1 Lectures 1-3

- Probability density function for x: $f(x)$
- Marginal density
  $f_i(x_i) = \int f(x)dx_1...dx_{i-1}dx_{i+1}...dx_p$
- Expected (mean) value $Ex = \int xf(x)dx$
- Covariance $\operatorname{cov}(x,y) = E\{(x-Ex)(y-Ey)\}$
- Correlation $\rho_{x,y} = \operatorname{corr}(x,y) = \frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y}$
- Variance $\operatorname{var}(x) = E\{(x-Ex)^2\} = \operatorname{cov}(x,x)$
- Relationships (a is a constant)
  - $E(x+a) = Ex + a$, $E(ax) = aEx$
  - $E(x+y) = Ex + Ey$
  - $\operatorname{cov}(x+a,y) = \operatorname{cov}(x,y)$
  - $\operatorname{cov}(x+z,y) = \operatorname{cov}(x,y) + \operatorname{cov}(z,y)$
  - $\operatorname{var}(ax) = a^2 \operatorname{var}(x)$

$$\begin{aligned} \text{uncorrelated} &\iff E(XY) = EX.EY \\ \text{independent} &\iff f_{X,Y}(x,y) = f_X(x).f_Y(y) \end{aligned}$$

- Autocovariance function

$$\gamma(s,t) = \operatorname{cov}(x_s,x_t) = E[(x_s-\mu_s)(x_t-\mu_t)]$$

Note: $\operatorname{var}(x_t) = \gamma(t,t)$

- Autocorrelation function (ACF)

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$$

Useful fact: If $U = \sum_{j=1}^{m} a_j x_j$ and

$$V = \sum_{k=1}^{r} b_k y_k$$

$$\operatorname{cov}(U,V) = \sum_{j=1}^m \sum_{k=1}^r a_j b_k \operatorname{cov}(x_j,y_k)$$

1.1 stationarity

- Time series x_t is weakly stationary (stationary) if
  - $Ex_t = const$
  - $\gamma(s,t) = \gamma(|s-t|)$
  - $\operatorname{var}(x_t) < \infty$
- $\gamma(t,t+h) = \gamma(|t+h-t|) = \gamma(h)$
  - Autocovariance depends on lag only!
- Autocovariance for stationary process
  $\gamma(h) = \operatorname{cov}(x_t,x_{t+h})$
- ACF for stationary process $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$

Properties of stationary process:

$$\gamma(h) = \gamma(-h) \qquad \rho(h) = \rho(-h)$$

$$|\gamma(h)| \le \gamma(0) \qquad \rho(h) \le 1, \rho(0) = 1$$

If x_t is stationary,

- Sample mean

$$Ex \approx \bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$$

- Sample autocovariance function

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(x_{t+h}-\bar{x})(x_t-\bar{x})$$

Theorem: Under weak conditions, if x_t is white noise and $n \to \infty$ then $\hat{\rho}(h)$ is approximately $N(0,\frac{1}{n})$

Consequence: If some $|\hat{\rho}(h)| > \frac{2}{\sqrt{n}}$ then the time series is not a white noise (with approximately 95 % confidence).

1.2 Backshift operator

- Backshift operator $Bx_t = x_{t-1}$,
  Powers $B^k x_t = x_{t-k}$
- Forward-shift operator $B^{-1}x_t = x_{t+1}$
- Note $BB^{-1}x_t = x_t$ (i.e. $BB^{-1} = 1$)
- Differencing $\nabla x_t = (1-B)x_t$
- Differences of order d: $\nabla^d = (1-B)^d$
- Property: Operators can be manipulated as polynomials
- Example Check that $\nabla^2 x_t = x_t - 2x_{t-1} + x_{t-2}$
- Property: Differencing of order p can remove polynomial trend of order p

1.3 MA, AR, ARMA

- **Moving average model of order q, MA(q)**

$$\begin{aligned} x_t &= w_t + \theta_1 w_{t-1} + ... + \theta_q w_{t-q} \\ &= \sum_{j=0}^{q} \theta_j w_{t-j} \end{aligned}$$

  - $w_t \sim wn(0, \sigma_w^2)$
  - $\theta_1, ...\theta_q$ constants, $\theta_q \neq 0$ and $\theta_0 = 1$

- **Moving average operator**

$$\theta(B) = \sum_{j=0}^{q} \theta_j B^j$$

- MA(q):

$$\boxed{x_t = \theta(B)w_t}$$

- **Autoregressive model of order** p**,** $AR(p)$

$$x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + w_t$$

  - x_t is stationary if x_0 is sampled from the stationary distribution
  - $w_t \sim wn(0, \sigma_w^2)$
  - $\phi_1, ...\phi_p$ constants, $\phi_p \neq 0$
  - $Ex_t = 0$

- **Autoregressive operator**

$$\phi(B) = 1 - \phi_1 B - ... - \phi_p B^p$$

- AR(p) model

$$\boxed{\phi(B)x_t = w_t}$$

- **ARMA(p,q)**

$$\begin{aligned} x_t =& \phi_1 x_{t-1} + ... + \phi_p x_{t-p} \\ &+ w_t + \theta_1 w_{t-1} + ... + \theta_q w_{t-q} \end{aligned}$$

  - $\phi_p \neq 0, \theta_q \neq 0$
  - Is stationary
  - $Ex_t = 0$

- p-autoregressive order, q-moving average order
- Alternative form

$$\boxed{\phi(B)x_t = \theta(B)w_t}$$

- Note: $x_t = \phi^{-1}(B)\theta(B)w_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$
  - But series might be non-convergent

1.4 Causality / invertiblity

A stationary process is **causal** if it is only dependent on the past values of the process

Def: A linear process is nonexplosive and causal if it can be written as a one-sided sum:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ and $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

Def: An MA process is **invertible** if it has a causal AR representation,

$$w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$$

Def: Linear process is **causal** and **nonexplosive** if

- $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$ (depends on the past only)
- $\sum_{j=0}^{\infty} |\psi_j| < \infty$
- We set $\psi_0 = 1$ by convention.

Property: ARMA(p,q) is **causal** iff roots $\phi(z') = 0$ are outside unit circle, i.e. $|z'| > 1$

$$\boxed{\phi(B) x_t = \theta(B) w_t}$$

Def: ARMA(p,q) is **invertible** if

- $w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$ (depends on the past only)
- $\sum_{j=0}^{\infty} |\pi_j| < \infty$

Property: ARMA(p,q) is **invertible** iff roots $\theta(z') = 0$ are outside unit circle, i.e. $|z'| > 1$

$$\boxed{\phi(B) x_t = \theta(B) w_t}$$