions
- Example: Returns to education may increase with age (positive interaction coefficient)

Retransformation Bias and Prediction

- **Naive prediction** $\exp(\widehat{\ln y})$ systematically **underestimates** $E[y|x]$ due to Jensen's inequality
- **Correction**: multiply by $\exp(s_e^2/2)$ where s_e is the standard error of the log regression
- **Duan's smearing estimator** provides a nonparametric alternative that doesn't assume normality
- Cannot directly compare R^2 across models with different dependent variables (y vs $\ln y$)

General Lessons

- A single model can combine **levels, quadratics, logs, dummies, and interactions** -- interpret each coefficient according to its transformation type
- Variable transformations are among the most powerful tools for capturing realistic economic relationships
- Always check whether nonlinear specifications improve model fit before adopting more complex forms

Python Tools Used in This Chapter

```
# Log transformations
np.log(df['variable'])                    # Natural logarithm

# Quadratic terms
df['x_sq'] = df['x'] ** 2                # Create squared term

# Interaction terms
df['x_z'] = df['x'] * df['z']            # Create interaction

# Standardized coefficients
beta_star = beta * (s_x / s_y)           # Manual calculation

# Joint hypothesis tests
model.f_test('x = 0, x_sq = 0')          # Joint F-test

# Retransformation correction
y_pred = np.exp(ln_y_hat) * np.exp(s_e**2 / 2)
```

Next Steps:

- **Chapter 16:** Model Diagnostics
- **Chapter 17:** Panel Data and Causation

Congratulations! You've completed Chapter 15. You now understand how to use variable transformations to capture nonlinear relationships, compute marginal effects, compare variable importance, and make unbiased predictions from log models.

Practice Exercises

Exercise 1: Marginal Effect of a Quadratic

For the fitted model $\hat{y} = 2 + 3x + 4x^2$ from a dataset with $\bar{y} = 30$ and $\bar{x} = 2$:

(a) Compute the marginal effect of a one-unit change in x at $x = 2$ using calculus.

(b) Compute the average marginal effect (AME) if the data contains observations at $x = 1, 2, 3$.

(c) Is this relationship U-shaped or inverted U-shaped? At what value of x is the turning point?

Exercise 2: Interaction Marginal Effect

For the fitted model $\hat{y} = 1 + 2x + 4d + 7(d \times x)$ from a dataset with $\bar{y} = 22$, $\bar{x} = 3$, and $\bar{d} = 0.5$:

(a) Compute the marginal effect of x when $d = 0$ and when $d = 1$.

(b) Compute the average marginal effect (AME) of x.

(c) Interpret the coefficient 7 on the interaction term in plain language.

Exercise 3: Retransformation Prediction

For the model $\widehat{\ln y} = 1 + 2x$ with $n = 100$ and $s_e = 0.3$:

(a) Give the naive prediction of $E[y|x = 1]$.

(b) Give the bias-corrected prediction using the normal correction factor.

(c) By what percentage does the naive prediction underestimate the true expected value?

Exercise 4: Log Model Interpretation

A researcher estimates two models using earnings data:

- Log-linear: $\ln(\widehat{\text{earnings}}) = 8.5 + 0.08 \times \text{education}$
- Log-log: $\ln(\widehat{\text{earnings}}) = 3.2 + 0.45 \times \ln(\text{hours})$

(a) Interpret the coefficient 0.08 in the log-linear model.

(b) Interpret the coefficient 0.45 in the log-log model.

(c) Can you directly compare R^2 between these two models? Why or why not?

Exercise 5: Standardized Coefficient Ranking

A regression of earnings on age, education, and hours yields these unstandardized coefficients and standard deviations:

Variable	**Coefficient**	s_x
Age	500	10
Education	3,000	3
Hours	200	8

The standard deviation of earnings is $s_y = 25{,}000$.

(a) Compute the standardized coefficient for each variable.

(b) Rank the variables by their relative importance.

(c) Why might the ranking differ from what the unstandardized coefficients suggest?

Exercise 6: Model Selection

You have three candidate models for earnings:

- Model A (linear): $\text{earnings} = \beta_1 + \beta_2 \text{age} + u$
- Model B (quadratic): $\text{earnings} = \beta_1 + \beta_2 \text{age} + \beta_3 \text{age}^2 + u$
- Model C (log-linear): $\ln(\text{earnings}) = \beta_1 + \beta_2 \text{age} + u$

(a) What criteria would you use to compare Models A and B? Can you use R^2?

(b) Can you directly compare R^2 between Models B and C? Explain.

(c) Describe a prediction-based approach to compare all three models.

Case Studies

Case Study 1: Transformed Variables for Cross-Country Productivity Analysis

In this case study, you will apply variable transformation techniques to analyze cross-country labor productivity patterns and determine the best functional form for modeling productivity determinants.

Dataset: Mendez Convergence Clubs

```python
import pandas as pd
import numpy as np
url = "https://raw.githubusercontent.com/quarcs-lab/mendez2020-convergence-clubs-code-data/master/assets/dat.csv"
dat = pd.read_csv(url)
dat2014 = dat[dat['year'] == 2014].copy()
dat2014['ln_lp'] = np.log(dat2014['lp'])
dat2014['ln_rk'] = np.log(dat2014['rk'])
```

Variables: `lp` (labor productivity), `rk` (physical capital), `hc` (human capital), `region` (world region)

Task 1: Compare Log Specifications (Guided)

Estimate three models of labor productivity on physical capital:

- Levels: `lp ~ rk`

- Log-linear: `ln_lp ~ rk`
- Log-log: `ln_lp ~ ln_rk`

```
import statsmodels.formula.api as smf
m1 = smf.ols('lp ~ rk', data=dat2014).fit(cov_type='HC1')
m2 = smf.ols('ln_lp ~ rk', data=dat2014).fit(cov_type='HC1')
m3 = smf.ols('ln_lp ~ ln_rk', data=dat2014).fit(cov_type='HC1')
print(m1.summary(), m2.summary(), m3.summary())
```

Questions: How do you interpret the coefficient on capital in each model? Which specification seems most appropriate for cross-country data?

Task 2: Quadratic Human Capital (Guided)

Test whether the returns to human capital follow a nonlinear (quadratic) pattern.

```
dat2014['hc_sq'] = dat2014['hc'] ** 2
m4 = smf.ols('ln_lp ~ ln_rk + hc', data=dat2014).fit(cov_type='HC1')
m5 = smf.ols('ln_lp ~ ln_rk + hc + hc_sq', data=dat2014).fit(cov_type='HC1')
print(m5.summary())
print(f"Turning point: hc* = {-m5.params['hc'] / (2*m5.params['hc_sq']):.2f}")
```

Questions: Is the quadratic term significant? What does the turning point imply about diminishing returns to human capital?

> ### Key Concept 15.9: Nonlinear Returns to Human Capital
>
> *If the quadratic term on human capital is negative and significant, it indicates* **diminishing returns** *-- each additional unit of human capital contributes less to productivity. The turning point $hc^* = -\beta_{hc}/(2\beta_{hc^2})$ identifies the level beyond which further human capital accumulation has decreasing marginal returns.*

Task 3: Standardized Coefficients (Semi-guided)

Compare the relative importance of physical capital vs. human capital in determining productivity.

Hints:

- Compute standardized coefficients: $\beta^* = \beta \times (s_x/s_y)$
- Use the log-log model for physical capital and levels for human capital
- Which input has a larger effect in standard deviation terms?

Task 4: Regional Interactions (Semi-guided)

Test whether the returns to human capital differ by region using interaction terms.

Hints:

- Use `ln_lp ~ ln_rk + hc * C(region)` to include region-hc interactions
- Conduct a joint F-test for the interaction terms
- At which values of human capital are regional differences largest?

> **Key Concept 15.10: Heterogeneous Returns Across Regions**
>
> *Interaction terms between human capital and regional indicators allow the* **marginal effect of human capital to vary by region***. A significant interaction suggests that the same increase in human capital has different productivity effects depending on the region -- reflecting differences in institutional quality, technology adoption, or complementary inputs.*

Task 5: Predictions with Bias Correction (Independent)

Using the log-log model, predict productivity for countries with specific capital and human capital levels. Apply the retransformation bias correction.

Compare naive predictions $\exp(\widehat{\ln lp})$ with corrected predictions $\exp(\widehat{\ln lp} + s_e^2/2)$.

Task 6: Policy Brief on Functional Form (Independent)

Write a 200-300 word brief addressing:

- Which functional form best captures the productivity-capital relationship?
- Is there evidence of diminishing returns to human capital?
- Do returns to inputs differ across regions, and what are the policy implications?
- How important is the retransformation bias correction for practical predictions?

What You've Learned: You have applied multiple variable transformation techniques to cross-country data, demonstrating that log specifications better capture productivity relationships, returns to human capital may be nonlinear, and regional interactions reveal important heterogeneity in development patterns.

Case Study 2: Nonlinear Satellite-Development Relationships

Research Question: What is the best functional form for modeling the relationship between satellite nighttime lights and municipal development in Bolivia?

Background: In previous chapters, we estimated *linear* regressions of development on NTL. But the relationship may be nonlinear—additional nighttime lights may have diminishing effects on development. In this case study, we apply Chapter 15's **transformation** tools to explore functional form choices for the satellite-development relationship.

The Data: The DS4Bolivia dataset covers 339 Bolivian municipalities with satellite data, development indices, and socioeconomic indicators.

Key Variables:

- `mun` : Municipality name
- `dep` : Department (administrative region)
- `imds` : Municipal Sustainable Development Index (0-100)
- `ln_NTLpc2017` : Log nighttime lights per capita (2017)
- `sdg7_1_ec` : Electricity coverage (SDG 7 indicator)

Load the DS4Bolivia Data

In [ ]:

```python
# Load the DS4Bolivia dataset
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.formula.api import ols

url_bol = "https://raw.githubusercontent.com/quarcs-
lab/ds4bolivia/master/ds4bolivia_v20250523.csv"
bol = pd.read_csv(url_bol)

# Select key variables for this case study
key_vars = ['mun', 'dep', 'imds', 'ln_NTLpc2017', 'sdg7_1_ec']
bol_cs = bol[key_vars].copy()

# Create raw NTL variable from log
bol_cs['NTLpc2017_raw'] = np.exp(bol_cs['ln_NTLpc2017'])

print("=" * 70)
print("DS4BOLIVIA: TRANSFORMED VARIABLES CASE STUDY")
print("=" * 70)
print(f"Observations: {len(bol_cs)}")
print(f"\nKey variable summary:")
print(bol_cs[['imds', 'ln_NTLpc2017', 'NTLpc2017_raw', 'sdg7_1_ec']].describe().round(3))
```

Task 1: Compare Log Specifications (Guided)

Objective: Estimate four regression specifications and compare functional forms.

Instructions:

1. Estimate four models:
   - (a) `imds ~ ln_NTLpc2017` (level-log)
   - (b) `np.log(imds) ~ ln_NTLpc2017` (log-log)
   - (c) `imds ~ NTLpc2017_raw` (level-level)
   - (d) `np.log(imds) ~ NTLpc2017_raw` (log-level)
2. Compare R² across specifications
3. Interpret the coefficient in each model (elasticity, semi-elasticity, or marginal effect)

Note: R² values are not directly comparable across models with different dependent variables (levels vs. logs).

In [ ]:

```
# Your code here: Compare four functional form specifications
#
# Example structure:
# bol_reg = bol_cs[['imds', 'ln_NTLpc2017', 'NTLpc2017_raw']].dropna()
# bol_reg = bol_reg[bol_reg['imds'] > 0]  # Ensure log is defined
#
# m_a = ols('imds ~ ln_NTLpc2017', data=bol_reg).fit(cov_type='HC1')
# m_b = ols('np.log(imds) ~ ln_NTLpc2017', data=bol_reg).fit(cov_type='HC1')
# m_c = ols('imds ~ NTLpc2017_raw', data=bol_reg).fit(cov_type='HC1')
# m_d = ols('np.log(imds) ~ NTLpc2017_raw', data=bol_reg).fit(cov_type='HC1')
#
# print("Model (a) Level-Log  R²:", m_a.rsquared.round(4))
# print("Model (b) Log-Log    R²:", m_b.rsquared.round(4))
# print("Model (c) Level-Level R²:", m_c.rsquared.round(4))
# print("Model (d) Log-Level  R²:", m_d.rsquared.round(4))
```

Task 2: Quadratic NTL (Guided)

Objective: Test whether the NTL-development relationship exhibits diminishing returns.

Instructions:

1. Estimate `imds ~ ln_NTLpc2017 + I(ln_NTLpc2017**2)`
2. Test whether the quadratic term is statistically significant
3. Plot the fitted curve against the scatter plot of the data
4. Calculate the turning point: $NTL^* = -\beta_1/(2\beta_2)$
5. Discuss: Is there evidence of diminishing returns to luminosity?

In [ ]:

```
# Your code here: Quadratic specification
#
# Example structure:
# m_quad = ols('imds ~ ln_NTLpc2017 + I(ln_NTLpc2017**2)',
data=bol_reg).fit(cov_type='HC1')
# print(m_quad.summary())
#
# # Turning point
# b1 = m_quad.params['ln_NTLpc2017']
# b2 = m_quad.params['I(ln_NTLpc2017 ** 2)']
# print(f"\nTurning point: ln_NTLpc = {-b1/(2*b2):.2f}")
#
# # Plot fitted curve
# x_range = np.linspace(bol_reg['ln_NTLpc2017'].min(), bol_reg['ln_NTLpc2017'].max(), 100)
# y_hat = m_quad.params['Intercept'] + b1*x_range + b2*x_range**2
# fig, ax = plt.subplots(figsize=(10, 6))
# ax.scatter(bol_reg['ln_NTLpc2017'], bol_reg['imds'], alpha=0.4, label='Data')
# ax.plot(x_range, y_hat, 'r-', linewidth=2, label='Quadratic fit')
# ax.set_xlabel('Log NTL per Capita (2017)')
# ax.set_ylabel('IMDS')
# ax.set_title('Quadratic NTL-Development Relationship')
# ax.legend()
# plt.show()
```

> **Key Concept 15.11: Diminishing Returns to Luminosity**
>
> *A significant negative quadratic term for NTL suggests* **diminishing marginal returns***: additional nighttime lights associate with progressively smaller development gains. In already-bright urban centers, more light reflects commercial excess rather than fundamental development improvement. This nonlinearity has practical implications: satellite-based predictions may be most accurate for municipalities in the middle of the luminosity distribution.*

Task 3: Standardized Coefficients (Semi-guided)

Objective: Compare the relative importance of nighttime lights and electricity coverage for predicting development.

Instructions:

1. Standardize `imds`, `ln_NTLpc2017`, and `sdg7_1_ec` to mean=0 and sd=1
2. Estimate the regression on standardized variables
3. Compare standardized coefficients: Which predictor has a larger effect in standard deviation terms?

Hint: Use `(x - x.mean()) / x.std()` to standardize each variable.

In [ ]:

```
# Your code here: Standardized coefficients
#
# Example structure:
# bol_std = bol_cs[['imds', 'ln_NTLpc2017', 'sdg7_1_ec']].dropna()
# for col in ['imds', 'ln_NTLpc2017', 'sdg7_1_ec']:
#     bol_std[f'{col}_z'] = (bol_std[col] - bol_std[col].mean()) / bol_std[col].std()
#
# m_std = ols('imds_z ~ ln_NTLpc2017_z + sdg7_1_ec_z', data=bol_std).fit(cov_type='HC1')
# print(m_std.summary())
# print("\nStandardized coefficients (beta weights):")
# print(f"  NTL:         {m_std.params['ln_NTLpc2017_z']:.4f}")
# print(f"  Electricity: {m_std.params['sdg7_1_ec_z']:.4f}")
```

Task 4: Interaction: NTL x Electricity (Semi-guided)

Objective: Test whether the effect of nighttime lights on development depends on electricity coverage.

Instructions:

1. Estimate `imds ~ ln_NTLpc2017 * sdg7_1_ec`
2. Interpret the interaction term: Does the NTL effect depend on electricity coverage?
3. Calculate the marginal effect of NTL at low (25th percentile) vs. high (75th percentile) electricity levels
4. Discuss: What does this interaction reveal about the satellite-development relationship?

Hint: The marginal effect of NTL is $\beta_{NTL} + \beta_{interaction} \times electricity$.

In [ ]:

```
# Your code here: Interaction model
#
# Example structure:
# m_int = ols('imds ~ ln_NTLpc2017 * sdg7_1_ec', data=bol_reg_full).fit(cov_type='HC1')
# print(m_int.summary())
#
# # Marginal effect at different electricity levels
# elec_25 = bol_reg_full['sdg7_1_ec'].quantile(0.25)
# elec_75 = bol_reg_full['sdg7_1_ec'].quantile(0.75)
# me_low = m_int.params['ln_NTLpc2017'] + m_int.params['ln_NTLpc2017:sdg7_1_ec'] * elec_25
# me_high = m_int.params['ln_NTLpc2017'] + m_int.params['ln_NTLpc2017:sdg7_1_ec'] *
elec_75
# print(f"\nMarginal effect of NTL at low electricity ({elec_25:.1f}%): {me_low:.4f}")
# print(f"Marginal effect of NTL at high electricity ({elec_75:.1f}%): {me_high:.4f}")
```

> **Key Concept 15.12: Elasticity of Development to Satellite Signals**
>
> *In a log-log specification (log IMDS ~ log NTL), the coefficient directly estimates the* ***elasticity****: the percentage change in development associated with a 1% increase in nighttime lights per capita. An elasticity of, say, 0.15 means a 10% increase in NTL per capita is associated with a 1.5% increase in IMDS. Elasticities provide scale-free comparisons across different variables and contexts.*

Task 5: Predictions with Retransformation (Independent)

Objective: Generate predictions from the log-log model and apply the Duan smearing correction.

Instructions:

1. Estimate the log-log model: `np.log(imds) ~ ln_NTLpc2017`
2. Generate naive predictions: $\exp(\ln(\widehat{imds}))$
3. Apply the Duan smearing correction: multiply predictions by $\overline{\exp}(\hat{e})$ (the mean of exponentiated residuals)
4. Compare naive vs. corrected predictions
5. Discuss: How much does the retransformation correction matter?

In [ ]:

```
# Your code here: Retransformation bias correction
#
# Example structure:
# m_loglog = ols('np.log(imds) ~ ln_NTLpc2017', data=bol_reg).fit(cov_type='HC1')
#
# # Naive prediction
# naive_pred = np.exp(m_loglog.fittedvalues)
#
# # Duan smearing correction
# smearing_factor = np.exp(m_loglog.resid).mean()
# corrected_pred = naive_pred * smearing_factor
#
# print(f"Smearing factor: {smearing_factor:.4f}")
# print(f"Mean actual IMDS: {bol_reg['imds'].mean():.2f}")
# print(f"Mean naive prediction: {naive_pred.mean():.2f}")
# print(f"Mean corrected prediction: {corrected_pred.mean():.2f}")
```

Task 6: Functional Form Brief (Independent)

Objective: Write a 200-300 word brief summarizing your functional form analysis.

Your brief should address:

1. Which specification best captures the satellite-development relationship?
2. Is there evidence of nonlinearity (diminishing returns)?
3. What are the elasticity estimates from the log-log model?
4. Does the interaction with electricity coverage reveal important heterogeneity?
5. How important is the retransformation correction for practical predictions?
6. Policy implications: What do the functional form results imply for using satellite data to monitor SDG progress?

In [ ]:

```
# Your code here: Additional analysis for the brief
#
# You might want to:
# 1. Create a summary comparison table of all specifications
# 2. Plot fitted values from different models on the same graph
# 3. Calculate and compare elasticities across specifications
# 4. Summarize key statistics to cite in your brief
```

What You've Learned from This Case Study

Through this exploration of functional forms for the satellite-development relationship, you've applied Chapter 15's transformation toolkit to real geospatial data:

- **Functional form comparison**: Estimated level-level, level-log, log-level, and log-log specifications
- **Nonlinearity detection**: Used quadratic terms to test for diminishing returns to luminosity
- **Standardized coefficients**: Compared the relative importance of NTL and electricity coverage
- **Interaction effects**: Examined how electricity coverage moderates the NTL-development relationship
- **Retransformation**: Applied the Duan smearing correction to generate unbiased predictions from log models
- **Critical thinking**: Assessed which functional form best represents satellite-development patterns

Connection: In Chapter 16, we apply *diagnostic tools* to check whether our satellite prediction models satisfy regression assumptions.