Assignment 12

$\{X_t\} \sim (0,1)$ μ_t: non constant function and σ_t: positive-non constant function

$Y_t = \mu_t + \sigma_t X_t$

a)

- $E(y_t) = E(\mu_t + \sigma_t X_t) = E(\mu_t) + \cancel{E(X_t)}^{0} \cdot \sigma_t = E(\mu_t) = \mu_t$

- $Cov_{Y_t}(t,h) = E\{(\mu_t + \sigma_t X_t - E(y_t)) \cdot (\mu_h + \sigma_h X_h - E(y_h))\} = E\{(\sigma_t X_t)(\sigma_h X_h)\} = \sigma_t \sigma_h \cdot E\{(X_t - \overset{0}{E(X_t)}) \cdot (X_h - \overset{0}{E(X_h)})\}$

$= \sigma_t \sigma_h \cdot Cov_X(t,h)$

$\overset{(E.1)}{=} \sigma_t \sigma_h \rho_x(h) \cdot \sqrt{\underbrace{\sigma X_t}_{1} \cdot \underbrace{\sigma X_h}_{1}}$

$= \sigma_t \sigma_h \rho_x(h)$.

we know that:

$\rho_x(h) = \dfrac{Cov_X(X_t, X_h)}{\sqrt{\sigma^2_{X_t} \cdot \sigma^2_{X_h}}} \Rightarrow$

$Cov_X(X_t, X_h) = \rho_x(h) \cdot \sqrt{\sigma^2_{X_t} \cdot \sigma^2_{X_h}}$. (E.1)

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b) $Corr_{Y_t}(t,h) = \dfrac{Cov_{Y_t}(t,h)}{\sqrt{\sigma^2_{Y_t} \sigma^2_{Y_h}}} \overset{(E.2)}{\underset{(E.3)}{=}} \dfrac{\sigma_t \sigma_h \rho_x(h)}{\sqrt{\sigma_t^2 \sigma_h^2}} = \rho_x(h)$.

so the ACF depends only on lag h

we know that

$\sigma^2_{Y_t} = E\{(y_t - E y_t)^2\}$

$= E\{(\cancel{\mu_t} + \sigma_t X_t - \cancel{\mu_t})^2\}$

$= E\{(\sigma_t X_t)^2\} = \sigma_t^2 \cdot \underbrace{E\{(X_t - E X_t)^2\}}_{\sigma^2_{X_t} = 1} = \sigma_t^2 \cdot (E.2)$

with the same steps

$\sigma^2_{Y_h} = \sigma_h^2$ (E.3).

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following from this assignment if we assume that $y_t = \sigma_t X_t$ then:

- $E y_t = 0$
- $Corr(y_t) = \rho_x(h)$
- $Cov(y_t, y_{t+h}) = \sigma_t \sigma_{t+h} \rho_x(h) \rightsquigarrow$ depends on t and h so not stationary.

Assignment 18

c) $x_t - 3x_{t-1} = w_t + 2w_{t-1} - 8w_{t-2} \Rightarrow (1-3B)x_t = (1+2B+8B^2)w_t$

- $1 - 3z = 0 \Rightarrow$

  $z = \frac{1}{3} < 1$

  not casual

- $\overset{c}{1} + \overset{b}{2z} + \overset{a}{8z^2} = 0$

  $\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{4-4\cdot1(-8)}}{-16} = \frac{-2\pm\sqrt{4+32}}{-16} = \frac{-2\pm 6}{-16} \to \frac{1}{2} = z_1, \; -\frac{1}{4} = z_2$

  $|z_1| = \frac{1}{2} < 1$

  $|z_2| = \frac{1}{4} < 1$

  not invertible

b) $x_t - 2x_{t-1} + 2x_{t-2} = w_t - \frac{8}{9}w_{t-1} \Rightarrow (1-2B+2B^2)x_t = (1-\frac{8}{9}B)w_t$

- $1 - 2z + 2z^2 = 0$

  $\frac{2\pm\sqrt{4-4\cdot1\cdot2}}{2\cdot2} = \frac{2\pm\sqrt{4-8}}{2} \to \frac{1+i}{2}, \; \frac{1-i}{2}$

  $|z_1| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = 0.7 < 1$

  $|z_2| = \sqrt{(\frac{1}{2})^2 + (-\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = 0.7 < 1$

  not casual

- $1 - \frac{8}{9}z = 0 \Rightarrow$

  $z = \frac{9}{8} = 1,125$

  invertible

g) $x_t - 4x_{t-2} = w_t - w_{t-1} + 0.5w_{t-2} \Rightarrow (1-4B^2)x_t = (1-B+0.5B^2)w_t$

- $1 - 4z^2 = 0 \Rightarrow$

  $z^2 = \frac{1}{4}$

  $z = \frac{1}{2} < 1$ not casual

- $1 - z + 0.5z^2 = 0 \Rightarrow$

  $\frac{1\pm\sqrt{1-4\cdot1\cdot\frac{1}{2}}}{2\cdot\frac{1}{2}} = \frac{1\pm\sqrt{-1}}{1} = 1\pm i$

  $|z_1| = \sqrt{1^2+1^2} = \sqrt{2} = 1.41 > 1$

  $|z_2| = \sqrt{1^2+(-1)^2} = \sqrt{2} = 1.41 > 1$

  invertible

) $x_t - \frac{9}{4}x_{t-1} - \frac{9}{4}x_{t-2} = w_t \Rightarrow (1 - \frac{9}{4}B + \frac{9}{4}B^2)x_t = w_t$

- $1 - \frac{9}{4}z - \frac{9}{4}z^2 = 0 -$

  $\frac{\frac{9}{4} \pm \sqrt{\frac{81}{16} - 4\cdot1\cdot(-\frac{9}{4})}}{2\cdot(-\frac{9}{4})} = \frac{\frac{9}{4} \pm \sqrt{\frac{81}{16}+9}}{-\frac{9}{2}}$

  $= \frac{\frac{9}{4} \pm \frac{15}{4}}{-\frac{9}{2}} \to -\frac{\frac{24}{4}}{-\frac{9}{2}} = \frac{2\cdot24}{4\cdot9} = \frac{24}{18} = 1,33 = z_1, \; -\frac{\frac{6}{4}}{-\frac{9}{2}} = \frac{12}{36} = 0.33 = z_2$

  $|z_1| = 1,33 > 1$

  $|z_2| = 0.33 < 1$ not casual

- invertible

time series H.A. 1

10) $E(X)=2,\ Var(X)=9,\ E(Y)=0,\ Var(Y)=4\quad Corr(X,Y)=0.25\qquad Corr(X+Y,X-Y)$

$$Corr(X+Y,X-Y)=\frac{Cov(X+Y),(X-Y)}{\sigma_{X+Y}\sigma_{X-Y}}\quad(5)$$

$$Cov((X+Y),(X-Y))=E\{(X-Y-\underbrace{E(X-Y)}_{E(X)-E(Y)=2})(X+Y-\underbrace{E(X+Y)}_{E(X)+E(Y)=2})\}=E\{(X-Y-2)(X+Y-2)\}=$$

$$E\{X^2+\cancel{XY}-2X-\cancel{XY}-Y^2+\cancel{2Y}-2X-\cancel{2Y}+4\}=\underbrace{E(X^2)}_{\sigma_x^2+E^2(x)=9+4}-\overbrace{4E(X)}^{4\cdot 2}-\underbrace{E(Y^2)}_{\sigma_y^2+E^2(y)=4}+4=9+4-8-4+4=5\quad(1)$$

$$\sigma^2_{X+Y}=E[(X+Y)^2]-E^2(X+Y)=\underbrace{E(X^2)}_{\sigma_x^2+E^2(x)}+2\underbrace{E(XY)}_{*}+\underbrace{E(Y^2)}_{\sigma_y^2+E^2(y)}-4=9+4+2\cdot\frac{3}{2}+4+0-4=16.\quad(2)$$

$$*\quad corr(X,Y)=\frac{1}{4}\Rightarrow\frac{E(XY)-E(X)E(Y)}{\sigma_x\sigma_y}=\frac{1}{4}\rightarrow E(XY)=\frac{1}{4}\sigma_x\sigma_y+E(X)\cdot E(Y)=\frac{1}{4}3\cdot 2+2\cdot 0=\frac{3}{2}$$

$$\sigma^2_{X-Y}=E[(X-Y)^2]-E^2(X-Y)=E(X^2)-2E(XY)+E(Y^2)-4=9+4-2\cdot\frac{3}{2}+4+0-4=10\quad(3)$$

$$(5)\xrightarrow[(3)]{(1),(2)}\ Corr[(X+Y),(X-Y)]=\frac{5}{\sqrt{10}\cdot\sqrt{16}}=\frac{5}{4}\frac{1}{\sqrt{10}}$$

11) $X_t=W_t+3W_{t-1}\qquad E(X_t)=0$

$$\gamma_x(k)=Cov(X_t\ X_{t-k})=E\{(X_t-E(X_t))(X_{t-k}-E(X_{t-k}))\}=E\{X_t X_{t-k}\}=E\{(W_t+3W_{t-1})(W_{t-k}+3W_{t-k-1})\}$$

$$=E\{W_tW_{t-k}\}+3E\{W_tW_{t-k-1}\}+3E\{W_{t-1}W_{t-k}\}+9E\{W_{t-1}W_{t-k-1}\}$$

for $k=0$ $\gamma_x(0)=\sigma_w^2+9\sigma_w^2=10\sigma_w^2\qquad\rho_x(0)=1$

for $k=1$ $\gamma_x(1)=\cancel{E\{W_tW_{t-1}\}}^{0}+3\cancel{E\{W_tW_{t-2}\}}^{0}+3E\{W_{t-1}W_{t-1}\}+9\cancel{E\{W_{t-1}W_{t-2}\}}^{0}=3\sigma_w^2\qquad\rho_x(1)=\frac{\gamma_x(1)}{\gamma_x(0)}=\frac{3\sigma_w^2}{10\sigma_w^2}=\frac{3}{10}$

$\forall k\geq 2\quad\rho_x(k)=0.$

same results with $\theta=\frac{1}{3}$.

Time Series A.A. 1

②

13) $y_t = (-1)^t x$ $\quad E(x)=0$ $\quad Var(x) = E(x^2) - E^2(x) = E(x^2) = \sigma_x^2$

$E(y_t) = (-1)^t E(x) = 0$ $\quad \gamma_y(y_t, y_{t+k}) = E\{y_t y_{t-k}\}$

$(-1)^t x \cdot (-1)^{t-k} x = (-1)^{2t-k} x^2 = (-1)^k \sigma_x^2$ doesn't depend on t

$t-k+t$ $\quad \sigma_x^2$

15)

$x_t = w_t + 0.5 w_{t-1} + 0.4 w_{t-2}$ $\quad E(x_t) = 0$

$\gamma_x(k) = E(x_t x_{t-k}) = E\{(w_t + 0.5 w_{t-1} + 0.4 w_{t-2})(w_{t-k} + 0.5 w_{t-k-1} + 0.4 w_{t-k-2})\}$

$= w_t w_{t-k} + 0.5 w_t w_{t-k-1} + 0.4 w_t w_{t-k-2} + 0.5 w_{t-1} w_{t-k} + 0.5^2 w_{t-1} w_{t-k-1} + 0.5 \cdot 0.4 w_{t-1} w_{t-k-2} + 0.4 w_{t-2} w_{t-k} + 0.4 \cdot 0.5 w_{t-2} w_{t-k-1}$

$+ 0.4^2 w_{t-2} w_{t-k-2}$

$\rho(0) = 1$. $\quad \gamma_x(0) = \sigma_w^2 + 0.5^2 \sigma_w^2 + 0.4^2 \sigma_w^2 = 1.41 \sigma_w^2$

for $k=1$ $\quad \gamma_x(1) = 0.5 \sigma_w^2 + 0.4 \cdot 0.5 \sigma_w^2 = 0.7 \sigma_w^2$ $\quad \rho_1 \simeq 0.5$

for $k=2$ $\quad \gamma_x(2) = 0.4 \sigma_w^2 \Rightarrow$ $\quad \rho$ $\quad \rho_2 \simeq 0.28$.

for $k>2$ $\rho_x = 0$

18) c) $x_t - 3x_{t-1} = w_t + 2w_{t-1} - 8w_{t-2} \Rightarrow (1-3B)x_t = (1+2B+8B^2)w_t$

$1-3z=0 \Rightarrow$ $\quad 1+2z+8z^2=0 \Rightarrow$

$z = \frac{1}{3} < 1$ $\quad \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{4-4\cdot1(-8)}}{-16} = \frac{-2 \pm \sqrt{4+32}}{-16} = \frac{-2 \pm 6}{-16}$ $\to 1/2$, $\to -1/4$

not casual $\quad |z_1| = \frac{1}{2} < 1$

$|z_2| = \frac{1}{4} < 1$ not invertible

d) $x_t - 2x_{t-1} + 2x_{t-2} = w_t - \frac{8}{9} w_{t-1} \Rightarrow (1-2B+2B^2)x_t = (1-\frac{8}{9}B)w_t$

$1-2z+2z^2=0 \Rightarrow$ $\quad 1-\frac{8}{9}z=0 \Rightarrow$

$\frac{2 \pm \sqrt{4-4\cdot1\cdot2}}{2\cdot2} = \frac{2 \pm \sqrt{4-8}}{2}$ $\to \frac{1+i}{2}$, $\to \frac{1-i}{2}$ $\quad z = \frac{9}{8} = 1{,}125$ invertible

$|z_1| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = 0.7 < 1$

$|z_2| = \dots = 0.7 < 1$ not casual.

e) $x_t - 4x_{t-2} = w_t - w_{t-1} + 0.5 w_{t-2} \Rightarrow (1-4B^2)x_t = (1-B+0.5B^2)w_t$

$1-4z^2=0 \Rightarrow$ $\quad 1-z+0.5z^2=0 \Rightarrow$

$z^2 = \frac{1}{4}$ $\quad \frac{1 \pm \sqrt{1-4\cdot1\cdot\frac{1}{2}}}{2\cdot\frac{1}{2}} = \frac{1 \pm \sqrt{-1}}{1} = 1 \pm i$

$z = \frac{1}{2} < 1$ not casual $\quad z_1 = \sqrt{1^2+1^2} = \sqrt{2} = 1.41 > 1$

$z_2 = \dots$ $\quad = 1{,}41 > 1$

invertible

$\frac{9}{4}x_{t-1} - \frac{9}{4}x_{t-2} = w_t \Rightarrow (1 - \frac{9}{4}B + \frac{9}{4}B^2)x_t = w_t$

$1 - \frac{9}{4}z - \frac{9}{4}z^2 = 0 \Rightarrow$

$$\frac{\frac{9}{4} \pm \sqrt{\frac{81}{16} - 4\cdot1\cdot(-\frac{9}{4})}}{2(-\frac{9}{4})} = \frac{\frac{9}{4} \pm \sqrt{\frac{81}{16}+9}}{-\frac{9}{2}} = \frac{\frac{9}{4} \pm \frac{15}{4}}{-\frac{9}{2}}$$

$\to \frac{\frac{24}{4}}{-\frac{9}{2}} = \frac{2\cdot 24}{4\cdot 9} = \frac{24}{18} = 1.33$

$\to \frac{-\frac{6}{4}}{-\frac{9}{2}} = \frac{12}{36} = 0.33$

$|z_1| = 1.33 > 1$

$|z_2| = 0.33 < 1$ not causal

but is invertible.

$x_t - 2x_{t-1} + 2x_{t-2} = w_t - \frac{8}{9}w_{t-1}$ we want it in the form $\phi(z)\cdot\psi(z) = \theta(z)$

$$\underbrace{(1 - 2z + 2z^2)}_{u}\underbrace{(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3)}_{v} = (1 - \tfrac{8}{9}z) \quad (\times)$$

$u\cdot v = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - 2z - 2z\psi_1 z - 2z\psi_2 z^2 - 2z\psi_3 z^3 + 2z^2 + 2z^2\psi_1 z + 2z^2\psi_2 z^2 + 2z^2\psi_3 z^3 =$

$1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - 2z - 2z^2\psi_1 - 2\psi_2 z^3 - 2\psi_3 z^4 + 2z^2 + 2\psi_1 z^3 + 2\psi_2 z^4 + 2\psi_3 z^5 =$

$1 + z(\psi_1 - 2) + z^2(\psi_2 - 2\psi_1 + 2) + z^3(\psi_3 - 2\psi_2 + 2\psi_1) + z^4(2\psi_3) + z^5 2\psi_3$

we match the coefficients with the right part of $(\times)$

$\psi_1 - 2 = -\frac{8}{9} \Rightarrow \psi_1 = \frac{8}{9} + 2 = \frac{-8+18}{9} = \frac{10}{9}$

$\psi_2 - 2\psi_1 + 2 = 0 \Rightarrow \psi_2 = -2 + 2\psi_1 = -2 + \frac{20}{9} = \frac{-18+20}{9} = \frac{2}{9}$

$\psi_3 - 2\psi_2 + 2\psi_1 = 0 \Rightarrow \psi_3 = 2\psi_2 - 2\psi_1 = 2\cdot\frac{2}{9} - 2\cdot\frac{10}{9} = \frac{4-20}{9} = -\frac{16}{9}$

$x_t - \frac{9}{4}x_{t-1} - \frac{9}{4}x_{t-2} = w_t - 3w_{t-1} + \frac{1}{9}w_{t-2} - \frac{1}{3}w_{t-3}$

$$\underbrace{(1 - \tfrac{9}{4}z - \tfrac{9}{4}z^2)}_{*}\underbrace{(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3)}_{\varepsilon} = (1 - 3z + \tfrac{1}{9}z^2 - \tfrac{1}{3}z^3) \quad (\delta)$$

$*\cdot\varepsilon = 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - \frac{9}{4}z - \frac{9}{4}z\psi_1 z - \frac{9}{4}z\psi_2 z^2 - \frac{9}{4}z\psi_3 z^3 - \frac{9}{4}z^2 - \frac{9}{4}z^2\psi_1 z - \frac{9}{4}z^2\psi_2 z^2 - \frac{9}{4}z^2\psi_3 z^3 =$

$1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 - \frac{9}{4}z - \frac{9}{4}\psi_1 z^2 - \frac{9}{4}\psi_2 z^3 - \frac{9}{4}\psi_3 z^4 - \frac{9}{4}z^2 - \frac{9}{4}\psi_1 z^3 - \frac{9}{4}\psi_2 z^4 - \frac{9}{4}\psi_3 z^5$

$1 + z(\psi_1 - \frac{9}{4}) + z^2(\psi_2 - \frac{9}{4}\psi_1 - \frac{9}{4}) + z^3(\psi_3 - \frac{9}{4}\psi_2 - \frac{9}{4}\psi_1) + z^4(-\frac{9}{4}\psi_3 - \frac{9}{4}\psi_2) + z^5(-\frac{9}{4})$

we match the coefficients with the right part of (δ)

$\psi_1 - \frac{9}{4} = -3 \Rightarrow \psi_1 = -3 + \frac{9}{4} = \frac{-12+9}{4} = -\frac{3}{4}$

$\psi_2 - \frac{9}{4}\psi_1 - \frac{9}{4} = \frac{1}{9} \Rightarrow \psi_2 = \frac{1}{9} + \frac{9}{4}\psi_1 + \frac{9}{4} = \frac{1}{9} + \frac{9}{4}\cdot(-\frac{3}{4}) + \frac{9}{4} = \frac{1}{9} + \frac{-27+36}{16} = \frac{1}{9} + \frac{9}{16}$

$\psi_3 - \frac{9}{4}\psi_2 - \frac{9}{4}\psi_1 = -\frac{1}{3} \Rightarrow \psi_3 = -\frac{1}{3} + \frac{9}{4}\psi_2 + \frac{9}{4}\psi_1 = -\frac{1}{3} + \frac{9}{4}(\frac{1}{9} + \frac{9}{16}) + \frac{9}{4}(-\frac{3}{4})$

$= \underbrace{-\frac{1}{3} + \frac{9}{36}} + \underbrace{\frac{81}{64} - \frac{27}{16}}$

$= -\frac{3}{36} \quad -\frac{27}{64} = -\frac{1}{12} - \frac{27}{64}$

Time series H.A.4

gnment 14

$x_t = \mu + w_t + w_{t-1}$

$E(x_t) = \mu$ suppose we have a sample $x_1, \dots x_n$ $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n} E(\sum x_i) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\cdot n \cdot \mu = \mu$$

$$\bar{x} = \frac{1}{n}(\mu + w_1 + w_0 + \mu + w_2 + w_1 + \dots + \mu + w_n + w_{n-1}) = \frac{1}{n}(\mu \cdot n + w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1})$$

$$= \mu + \frac{w_0 + w_n}{2} + 2\cdot\sum_{i=1}^{n-1} w_i$$

$$\cancel{Var(x_t) = E\{(x_t - E(x_t))^2\} = E\left\{\left(\frac{w_0 + w_n}{2} + 2\sum_{i=1}^{n-1} w_i\right)^2\right\} = E\left\{\left(\frac{w_0 + w_n}{2}\right)^2 + 4\left(\frac{w_0 + w_n}{2}\right)\sum_{i=1}^{n-1} w_i + 4\left(\sum_{i=1}^{n-1} w_i\right)^2\right\}}$$

$$\cancel{= \frac{1}{4}(w_0^2 + 2w_0 w_n + w}$$

$$\bar{x} - E(\bar{x}) = \frac{1}{n}(\mu n + w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1}) - \mu = \frac{1}{n}(w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1})$$

$$Var(x_t) = E\{(\bar{x} - E(\bar{x}))^2\} = E\left\{\left(\frac{1}{n}(w_0 + w_n + 2w_1 + 2w_2 + \dots + 2w_{n-1})\right)^2\right\}$$

$$= E\left\{\frac{1}{n^2}(w_0^2 + w_n^2 + \underbrace{4w_1^2 + 4w_2^2 + \dots + 4w_{n-1}^2}_{4(n-1)\sigma_w^2} + \underbrace{2w_0 w_n + 4w_0 w_1 + \cdots}_{0})\right\}$$

$$= \frac{1}{n^2}\{\underbrace{E(w_0^2) + E(w_n^2)}_{2\sigma_w^2} + 4E(w_1^2) + \dots + 4E(w_{n-1}^2)\}$$

$$= \frac{1}{n^2}(2\sigma_w^2 + 4(n-1)\sigma_w^2) = \frac{2\sigma_w^2 + 4n\sigma_w^2 - 4\sigma_w^2}{n^2} = \frac{4\sigma_w^2 - 2\sigma_w^2}{n^2} = \frac{2\sigma_w^2(2n-1)}{n^2}$$

$$\frac{1}{n^2}(\underbrace{\sigma_w^2 + \sigma_w^2}_{2\sigma_w^2} + \underbrace{4\sigma_w^2 + \dots + 4\sigma_w^2}_{4(n-1)\sigma_w^2}$$

$$(w_0 + w_1 + w_2)^2$$

$$w_0^2 + w_1^2 + w_1^2 + 2w_0 w_1 + \dots$$

$$E(w_0 w_1) = 0$$

[illegible]

<u>Assignment 17</u> $x_t = \phi x_{t-1} + w_t$

$\{x_t\} \sim AR(1)$ $\quad -1 < \phi < +1$ $\quad y_t = \nabla x_t = x_t - x_{t-1}$

a) $E(x_t) = 0$ $\quad \gamma_x(h) = \phi^h \dfrac{\sigma_w^2}{1-\phi^2}$ $\quad E(y_t) = E(x_t) - E(x_{t-1}) = 0$

$$\gamma_y(h) = E\{\underbrace{(y_t - E(y_t))}_{x_t - x_{t-1}}\underbrace{(y_{t-h} - E(y_{t-h}))}_{x_{t-h} - x_{t-h-1}}\} = E\{(x_t - x_{t-1})(x_{t-h} - x_{t-h-1})\} = E\{\underbrace{x_t x_{t-h}}_{h\text{ distance}} - \overbrace{x_t x_{t-h-1}}^{t-(t-h-1)=(h+1)\text{ distance}} \underbrace{- x_{t-1} x_{t-h}}_{\substack{t-1-t+h= \\ (h+1)\text{ distance}}} + \overbrace{x_{t-1} x_{t-h-1}}^{t-1-t+h+1=h\text{ distance}}\}$$

$$= \gamma_x(h) - \gamma_x(h+1) - \gamma(h-1) + \gamma(h) = 2\gamma_x(h) - \gamma_x(h+1) - \gamma_x(h-1)$$

$$= \frac{2\phi^h \sigma_w^2}{1-\phi^2} - \frac{\phi^{h+1}\sigma_w^2}{1-\phi^2} - \frac{\phi^{h-1}\sigma_w^2}{1-\phi^2} = \frac{(2-\phi-\frac{1}{\phi})}{(1-\phi)(1+\phi)}\sigma_w^2\phi^h = -\frac{(1-\phi)^2}{\phi(1+\phi)(1+\phi)}\sigma_w^2\phi^h = -\frac{(1-\phi)}{(1+\phi)}\sigma_w^2\phi^{h-1}$$

b) $$\mathrm{Var}(y_t) = E(y_t^2) - E^2(y_t) = E\{(x_t - x_{t-1})^2\} - E^2(y_t) = E\{x_t^2 - 2x_t x_{t-1} + x_{t-1}^2\} - \cancel{E^2(x_t - x_{t-1})}^{\,0}$$

$$= \gamma_x(0) - 2\gamma_x(1) + \gamma_x(0) = -2\frac{\sigma_w^2\phi}{(1+\phi^2)} + \frac{2\sigma_w^2}{(1-\phi^2)} = \frac{2\sigma_w^2(1-\phi)}{1-\phi^2} = \frac{2\sigma_w^2(1-\phi)}{(1-\phi)(1+\phi)} = \frac{2\sigma_w^2}{1+\phi}$$

Teaching Session II

Assignment 1

we are asked to find the state space representation of the ARIMA$(p,d,q)\times(P,D,Q)_s$.

$$\Phi^{P}(B^s)\varphi^{p}(B)(1-B^s)^D(1-B)^d x_t = \Theta^{Q}(B^s)\vartheta^{q}(B)w_t. \quad (E.1)$$

we start from the simple case with $p=3, d=2, q=1, P=2, D=1, Q=1, s=5$.

the model is then:

$$\Phi^2(B^5)\ \varphi^3(B)\ (1-B^5)^1\ (1-B)^2 x_t = \Theta^1(B^5)\ \vartheta^1(B)\ w_t$$

where:

- $\Phi^2(B^5) = 1-\Phi_1 B^5 - \Phi_2 B^{10}$
- $\varphi^3(B) = 1-\varphi_1 B - \varphi_2 B^2 - \varphi_3 B^3$
- $\Theta^1(B^5) = 1-\Theta_1 B^5$
- $\vartheta^1(B) = 1-\vartheta_1 B$

the higher order of the: $\Phi^2(B^5)\cdot\varphi^3(B)\cdot(1-B^5)^1\cdot(1-B)^2$ or $(1-\Phi_1B^5-\Phi_2B^{10})\cdot(1-\varphi_1B-\varphi_2B^2-\varphi_3B^3)\cdot(1-B^5)^1(1-B)^2$

is $\tilde{p}=20$.

the higher order of the: $\Theta^1(B^5)\cdot\vartheta^1(B)$ or $(1-\Theta_1B^5)\cdot(1-\vartheta_1B)$

is $\tilde{q}=6$.

so $r=\max(\tilde{p},\tilde{q}+1) = \max(20,7) = 20$.

Hence:

$x_t = \vartheta^r(B)\cdot z_t$ and $\varphi^r(B)z_t = w_t$.

$\varphi^{20}(B) = 1-\varphi_1B - \dots - \varphi_{20}B^{20}$

$\vartheta^{r}(B) = 1+\vartheta_1B + \dots + \vartheta_{19}B^{19}$

so

$$z_t = \begin{bmatrix} z_t \\ z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-19} \end{bmatrix}_{(20\times1)} \quad\text{and}\quad Z_t = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_{20} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots\ 1 & 0 \end{bmatrix}_{(20\times20)} \cdot Z_{t-1} + \begin{bmatrix} w_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(20\times1)}$$

$$x_t = [1\ \vartheta_1\ \vartheta_2\ \cdots\ \vartheta_{19}]\,Z_t$$

for the general case (E.1). following the same steps.

$$r = \max\left((P\cdot s) + p + (s\cdot D) + d,\ (Q\cdot s + q) + 1 \right).$$

$$z_t = \vartheta^r(B)\, z_t \qquad \text{and} \qquad \varphi^r(B)\, z_t = w_t$$

$$z_t = \begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ \vdots \\ z_{t-(q)} \end{bmatrix}_{(r\times 1)} \qquad \text{and} \qquad z_t = \underbrace{\begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_r \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}}_{(r\times r)} z_{t-1} + \begin{bmatrix} w_t \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}_{(r\times 1)}.$$

$$x_t = [1\ \vartheta_1\ \vartheta_2 \cdots \vartheta_{r-1}]\, z_t$$

we manage to decompose the $ARIMA(p,d,q)\times(P,D,Q)_s$ to an AR process.

Teaching Session III

Assignment 1

Prove the Kalman filtering recursion for the following state space model with initial prior on the state $f(z_1) = N(z_1; m_0, P_0)$ where $e_t \sim N(0, Q_t)$ and $v_t \sim N(0, R_t)$

$$z_t = A_{t-1} z_{t-1} + e_t \qquad (1)$$

$$x_t = C_t z_t + v_t \qquad (2)$$

- Particularly, show the given $f(z_t / x_t) = N(z_t; m_{t|t}, P_{t|t})$, the predicted density $f(z_{t+1} / x_{1:t})$ is given by:

$$f(z_{t+1} / x_{1:t}) = N(z_{t+1}; A_t m_{t|t}, A_t P_{t|t} A_t^T + Q_{t+1})$$

- Also, show that given $f(z_t / x_{1:t-1}) = N(z_t; m_{t|t-1}, P_{t|t-1})$, the observation updated density $f(z_t / x_{1:t})$ is given by: $f(z_t / x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$ where

$$\begin{cases} m_{t|t} = m_{t|t-1} + K_t (x_t - C_t m_{t|t-1}) \\ P_{t|t} = (I - K_t C_t) P_{t|t-1} \\ K_t = P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_t)^{-1} \end{cases}$$

- The joint posterior $f(z_t, z_{t+1} / x_{1:t})$ can be given by:

$$f(z_t, z_{t+1} / x_{1:t}) = \underbrace{N(z_t; m_{t|t}, P_{t|t})}_{\text{given}} \cdot \underbrace{N(z_{t+1}, A_t z_t, Q_{t+1})}_{\text{from (1)} \Rightarrow z_{t+1} = A_t z_t + e_{t+1},\ e_{t+1} \sim N(0, Q_{t+1})}$$

$$= N\left(\begin{bmatrix} z_t \\ z_{t+1} \end{bmatrix}; \begin{bmatrix} m_{t|t} \\ A_t m_{t|t} \end{bmatrix}, \begin{bmatrix} P_{t|t} & P_{t|t} A_t^T \\ A_t P_{t|t} & A_t P_{t|t} A_t^T + Q_{t+1} \end{bmatrix} \right)$$

Then we marginalize z_t out $\left[\text{property } f(y_2) = N(y_2; \mu_2, \Sigma_{22}), \text{ where } y_2 = z_{t+1} / x_{1:t}\right]$ so we have

$$f(z_{t+1} / x_{1:t}) = N(z_{t+1}; A_t m_{t|t}, A_t P_{t|t} A_t^T + Q_{t+1}).$$

- $$f(z_t | x_{1:t}) = \frac{f(z_t, x_t | x_{1:t-1})}{f(x_t | x_{1:t-1})} = \frac{f(z_t | x_{1:t-1}) \cdot f(x_t | z_t, x_{1:t-1})}{f(x_t | x_{1:t-1})} \overset{\substack{\text{using conditional} \\ \text{independence of } x_t \text{ given} \\ z_t \text{ on } x_{1:t-1}}}{=} \frac{\overbrace{f(z_t | x_{1:t-1})}^{\text{given}} \overbrace{f(x_t | z_t)}^{\substack{\text{from (2)} \\ x_t = C_t z_t + v_t,\ v_t \sim N(0, R_t)}}}{\underbrace{f(x_t | x_{1:t-1})}_{\text{constant}}}$$

working in the numerator:

$$N(z_t; m_{t|t-1}, P_{t|t-1}) N(x_t; C_t m_{t|t-1}, R_t) = N\left(\begin{bmatrix} z_t \\ x_t \end{bmatrix}; \begin{bmatrix} m_{t|t-1} \\ C_t m_{t|t-1} \end{bmatrix}, \begin{bmatrix} \overset{\Sigma_{11}}{P_{t|t-1}} & \overset{\Sigma_{12}}{P_{t|t-1} C_t^T} \\ \underset{\Sigma_{21}}{C_t P_{t|t-1}} & \underset{\Sigma_{22}}{C_t P_{t|t-1} C_t^T + R_t} \end{bmatrix} \right)$$

from property $f(y_1/y_2) = N(y_1; \underbrace{\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2 - \mu_2)}_{w}, \underbrace{\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}_{J})$.

start with $w \Rightarrow m_{t|t-1} + \underbrace{\left[P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_t)^{-1} \right]}_{K_t} (x_t - C_t m_{t|t-1})$

$= m_{t|t-1} + K_t(x_t - C_t m_{t|t-1})$ and setting it ~~equal~~ to $m_{t|t}$

now for $J \Rightarrow P_{t|t-1} - \underbrace{\left[P_{t|t-1} C_t^T (C_t P_{t|t-1} C_t^T + R_t)^{-1} \right]}_{K_t} \cdot C_t \cdot P_{t|t-1} =$

$P_{t|t-1} - K_t C_t P_{t|t-1} = P_{t|t-1}(I - K_t C_t)$ and setting it to $P_{t|t}$.

finally $f(z_t / x_{1:t}) = N(z_t; m_{t|t}, P_{t|t})$.