Lab3

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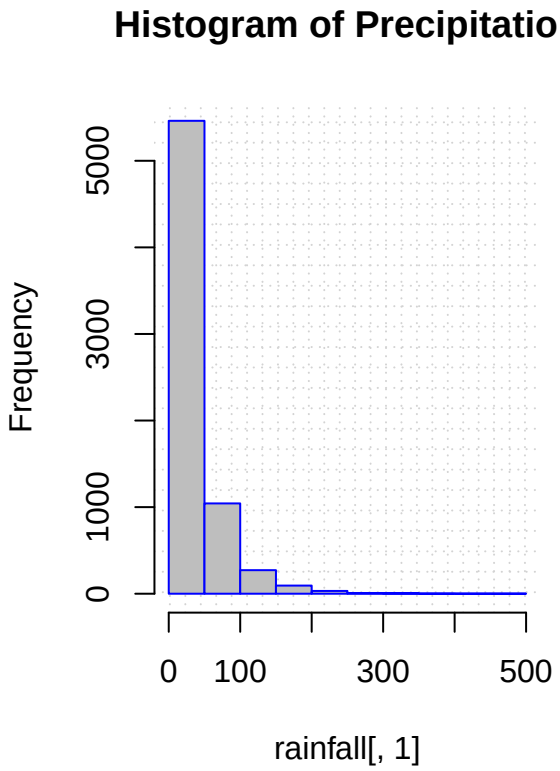
1.Normal model, mixture of normal model with semi-conjugate prior

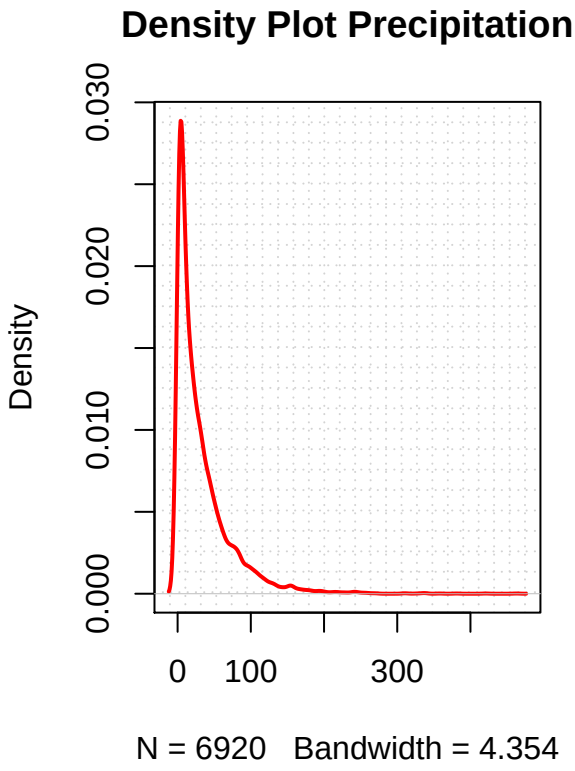
a)Normal model

Assume the daily precipitation $y_1, ..., y_n$ are independent normally distributed,$y_1, ..., y_n|\mu, \sigma^2 \sim N(\mu, \sigma^2)$ where both μ and σ^2 are unknown. Let $\mu \sim N(\mu_0, \tau_0^2)$ independently of $\sigma^2 \sim Inv - \chi^2(\nu_0, \sigma_0^2)$

Plot of Histogram and Denstity

```
#  i)
# read data
rainfall<-read.table('rainfall.dat',head=F)

par(mfrow=c(1,2))
hist(rainfall[,1],col="gray",border="blue",
     main="Histogram of Precipitation",panel.first=grid(25,25))
plot(density(rainfall[,1]),col="red",
     main="Density Plot Precipitation",
     lwd=2,panel.first=grid(25,25))
```





i)

Implement (code!) a Gibbs sampler that simulates from the joint posterior $p(\mu, \sigma^2|y_1, ..., y_n$.The full conditional posteriors are given on the slides from Lecture 7.

```
# inverse chi-sqare function
rInvChi2<-function(v0,sigma_sq0){
  # returns one sample from Inv-chi square for given (df,sigma)
  inverse.chi<-(v0*sigma_sq0)/rchisq(1,v0)
  return(inverse.chi)
}
# Gibbs sampler function
GibbsSampler<-function(nIter,mu0,tau0_squared,Sigma0_squared,v0,y){
  n=length(y)
  muPost<-rep(0,nIter) # posterior mu
  SigmaPost<-rep(0,nIter) # posterior sigma
  #ySample<-rep(0,nIter)

  numerator= ( n/Sigma0_squared )+( 1/tau0_squared)
  w=(n/Sigma0_squared)/numerator
  mu_n=w*mean(y)+(1-w)*mu0
  tau_n_squared=1/numerator
  # filling the posterior vectors
  for (i in 1:nIter){

    muPost[i]<-rnorm(n = 1,mean = mu_n,sd = tau_n_squared)
```

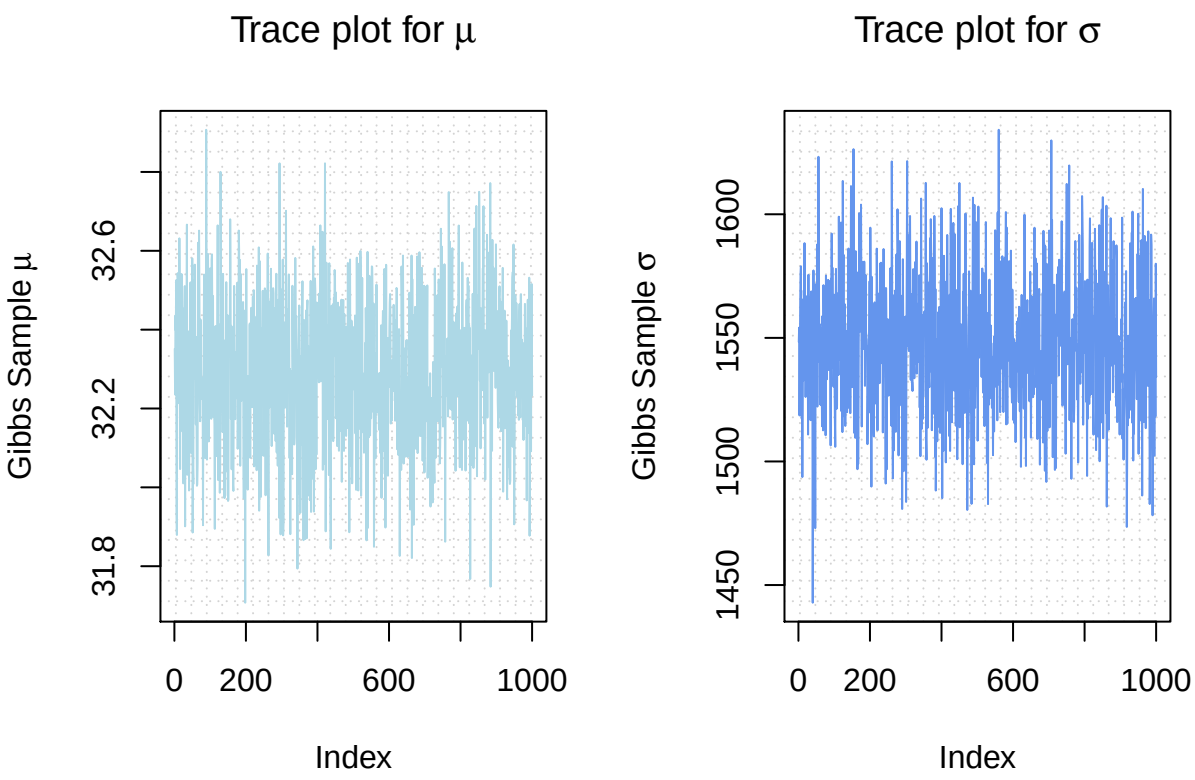
```
    SigmaPost[i]<-rInvChi2(v0+n, (v0*Sigma0_squared+sum( (y-muPost[i])^2)) / (v0+n) )
    #ySample[i]<-rnorm(n=1,muPost[i],SigmaPost[i])
  }

  return(list('muPost'=muPost,'SigmaPost'=SigmaPost)) #,'samplePost'=ySample))

}
# setting priors
mu0=mean(rainfall[,1])
tau0=1
v0=1
Sigma2=var(rainfall[,1])
# take sample
gibbs_sample=GibbsSampler(1000,mu0,tau0,Sigma2,v0,rainfall[,1])
```

ii)

Analyze the daily precipitation using your Gibbs sampler in (a)-i. Evaluate the convergence of the Gibbs sampler by suitable graphical methods, for example by plotting the trajectories of the sampled Markov chains.

Trace Plots

```
par(mfrow=c(1,2))
plot(gibbs_sample$muPost,type = "l",main = expression(paste("Trace plot for ", mu)),
     col="lightblue",ylab=expression(paste("Gibbs Sample ",mu)),
     panel.first = grid(25,25))
plot(gibbs_sample$SigmaPost,type = 'l',
     main = expression(paste("Trace plot for ",sigma)),
     col="cornflowerblue",ylab=expression(paste("Gibbs Sample ",sigma)),
     panel.first = grid(25,25))
```



Converdence Plots

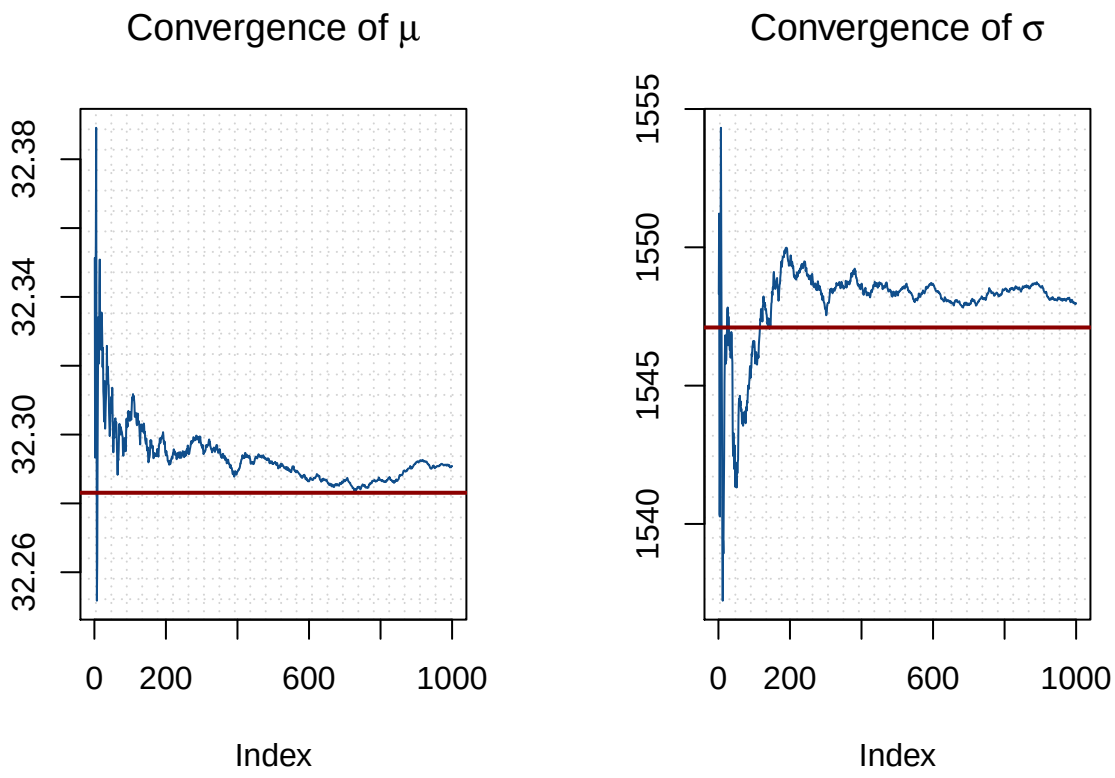
```
library(dplyr)
```

```
##
## Attaching package: 'dplyr'
```

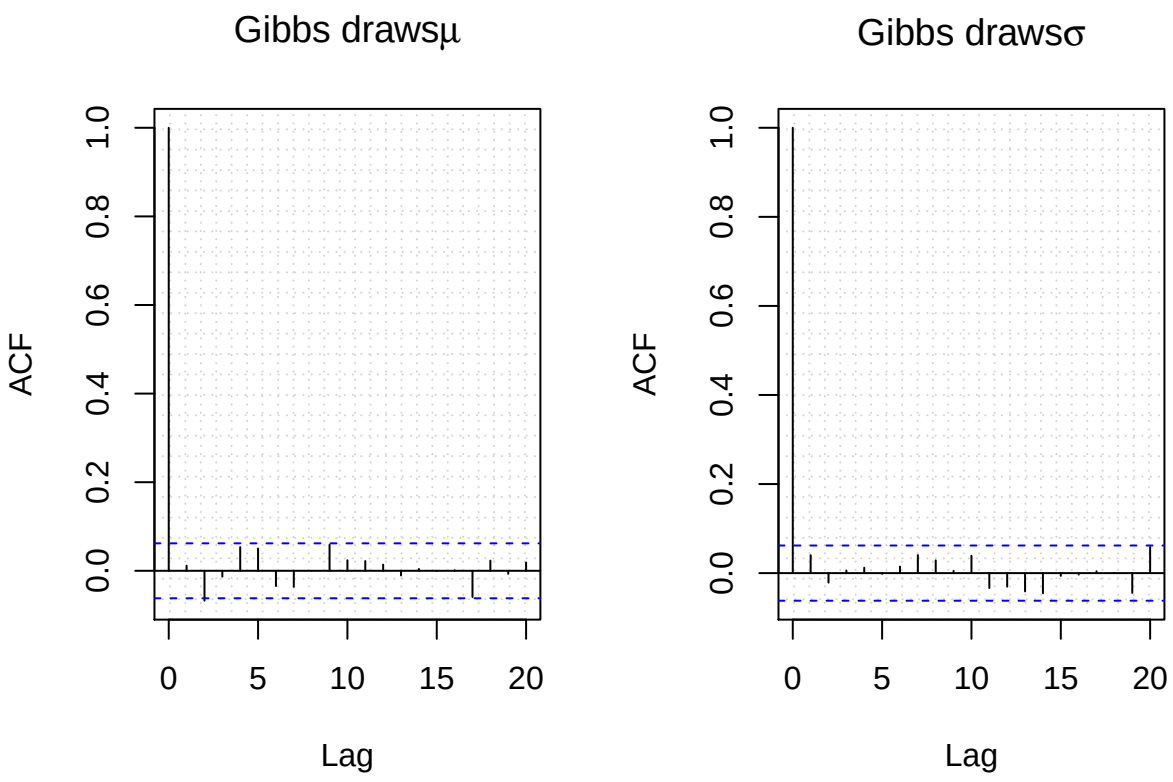
```
## The following objects are masked from 'package:stats':
##
##     filter, lag
```

```
## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

```
par(mfrow=c(1,2))
plot(cummean(gibbs_sample$muPost),type="l",col="dodgerblue4",
     main=expression(paste("Convergence of ",mu)),ylab = " ",
     panel.first = grid(25,25))
abline(h=mean(rainfall[,1]),col="darkred",lwd=2)

plot(cummean(gibbs_sample$SigmaPost),type="l",col="dodgerblue4",
     main = expression(paste("Convergence of ",sigma)),ylab=" ",
     panel.first = grid(25,25))
abline(h=var(rainfall[,1]),col="darkred",lwd=2)
```




```
par(mfrow=c(1,2))
acf(gibbs_sample$muPost, main=expression(paste('Gibbs draws',mu)),
    lag.max = 20,panel.first = grid(25,25))
acf(gibbs_sample$SigmaPost, main= expression(paste('Gibbs draws',sigma)),
    lag.max = 20,panel.first = grid(25,25))
```



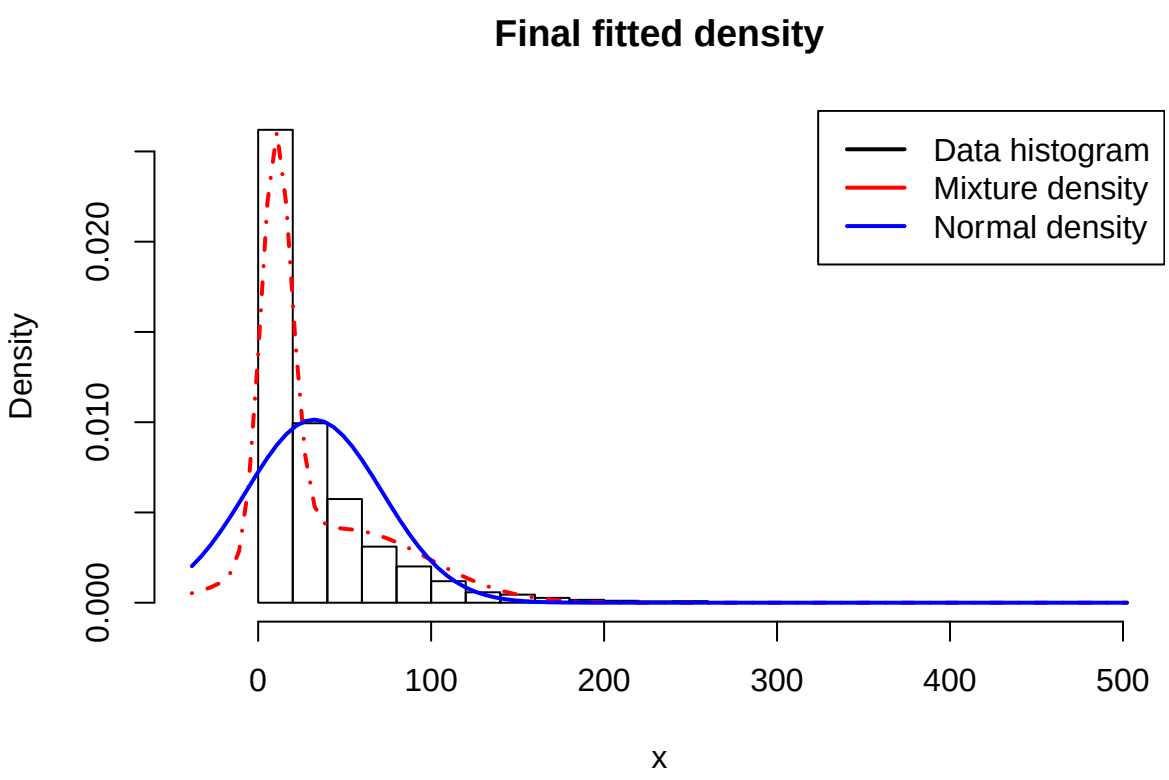
b)Mixture normal model

```
###Q1T2
# Estimating a simple mixture of normals
# Author: Mattias Villani, IDA, Linkoping University. http://mattiasvillani.com

##########    BEGIN USER INPUT #################
# Data options
rawData <- rainfall
x <- as.matrix(rainfall[,1])

# Model options
nComp <- 2    # Number of mixture components

# Prior options
alpha <- 10*rep(1,nComp) # Dirichlet(alpha)
muPrior <- rep(0,nComp) # Prior mean of mu
tau2Prior <- rep(10,nComp) # Prior std of mu
sigma2_0 <- rep(var(x),nComp) # s20 (best guess of sigma2)
nu0 <- rep(4,nComp) # degrees of freedom for prior on sigma2


# MCMC options
nIter <- 100 # Number of Gibbs sampling draws
```

```r
# Plotting options
plotFit <- TRUE
lineColors <- c("blue", "green", "magenta", 'yellow')
#sleepTime <- 0.1 # Adding sleep time between iterations for plotting
################   END USER INPUT ###############

###### Defining a function that simulates from the
rScaledInvChi2 <- function(n, df, scale){
  return((df*scale)/rchisq(n,df=df))
}

####### Defining a function that simulates from a Dirichlet distribution
rDirichlet <- function(param){
  nCat <- length(param)
  piDraws <- matrix(NA,nCat,1)
  for (j in 1:nCat){
    piDraws[j] <- rgamma(1,param[j],1)
  }
  piDraws = piDraws/sum(piDraws) # Diving every column of piDraws by the sum of the elements in that co
  return(piDraws)
}

# Simple function that converts between two different representations of the mixture allocation
S2alloc <- function(S){
  n <- dim(S)[1]
  alloc <- rep(0,n)
  for (i in 1:n){
    alloc[i] <- which(S[i,] == 1)
  }
  return(alloc)
}

# Initial value for the MCMC
nObs <- length(x)
S <- t(rmultinom(nObs, size = 1 , prob = rep(1/nComp,nComp))) # nObs-by-nComp matrix with component all
mu <- quantile(x, probs = seq(0,1,length = nComp))
sigma2 <- rep(var(x),nComp)
probObsInComp <- rep(NA, nComp)

# Setting up the plot
xGrid <- seq(min(x)-1*apply(x,2,sd),max(x)+1*apply(x,2,sd),length = 100)
xGridMin <- min(xGrid)
xGridMax <- max(xGrid)
mixDensMean <- rep(0,length(xGrid))
effIterCount <- 0
#ylim <- c(0,2*max(hist(x)$density))

for (k in 1:nIter){
  #message(paste('Iteration number:',k))
  alloc <- S2alloc(S) # Just a function that converts between different representations of the group al
  nAlloc <- colSums(S)
  #print(nAlloc)
  # Update components probabilities
```

```
  pi <- rDirichlet(alpha + nAlloc)

  # Update mu's
  for (j in 1:nComp){
    precPrior <- 1/tau2Prior[j]
    precData <- nAlloc[j]/sigma2[j]
    precPost <- precPrior + precData
    wPrior <- precPrior/precPost
    muPost <- wPrior*muPrior + (1-wPrior)*mean(x[alloc == j])
    tau2Post <- 1/precPost
    mu[j] <- rnorm(1, mean = muPost, sd = sqrt(tau2Post))
  }

  # Update sigma2's
  for (j in 1:nComp){
    sigma2[j] <- rScaledInvChi2(1, df = nu0[j] + nAlloc[j], scale = (nu0[j]*sigma2_0[j] + sum((x[alloc =
  }

  # Update allocation
  for (i in 1:nObs){
    for (j in 1:nComp){
      probObsInComp[j] <- pi[j]*dnorm(x[i], mean = mu[j], sd = sqrt(sigma2[j]))
    }
    S[i,] <- t(rmultinom(1, size = 1 , prob = probObsInComp/sum(probObsInComp)))
  }

  # Printing the fitted density against data histogram
  if (plotFit && (k%%1 ==0)){
    effIterCount <- effIterCount + 1
    #hist(x, breaks = 20, freq = FALSE, xlim = c(xGridMin,xGridMax), main = paste("Iteration number",k)
    mixDens <- rep(0,length(xGrid))
    components <- c()
    for (j in 1:nComp){
      compDens <- dnorm(xGrid,mu[j],sd = sqrt(sigma2[j]))
      mixDens <- mixDens + pi[j]*compDens
      #lines(xGrid, compDens, type = "l", lwd = 2, col = lineColors[j])
      components[j] <- paste("Component ",j)
    }
    mixDensMean <- ((effIterCount-1)*mixDensMean + mixDens)/effIterCount

    #lines(xGrid, mixDens, type = "l", lty = 2, lwd = 3, col = 'red')
    #legend("topleft", box.lty = 1, legend = c("Data histogram",components, 'Mixture'),
    #       col = c("black",lineColors[1:nComp], 'red'), lwd = 2)
  }
}
```

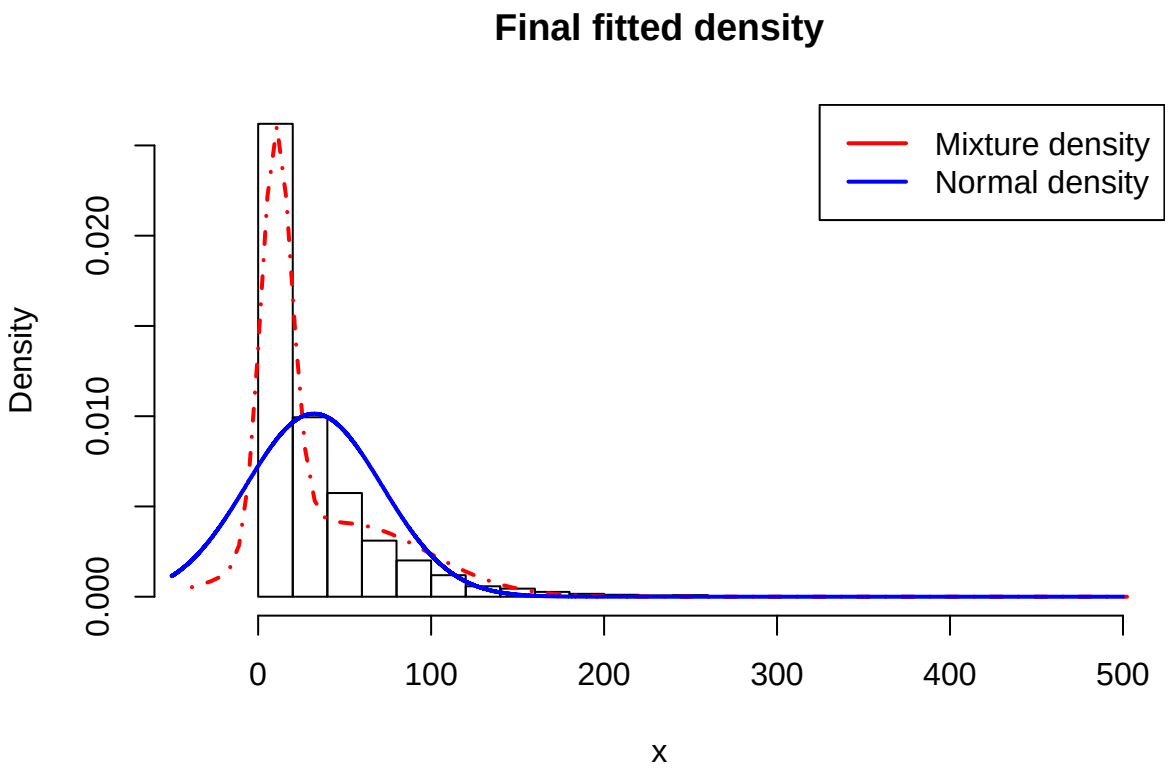
```
hist(x, breaks = 20, freq = FALSE, xlim = c(xGridMin,xGridMax), main = "Final fitted density")
lines(xGrid, mixDensMean, type = "l", lwd = 2, lty = 4, col = "red")
lines(xGrid, dnorm(xGrid, mean = mean(x), sd = apply(x,2,sd)), type = "l", lwd = 2, col = "blue")
legend("topright", box.lty = 1, legend = c("Data histogram","Mixture density","Normal density"), col=c(
```




```
#########################    Helper functions    ##############################################
```

c)Graphical comparison.

```
###Q1T3
hist(x, breaks = 20, freq = FALSE, xlim = c(xGridMin,xGridMax), main = "Final fitted density")
lines(xGrid, mixDensMean, type = "l", lwd = 2, lty = 4, col = "red")
lines(seq(-50,500,0.01),dnorm(seq(-50,500,0.01),mean(gibbs_sample$muPost),sqrt(mean(gibbs_sample$SigmaP
legend("topright", box.lty = 1, legend = c("Mixture density","Normal density"), col=c("red","blue"), lw
```



2.Metropolis Random Walk for Poisson regression

Consider the following Poisson regression model $y_i|\beta \sim Poisson[exp(X_i^T\beta)], i = 1, 2, ..., n$, where y_i is the count for the ith observation in the sample and x_i is the p-dimensional vector with covariate observations for the ith observation. Use the data set eBayNumberOfBidderData.dat. This dataset contains observations from 1000 eBay auctions of coins. The response variable is nBids and records the number of bids in each auction. The remaining variables are features/covariates (x):

a)

Obtain the maximum likelihood estimator of β in the Poisson regression model for the eBay data [Hint: glm.R, don €™t forget that glm() adds its own intercept so don €™t input the covariate Const]. Which covariates are significant?

```
# a)
# read data
ebay_data=read.table('eBayNumberOfBidderData.dat',head=T)
#dim(ebay_data)
# fit glm model
glmModel <-glm(nBids ~ 0 + ., data = ebay_data, family = poisson(link = "log"))

# create a boolen for the significant coefficients a=0.05
coeffs_toselect<-summary(glmModel)$coefficients[-1,4]<0.05
# select sig. variables
sign_coeffs<- names(coeffs_toselect)[coeffs_toselect == TRUE]
```

```
cat('The significant coefficients are :',sign_coeffs)
```

```
## The significant coefficients are : VerifyID Sealed MajBlem LogBook MinBidShare
```

b)

Let €™s now do a Bayesian analysis of the Poisson regression. Let the prior be $\beta \sim N[0, 100 * (X^T X)^-1]$ where X is the nxp covariate matrix. This is a commonly used prior which is called Zellner €™s g-prior. Assume first that the posterior density is approximately multivariate normal:

$$\beta|y \sim N(\tilde{\beta}, J_y^- 1(\tilde{\beta})),$$

where $\tilde{\beta}$ is the posterior mode and $J_y(\tilde{\beta})$ is the negative Hessian at the posterior mode.$\tilde{\beta}$ and $J_y(\tilde{\beta})$ can be obtained by numerical optimization (optim.R) exactly like you already did for the logistic regression in Lab 2 (but with the log posterior function replaced by the corresponding one for the Poisson model,which you have to code up.).

```
# b)

library("mvtnorm")
```

```
## Warning: package 'mvtnorm' was built under R version 3.5.2
```

```
# Data from the read.table function is a data frame. Let's convert y and X to vector and matrix.
y <- as.vector(ebay_data[,which(names(ebay_data)=="nBids")]);length(y) # response variable
```

```
## [1] 1000
```

```
X <- as.matrix(ebay_data[,-which(names(ebay_data)=='nBids')]);dim(X) # covariates
```

```
## [1] 1000    9
```

```
covNames <- names(X)
nPara <- dim(X)[2]; # number of covariates
# setting priors
tau <- 10;
Sigma <- tau^2*solve(t(X)%*%X);
# LOgposterior function
LogPostPoison <- function(betaVect,y,X,Sigma){

  nPara <- length(betaVect);
  linPred <- X%*%betaVect;

  # evaluating the log-likelihood
  logLik <- sum( linPred*y -exp(linPred) );
  if (abs(logLik) == Inf) logLik = -20000; # Likelihood is not finite, stear the optimizer away from he

  # evaluating the prior
  logPrior <- dmvnorm(betaVect, matrix(0,nPara,1), Sigma, log=TRUE);

  # add the log prior and log-likelihood together to get log posterior
  return(logLik + logPrior)
}
# initial value
initVal <- as.vector(rep(0,dim(X)[2]));
# use optim to minimize
```

```
OptimResults<-optim(initVal,LogPostPoison,gr=NULL,y,X,Sigma,method=c("BFGS"),control=list(fnscale=-1),h
```

```
# Printing the results to the screen
postMode <- OptimResults$par
#  Posterior covariance matrix is -inv(Hessian)
postCov <- -solve(OptimResults$hessian)
# Naming the coefficient by covariates
names(postMode) <- covNames
# Computing approximate standard deviations.
approxPostStd <- sqrt(diag(postCov))
# Naming the coefficient by covariates
names(approxPostStd) <- covNames

cat('The posterior mode is:')
```

```
## The posterior mode is:
```

```
cat("\n")
```

```
postMode
```

```
## [1]  1.06984118 -0.02051246 -0.39300599  0.44355549 -0.05246627 -0.22123840
## [7]  0.07069683 -0.12021767 -1.89198501
```

```
cat("-------------------------------------------------")
```

```
## -------------------------------------------------
```

```
cat("\n")
```

```
cat('The approximate posterior standard deviation is:')
```

```
## The approximate posterior standard deviation is:
```

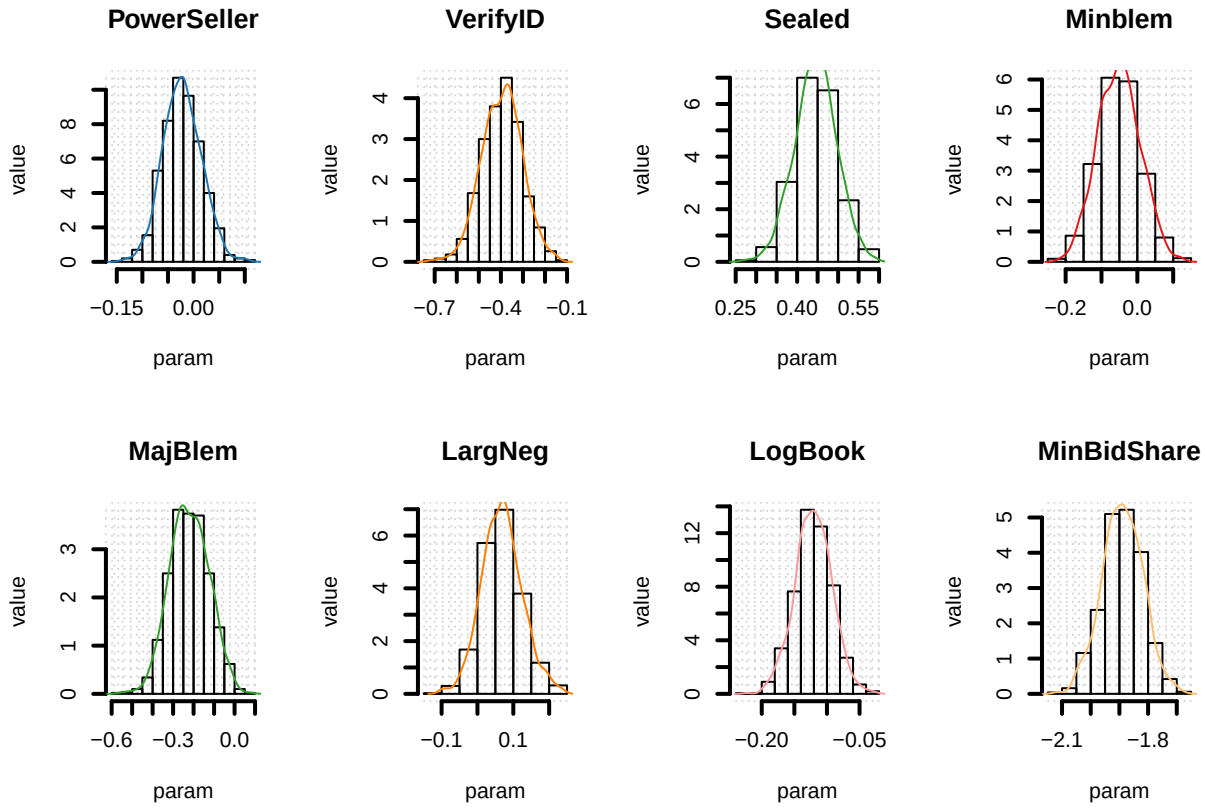
```
cat("\n")
```

```
approxPostStd
```

```
## [1] 0.03074837 0.03678418 0.09227871 0.05057448 0.06020470 0.09146070
## [7] 0.05634767 0.02895635 0.07109682
```

Plot of posterior covariates

```
library(MASS)
require(RColorBrewer)
# create pallete for sampling
pal<-brewer.pal(9, "Paired")
# take sample
samp<-mvrnorm(1000,postMode,postCov)

par(mfrow=c(2,4))
for(i in 2:9){
  hist(samp[,i],freq=F,
       main=paste(colnames(X)[i]),
       xlab='param',ylab='value',lwd=2,panel.first = grid(25,25))
  lines(density(samp[,i]),col =sample(pal,1))
}
```



c)

Now, let €™s simulate from the actual posterior of β using the Metropolis algorithm and compare with the approximate results in b). Program a general function that uses the Metropolis algorithm to generate random draws from an arbitrary posterior density. In order to show that it is a general function for any model, I will denote the vector of model parameters by θ. Let the proposal density be the multivariate normal density mentioned in Lecture 8 (random walk Metropolis):

$$\theta_p|\theta^{(i-1)} \sim N(\theta^{(i-1)}, c\ \Sigma)$$

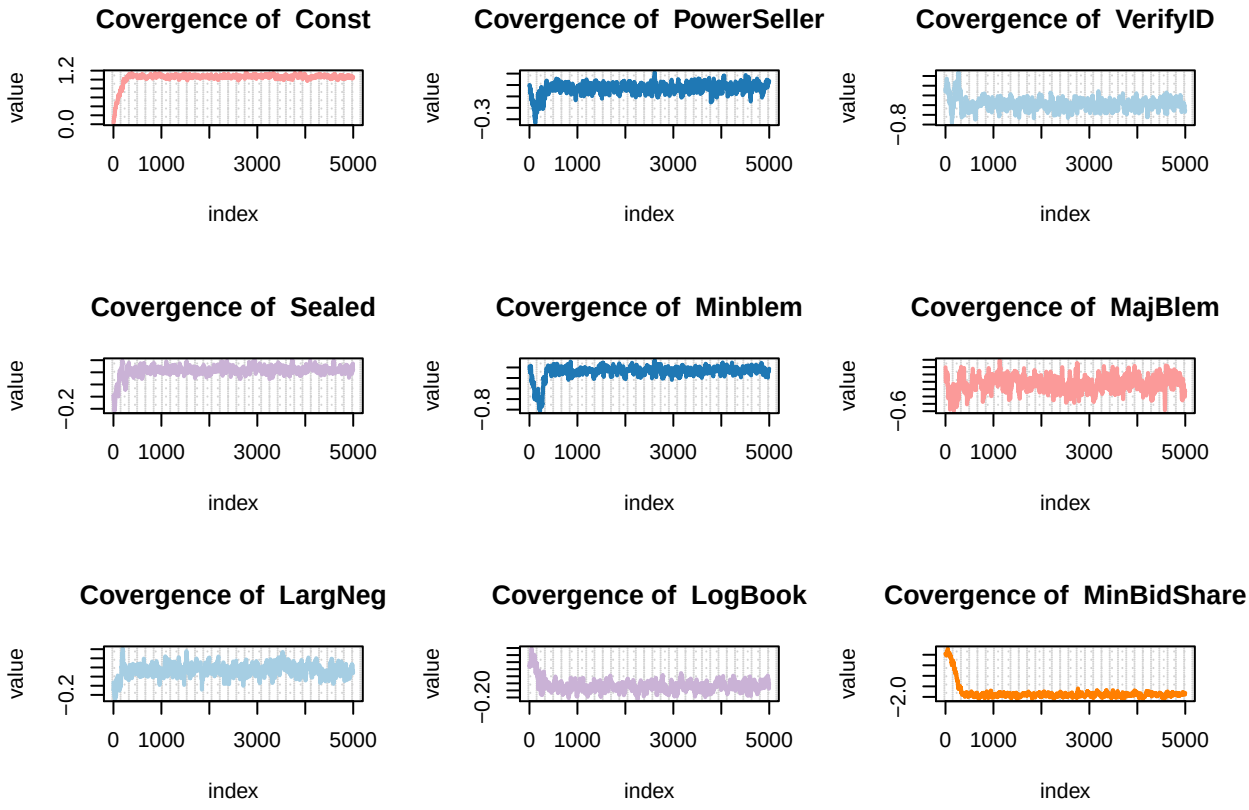
where $\Sigma = J_y^-1(\tilde{\beta})$ obtained in b). The value c is a tuning parameter and should be an input to your Metropolis function. The user of your Metropolis function should be able to supply her own posterior density function, not necessarily for the Poisson regression, and still be able to use your Metropolis function. This is not so straightforward, unless you have come across function objects in R and the triple dot (…) wildcard argument. I have posted a note (HowToCodeRWM.pdf) on the course web page that describes how to do this in R. Now, use your new Metropolis function to sample from the posterior of β in the Poisson regression for the eBay dataset. Assess MCMC convergence by graphical methods.

```
#  c)
library(mvtnorm)
# Metropolis Sampling function
RWMSampler<-function(logPostFunc,n.sim,cx,betaVect,SigmaP,...){
  # initial sample
  initSample=mvrnorm(n = 1,betaVect,SigmaP)
  # matrix to store the samples
  mat_betas<-matrix(nrow=n.sim,ncol=length(initSample))
```

```
  # store the initial sample to matrix
  mat_betas[1,]<-initSample
  for(i in 2:n.sim){
    # proposal sample
    propSample<-as.vector( mvrnorm(n=1,mat_betas[i-1,],cx*SigmaP) )
    # calculate the approval ratio
    r=exp( logPostFunc(propSample,...)-logPostFunc(as.vector(mat_betas[i-1,]),...) )
    # check the acceptance condition
    if( runif(1)<min(1,r) ){
      mat_betas[i,]<-propSample
    }else{
      mat_betas[i,]<-mat_betas[i-1,]
    }
  }
  return(mat_betas)
}
# setting the input
betaVect=as.vector(rep(0,dim(X)[2])); y=y; X=X; SigmaP=postCov; n.sim=5000; cx=0.5
res<-RWMSampler(LogPostPoison,n.sim,cx,betaVect,SigmaP,y,X,Sigma) # sample from Metropolis
```

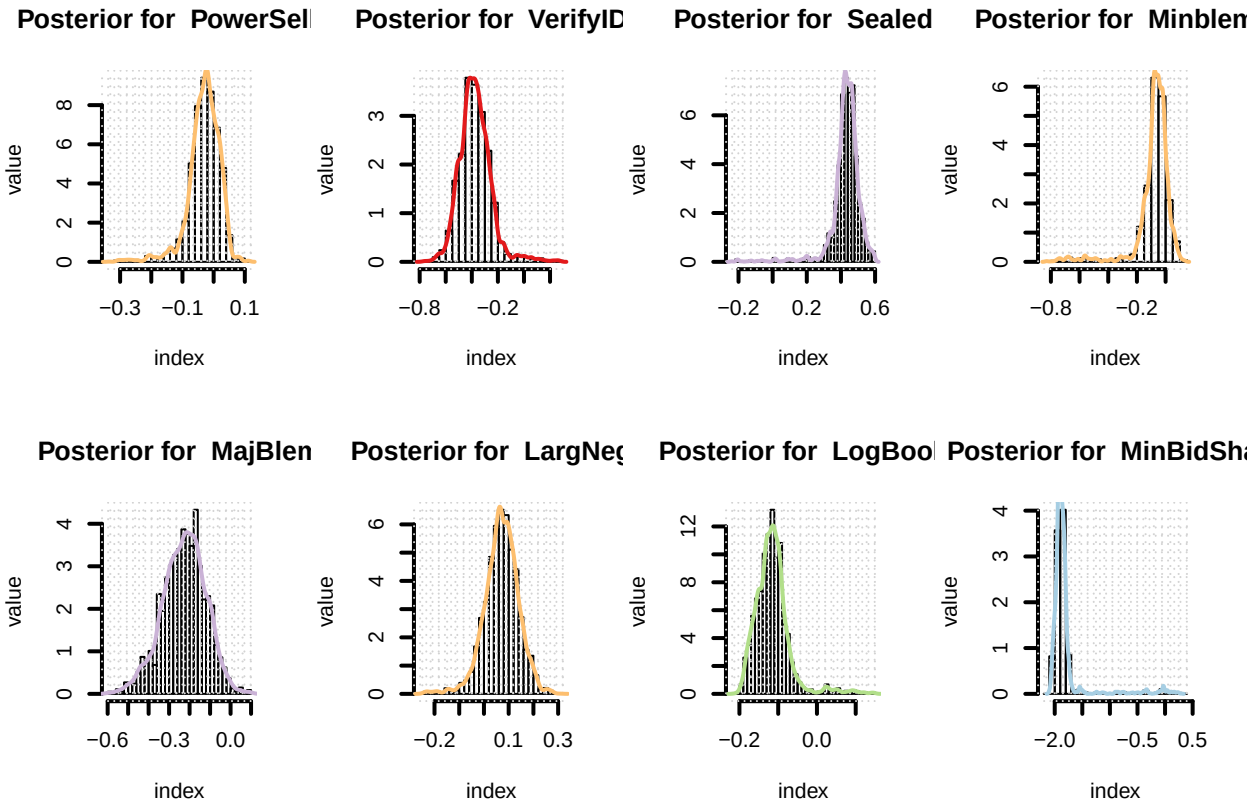
Convergence Plots

```
par(mfrow=c(3,3))
for(i in 1:9){
  plot(res[,i],type="l",
       main=paste('Covergence of ',colnames(X)[i]),
       col =sample(pal,1),
       xlab='index',ylab='value',lwd=2,panel.first = grid(25,25))
}
```



Plot of Histograms and Posteriors

```
par(mfrow=c(2,4))
for(i in 2:9){
  hist(res[,i],freq=F,breaks=30,
       main=paste('Posterior for ',colnames(X)[i]),
       xlab='index',ylab='value',lwd=2)
  lines(density(res[,i]),col =sample(pal,1),lwd=2,panel.first = grid(25,25))
}
```


Posterior for PowerSel
Posterior for VerifyID
Posterior for Sealed
Posterior for Minblen
Posterior for MajBlen
Posterior for LargNeg
Posterior for LogBoo
Posterior for MinBidSh
value
index