3.1 Motion

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1 Definitions

Kinematics is the study of motion. It refers only the motion of objects and the positions, without considering their masses or forces.

Definition 1.1. Displacement is the distance moved in a particular direction from a reference point. It is a vector quantity.

Definition 1.2. Velocity is displacement per unit time. It is the first derivative of displacement with respect to time ($\frac{ds}{dt}$), and is a vector quantity.

Definition 1.3. Acceleration is change in velocity per unit time. It is the second derivative of displacement w.r.t. time ($\frac{d^2s}{dt^2}$), and is a vector quantity.

2 Graphs of motion

Imagine throwing an object vertically into the air. These are its displacement, velocity, and acceleration graphs:

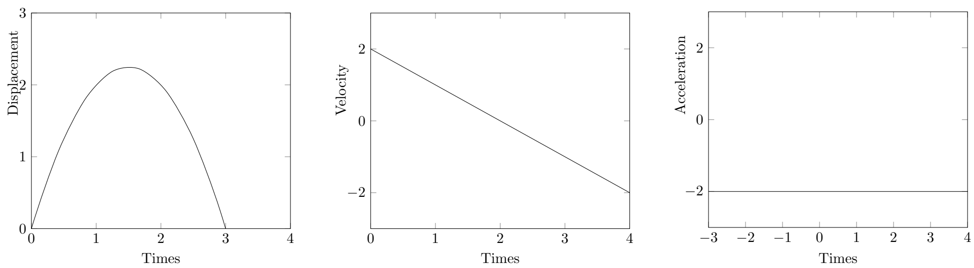

Figure 1: Graphs for a ball being thrown

We can see from this that when the displacement graph is a curve, the velocity is changing at a constant rate. If the velocity is a curve, acceleration is changing at a constant rate. The gradient of a displacement-time graph gives velocity, and the gradient of a velocity time graph gives acceleration.

The reverse is also true: the area under the curve of an acceleration-time graph gives velocity, and the area under a velocity-time graph gives displacement. In calculus terms, the integral of acceleration with respect to time is velocity.

3 Constant Acceleration Equations

Motion with invariant acceleration can be described using a family of equations. Each of these use a combination of five variables:

- s: displacement.
- u: initial velocity.
- v: final velocity.
- a: acceleration.
- t: time.

Each suvat equation uses four of these, meaning that all parameters can be calculated if we know only three.

Definition 3.1. Suvat equations: $v = u + at$, $s = ut + \frac{1}{2}at^2$, $s = vt - \frac{1}{2}at^2$, $v^2 = u^2 + 2as$, and $s = \frac{1}{2}(u + v)t$.

We can combine, rearrange and evaluate these for different systems. It's important to note that sign is important. If you say that $a = -9.81$ then downward displacement should be negative.

4 Projectile Motion

Imagine a ball thrown horizontally from a height (maybe a cliff or a tower). It has some initial velocity u and an initial height. The following is a diagram of the scenario.

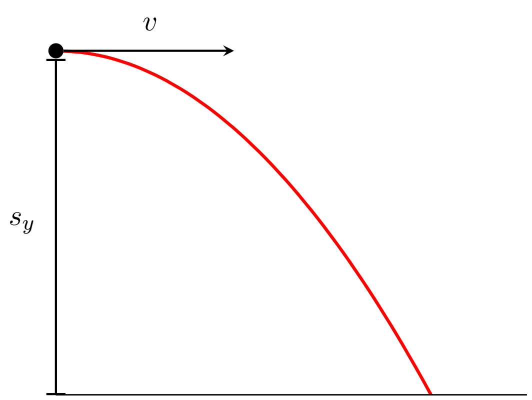

Figure 2: A ball is thrown with a horizontal velocity v from a height s_y.

The time that the ball is in the air is, counter-intuitively, solely dependent on s_y and on a, the acceleration. The time that it will take the ball to hit the ground is calculated as follows:

$$s_y = ut + \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2s_y}{a}}$$

And the horizontal distance to the point of impact with the ground is given by:

$$v = \frac{s}{t} \Rightarrow s = vt$$

Now, consider a different scenario. The ball is now projected from the ground, with an initial velocity v at an angle θ from the horizontal.

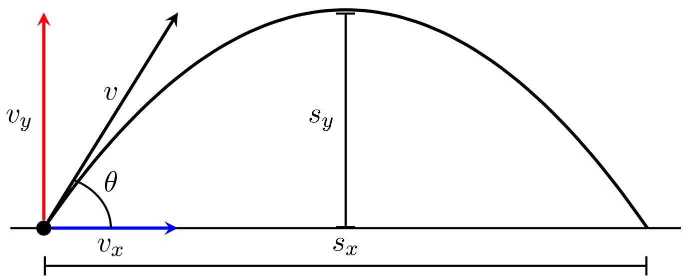

Figure 3: A ball is thrown with a horizontal velocity v from a height h.

A fundamental principle is that the perpendicular components of velocity are independent of each other. The horizontal velocity v_x is constant, and the vertical velocity v_y is dependent on acceleration. This means that, again, time is solely dependent on v_y, and the horizontal distance on v_x. First, we need to resolve the velocity. Using trigonometry:

$$v_x = v\cos\theta$$

$$v_y = v\sin\theta$$

At the maximum of the curve, or the highest point, $v_y = 0$. This makes sense, since it is the moment where the ball changes direction (and is when $\frac{ds}{dt} = 0$). Knowing this fact, we can use $v^2 = u^2 + 2as$ to calculate s_y:

$$v^2 = u^2 + 2as \Rightarrow s_y = \frac{0 - u^2}{2a}$$

Now that we know the maximum height, notice that the second half of this trajectory is identical to the scenario with horizontal motion. We can now use the same logic to find the time for the ball to drop from this point to the ground (i.e. the time for the second half of the trajectory)

$$s_y = ut + \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2s_y}{a}}$$

Notice that when the start and end height are the same, the trajectory is symmeterical. This means that the time for the ball to reach the maximum point is the same as how long it takes for it to fall. Therefore, its total flight time is $2t$. Knowing this, we can calculate s_x. Horizontal velocity is constant so:

$$s_x = 2tv$$