732A62 Time Series Analysis
Division of Statistics and Machine Learning
Department of Computer and Information Science

Teaching session III

Instructions

The hand-in assignment should be solved individually and should be submitted via LISAM in pdf format before the deadline also specified in LISAM. For the best learning outcome, you are encouraged to solve the problem by pen and paper and take a photo in pdf format and submit. However, other formats are equally accepted by the teacher. The solutions are graded pass / insufficient. An insufficient solution can be completed and resubmitted

Introduction

Useful properties of the normal density function for this assignment are listed here.
Property 1: $f(\mathbf{y}_1)f(\mathbf{y}_2|\mathbf{y}_1) = f(\mathbf{y}_1, \mathbf{y}_2)$

$$N(\mathbf{y}_1; \mu, \Sigma)N(\mathbf{y}_2; B\mathbf{y}_1, R) = N\left(\begin{bmatrix}\mathbf{y}_1 \\ \mathbf{y}_2\end{bmatrix}; \begin{bmatrix}\mu \\ B\mu\end{bmatrix}, \begin{bmatrix}\Sigma & \Sigma B^{\mathrm{T}} \\ B\Sigma & B\Sigma B^{\mathrm{T}} + R\end{bmatrix}\right)$$

Property 2: marginalization and conditioning
If $\mathbf{y}_1$, $\mathbf{y}_2$ were jointly normal:

$$f(\mathbf{y}_1, \mathbf{y}_2) = N\left(\begin{bmatrix}\mathbf{y}_1 \\ \mathbf{y}_2\end{bmatrix}; \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}, \begin{bmatrix}\Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\right)$$

then

$$\begin{aligned} f(\mathbf{y}_1) &= N(\mathbf{y}_1; \mu_1, \Sigma_{11}) \\ f(\mathbf{y}_2) &= N(\mathbf{y}_2; \mu_2, \Sigma_{22}) \\ f(\mathbf{y}_1|\mathbf{y}_2) &= N(\mathbf{y}_1; \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{y}_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) \\ f(\mathbf{y}_2|\mathbf{y}_1) &= N(\mathbf{y}_2; \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(\mathbf{y}_1 - \mu_1), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}) \end{aligned}$$

Assignment 1
Prove the Kalman filtering recursion for the following state space model with initial prior on the state $f(\mathbf{z}_1) = N(\mathbf{z}_1; m_0, P_0)$ where $e_t \sim N(0, Q_t)$ and $\nu_t \sim N(0, R_t)$

$$\mathbf{z}_t = A_{t-1}\mathbf{z}_{t-1} + e_t, \tag{1}$$
$$\mathbf{x}_t = C_t\mathbf{z}_t + \nu_t, \tag{2}$$

Particularly, show that given $f(\mathbf{z}_t|\mathbf{x}_{1:t}) = N(\mathbf{z}_t; m_{t|t}, P_{t|t})$, the predicted density $f(\mathbf{z}_{t+1}|\mathbf{x}_{1:t})$ is given by

$$f(\mathbf{z}_{t+1}|\mathbf{x}_{1:t}) = N(\mathbf{z}_{t+1}; A_t m_{t|t}, A_t P_{t|t} A_t^{\mathrm{T}} + Q_{t+1}).$$

Also, show that given $f(\mathbf{z}_t|\mathbf{x}_{1:t-1}) = N(\mathbf{z}_t; m_{t|t-1}, P_{t|t-1})$, the observation updated density $f(\mathbf{z}_t|\mathbf{x}_{1:t})$ is given by

$$f(\mathbf{z}_t|\mathbf{x}_{1:t}) = N(\mathbf{z}_t; m_{t|t}, P_{t|t})$$

where

$$\begin{aligned}
m_{t|t} =& m_{t|t-1} + K_t(\mathbf{x}_t - C_t m_{t|t-1}) \\
P_{t|t} =& (I - K_t C_t) P_{t|t-1} \\
K_t =& P_{t|t-1} C_t^{\mathrm{T}} (C_t P_{t|t-1} C_t^{\mathrm{T}} + R_t)^{-1}.
\end{aligned}$$

Table 1: Kalman filtering recursion

Inputs: A_t, C_t, Q_t, R_t, m_0, P_0 and $\mathbf{x}_{1:T}$.
initialization
$m_{1|0} \leftarrow m_0$, $P_{1|0} \leftarrow P_0$
for $t = 1$ to T **do**
    observation update step
    $K_t \leftarrow P_{t|t-1} C_t^{\mathrm{T}} (C_t P_{t|t-1} C_t^{\mathrm{T}} + R_t)^{-1}$
    $m_{t|t} \leftarrow m_{t|t-1} + K_t(\mathbf{x}_t - C_t m_{t|t-1})$
    $P_{t|t} \leftarrow (I - K_t C_t) P_{t|t-1}$
    prediction step
    $m_{t+1|t} \leftarrow A_t m_{t|t}$
    $P_{t+1|t} \leftarrow A_t P_{t|t} A_t^{\mathrm{T}} + Q_{t+1}$
end for
Outputs: $m_{t|t}$, $P_{t|t}$ for $t = 1 : T$
