Time series H.A. 1

10) $E(X)=2,\ Var(X)=9,\ E(Y)=0,\ Var(Y)=4\quad Corr(X,Y)=0.25\qquad Corr(X+Y,X-Y)$

$$Corr(X+Y,X-Y)=\frac{Cov(X+Y),(X-Y)}{\sigma_{X+Y}\sigma_{X-Y}}\quad(5)$$

$$Cov((X+Y),(X-Y))=E\{(X-Y-\underbrace{E(X+Y)}_{E(X)+E(Y)=2})(X+Y-\underbrace{E(X+Y)}_{E(X)+E(Y)=2})\}=E\{(X-Y-2)(X+Y-2)\}=$$

$$E\{X^2+\cancel{XY}-2X-\cancel{XY}-Y^2+\cancel{2Y}-2X-\cancel{2Y}+4\}=\underbrace{E(X^2)}_{\sigma_X^2+E^2(X)=9+4}-\underbrace{4E(X)}_{4\cdot2}-\underbrace{E(Y^2)}_{\sigma_Y^2+E^2(Y)=4}+4=9+4-8-4+4=5\quad(1)$$

$$\sigma^2_{X+Y}=E[(X+Y)^2]-E^2(X+Y)=\underbrace{E(X^2)}_{\sigma_X^2+E^2(X)}+2\underbrace{E(XY)}_{*}+\underbrace{E(Y^2)}_{\sigma_Y^2+E^2(Y)}-4=9+4+\cancel{2}\cdot\frac{3}{\cancel{2}}+4+0-4=16.\quad(2)$$

$$*\quad corr(X,Y)=\frac{1}{4}\Rightarrow\frac{E(XY)-E(X)E(Y)}{\sigma_X\sigma_Y}=\frac{1}{4}\to E(XY)=\frac{1}{4}\sigma_X\sigma_Y+E(X)\cdot E(Y)=\frac{1}{\cancel{4}_2}3\cdot\cancel{2}+2\cdot0=\frac{3}{2}$$

$$\sigma^2_{X-Y}=E[(X-Y)^2]-E^2(X-Y)=E(X^2)-2E(XY)+E(Y^2)-4=9+4-\cancel{2}\cdot\frac{3}{\cancel{2}}+4+0-4=10\quad(3)$$

$$(5)\xRightarrow[(3)]{(1),(2)}\ Corr[(X+Y),(X-Y)]=\frac{5}{\sqrt{10}\cdot\sqrt{16}}=\frac{5}{4}\frac{1}{\sqrt{10}}$$

11) $X_t=W_t+3W_{t-1}\qquad E(X_t)=0$

$$\gamma_X(k)=Cov(X_t\ X_{t-k})=E\{(X_t-E(X_t))(X_{t-k}-E(X_{t-k}))\}=E\{X_t X_{t-k}\}=E\{(W_t+3W_{t-1})(W_{t-k}+3W_{t-k-1})\}$$

$$=E\{W_tW_{t-k}\}+3E\{W_tW_{t-k-1}\}+3E\{W_{t-1}W_{t-k}\}+9E\{W_{t-1}W_{t-k-1}\}$$

for $k=0$ $\gamma_X(0)=\sigma_W^2+9\sigma_W^2=10\sigma_W^2\qquad\rho_X(0)=1$

for $k=1$ $\gamma_X(1)=\cancel{E\{W_tW_{t-1}\}}^{0}+\cancel{3E\{W_tW_{t-2}\}}^{0}+3E\{W_{t-1}W_{t-1}\}+\cancel{9E\{W_{t-1}W_{t-2}\}}^{0}=3\sigma_W^2\qquad\rho_X(1)=\frac{\gamma_X(1)}{\gamma_X(0)}=\frac{3\sigma_W^2}{10\sigma_W^2}=\frac{3}{10}$

$\forall k\geq2\quad\rho_X(k)=0.$

same results with $\theta=\frac{1}{3}$.

Time Series AA. 1

13) $y_t = (-1)^t x$ $\quad E(x) = 0$ $\quad Var(x) = E(x^2) - E^2(x) = E(x^2) = \sigma_x^2$

$E(y_t) = (-1)^t E(x) = 0$ $\quad \gamma_y(y_t, y_{t+k}) = E\{y_t\, y_{t-k}\}$

$(-1)^t x \cdot (-1)^{t-k} x = (-1)^{2t-k} x^2 = (-1)^k \sigma_x^2$ doesn't depend on t

$t-k+t$ $\quad \sigma_x^2$

15)

$x_t = w_t + 0.5 w_{t-1} + 0.4 w_{t-2}$ $\quad E(x_t) = 0$

$\gamma_x(k) = E(x_t x_{t-k}) = E\{(w_t + 0.5 w_{t-1} + 0.4 w_{t-2})(w_{t-k} + 0.5 w_{t-k-1} + 0.4 w_{t-k-2})\}$

$= w_t w_{t-k} + 0.5 w_t w_{t-k-1} + 0.4 w_t w_{t-k-2} + 0.5 w_{t-1} w_{t-k} + 0.5^2 w_{t-1} w_{t-k-1} + 0.5 \cdot 0.4 w_{t-1} w_{t-k-2} + 0.4 w_{t-2} w_{t-k} + 0.4 \cdot 0.5 w_{t-2} w_{t-k-1}$

$+ 0.4^2 w_{t-2} w_{t-k-2}$

$\rho(0) = 1$. $\quad \gamma_x(0) = \sigma_w^2 + 0.5^2 \sigma_w^2 + 0.4^2 \sigma_w^2 = 1.41 \sigma_w^2$

for $k=1$ $\quad \gamma_x(1) = 0.5\sigma_w^2 + 0.4 \cdot 0.5 \sigma_w^2 = 0.7 \sigma_w^2$ $\quad \rho_1 \simeq 0.5$

for $k=2$ $\quad \gamma_x(2) = 0.4 \sigma_w^2$ $\quad \rho_2 \simeq 0.28$.

for $k > 2$ $\rho_x = 0$

18) c) $x_t - 3x_{t-1} = w_t + 2w_{t-1} - 8w_{t-2} \Rightarrow (1-3B)x_t = (1+2B+8B^2)w_t$

$1 - 3z = 0 \Rightarrow$ $\quad 1 + 2z + 8z^2 = 0 \Rightarrow$

$z = \frac{1}{3} < 1$ $\quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \cdot 1(-8)}}{-16} = \frac{-2 \pm \sqrt{4+32}}{-16} = \frac{-2 \pm 6}{-16}$ $\to 1/2$, $\to -1/4$

not casual $\quad |z_1| = \frac{1}{2} < 1$

$|z_2| = \frac{1}{4} < 1$ not invertible

d) $x_t - 2x_{t-1} + 2x_{t-2} = w_t - \frac{8}{9} w_{t-1} \Rightarrow (1 - 2B + 2B^2)x_t = (1 - \frac{8}{9}B) w_t$

$1 - 2z + 2z^2 = 0 \Rightarrow$ $\quad 1 - \frac{8}{9} z = 0 \Rightarrow$

$\frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 2}}{2 \cdot 2} = \frac{2 \pm \sqrt{4-8}}{2}$ $\to \frac{1+i}{2}$, $\to \frac{1-i}{2}$ $\quad z = \frac{9}{8} = 1.125$ invertible

$|z_1| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = 0.7 < 1$

$|z_2| = \ldots = 0.7 < 1$ not casual.

e) $x_t - 4x_{t-2} = w_t - w_{t-1} + 0.5 w_{t-2} \Rightarrow (1 - 4B^2)x_t = (1 - B + 0.5B^2) w_t$

$1 - 4z^2 = 0 \Rightarrow$ $\quad 1 - z + 0.5z^2 = 0 \Rightarrow$

$z^2 = \frac{1}{4}$ $\quad \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot \frac{1}{2}}}{2 \cdot \frac{1}{2}} = \frac{1 \pm \sqrt{-1}}{1} = 1 \pm i$

$z = \frac{1}{2} < 1$ not casual $\quad z_1 = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.41 > 1$

$z_2 = \ldots$ $\quad = 1.41 > 1$

invertible

$\frac{9}{4}x_{t-1} - \frac{9}{4}x_{t-2} = w_t \Rightarrow (1 - \frac{9}{4}B + \frac{9}{4}B^2)x_t = w_t$

$1 - \frac{9}{4}z - \frac{9}{4}z^2 = 0 \Rightarrow$

$\frac{\frac{9}{4} \pm \sqrt{\frac{81}{16} - 4 \cdot 1 \cdot (-\frac{9}{4})}}{2(-9/4)} = \frac{\frac{9}{4} \pm \sqrt{\frac{81}{16} + 9}}{-\frac{9}{2}} = \frac{\frac{9}{4} \pm \frac{15}{4}}{-\frac{9}{2}}$

$\rightarrow \frac{\frac{24}{2}}{-\frac{9}{2}} = \frac{2 \cdot 24}{4 \cdot 9} = \frac{24}{18} = 1{,}33$

$\rightarrow \frac{-\frac{6}{4}}{-\frac{9}{2}} = \frac{12}{36} = 0.33$

$|z_1| = 1{,}33 > 1$

$|z_2| = 0.33 < 1$ not causal

But is invertible.

$x_t - 2x_{t-1} + 2x_{t-2} = w_t - \frac{8}{9}w_{t-1}$ we want it in the form $\phi(z)\cdot\psi(z) = \theta(z)$

$$\underbrace{(1-2z+2z^2)}_{u}\underbrace{(1+\psi_1 z+\psi_2 z^2+\psi_3 z^3)}_{v} = (1 - \frac{8}{9}z) \quad (\times)$$

$u \cdot v = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - 2z - 2z\psi_1 z - 2z\psi_2 z^2 - 2z\psi_3 z^3 + 2z^2 + 2z^2\psi_1 z + 2z^2\psi_2 z^2 + 2z^2\psi_3 z^3 =$

$1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - 2z - 2z^2\psi_1 - 2\psi_2 z^3 - 2\psi_3 z^4 + 2z^2 + 2\psi_1 z^3 + 2\psi_2 z^4 + 2\psi_3 z^5 =$

$1 + z(\psi_1 - 2) + z^2(\psi_2 - 2\psi_1 + 2) + z^3(\psi_3 - 2\psi_2 + 2\psi_1) + z^4(2\psi_3) + z^5 2\psi_3$

we match the coefficients with the right part of $(\times)$

$\psi_1 - 2 = -\frac{8}{9} \Rightarrow \psi_1 = \frac{8}{9} + 2 = \frac{-8+18}{9} = \frac{10}{9}$

$\psi_2 - 2\psi_1 + 2 = 0 \Rightarrow \psi_2 = -2 + 2\psi_1 = -2 + \frac{20}{9} = \frac{-18+20}{9} = \frac{2}{9}$

$\psi_3 - 2\psi_2 + 2\psi_1 = 0 \Rightarrow \psi_3 = 2\psi_2 - 2\psi_1 = 2\cdot\frac{2}{9} - 2\cdot\frac{10}{9} = \frac{4-20}{9} = -\frac{16}{9}$

$x_t - \frac{9}{4}x_{t-1} - \frac{9}{4}x_{t-2} = w_t - 3w_{t-1} + \frac{1}{9}w_{t-2} - \frac{1}{3}w_{t-3}$

$$\underbrace{(1 - \frac{9}{4}z - \frac{9}{4}z^2)}_{*}\underbrace{(1+\psi_1 z+\psi_2 z^2+\psi_3 z^3)}_{\varepsilon} = (1 - 3z + \frac{1}{9}z^2 - \frac{1}{3}z^3) \quad (\varepsilon)$$

$* \cdot \varepsilon = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - \frac{9}{4}z - \frac{9}{4}z\psi_1 z - \frac{9}{4}z\psi_2 z^2 - \frac{9}{4}z\psi_3 z^3 - \frac{9}{4}z^2 - \frac{9}{4}z^2\psi_1 z - \frac{9}{4}z^2\psi_2 z^2 - \frac{9}{4}z^2\psi_3 z^3 =$

$1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - \frac{9}{4}z - \frac{9}{4}\psi_1 z^2 - \frac{9}{4}\psi_2 z^3 - \frac{9}{4}\psi_3 z^4 - \frac{9}{4}z^2 - \frac{9}{4}\psi_1 z^3 - \frac{9}{4}\psi_2 z^4 - \frac{9}{4}\psi_3 z^5$

$1 + z(\psi_1 - \frac{9}{4}) + z^2(\psi_2 - \frac{9}{4}\psi_1 - \frac{9}{4}) + z^3(\psi_3 - \frac{9}{4}\psi_2 - \frac{9}{4}\psi_1) + z^4(-\frac{9}{4}\psi_3 - \frac{9}{4}\psi_2) + z^5(-\frac{9}{4})$

we match the coefficients with the right part of (ε)

$\psi_1 - \frac{9}{4} = -3 \Rightarrow \psi_1 = -3 + \frac{9}{4} = \frac{-12+9}{4} = -\frac{3}{4}$

$\psi_2 - \frac{9}{4}\psi_1 - \frac{9}{4} = \frac{1}{9} \Rightarrow \psi_2 = \frac{1}{9} + \frac{9}{4}\psi_1 + \frac{9}{4} = \frac{1}{9} + \frac{9}{4}\cdot(-\frac{3}{4}) + \frac{9}{4} = \frac{1}{9} + \frac{-27+36}{16} = \frac{1}{9} + \frac{9}{16}$

$\psi_3 - \frac{9}{4}\psi_2 - \frac{9}{4}\psi_1 = -\frac{1}{3} \Rightarrow \psi_3 = -\frac{1}{3} + \frac{9}{4}\psi_2 + \frac{9}{4}\psi_1 = -\frac{1}{3} + \frac{9}{4}(\frac{1}{9} + \frac{9}{16}) + \frac{9}{4}(-\frac{3}{4})$

$= \underbrace{-\frac{1}{3} + \frac{9}{36}} + \underbrace{\frac{81}{64} - \frac{27}{16}}$

$= -\frac{3}{36} \quad -\frac{27}{64} = -\frac{1}{12} - \frac{27}{64}$

gnment 14

$x_t = \mu + w_t + w_{t-1}$

$E(x_t) = \mu$ suppose we have a sample $x_1, \dots x_n$ $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n} E\left(\sum x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\cdot n \cdot \mu = \mu$$

$$\bar{x} = \frac{1}{n}(\mu + w_1 + w_0 + \mu + w_2 + w_1 + \dots + \mu + w_n + w_{n-1}) = \frac{1}{n}(\mu \cdot n + w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1})$$

$$= \mu + \frac{w_0 + w_n}{2} + 2\cdot\sum_{i=1}^{n-1} w_i$$

$$\cancel{Var(x_t) = E\{(x_t - E(x_t))^2\}} = E\left\{\left(\frac{w_0+w_n}{2} + 2\sum_{i=1}^{n-1} w_i\right)^2\right\} = E\left\{\left(\frac{w_0+w_n}{2}\right)^2 + 4\left(\frac{w_0+w_n}{2}\right)\sum_{i=1}^{n-1} w_i + 4\left(\sum_{i=1}^{n-1} w_i\right)^2\right\}$$

$$\cancel{= \frac{1}{4}(w_0^2 + 2w_0 w_n + w}$$

$$\bar{x} - E(\bar{x}) = \frac{1}{n}(\mu n + w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1}) - \mu = \frac{1}{n}(w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1})$$

$$Var(x_t) = E\{(\bar{x} - E(\bar{x}))^2\} = E\left\{\left(\frac{1}{n}(w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1})\right)^2\right\}$$

$$= E\left\{\frac{1}{n^2}(\underbrace{w_0^2 + w_n^2 + 4w_1^2 + 4w_2^2 + \dots + 4w_{n-1}^2}_{4(n-1)\sigma_w^2} + \underbrace{2w_0 w_n + 4w_0 w_1 + \dots}_{0})\right\}$$

$$= \frac{1}{n^2}\{\underbrace{E(w_0^2) + E(w_n^2)}_{2\sigma_w^2} + 4E(w_1^2) + \dots + 4E(w_{n-1}^2)\}$$

$$= \frac{1}{n^2}(2\sigma_w^2 + 4(n-1)\sigma_w^2) = \frac{2\sigma_w^2 + 4n\sigma_w^2 - 4\sigma_w^2}{n^2} = \frac{4\sigma_w^2 - 2\sigma_w^2}{n^2} = \frac{2\sigma_w^2(2n-1)}{n^2}$$

$$\frac{1}{n^2}(\underbrace{\sigma_w^2 + \sigma_w^2}_{2\sigma_w^2} + \underbrace{4\sigma_w^2 + \dots + 4\sigma_w^2}_{4(n-1)\sigma_w^2}$$

$$(w_0 + w_1 + w_2)^2$$

$$w_0^2 + w_1^2 + w_1^2 + \underline{2w_0 w_1 +} \dots$$

$$E(w_0 w_1) = 0$$

Assignment 17 $\quad x_t = \overset{\text{[illegible]}}{\varphi} x_{t-1} + w_t$

$\{x_t\} \sim AR(1) \qquad -1<\varphi<+1 \qquad y_t = \nabla x_t = x_t - x_{t-1}$

a) $E(x_t) = 0 \qquad \gamma_x(h) = \varphi^h \dfrac{\sigma_w^2}{1-\varphi^2} \qquad E(y_t) = E(x_t) - E(x_{t-1}) = 0$

$$\gamma_y(h) = E\{\underbrace{(y_t - E(y_t))}_{x_t - x_{t-1}}\underbrace{(y_{t-h} - E(y_{t-h}))}_{x_{t-h} - x_{t-h-1}}\} = E\{(x_t - x_{t-1})(x_{t-h} - x_{t-h-1})\} = E\{\underbrace{x_t x_{t-h}}_{h\text{ distance}} - \overbrace{x_t x_{t-h-1}}^{t-(t-h-1)=(h+1)\text{ distance}} \underbrace{- x_{t-1} x_{t-h}}_{t-1-t+h=(h-1)\text{ distance}} + \overbrace{x_{t-1} x_{t-h-1}}^{t-1-t+h+1=h\text{ distance}}\}$$

$$= \gamma_x(h) - \gamma_x(h+1) - \gamma(h-1) + \gamma(h) = 2\gamma_x(h) - \gamma_x(h+1) - \gamma_x(h-1)$$

$$= \frac{2\varphi^h \sigma_w^2}{1-\varphi^2} - \frac{\varphi^{h+1}\sigma_w^2}{1-\varphi^2} - \frac{\varphi^{h-1}\sigma_w^2}{1-\varphi^2} = \frac{(2-\varphi-\frac{1}{\varphi})}{(1-\varphi)(1+\varphi)}\sigma_w^2\varphi^h = -\frac{(1-\varphi)^2}{\varphi(1-\varphi)(1+\varphi)}\sigma_w^2\varphi^h = -\frac{(1-\varphi)}{(1+\varphi)}\sigma_w^2\varphi^{h-1}$$

b) $$\mathrm{Var}(y_t) = E(y_t^2) - E^2(y_t) = E\{(x_t - x_{t-1})^2\} - E^2(y_t) = E\{x_t^2 - 2x_t x_{t-1} + x_{t-1}^2\} - E^2\cancelto{0}{(x_t - x_{t-1})}$$

$$= \gamma_x(0) - 2\gamma_x(1) + \gamma_x(0) = -2\frac{\sigma_w^2\varphi}{(1+\varphi^2)} + \frac{2\sigma_w^2}{(1-\varphi^2)} = \frac{2\sigma_w^2(1-\varphi)}{1-\varphi^2} = \frac{2\sigma_w^2(1-\varphi)}{(1-\varphi)(1+\varphi)} = \frac{2\sigma_w^2}{1+\varphi}$$