Time Series Analysis

Lecture 8: State Space Model
Stochastic Volatility

Tohid Ardeshiri

Linköping University
Division of Statistics and Machine Learning

October 4, 2019

Remaining Course topics

- ARIMA models
- State space models (2 lectures, 1 teaching session with hand-in, 1 computer lab with short report)
  - Linear and Gaussian state space models (Chapter 6.1)
  - Kalman filtering, Kalman smoothing and Forecasting (Chapter 6.2)
  - Maximum likelihood estimate of the state space models (Chapter 6.3)
  - Stochastic volatility (Chapter 6.11)
- Recurrent Neural Networks (RNNs) (1 lecture and 1 Computer lab No examination)
- Summary lecture

Why Stochastic volatility

$$\begin{aligned} \mathbf{z}_t &= A\mathbf{z}_{t-1} + e_t, & e_t &\sim N(0, Q) \\ \mathbf{x}_t &= C\mathbf{z}_t + \nu_t, & \nu_t &\sim N(0, R) \end{aligned}$$

- Filtering: Kalman filtering, $f(\mathbf{z}_t|\mathbf{x}_{1:t})$
- Smooting: Kalman smoothing, $f(\mathbf{z}_t|\mathbf{x}_{1:T})$
- Modelling: Maximum likelihood and EM, $\widehat{\theta} = \arg\max_\theta f(\mathbf{x}_{1:T}|\theta)$
- Case study on Stochastic volatility via a generalization of the above tools

Stochastic Volatility

In finance, **return** is a profit on an investment. It comprises any change in value of the investment, and/or cash flows which the investor receives from the investment, such as interest payments or dividends.

Stochastic volatility models are those in which the variance of a stochastic process is itself randomly distributed.

In the following:

- r_t denote the return of some financial asset. A common model for the return is

$$r_t = \beta \sigma_t \epsilon_t$$

- σ_t is the volatility process and
- ϵ_t is an iid sequence and $\epsilon_t \sim iid(0,1)$ and ϵ_t is independent of past σ_s $(s \leq t)$

Stochastic Volatility

In the following:

- r_t denote the return of some financial asset. A common model for the return is

$$r_t = \beta \sigma_t \epsilon_t$$

- σ_t is the volatility process and
- ϵ_t is an iid sequence and $\epsilon_t \sim iid(0,1)$ and ϵ_t is independent of past σ_s $(s \leq t)$
- Let $\mathbf{z}_t = \log \sigma_t^2$ and consider the hidden autoregressive model

$$\begin{aligned} \mathbf{z}_t &= \phi \mathbf{z}_{t-1} + w_t \\ r_t &= \beta \exp(\mathbf{z}_t/2)\epsilon_t \end{aligned}$$

Stochastic Volatility **Whiteboard**

In this model $w_t \sim iidN(0, \sigma_w^2)$ and ϵ_t is iid noise with finite moments.

$$\begin{aligned} \mathbf{z}_t &= \phi \mathbf{z}_{t-1} + w_t \\ r_t &= \beta \exp(\mathbf{z}_t/2)\epsilon_t \end{aligned}$$

Furthermore, let $\mathbf{x}_t = \log r_t^2$ and $\nu_t = \log \epsilon_t^2$. We obtain

$$\mathbf{x}_t = \alpha + \mathbf{z}_t + \nu_t$$

We can move the α to the state equation and rewrite it as

$$\begin{aligned} \mathbf{z}_t &= \phi_0 + \phi_1 \mathbf{z}_{t-1} + w_t \\ \mathbf{x}_t &= \mathbf{z}_t + \nu_t \end{aligned}$$

where the ϕ_0 is called the leverage effect.

Stochastic Volatility

The distribution of ν_t is not Gaussian because

$$\nu_t = \log \epsilon_t^2 \text{ and}$$
$$\epsilon_t \sim iidN(0,1)$$

Hence, ν is distributed as a log of a chi-squared distribution with defree of freedom 1 with density

$$f(\nu) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(e^{\nu} - \nu)\} \qquad -\infty < \nu < \infty$$

Stochastic Volatility - Gaussian mixture approximation

Instead let us approximate $f(\nu)$ by a Gaussian mixture

$$f(\eta) = \pi_0 N(\eta; 0, \sigma_0^2) + \pi_1 N(\eta; \mu_1, \sigma_1^2)$$

That is,

$$\eta_t = I_t n_{t0} + (1 - I_t) n_{t1}$$

where I_t is an iid Bernoulli process where $Pr\{I = 0\} = \pi_0$ and $Pr\{I = 1\} = \pi_1$, $\pi_0 + \pi_1 = 1$. Also,

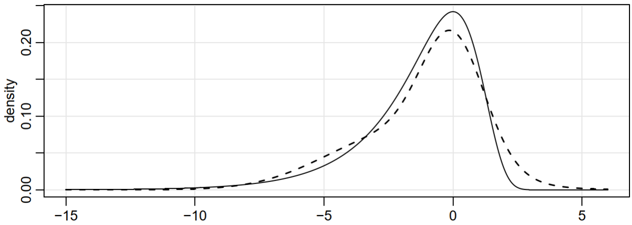
$$n_{t0} \sim N(0, \sigma_0^2)$$
$$n_{t1} \sim N(\mu_1, \sigma_1^2)$$

Stochastic Volatility - Gaussian sum approximation

$$f(\eta) = \pi_0 N(\eta; 0, \sigma_0^2) + \pi_1 N(\eta; \mu_1, \sigma_1^2) \qquad -\infty < \eta < \infty$$

$$f(\nu) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(e^{\nu} - \nu)\} \qquad -\infty < \nu < \infty$$

$f(\nu)$ and $f(\eta)$ are plotted for comparison. The dashed line is the Gaussian sum approximation, $f(\eta)$.



Stochastic Volatility - Gaussian sum formulation

The problem is finding the filtering distribution of $\mathbf{z}_t | \mathbf{x}_{1:t}$ when

$$\begin{aligned} \mathbf{z}_t &= \phi_0 + \phi_1 \mathbf{z}_{t-1} + w_t \\ \mathbf{x}_t &= \mathbf{z}_t + \eta_t \end{aligned}$$

and

$$\begin{aligned} w_t &\sim iid N(0, \sigma_w^2) \\ \eta_t &\sim \pi_0 N(0, \sigma_0^2) + \pi_1 N(\mu_1, \sigma_1^2) \end{aligned}$$

where $\pi_0 + \pi_1 = 1$

Gaussian sum filter - derivation **Whiteboard**

The problem is finding the filtering distribution of $\mathbf{z}_t|\mathbf{x}_{1:t}$ when

$$\mathbf{z}_t = A\mathbf{z}_{t-1} + w_t$$
$$\mathbf{x}_t = C\mathbf{z}_t + \eta_t$$

and

$$w_t \sim iidN(0, Q)$$
$$\eta_t \sim \pi_0 N(\mu_0, R_1) + \pi_1 N(\mu_1, R_2)$$

where $\pi_0 + \pi_1 = 1$

Read home

- Shumway and Stoffer, Chapter 6.11