# Linear regression without correspondence

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## Problem definition

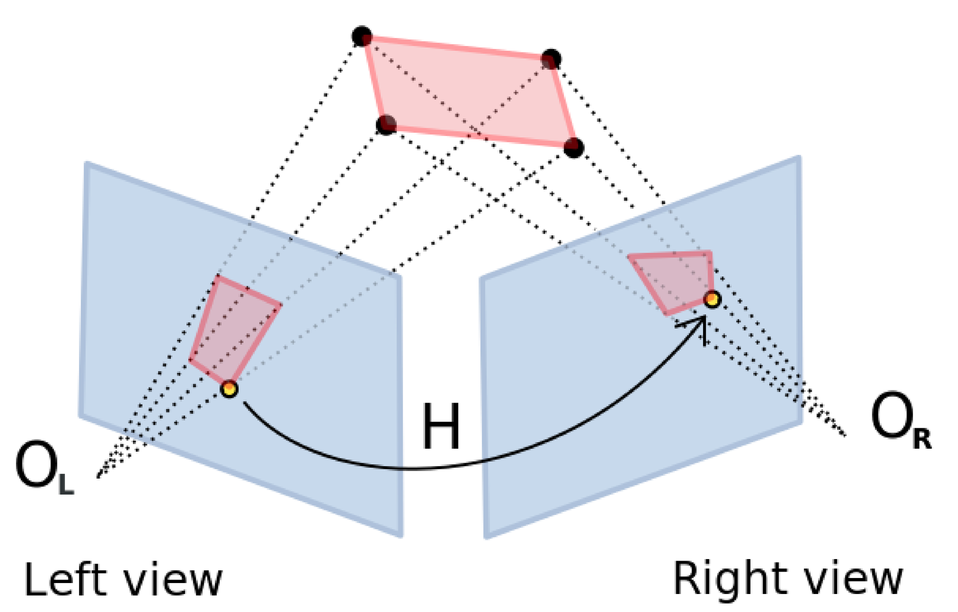
- **Covariate vectors**: $x_1, x_2, \dots, x_n \in \mathbb{R}^d$
- **Responses**: $y_1, y_2, \dots, y_n \in \mathbb{R}$
- **Model**:

$$y_i \;=\; \bar{w}^\top x_{\bar{\pi}(i)} + \varepsilon_i\,, \quad i \in [n]$$

- Unknown linear function: $\bar{w} \in \mathbb{R}^d$
- Unknown permutation: $\bar{\pi} \in S_n$
- Measurement errors: $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \mathbb{R}$ e.g., $(\varepsilon_i)_{i=1}^n$ iid from $\mathrm{N}(0, \sigma^2))$

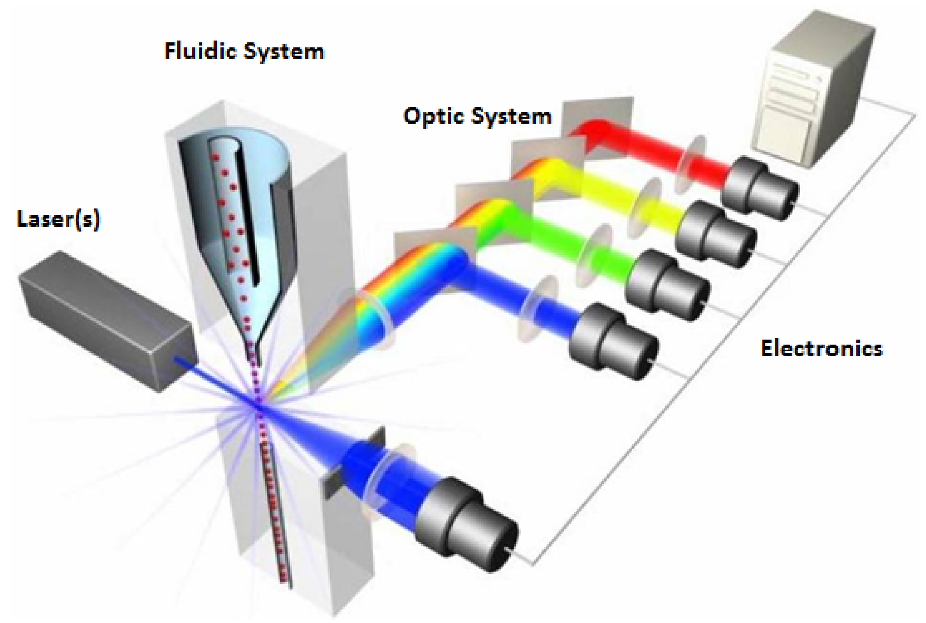
## Examples

### Multi-view geometry




- Unknown correspondence between keypoints

### Flow cytometry




- Observe the entire emission spectrum at once

## Strong NP-hardness

**Definition 1 (Permuted Linear System).**
Given $X \in \mathbb{Z}^{n \times d}, Y \in \mathbb{Q}^n$, decide if there exists a vector $w \in \mathbb{Q}^d$ and a permutation $\pi \in S_n$ such that $Xw = Y_\pi$

**Proposition 1.** *Permuted Linear System is strongly NP-complete by a reduction from 3-Partition.*

## Approximation guarantee for least-squares

**Definition 2 (Least-squares recovery).**
Given $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ from $\mathbb{R}$, find

$$(\hat{w}_{mle}, \hat{\pi}_{mle}) := \operatorname*{arg\,min}_{w \in \mathbb{R}, \pi \in S_n} \sum_{i=1}^n \left(y_i - w x_{\pi(i)}\right)^2$$

**Theorem 1.** *There is an algorithm that given any inputs $(x_i)_{i=1}^n$, $(y_i)_{i=1}^n$, and $\epsilon \in (0,1)$, returns a $(1+\epsilon)$-approximate solution to the least squares problem in time $(n/\epsilon)^{O(k)} + \mathrm{poly}(n,d)$, where $k = \dim(\mathrm{span}(x_i)_{i=1}^n)$. ;*

## Approximation algorithm

This uses the following coreset result for linear systems:

**Proposition 2 (Boutsidis, Drineas, Magdon-Ismail).** *Given a matrix $A \in \mathbb{R}^{n \times k}$, there exists a weighted subset of $4k$ rows determined by a matrix $S \in \mathbb{R}^{4k \times n}$ such that for any $b$, every minimizer of the subsampled linear system*

$$w' \in \operatorname*{arg\,min}_{w} \|S(Aw - b)\|_2^2$$

*also satisfies*

$$\|Aw' - b\|_2^2 \leq c$$

*for $c = O(n/k)$. Morever, there exists an efficient algorithm which returns a matrix $S$ in time poly$(n,k)$.*

**Algorithm 1** Approximation algorithm

**input** Covariate matrix $X = [x_1|x_2|\cdots|x_n]^\top \in \mathbb{R}^{n \times k}$; response vector $y = (y_1, y_2, \dots, y_n)^\top \in \mathbb{R}^n$; approximation parameter $\epsilon \in (0,1)$.

Compute the matrix $S \in \mathbb{R}^{r \times n}$ from input matrix $X$.
Let $\mathcal{B}$ be the set of all permutations of $y$
Let $c := 1 + 4(1 + \sqrt{n/(4k)})^2$.
**for** each $b \in \mathcal{B}$ **do**
 Compute $\tilde{w}_b \in \operatorname*{arg\,min}_{w \in \mathbb{R}^k} \|[\|0]S(Xw - b)_2^2$, and let $r_b := \min_{\Pi \in \mathcal{P}_n} \|[\|0]X\tilde{w}_b - \Pi^\top y_2^2$.
 Construct a $\sqrt{\epsilon r_b / c}$-net $\mathcal{N}_b$ for the Euclidean ball of radius $\sqrt{c r_b}$ around $\tilde{w}_b$, so that for each $v \in \mathbb{R}^k$ with $\|[\|0]v - \tilde{w}_{b2} \leq \sqrt{c r_b}$, there exists $v' \in \mathcal{N}_b$ such that $\|[\|0]v - v'_2 \leq \sqrt{\epsilon r_b / c}$.
**end for**
**return**

$$\hat{w} \in \operatorname*{arg\,min}_{w \in \bigcup_{b \in \mathcal{B}} \mathcal{N}_b} \min_{\Pi \in \mathcal{P}_n} \|Xw - \Pi^\top y\|_2^2$$

and

$$\hat{\Pi} \in \operatorname*{arg\,min}_{\Pi \in \mathcal{P}_n} \|X\hat{w} - \Pi^\top y\|_2^2$$

## Polynomial time recovery in the random setting

**Theorem 2.** *Fix any $\bar{w} \in \mathbb{R}^d$ and $\bar{\pi} \in S_n$, and assume $n \geq d$. Suppose $(x_i)_{i=0}^n$ are drawn iid from $\mathrm{N}(0, I_d)$, and $(y_i)_{i=0}^n$ satisfy*

$$y_0 \;=\; \bar{w}^\top x_0\,; \qquad y_i \;=\; \bar{w}^\top x_{\bar{\pi}(i)}\,, \quad i \in [n]\,.$$

*There is an algorithm that, given inputs $(x_i)_{i=0}^n$ and $(y_i)_{i=0}^n$, returns $\bar{\pi}$ and $\bar{w}$ with high probability.*

## Reduction to (random) subset sum

Given $d+1$ measurements and one correspondence $y_0 = \bar{w}^T x_0$, for orthogonal $(x_i)_{i=0}^n$, can write:

$$y_0 = \sum_{j=1}^d (\bar{w}^\top x_j)\,(x_j^\top x_0) = \sum_{j=1}^d y_{\bar{\pi}^{-1}(j)}\,(x_j^\top x_0)$$

$$= \sum_{i=1}^d \sum_{j=1}^d \mathbb{1}\{\bar{\pi}(i) = j\} \cdot \underbrace{y_i\,(x_j^\top x_0)}_{c_{i,j}}$$

- $\{c_{i,j}\}$ and $y_0$ define a subset sum problem whose solution recovers the underlying correspondence.
- In general $(x_i)_{i=0}^n$ are close to orthogonal; use the Moore-Penrose pseudoinverse.
- The one given correspondence can be brute-forced, creating $d+1$ subset sum instances of which only one has a solution

## Solving random subset-sum instances

**Proposition 3 (Lagarias and Odlyzko).**
*Random instances of subset sum are efficiently solvable when the $c_{i,j}$'s are independently and uniformly distributed over a large enough subinterval of $\mathbb{Z}$.*

This relies on the following inequality which lower bounds the closeness to the target sum of incorrect solutions.

**Lemma 1.** *For any vector $(z_{i,j})$ which is not the correct correspondence,*

$$\left| y_0 - \sum_{i,j} z_{i,j} c_{i,j} \right| \geq \frac{1}{2^{poly(d)}} \|\bar{w}\|_2$$

- We show this bound holds under other distributions satisfying general anticoncentration bounds and even if the $c_{i,j}$'s are not independent

## Reduction to shortest vector problem

**Definition 3 (Shortest vector problem).**
Given a lattice basis $\mathbf{B} \subset \mathbb{R}^d$, output a lattice vector $\mathbf{B}z \in \Lambda\mathbf{B}$ where

$$z = \operatorname*{arg\,min}_{z \in \mathbb{Z} - \{0\}} \|\mathbf{B}z\|_2^2$$

**Lemma 2 (LLL Lattice Basis Reduction).**
*There is an efficient approximation algorithm for solving the Shortest Vector Problem with*

- *Approximation factor: $2^{d/2}$*
- *Running time: poly$(d, \log\lambda(\mathbf{B}))$*

**Algorithm 2** Lattice algorithm for subset sum

**input** Source numbers $\{c_i\}_{i \in \mathcal{I}} \subset \mathbb{R}$; target sum $t \in \mathbb{R}$; lattice parameter $\beta > 0$.

Construct lattice basis $B \in \mathbb{R}^{(|\mathcal{I}|+2) \times (|\mathcal{I}|+1)}$ where

$$B \;:=\; \begin{bmatrix} I_{|\mathcal{I}|+1} \\ \beta t \,|\, -\beta c_i : i \in \mathcal{I} \end{bmatrix} \;\in\; \mathbb{R}^{(|\mathcal{I}|+2) \times (|\mathcal{I}|+1)}\,.$$

Run LLL Lattice Basis Reduction to find non-zero lattice vector $v$ of length at most $2^{|\mathcal{I}|/2} \cdot \lambda_1(B)$.

## Information-theoretic lower bounds on SNR

**Definition 4 (Uniform model with Gaussian noise).**
Observe

$$y_i = \bar{w}^T x_{\bar{\pi}(i)} + \epsilon_i$$

where

- $\epsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$ is the measurement noise
- $x_i \overset{iid}{\sim}$ Uniform$([-1,1]^d)$ are the covariates

**Definition 5 (SNR).**
The signal-to-noise ratio for this model is $\|\bar{w}\|_2 / \sigma$

**Theorem 3.** *If $(x_i)_{i=1}^n$ are iid draws from Uniform$([-1,1]^d)$, $(y_i)_{i=1}^n$ follow the linear model with $\mathcal{N}(0, \sigma^2)$ noise, and*

$$SNR \;\leq\; (1 - 2c)^2$$

*for some $c \in (0, 1/2)$, then for any estimator $\hat{w}$, there exists $\bar{w} \in \mathbb{R}^d$ such that*

$$\mathbb{E}\,\|\hat{w} - \bar{w}\|_2 \geq c\|\bar{w}\|_2$$

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