A Treatise on ReMan

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Abstract

The goal of this work is to develop a complete model to study the benefits of depopulating heads in the field, which is known as ReMan. These benefits include the extension of the device lifetime and the gain in the data durability for erasure coded systems. We will describe the devices with two distinct failure modes: the first one is head related, causing individual heads to fail, and the other will be at the device level, failing the whole device. We will also provide simulation results to support the model.

Index Terms

Depop, Weibull Distribution, reliability

I. Modeling failure distributions

Weibull distribution [1] is a versatile function due to its flexibility to fit distributions of various shapes. We first consider a particular device failure mode, and assume it can be accurately fit by a Weibull distribution function. A Weibull distribution has two parameters: α parameter sets the time scale whereas the β parameter sets the shape of the distribution. The failure probability density, cumulative failure distributions, and the hazard rate (failure rate) are defined as follows:

$$f_{\alpha,\beta}(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}, \tag{1}$$

$$F_{\alpha,\beta}(t) = \int_0^t d\tau f_{\alpha,\beta}(\tau) = 1 - e^{-\left(\frac{t}{\alpha}\right)^{\beta}}, \tag{2}$$

$$h_{\alpha,\beta}(t) = \frac{f_{\alpha,\beta}(t)}{1 - F_{\alpha,\beta}(t)} = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}. \tag{3}$$

The hazard rate function, $h_{\alpha,\beta}(t)$, is the event rate normalized with respect the to the population that survived until t. The actual values of α and β vary from product to product: β is expected to be between 1 and 2. When $\beta \neq 1$, the failure rates are time dependent. It is certainly possible to model ReMan with $\beta \neq 1$, but $\beta = 1$ is typically a good fit, and that is what we will assume for most of the modeling work here. The case of $\beta \neq 1$ is treated in the last section. except for the last section

Figure 1. Animated Weibull functions as β is varied. The animation requires Adobe Acrobat reader (it will not work in a browser viewer.)

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the exponential distribution, which has completely random head failure times with a fixed failure(hazard) rate: $h_{\alpha,\beta}(t) = \frac{1}{\alpha} \equiv \lambda$. This simplifies Eqs. (1) , (2) and (3) to:

$$f_\lambda(t) = \lambda e^{-\lambda t}, \quad F_\lambda(t) = 1 - e^{-\lambda t}, \text{ and } \quad h_\lambda(t) = \lambda. \tag{4}$$

It is important to understand how the storage devices fail. Weibull is typically a good fit. No matter how reliable individual devices are, failures are inevitable. Therefore, redundancies are needed to preserve data in case of drive failures.

II. Modeling One-Head ReMan

The probability of having no failures in an N-header drive is:

$$p_{drive}^{pass} = p_{head}^{all\,pass} = (p_{head}^{pass})^N = e^{-N\left(\frac{t}{\alpha}\right)^\beta}, \tag{5}$$

where α and β are head-level Weibull parameters. The probability of having **exactly** one head failure is

$$p_{drive}^{exactly\,one\,head\,fail} = N\left(p_{head}^{fail}\right)(p_{head}^{pass})^{N-1} = N\left(1 - e^{-\left(\frac{t}{\alpha}\right)^\beta}\right)e^{-(N-1)\left(\frac{t}{\alpha}\right)^\beta}. \tag{6}$$

Then the cumulative probability of having two or more failures in an N-header drive (CDF_R) reads:

$$CDF_R = N\left(1 - e^{-(N-1)\left(\frac{t}{\alpha}\right)^\beta}\right) - (N-1)\left(1 - e^{-N\left(\frac{t}{\alpha}\right)^\beta}\right). \tag{7}$$

Eq. (7) is a mixture of two Weibull modes. The first term describes $N-1$ header drives, and the second describes N header drives. This equation still uses the time per head, and we need to convert that to drive level time. After the first head failure, the head time will increase as $t_d/(N-1)$. Scaling the time in Eq. (7), we get

$$CDF_R^{drive}(t_d) = 1 - p_{drive}^{pass} - p_{drive}^{exactly\,one\,head\,fail} = N\left(1 - e^{-(N-1)\left(\frac{t_d}{\alpha(N-1)}\right)^\beta}\right) - (N-1)\left(1 - e^{-N\left(\frac{t_d}{\alpha(N-1)}\right)^\beta}\right) \tag{8}$$

III. Modeling k-Head ReMan

Consider the case where at most k-head failure is acceptable. The probability of having up to and including k failures is

$$p_{drive}^{up\,to\,k\,head\,failures} = \sum_{j=0}^{k}\binom{N}{j}\left(p_{head}^{fail}\right)^j(p_{head}^{pass})^{N-j} = \sum_{j=0}^{k}\binom{N}{j}\left(1 - e^{-\left(\frac{t}{\alpha}\right)^\beta}\right)^j e^{-(N-j)\left(\frac{t}{\alpha}\right)^\beta}. \tag{9}$$

Then the CDF for having more than k head failures becomes

$$CDF(\text{k} - \text{ReMAN}) = p_{drive}^{more\,than\,k\,head\,failures} = 1 - p_{drive}^{up\,to\,k\,head\,failures}$$
$$= 1 - \sum_{j=0}^{k}\binom{N}{j}\left(1 - e^{-\left(\frac{t}{\alpha}\right)^\beta}\right)^j e^{-(N-j)\left(\frac{t}{\alpha}\right)^\beta}. \tag{10}$$

IV. ReMan eliminates head related failures

There are two types of failures:

Head specific failures, which can be handled by ReMan: Just depop the failing head, and move on. We will show that this practically eliminates head related failures.

Entire device failures: motor/PCBA/HSA failures, etc. These cannot be addressed by ReMan.

The overall reliability of a device can be computed by the multiplying the individual reliability functions, R_i:

$$R(t) = \prod_i R_i = R_{\text{ReMan'able}} \times R_{\text{non-ReMan'able}}. \tag{11}$$

$R_{\text{non-ReMan'able}}$ is the reliability associated with the entire device failures, and it can simply be defined by a failure rate λ_{NR}. $R_{\text{ReMan'able}}$, on the other hand, is the associated with the head-specific failures. They are fixable by depoping heads. $R_{\text{ReMan'able}}$ can be derived from Eq. (8) as $1 - CDF_R^{drive}(t_d)$.

See Fig. 2 for an example simulation. See the caption for the parameters.
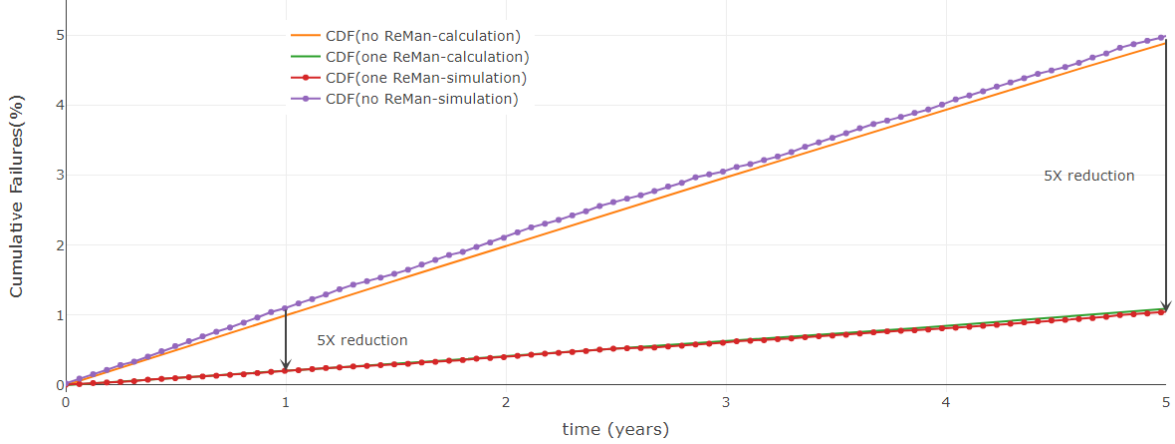

Figure 2. Comparison of cumulative failures with and without ReMan. Total failure rate is taken as 1% split into 0.8% ReManable and 0.2% Non-ReMan'able. We also take $\beta_R = 1$, and $\beta_{NR} = 1$.

Even one-head ReMan absorbs practically all head specific failures.

In this example, the Annualized Failure Rate (AFR) is effectively reduced down to 0.2% from the original value of 1%.

V. ReMan preserves capacity at the edge

Consider an edge application in which failed drives are not replaced immediately. ReMan will be an indispensable tool to preserve the capacity of the storage system:

Entire device failures will take down the full drive causing the loss of the full capacity.

Head specific failures will be ReMan'ed, losing $1/N$ of the full cap (5% for a 20-header drive).

ReMan helps preserve capacity much longer.

When no ReMan is allowed the average capacity will drop with drive failures, which can be written as follows:

$$\mathcal{CAP}_{0R} = \mathcal{CAP}_0(1 - CDF_{0R}) \tag{12}$$

When at most one ReMan is allowed, the average capacity will be

$$\mathcal{CAP}_{1R} = \mathcal{CAP}_0(1 - CDF_{1R}) - \mathcal{CAP}_0(CDF_{0R} - CDF_{1R})/N, \tag{13}$$

where the second term is the ratio of the drives with exactly one head failure [they lost a capacity of $\mathcal{CAP}_0/N$].
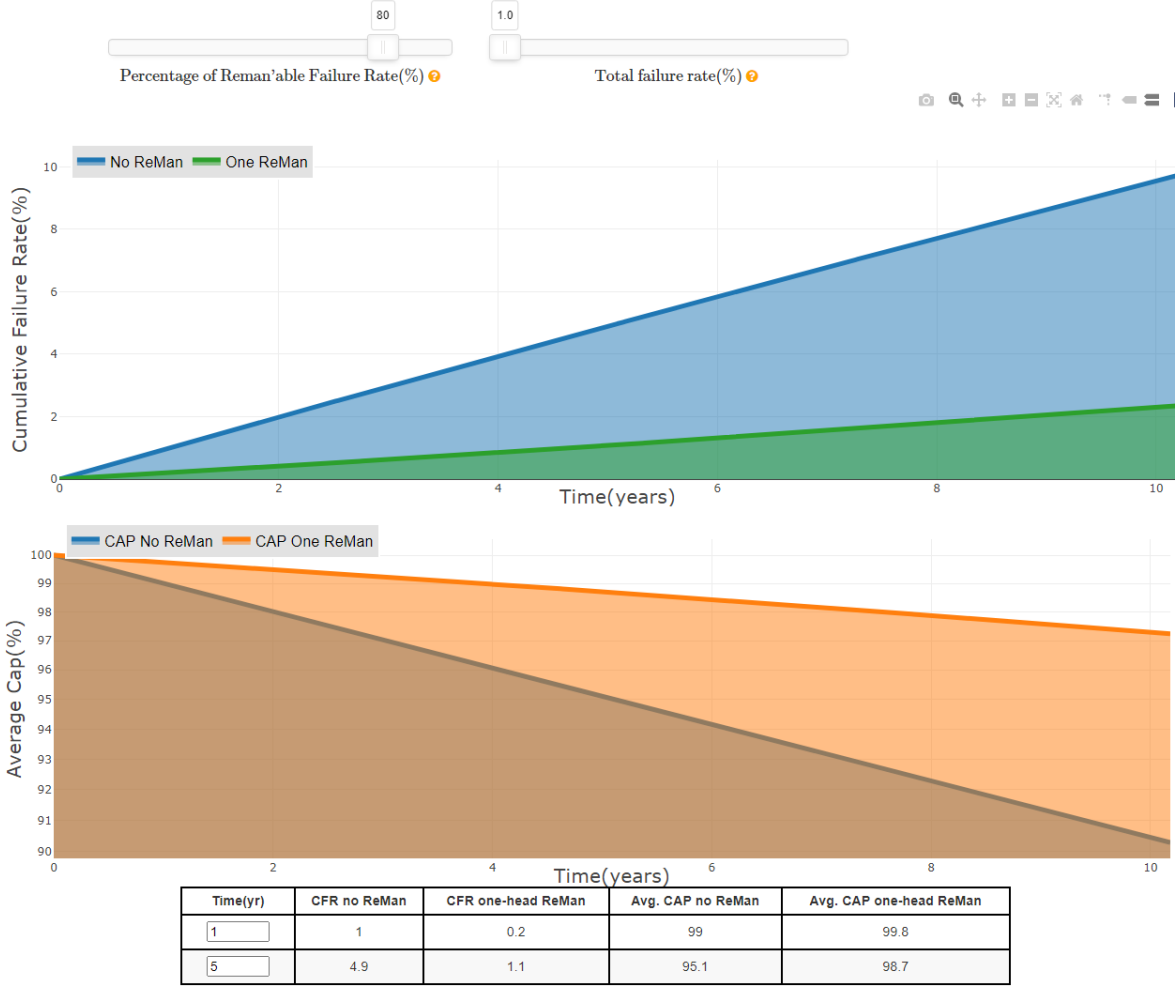
Figure 3 shows the CDF and capacity.

Figure 3. Cumulative Failures/Average Capacity vs time with number of heads $N = 20$. We also take $\beta_R = 1$, and $\beta_{NR} = 1$

VI. ReMan reduces swap rate and TCO in data centers

As shown earlier, ReMan substantially reduces drive failures and, in turn, the cost of replacing them. The only drawback of ReMan is that ReMan'ed drives will have one-fewer heads, hence slightly reduced capacity. In the case of maximum one-head ReMan, the overall capacity of the system will reduce only slightly. Qualitatively speaking, the ratio of ReMan'ed drives will increase slowly, because the increase is suppressed by two factors:

As drives fail for the NonReMan'able mode, they are replaced. This removes already ReMan'ed drives out of the population.

If a second head in an already-ReMan'ed drive fails, the drive is replaced. This also removes ReMan'ed drives out of the population.

The precise mathematical model is not trivial[2] due to the factors listed above. The final expression for the ReMan'ed drive density is given in Eq. (18) and it is plotted and compared with simulation results in Figure 4 for illustation purposes[see the caption for details]. Once the ReMan'ed drive density is calculated, the degradation in the overall capacity can be computed as in Eq. (20) . The change of average capacity with time is shown in Fig. 5.

When a head fails, it is "turned off," and the drive continues to operate with reduced capacity. We will denote the ratio of devices that lost i heads as r_i, and compute how these ratios evolve with time. We first

consider a replacement policy which allows for maximum one-head ReMan, which implies that

$$r_0(t) + r_1(t) = 1. \tag{14}$$

Furthermore, since the system starts with new devices, the initial values of the ratios are

$$r_0(0) = 1 \quad \text{and} \quad r_1(0) = 0. \tag{15}$$

We need to calculate how r_0 evolves with time given that devices have two distinct failure modes: **i)** a failure mode that affects individual heads, **ii)** another failure mode that affects the whole device and results in device replacement. We define the failure rate that affects heads as λ_{R}, where R stands for *Re-manufacturable*, and the other mode as λ_{NR}, where NR stands for *NonRe-manufacturable*. As heads fail, the ratio of the devices with no head failures (r_0) will decrease: $\dot{r}_0 \propto -\lambda_{\text{R}} r_0$. On the other hand, devices that already have one head failure (r_1) will be replaced with new devices once another head fails. The replacement will move these devices to the r_0 bucket. The rate of this event is proportional to the total number of heads in r_1 bucket. Adding this term yields $\dot{r}_0 \propto -\lambda_{\text{R}} r_0 + \lambda_{\text{R}} \frac{N-1}{N} r_1$, where the factor $\frac{N-1}{N}$ normalizes the failure rate since there are $N-1$ heads per device in the r_1 bucket. There is one more term to add: the device level failures with rate λ_{NR}. Such failures require device replacements, increasing r_0 only if the failure happens to be in the r_1 bucket. Therefore the rate of such events will be $r_1 \lambda_{\text{NR}}$. Adding all the terms, and using $r_1 = 1 - r_0$ result in the following differential equation:

$$\dot{r}_0 \;=\; -\lambda_{\text{R}} r_0 + \lambda_{\text{R}} \frac{N-1}{N}(1 - r_0) + \lambda_{\text{NR}}(1 - r_0). \tag{16}$$

Eq. (16) has the following solution:

$$r_0(t) = \frac{N - 1 + N\kappa + N e^{-(2+\kappa-1/N)\lambda_{\text{R}}t}}{2N - 1 + N\kappa} = 1 - \frac{1}{2 + \kappa - 1/N}\left(1 - e^{-(2+\kappa-1/N)\lambda_{\text{R}}t}\right), \tag{17}$$

where we defined $\kappa = \frac{\lambda_{\text{NR}}}{\lambda_{\text{R}}}$. Since $r_0 + r_1 = 1$, we get

$$r_1(t) = 1 - r_0(t) = \frac{1}{2 + \kappa - 1/N}\left(1 - e^{-(2+\kappa-1/N)\lambda_{\text{R}}t}\right), \tag{18}$$

which is the relative ratio of the ReMan'ed drives. We can now write the average capacity of the system:

$$\overline{CAP} = C(r_0 + r_1\frac{N-1}{N}) = C(r_0 + r_1 - \frac{r_1}{N}) = C(1 - \frac{r_1}{N}) \tag{19}$$

Equivalently, we can define the relative the loss in the capacity:

$$\frac{\Delta\overline{CAP}}{C} = \frac{C - \overline{CAP}}{C} = \frac{r_1}{N}, \tag{20}$$

which basically is the density of the ReMan'ed drives divided by N, since they lose $1/N$ of their capacity.
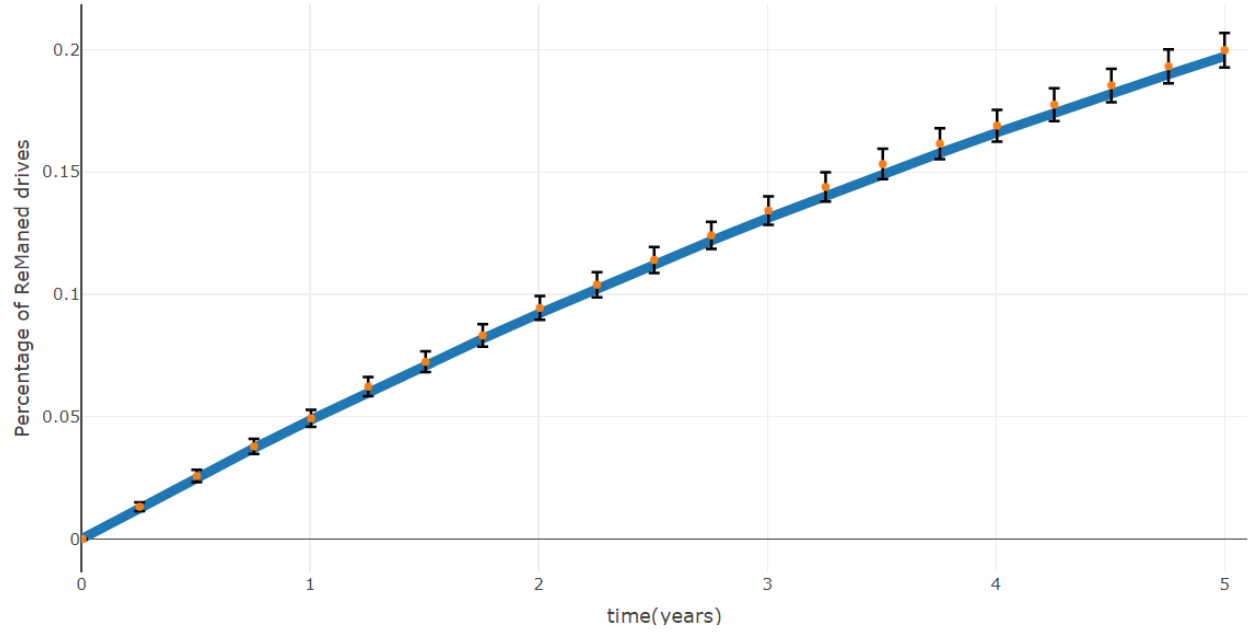
Figure 4. Cumulative Failures/Average Capacity vs time with number of heads $N = 20$. We also take $\beta_R = 1$, and $\beta_{NR} = 1$

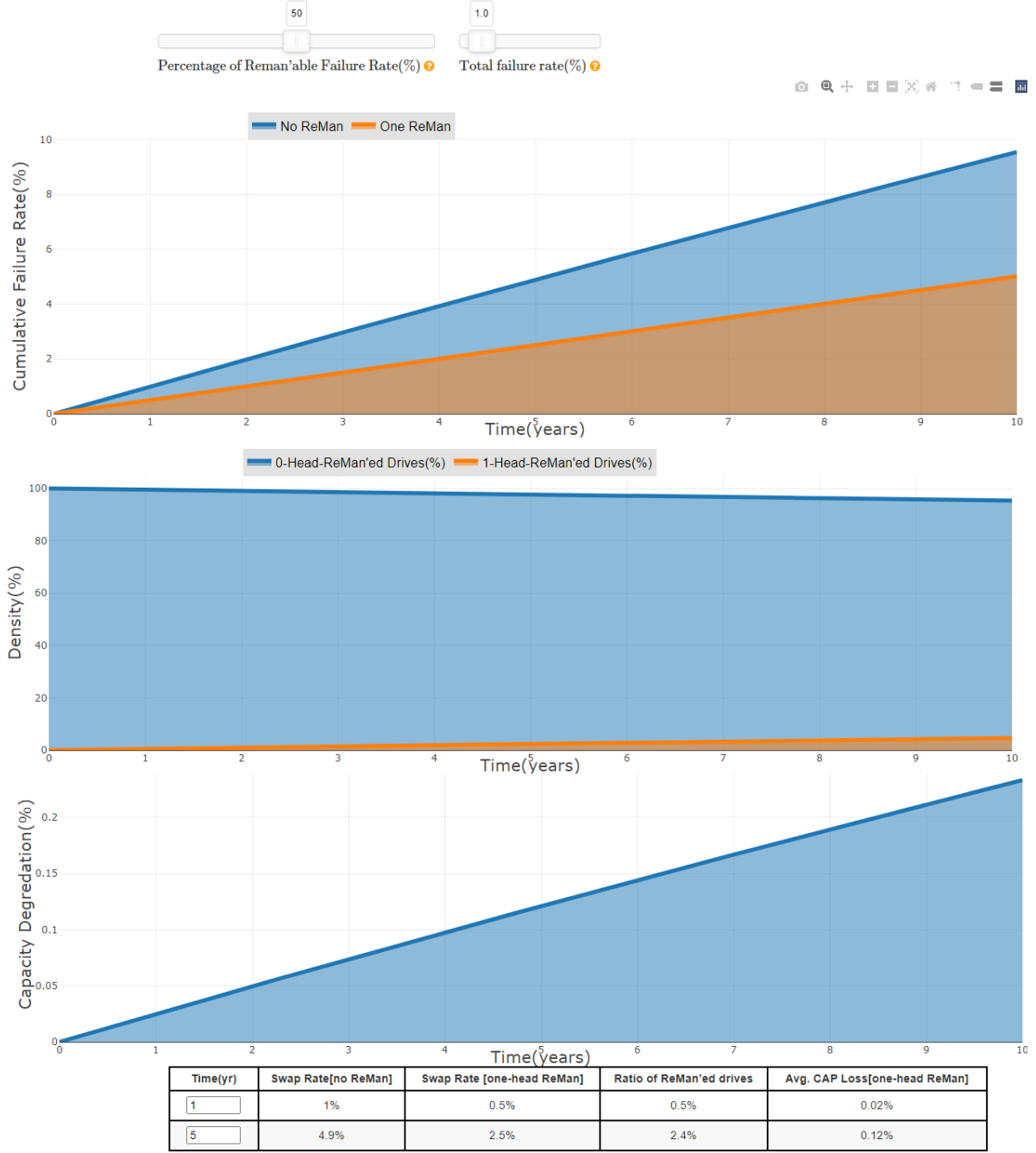Figure 5. CFR/Reman Density/Capacity degredation vs time, $\beta_R = 1$, and $\beta_{NR} = 1$.

VII. ReMan improves data durability

Consider the simplest case: one redundancy. Given a head/drive failure, there are two paths to losing data:

A second failure while recovering from a head failure: less likely due to faster recovery.

A second failure while recovering from a drive failure: more likely due to longer recovery.
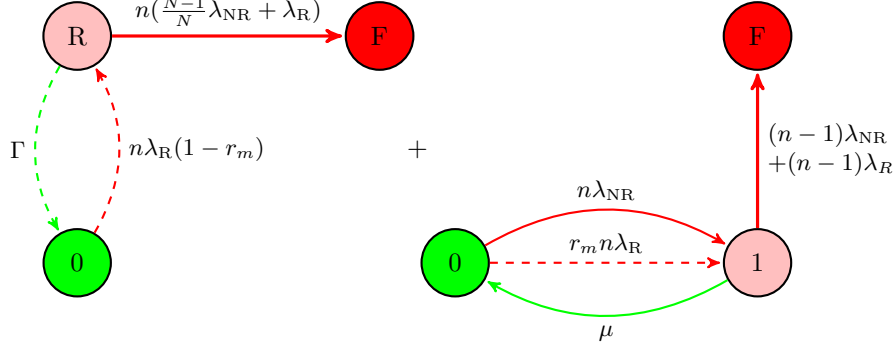
Figure 6. Markov chain split into two parallel paths: recovery from subcomponent failures and recovery from device failures.

The plot on the left shows two parallel Markov chains:

Left-most shows data recovery for a ReMan'able failure.

Right-most shows the standard recovery when a drive fails.

r_m term represents the maximally ReMan'ed drive population.

ReMan'able failures on maximal drives trigger replacement.

This is represented by the dashed red curves.

The transition rates are time dependent, and involve r_m functions.

$\mathcal{R}_1^{(1)}(t) = exp\left(-\frac{\mathcal{C}}{2\Gamma}\left[t + \frac{1-e^{-2\lambda_\mathrm{R} t}}{2\lambda_\mathrm{R}} \right] - \frac{\mathcal{C}}{2\mu}\left[(1+2\kappa)t - \frac{1-e^{-2\lambda_\mathrm{R} t}}{2\lambda_\mathrm{R}} \right] \right)$ where $\mathcal{C} \equiv n^2\lambda_\mathrm{R}^2(1+\kappa)$, and $\kappa \equiv \frac{\lambda_\mathrm{NR}}{\lambda_\mathrm{R}}$

n : number of drives in the EC scheme, N : number of heads per drive,

Γ : ReMan Repair rate, μ : drive repair rate, λ_R: ReMan'able failure rate, λ_{NR}: non-ReMan'able failure rate.

This is the expression when we allow for 1 ReMan/drive. Similar formulas are calculated for $R_{max} = 2, 3$. See [2] for the details.

We have an analytical model of Erasure Coded systems that support ReMan.

The closed mathematical form can be computed instantly enabling a real-time web application.

VIII. Simulation with online ReMan

- Figure 7 shows the comparison of Monte Carlo Simulation and theoretical results[Assuming 1-ReMan/drive is allowed.]:
- ReMan'able AFR= 8% and non ReMan'able AFR=2%.[Taken unrealistically high to show the functional behavior]
- The plot shows the theoretical prediction is remarkably accurate: The gain in durability is about 4x.
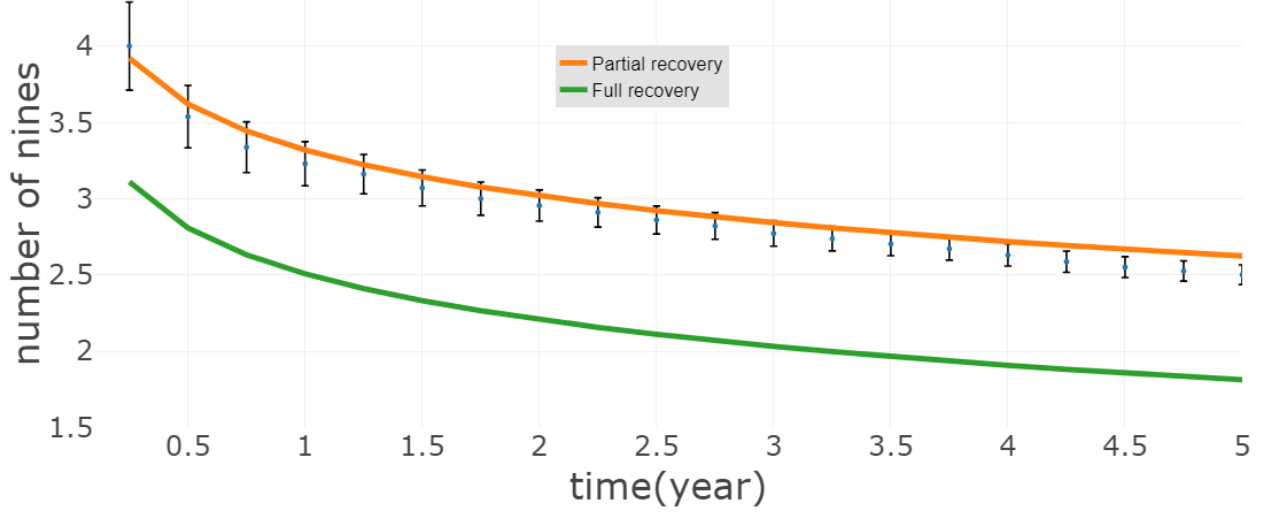
Figure 7. The calculated values of number of nines without ReMan(green curve), with ReMan(orange curve) and the simulation results for number of nines with ReMan (blue dots). Input parameters are taken as $\lambda_R = 1/12$ and $\lambda_{NR} = 1/50$ (1/year), which correspond to 8% and 2% AFR respectively.

IX. MODELING VARIABLE FAILURE RATES

So far we have assumed that λ_R and λ_{NR} are constants, and in this chapter we will generalize the analysis to generic, time dependent λ. The main difficulty in modeling is that the differential equation will have time dependent coefficients, and it will be harder to solve analytically. However, it turns out that if we assume that the ratio of the ReMan'able failure rate and the non-Reman'able one stays constant, we can still solve the problem analytically.

We start from Eq. (16) , and rewrite it as follows:

$$\dot{r}_0(t) = -\left[\frac{2N-1}{N}\lambda_R + \lambda_{NR}\right]r_0 + \lambda_{NR} + \frac{N-1}{N}\lambda_R \equiv a(t)r_0(t) + b(t), \tag{21}$$

with

$$a(t) = -\left[\frac{2N-1}{N}\lambda_R + \lambda_{NR}\right], \quad \text{and} \quad b(t) = \lambda_{NR} + \frac{N-1}{N}\lambda_R. \tag{22}$$

The solution to the ordinary differential equation in Eq. (21) is given by

$$r_0(t) = e^{A(t)}\left[1 + \int_0^t d\tau\, e^{-A(\tau)}b(\tau)\right], \tag{23}$$

where we defined $A(t)$ as

$$A(t) = \int_0^t d\tau\, a(\tau). \tag{24}$$

Whether the integrals in Eq. (23) can be evaluated analytically will depend on the functional forms of λ_R and λ_{NR}.

Proportional failure rates

Let us first consider a special case for which $\lambda_{NR} = \kappa\lambda_R$, i.e., the ReMan'able failure rate and the non-ReMan'able one have the same functional form. This will enable us to combine the terms and compute the integrals. To simplify the notation, we introduce a pair of new functions:

$$\lambda_R(t) \equiv \frac{d\omega_R}{dt}, \quad \text{and} \quad \lambda_{NR}(t) \equiv \frac{d\omega_{NR}}{dt}, \tag{25}$$

and compute $A(t)$ in Eq. (24)

$$A(t) = -\int_0^t d\tau\, \frac{d}{d\tau}\left[\frac{2N-1}{N}\omega_R + \omega_{NR}\right] = -\frac{2N-1+N\kappa}{N}\omega_R. \tag{26}$$

The integral in Eq. (23) can be evaluated as

$$
\begin{aligned}
I & \equiv \int_0^t d\tau \, e^{-A(\tau)} b(\tau) = \frac{N-1+N\kappa}{N} \int_0^t d\tau \, e^{\frac{2N-1+N\kappa}{N}\omega_{\mathrm{R}}} \frac{d\omega_{\mathrm{R}}}{d\tau} \\
& = \left[e^{-\frac{2N-1+N\kappa}{N}\omega_{\mathrm{R}}(t)} - 1 \right] \frac{N-1+N\kappa}{2N-1+N\kappa}.
\end{aligned}
\tag{27}
$$

Putting it back in Eq. (23) yields

$$
r_0(t) = \frac{N-1+N\kappa + Ne^{-\frac{2N-1+N\kappa}{N}\omega_{\mathrm{R}}(t)}}{2N-1+N\kappa} = 1 - \frac{1}{2+\kappa-1/N}\left(1 - e^{-(2+\kappa-1/N)\omega_{\mathrm{R}}(t)}\right),
\tag{28}
$$

and using $r_0 + r_1 = 1$ we get:

$$
r_1(t) = \frac{1}{2+\kappa-1/N}\left(1 - e^{-(2+\kappa-1/N)\omega_{\mathrm{R}}(t)}\right).
\tag{29}
$$

It may look surprising that Eq. (28) is very similar to Eq. (17), the only difference being the change from $\lambda_{\mathrm{R}}t$ to $\omega_{\mathrm{R}}(t)$. However, one could see this coming: if $\lambda_{\mathrm{NR}} = \kappa\lambda_{\mathrm{R}}$, all the explicit time dependence can simply be absorbed by defining a time parameter $\xi \equiv \omega_{\mathrm{R}}(t)$. This will transform $\frac{d}{dt}$ to $\frac{d\xi}{dt}\frac{d}{d\xi} = \frac{d\omega_{\mathrm{R}}}{dt}\frac{d}{d\xi} = \lambda_{\mathrm{R}}\frac{d}{d\xi}$, and λ_{R} factor on the left of the derivative operator cancels out with the same factor on the right hand side of Eq. (16) and hides away the variable rates into the warped time parameter ξ. Equation (29) describes how the density of ReMan'ed drives will increase given any generic $w_{\mathrm{R}}(t)$. In the case of Weibull distribution, we will have

$$
\omega_{\mathrm{R}}(t) = \left(\frac{t}{\alpha}\right)^{\beta}.
\tag{30}
$$

Figure 8 shows various functions where the failure rates are time dependent but the ratio of the ReMan'able failure rate and the non-Reman'able one stays constant, i.e., $\lambda_{\mathrm{NR}} = \kappa\lambda_{\mathrm{R}}$. The parameter β sets the shape of the failure distribution, as in Eqs (1), (2), (3), and Fig 1. Note that we will use a more intuitive parameter to replace α, and call it "Total Failure Rate in the first year," which is, as the name suggests, the total failure at the end of the first year, i.e., CDF at 1 year. Note that the failure rate is time dependent, and it will be a different number in the following years. In fact, it will get larger if $\beta > 1$. Once we define the first year failure rate and the shape parameter β, the failure distribution is completely fixed. The value of α can be expressed in "Total Failure Rate in the first year" by inverting Eq. (2) at $t = 1$(year) as follows

$$
\alpha = (-ln[1 - \text{Total Failure Rate in the first year}])^{-1/\beta},
\tag{31}
$$

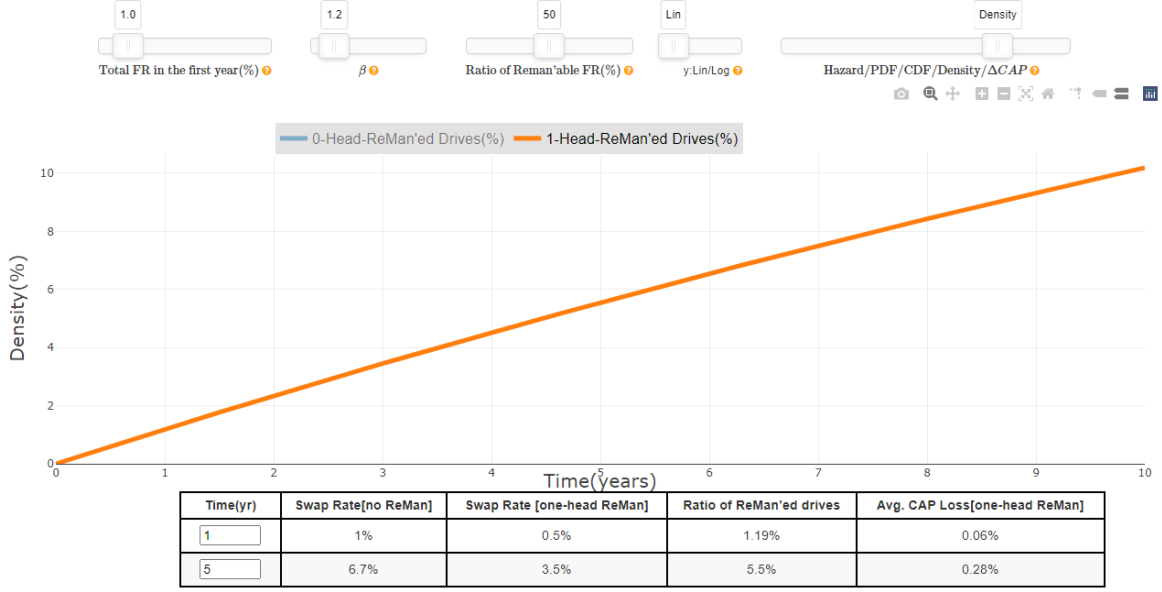which is the number that enters into the Weibull functions.

Time(yr)	Swap Rate[no ReMan]	Swap Rate [one-head ReMan]	Ratio of ReMan'ed drives	Avg. CAP Loss[one-head ReMan]
1	1%	0.5%	1.19%	0.06%
5	6.7%	3.5%	5.5%	0.28%

Figure 8. Capacity degredation vs time

Generic Failure Rates

For completeness, let us consider the case where λ_{NR} and λ_{R} are not necessarily proportional to each other. They might be certain empirical distributions extracted from data. It really comes down to evaluating the integral Eq. (23) , and that can be done numerically. However, we can at least evaluate the dominant portion analytically, and refer to numerical methods for the rest.

We start from Eq. (23) and split it into two parts as follows:

$$
\begin{aligned}
I &= \int_0^t d\tau \, e^{\left[\frac{2N-1}{N}\omega_{\mathrm{R}}(\tau)+\omega_{\mathrm{NR}}(\tau)\right]} \frac{d}{d\tau}\left[\omega_{\mathrm{NR}}+\frac{N-1}{N}\omega_{\mathrm{R}}\right] \\
&= \frac{N-1}{2N-1}\int_0^t d\tau \, e^{\left[\frac{2N-1}{N}\omega_{\mathrm{R}}(\tau)+\omega_{\mathrm{NR}}(\tau)\right]} \frac{d}{d\tau}\left[\frac{2N-1}{N}\omega_{\mathrm{R}}+\omega_{\mathrm{NR}}+\frac{N}{N-1}\omega_{\mathrm{NR}}\right] \\
&= \frac{N-1}{2N-1}\left(e^{\left[\frac{2N-1}{N}\omega_{\mathrm{R}}(t)+\omega_{\mathrm{NR}}(t)\right]}-1\right)+\frac{N}{2N-1}\int_0^t d\tau \, e^{\left[\frac{2N-1}{N}\omega_{\mathrm{R}}(\tau)+\omega_{\mathrm{NR}}(\tau)\right]}\frac{d\omega_{\mathrm{NR}}}{d\tau}. \quad (32)
\end{aligned}
$$

It may look like we have not accomplished much since we still have an integral left, however, in practice λ_{NR} will give a smaller contribution to the overall failures [it can be as low as 20% of the total failure rate.] We can still evaluate the remaining integral numerically. Alternatively, we can approximate the integral as $\omega_{\mathrm{NR}}(t)\ll 1$ and $\omega_{\mathrm{NR}}(t)\ll 1$:

$$
\begin{aligned}
I_2 &\equiv \frac{N}{2N-1}\int_0^t d\tau \, e^{\left[\frac{2N-1}{N}\omega_{\mathrm{R}}(\tau)+\omega_{\mathrm{NR}}(\tau)\right]}\frac{d\omega_{\mathrm{NR}}}{d\tau} \simeq \frac{N}{2N-1}\int_0^t d\tau\left[1+\frac{2N-1}{N}\omega_{\mathrm{R}}(\tau)+\omega_{\mathrm{NR}}(\tau)\right]\frac{d\omega_{\mathrm{NR}}}{d\tau} \\
&\simeq \frac{N}{2N-1}\omega_{\mathrm{NR}}(t)+\frac{N}{2(2N-1)}\omega_{\mathrm{NR}}^2(t)+\frac{N}{2N-1}\int_0^t d\tau \, \omega_{\mathrm{R}}(\tau)\frac{d\omega_{\mathrm{NR}}}{d\tau}. \quad (33)
\end{aligned}
$$

If we use Weibull expressions for ω, as given in (30) :

$$
\begin{aligned}
\int_0^t d\tau \, \omega_{\mathrm{R}}(\tau)\frac{d\omega_{\mathrm{NR}}}{d\tau} &= \frac{\beta_{\mathrm{NR}}}{\alpha_{\mathrm{NR}}}\int_0^t d\tau\left(\frac{t}{\alpha_{\mathrm{R}}}\right)^{\beta_{\mathrm{R}}}\left(\frac{t}{\alpha_{\mathrm{NR}}}\right)^{\beta_{\mathrm{NR}}-1} = \frac{\beta_{\mathrm{NR}}}{(\alpha_{\mathrm{NR}})^{\beta_{\mathrm{NR}}}(\alpha_{\mathrm{R}})^{\beta_{\mathrm{R}}}}\int_0^t d\tau \, \tau^{\beta_{\mathrm{R}}+\beta_{\mathrm{NR}}-1} \\
&= \frac{\tau^{\beta_{\mathrm{R}}+\beta_{\mathrm{NR}}}}{(\alpha_{\mathrm{NR}})^{\beta_{\mathrm{NR}}}(\alpha_{\mathrm{R}})^{\beta_{\mathrm{R}}}} = \frac{\beta_{\mathrm{NR}}}{\beta_{\mathrm{R}}+\beta_{\mathrm{NR}}}\omega_{\mathrm{R}}(t)\omega_{\mathrm{NR}}(t). \quad (34)
\end{aligned}
$$

Compiling the terms yields:

$$
I \simeq \frac{N-1}{2N-1}\left(e^{\left[\frac{2N-1}{N}\omega_{\mathrm{R}}(t)+\omega_{\mathrm{NR}}(t)\right]}-1\right)+\frac{N}{2N-1}\omega_{\mathrm{NR}}(t)+\frac{N}{2(2N-1)}\omega_{\mathrm{NR}}^2(t)+\frac{N}{2N-1}\frac{\beta_{\mathrm{NR}}}{\beta_{\mathrm{R}}+\beta_{\mathrm{NR}}}\omega_{\mathrm{R}}(t)\omega_{\mathrm{NR}}(t). \quad (35)
$$

Finally we put the results back in Eq. (23) and get

$$r_0(t) \simeq \frac{N-1}{2N-1} + \frac{N}{2N-1} e^{-\left[\frac{2N-1}{N}\omega_{\text{R}}(t)+\omega_{\text{NR}}(t)\right]} \left[1 + \omega_{\text{NR}}(t) + \frac{1}{2}\omega_{\text{NR}}^2(t) + \frac{\beta_{\text{NR}}}{\beta_{\text{R}}+\beta_{\text{NR}}}\omega_{\text{R}}(t)\omega_{\text{NR}}(t)\right] (36)$$

It is important to note that this equation holds when failure modes are defined by Weibull distributions. For other distributions, one needs to evaluate the integral in Eq. (32) .

X. References

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