Teaching Session III

Assignment 1

Prove the Kalman filtering recursion for the following state space model with initial prior on the state $f(z_1) = N(z_1; m_0, P_0)$ where $e_t \sim N(0, Q_t)$ and $v_t \sim N(0, R_t)$

$$z_t = A_{t-1} z_{t-1} + e_t \qquad (1)$$

$$x_t = C_t z_t + v_t \qquad (2)$$

- Particularly, show the given $f(z_t / x_t) = N(z_t; m_{t|t}, P_{t|t})$, the predicted density $f(z_{t+1} / x_{1:t})$ is given by:

$$f(z_{t+1} / x_{1:t}) = N(z_{t+1}; A_t m_{t|t}, A_t P_{t|t} A_t^T + Q_{t+1})$$

- Also, show that given $f(z_t / x_{1:t-1}) = N(z_t; m_{t|t-1}, P_{t|t-1})$, the observation updated density $f(z_t / x_{1:t})$ is given by: $f(z_t / x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$ where

$$\begin{cases} m_{t|t} = m_{t|t-1} + K_t (x_t - C_t m_{t|t-1}) \\ P_{t|t} = (I - K_t C_t) P_{t|t-1} \\ K_t = P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_t)^{-1} \end{cases}$$

- The joint posterior $f(z_t, z_{t+1} / x_{1:t})$ can be given by:

$$f(z_t, z_{t+1} / x_{1:t}) = \underbrace{N(z_t; m_{t|t}, P_{t|t})}_{\text{given}} \cdot \underbrace{N(z_{t+1}, A_t z_t, Q_{t+1})}_{\text{from (1)} \Rightarrow z_{t+1} = A_t z_t + e_{t+1},\ e_{t+1} \sim N(0, Q_{t+1})}$$

$$= N\left(\begin{bmatrix} z_t \\ z_{t+1} \end{bmatrix}; \begin{bmatrix} m_{t|t} \\ A_t m_{t|t} \end{bmatrix}, \begin{bmatrix} P_{t|t} & P_{t|t} A_t^T \\ A_t P_{t|t} & A_t P_{t|t} A_t^T + Q_{t+1} \end{bmatrix} \right)$$

Then we marginalize z_t out [property $f(y_2) = N(y_2; \mu_2, \Sigma_{22})$, where $y_2 = z_{t+1} / x_{1:t}$] so we have

$$f(z_{t+1} / x_{1:t}) = N(z_{t+1}; A_t m_{t|t}, A_t P_{t|t} A_t^T + Q_{t+1}).$$

- $$f(z_t | x_{1:t}) = \frac{f(z_t, x_t / x_{1:t-1})}{f(x_t | x_{1:t-1})} = \frac{f(z_t | x_{1:t-1}) \cdot f(x_t / z_t, x_{1:t-1})}{f(x_t / x_{1:t-1})} \overset{\substack{\text{using conditional} \\ \text{independence of } x_t \text{ given} \\ z_t \text{ on } x_{1:t-1}}}{=} \frac{\overset{\text{given}}{f(z_t | x_{1:t-1})}\ \overset{\substack{\text{from (2)} \\ x_t = C_t z_t + v_t,\ v_t \sim N(0, R_t)}}{f(x_t / z_t)}}{\underbrace{f(x_t / x_{1:t-1})}_{\text{constant}}}$$

working in the numerator:

$$N(z_t; m_{t|t-1}, P_{t|t-1}) N(x_t; C_t m_{t|t-1}, R_t) = N\left(\begin{bmatrix} z_t \\ x_t \end{bmatrix}; \begin{bmatrix} m_{t|t-1} \\ C_t m_{t|t-1} \end{bmatrix}, \begin{bmatrix} \overset{\Sigma_{11}}{P_{t|t-1}} & \overset{\Sigma_{12}}{P_{t|t-1} C_t^T} \\ \underset{\Sigma_{21}}{C_t P_{t|t-1}} & \underset{\Sigma_{22}}{C_t P_{t|t-1} C_t^T + R_t} \end{bmatrix} \right)$$

from property $f(y_1/y_2) = N(y_1; \underbrace{\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2-\mu_2)}_{w}, \underbrace{\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}_{\mathcal{J}})$.

start with $w \Rightarrow m_{t|t-1} + \underbrace{\left[P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_t)^{-1} \right]}_{K_t} (x_t - C_t m_{t|t-1})$

$= m_{t|t-1} + K_t (x_t - C_t m_{t|t-1})$ and setting it ~~equal~~ to $m_{t|t}$

now for $\mathcal{J} \Rightarrow P_{t|t-1} - \underbrace{\left[P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_t)^{-1} \right]}_{K_t} \cdot C_t \cdot P_{t|t-1} =$

$P_{t|t-1} - K_t C_t P_{t|t-1} = P_{t|t-1}(I - K_t C_t)$ and setting it to $P_{t|t}$.

finally $f(z_t / x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$.