QUANTUM DECOHERENCE

Project Report

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Abstract

Quantum theory, governing microscopic objects, exhibit unique features like quantum entanglement and superpositions, which conflict with our intuitions of the classical macroscopic world. Decoherence is a potential explanation for this quantum to classical transition. This report presents a comprehensive study on Quantum Decoherence, exploring the concepts of von Neumann Measurement (addressing the measurement problem in quantum mechanics), Environment-Induced Decoherence and decoherence effects of simple two level spin model. Then we further examine the Quantum Brownian Motion model, a classic decoherence model, and extend it to include both position and momentum coupling with the environment, providing some insights on simultaneous decoherence of canonical conjugate variables. Further we would like to explore the attainment of classicality and thus emergence of classical trajectories after decoherence.

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Contents

1 Introduction

What we experience in this macroscopic world is a result of microscopic objects that are governed by quantum theory. Quantum theory has some exceptional features like quantum entanglement and quantum superpositions which led to lots of confusion for decades. They are the reasons why quantum theory seems to be in conflict with our intuitions of our macroscopic (classical) world. The classical macroscopic world does not show any of these quantum features. However, a crucial question emerges: How can quantum physics give rise to the classical world? (i.e Quantum to classical transitions). Decoherence is one of the possible answers that we will attempt to explore here.

Basically, from the ideas of classical physics of idealised isolated systems we had incorporated into quantum mechanics easily without much scrutiny. However, isolated quantum systems hinder us from understanding the quantum-to-classical transition. We therefore require open quantum systems which interact with the environment not just as a fix but an essential and more complete description of many systems of interest today, from quantum optics to quantum computing.

1.1 Density matrix and reduced density matrix

For the study of decoherence, density matrix formulation, especially, reduced density matrix plays an important role since system cannot be described by pure states due to the entanglement of system and environment.

Defining the density matrix for pure states $|\psi\rangle$ to be

$$\hat{\rho} = |\psi\rangle \langle\psi| . \tag{1}$$

Any state $|\psi\rangle$ can be expressed as a superposition of some orthonormal basis states $|\psi_i\rangle$

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle \tag{2}$$

then the density matrix is

$$\hat{\rho} = \sum_{i,j} c_i c_j^* |\psi_i\rangle \langle\psi_j| \tag{3}$$

From here the off-diagonal $(i \neq j)$ terms are the reasons for the interference effects. So, usually referred to as interference terms. Note the density operator ρ can be diagonal or not based on the basis we choose to represent them. With this in hands the quantity of interest is the expectation values of hermitian operator $(\hat{O})$ which can be obtained by

$$Tr(\hat{\rho}\hat{O}) = \sum_i \langle o_i|(|\psi\rangle \langle\psi|)\hat{O}|o_i\rangle \tag{4}$$

here $|o_i\rangle$ are eigenstates of $\hat{O}$

$$= \sum_i o_i|\langle o_i|\psi\rangle|^2 \tag{5}$$

$$= \langle\hat{O}\rangle. \tag{6}$$

We have seen that $\hat{\rho}$ is completely equivalent to state $|\psi\rangle$ and of not much use. But, $\hat{\rho}$ becomes handy when we insufficient information about the state of the system (mixed states). What it means is the classical ensemble of set of possible pure states $|\psi_i\rangle$ with probabilities $p_i's$, then this density matrix is given by

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle\psi_i| \tag{7}$$

with $p_i's \geq 0$ and $\sum_i p_i = 1$. This now is a classical probability distribution of pure-state density matrices $\hat{\rho}_i = |\psi_i\rangle \langle\psi_i|$.

Reduced density matrix: Consider two systems $\mathcal{S}$ and $\mathcal{A}$ now the reduced density matrix of system $\mathcal{S}$ is

$$\hat{\rho}_{\mathcal{S}} = Tr_{\mathcal{A}}(\hat{\rho}). \tag{8}$$

where $\hat{\rho}$ is of composite system $\mathcal{S}$ and $\mathcal{A}$. Similarly expectation value of any observable of system $\mathcal{S}$ can be derived from reduced density matrix as

$$\langle\hat{O}_{\mathcal{S}}\rangle = Tr_{\mathcal{S}}(\hat{\rho}_{\mathcal{S}}\hat{O}_{\mathcal{S}}). \tag{9}$$

1.2 The von Neumann Measurement

John von Neumann's idea was to describe the measurement process in entirely quantum mechanical terms. This means that not only the measured system but also the apparatus should be treated as a quantum mechanical object. Note that this approach is in sharp contrast to the Copenhagen interpretation, where the measurement apparatuses are excluded from the quantum mechanical description and are postulated as intrinsically classical. This ideal quantum measurement is the easiest way to understand how quantum entanglement arises.

Let us consider a quantum mechanical system $\mathcal{S}$ in the Hilbert space $\mathcal{H}_\mathcal{S}$ with basis vectors $|s_i\rangle$, and a measuring apparatus $\mathcal{A}$ with basis vectors $|a_i\rangle$ in a Hilbert space $\mathcal{H}_\mathcal{A}$. The purpose of the apparatus is now to measure the state of the system $\mathcal{S}$. We can think of the apparatus as having some kind of pointer that moves to the position $'i'$, represented by the state $|a_i\rangle$, if the system is measured to be in the state $|s_i\rangle$ (the main idea here is to establish a one-to-one correspondence between the state of the system and the state of the apparatus). Initially the apparatus is in some initial "ready" state $|a_r\rangle$, the dynamical measurement interaction between the system and the apparatus will then be of the form

$$|s_i\rangle\,|a_r\rangle \;\longrightarrow\; |s_i\rangle\,|a_i\rangle \tag{10}$$

There is an assumption that the measurement interaction does not change the state of the system (ideal). Let us assume that the initial (more general) state of the system is in superposition of the states $|s_i\rangle$

$$|\psi_s\rangle = \sum_i c_i\,|s_i\rangle \tag{11}$$

Then the initial state of the composite $\mathcal{S} + \mathcal{A}$ is given by

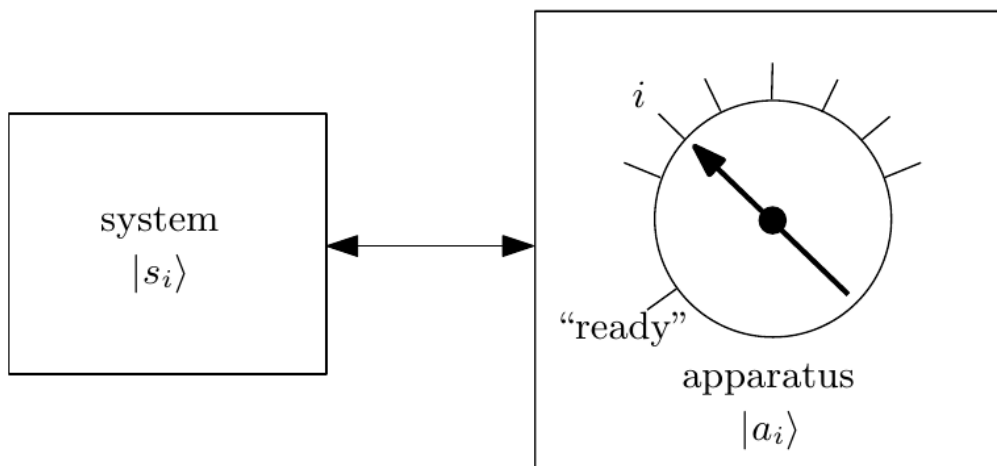$$|\psi\rangle = \sum_i c_i\,|s_i\rangle\,|a_r\rangle \tag{12}$$

Figure 1: Illustration of establishment of one-to-one correspondence between the state of the system and the state of the apparatus [1].

under the unitary evolution of the whole system the transformation must be of the form

$$\mathcal{U}(t)\,|s_i\rangle\,|a_r\rangle \;\longrightarrow\; |s_i\rangle\,|a_i\rangle \tag{13}$$

The total Hamiltonian of the system-apparatus combine as $H = H_{\mathcal{S}} + H_{\mathcal{A}} + H_{int}$, and to satisfy (4) we can consider the interaction of the von Neumann form[2]

$$H_{int} = \sum_i |s_i\rangle\,\langle s_i| \otimes A_i \tag{14}$$

Here the $A_i's$ are operators acting on $\mathcal{H}_{\mathcal{A}}$, and during the measurement time we will consider that the interaction term in the Hamiltonian dominates over the other two terms (so that $H \simeq H_{int}$). This leads to

$$e^{-iH_{int}t/\hbar}(\sum_i c_i\,|s_i\rangle\,|a_r\rangle) \tag{15}$$

$$= \sum_i c_i\,|s_i\rangle \otimes e^{-iA_it/\hbar}\,|a_r\rangle \tag{16}$$

Let

$$|a_i\rangle \;=\; e^{-iA_it/\hbar}\,|a_r\rangle$$

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which implies that[1]

$$|\psi(t)\rangle \;=\; \sum_i c_i \, |s_i\rangle \, |a_i\rangle \qquad (17)$$

Note that as the apparatus state entangles with the system the superposition is also carried on to the apparatus. And also the measurement is not complete yet, so often it is referred to as pre-measurement.

2 The Measurement Problem in Quantum Mechanics

Nearly a century has passed since the foundation of quantum mechanics and despite this, even today, standard textbooks still do not address the measurement problem.

The measurement problem itself consists of three different parts as mentioned below[1] :

1. *The problem of the preferred basis* : What chooses the preferred physical quantities (observables) to be observed in nature?

2. *The problem of the nonobservability of interference* : Why do quantum interference effects vanish at macroscopic level?

3. *The problem of outcomes* : What selects the particular outcome (eigenstate of the observable) of different outcomes available given by quantum probability distribution.

2.1 The problem of preferred basis

From the von Neumann scheme above the final state after the premeasurement can be expressed in many ways by the choice of the basis

$$\sum_i c_i \, |s_i\rangle \, |a_i\rangle = \sum_i c_i' \, |s_i'\rangle \, |a_i'\rangle \qquad (18)$$

Now, for the detector to give us a classical measurement (macroscopically distinguishable) we require its basis states to be orthogonal $\langle a_i | a_j \rangle = 0$ for $i \neq j$.

Coming back to the question of the given infinite number of ways to express the states in different basis, how does it select an unique (preferred) basis? Let's see an example of the Stern-Gerlach experiment and infer from it. If we choose to measure spin in z-direction which fixes the basis of the apparatus to $|0_z\rangle_{\mathcal{A}}, |1_z\rangle_{\mathcal{A}}$ for example[3],

$$|\psi_f\rangle \; = \; \frac{1}{\sqrt{2}}(|z, +\rangle_{\mathcal{S}} |z, -\rangle_{\mathcal{A}} - |z, -\rangle_{\mathcal{S}} |z, +\rangle_{\mathcal{A}}) \qquad (19)$$

or

$$|\psi_f\rangle \; = \; \frac{1}{\sqrt{2}}(|x, +\rangle_{\mathcal{S}} |x, -\rangle_{\mathcal{A}} - |x, -\rangle_{\mathcal{S}} |x, +\rangle_{\mathcal{A}}) \qquad (20)$$

But this suggests that the apparatus $\mathcal{A}$ could simultaneously measure the two non-commuting operators $\hat{\sigma_x}$ and $\hat{\sigma_z}$, which is impossible as the apparatus can only be set up one way at a time.

Thus, the selection of the configuration of $\mathcal{A}$ uniquely determines the basis of $\mathcal{S}$ that we observe. Furthermore, this can be extended beyond spin directions to any non-commuting values. This solves the problem of preferred basis.

3 Decoherence

3.1 Basic concept of decoherence

Decoherence is the most commonly used approach to tackle the problem of *quantum to classical transition*. As the theory of idealised isolated system had great success in classical physics it got incorporated into quantum mechanics easily without much scrutiny. But the *key idea* of the decoherence theory is that no system can be perfectly isolated[4]. When the system interacts with the environment (have a very large number of degrees of freedom, inaccessible for all practical purposes), it monitors the macroscopic system and

therefore locally suppresses interference of macroscopically distinct states. Thus we have to study it under open quantum systems.

Around 1980s, Wojciech Zurek pioneered an approach to the quantum-classical transition that focused on a property called quantum coherence. When a system possesses coherence, it is able to behave in a fundamentally quantum way, maintaining superpositions between classically distinct states.

3.2   Environment-Induced Decoherence

Let's consider a system of interest $\mathcal{S}$ interacting with its environment $\mathcal{E}$ in the Hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{E}}$ with total hamiltonian

$$H_T = H_{\mathcal{S}} \otimes I_{\mathcal{E}} + I_{\mathcal{S}} \otimes H_{\mathcal{E}} + H_{int} \tag{21}$$

Furthermore, we assume the interaction to be of the von Neumann type (ideal).

$$H_{int} = \sum_i |s_i\rangle \langle s_i| \otimes \hat{E}_i \tag{22}$$

Initially system-environment are in seperable states as

$$|\psi\rangle = \sum_i c_i |s_i\rangle |E_0\rangle \tag{23}$$

under the unitary evolution of the whole $\mathcal{S} + \mathcal{E}$

$$\mathcal{U}(t) |s_i\rangle |E_0\rangle \longrightarrow |s_i\rangle |E_i(t)\rangle \tag{24}$$

the interaction term in the Hamiltonian dominates over the other two terms (so that $H \simeq H_{int}$). This leads to

$$e^{-iH_{int}t/\hbar} \left(\sum_i c_i |s_i\rangle |E_0\rangle \right) \tag{25}$$

$$|\psi(t)\rangle = \sum_i c_i |s_i\rangle |E_i\rangle \tag{26}$$

Since the interaction is of von Neumann type we can say that 'the environment measures(monitors) the system'. As mentioned earlier, the superposition, initially carried by state of the system, is now carried by the system-environment state. Density matrix of the pure state

$$\rho(t) = \sum_{i,j} c_i c_j^* (|s_i\rangle \langle s_j|) \otimes (|E_i\rangle \langle E_j|) \qquad (27)$$

Our observations are restricted to the system $\mathcal{S}$ which is of our interest[5] and unfortunately, due to the large number of degrees of freedom carried by the environment (inaccessible for all practical purposes), we have to trace out the environment.

The reduced density operator to the density operator of Eq.(18) is

$$Tr_{\mathcal{E}}(\rho_{\mathcal{SE}}) = \sum_{i,j} c_i c_j^* (|s_i\rangle \langle s_j|) \otimes \sum_k \langle k|E_i\rangle \langle E_j|k\rangle \qquad (28)$$

Typically for macroscopically distinct states, the system-environment interaction will lead to distinguishable environmental states[1], i.e.

$$\langle E_i|E_j\rangle \longrightarrow \delta_{ij}. \qquad (29)$$

therefore the system can be described by the density matrix

$$\rho_{\mathcal{S}}(t) \longrightarrow \sum_i |c_i|^2 |s_i\rangle \langle s_i| \qquad (30)$$

$$\rho = \underbrace{\sum_i |c_i|^2 |\psi_i\rangle \langle \psi_i|}_{\text{diagonal}} + \underbrace{\sum_{i \neq j} c_i c_j^* |\psi_i\rangle \langle \psi_j|}_{\text{interference}} \qquad (31)$$

From (21) and (22) it's easy to see that the reduced density matrix has turned diagonal and the interference terms are no longer there because the superposition of states has been destroyed locally. This reduced density matrix represents an ensemble of classical mixture. Thus, decoherence could explain why it is so difficult to observe interference on a macroscopic scale.

3.3 Precise and imprecise measurements

Take the von-Neumann type measurement with dominant interaction Hamiltonian,

$$H_{int} = \int dx\,|x\rangle\langle x| \otimes \frac{x.\hat{A}}{\tau} \tag{32}$$

in the continuous basis. With $\hat{A}$ be canonical conjugate momentum operator of Environment $\mathcal{E}$ position states given by $|Y\rangle$.

Initially assume system-environment are in seperable states as

$$|\psi\rangle = \int dx\,\psi(x)\,|x\rangle\,|Y\rangle \tag{33}$$

under the unitary evolution of the whole $\mathcal{S} + \mathcal{E}$

$$e^{-iH_{int}t/\hbar}\left(\int dx\,\psi(x)\,|x\rangle\,|Y\rangle\right) \tag{34}$$

the interaction term in the Hamiltonian dominates over the other two terms (so that $H \simeq H_{int}$). This leads to

$$|\psi(t)\rangle = \int dx\,\psi(x)\,|x\rangle \otimes e^{\frac{-ixt}{\tau\hbar}\hat{A}}\,|Y\rangle \tag{35}$$

$$|\psi(t)\rangle = \int dx\,\psi(x)\,|x\rangle\,|Y + \frac{xt}{\tau}\rangle \tag{36}$$

Calculating the reduced density matrix,

$$\rho_r = \int dx\,d\bar{x}\,\psi(x)\psi^*(\bar{x})\delta(\frac{t}{\tau}(x - \bar{x}))\,|x\rangle\langle\bar{x}| \tag{37}$$

Decoherence has caused the reduced matrix to be diagonal in $|x\rangle$ almost instantaneously. This usually referred as precise measurement.

$$\rho_r = \int dx\,\psi(x)\psi^*(x)\,|x\rangle\langle x| \tag{38}$$

Lets investigate if we have interaction Hamiltonian of the form in Eqn(32), but operator $\hat{A}$ has smeared projective operator as,

$$\hat{A} = \int dP\,d\bar{P}\,e^{-\left(\frac{P-\bar{P}}{\sigma}\right)^2}P\,|P\rangle\langle\bar{P}| \tag{39}$$

then we can explicitly show the reduced density matrix as,

$$\rho_r = \int dx d\bar{x} \psi(x) \psi^*(\bar{x}) F(x, \bar{x}, t) \ket{x} \bra{\bar{x}} \tag{40}$$

here the function F should approach,

$$F(x, \bar{x}, t) \longrightarrow \delta(x - \bar{x}) \quad as \quad t \longrightarrow \infty \tag{41}$$

for the density matrix to be diagonal(decohered) in the $\ket{x}$ basis. This type of measurement is imprecise measurement which might be useful for realistic model than just the precise (ideal) measurement.

3.4 Simple model for Decoherence

Let's study the decoherence in simple model[1] of two state system (with basis as $\ket{0}$ and $\ket{1}$) and environment as N two level states as $\ket{u}_i$ and $\ket{d}_i$ which forms basis of $\mathcal{E}$. The total system environment combination by 2^{N+1} dimensional tensor product Hilbert space. Assuming that the interaction Hamiltonian H_{int} governing the interaction between the system and the environment completely dominates the evolution of system with the form given by[3],

$$H_{int} = (\ket{1}\bra{1} - \ket{0}\bra{0}) \otimes$$
$$\left[\sum_{i=1}^{N} g_i(\ket{u_i}\bra{u_i} - \ket{d_i}\bra{d_i}) \otimes_{i' \neq i} I_{i'} \right]. \tag{42}$$

Here $I_i = \ket{u_i}\bra{u_i} + \ket{d_i}\bra{d_i}$ denotes the Identity operator for i^{th} states. The initial state

$$\ket{\psi(0)} = (a\ket{1} + b\ket{0}) \otimes_{i=1}^{N} (\alpha_i \ket{u}_i + \beta_i \ket{u}_i) \tag{43}$$

under time evolution with H_{int}, it evolves to

$$\ket{\psi(t)} = a\ket{1}\ket{e_u(t)} + b\ket{0}\ket{e_d(t)} \tag{44}$$

where

$$\ket{e_u(t)} = \ket{e_d(-t)} = \otimes_{k=1}^{N}(\alpha_k e^{ig_k t}\ket{u}_k + \beta_k e^{-ig_k t}\ket{d}_k). \tag{45}$$

The reduced density matrix by tracing out the $\mathcal{E}$ states is

$$\rho_{\mathcal{S}}(t) = Tr_{\mathcal{E}}(|\psi(t)\rangle \langle \psi(t)|)$$

$$= |a|^2 |1\rangle \langle 1| + |b|^2 |0\rangle \langle 0| + z(t)ab^* |1\rangle \langle 0| + z^*(t)a^*b |0\rangle \langle 1| \quad (46)$$

where

$$z(t) = \langle e_u(t)|e_u(t)\rangle = \Pi_{k=1}^{N}(|\alpha_k|^2 e^{ig_k t} + |\beta_k|^2 e^{-ig_k t}). \quad (47)$$

This gives

$$|z(t)|^2 = \Pi_{k=1}^{N}\{1 + [(|\alpha_k|^2 - |\beta_k|^2)^2 - 1]sin^2 2g_k t\}. \quad (48)$$

We have z(0) =1. For those initial environment states which make the square bracket in Eq.(48) vanish for each k, z(t) =1 for all t and the interference terms between $|1\rangle$ and $|0\rangle$ in Eq.(46) remain present for all times. For generic environments, however, these terms are generally nonzero. Note that

$$|z(t)| = | \langle e_u(t)|e_d(t)\rangle | \leq 1. \quad (49)$$

$$\langle z(t)\rangle \equiv lim_{T\to\infty}T^{-1} \int_{t-T/2}^{t+T/2} z(t')dt' = 0. \quad (50)$$

$$\langle |z(t)|^2\rangle = 2^{-N}\Pi_{k=1}^{N}[1 + (|\alpha_k|^2 - |\beta_k|^2)^2] \quad (51)$$

$$\langle |z(t)|^2\rangle \to 0 \quad as \quad N \to \infty. \quad (52)$$

This shows that, for large generic environments, the interference terms in Eq.(46) are strongly suppressed.

3.5   Quantum Brownian Motion

Quantum Brownian motion constitutes the most important and most studied decoherence model for its success in estimating decoherence effects especially on macroscopic scales. Being a classic and extensively studied model of decoherence and dissipation is the one-dimensional motion of a particle linearly coupled to a thermal bath of non-interacting harmonic oscillators in thermal equilibrium

at a temperature T.

The Hamiltonian of the total system consisting of a system $\mathcal{S}$ of harmonic oscillator of mass M and frequency Ω interacting with the environment (bath) of N harmonic oscillators of masses m and frequencies ω_i in an equilibrium state at a finite temperature T can be written as,

$$H_{\text{tot}} = H_{\mathcal{S}} + H_{\mathcal{E}} + H_{\text{int}} \tag{53}$$

where

$$H_{\text{sys}} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega_S^2\hat{X}^2 \tag{54}$$

is the system Hamiltonian with frequency ω_S,

$$H_{\mathcal{E}} = \sum_i \frac{\hat{p}_i^2}{2m} + \frac{1}{2}m\omega_i^2\hat{q}_i^2 \tag{55}$$

where m and ω_i denote the mass and natural frequency of the ith oscillator, and q_i and p_i are the canonical position and momentum operators, and

$$H_{\text{int}} = \lambda \sum_i \hat{X} \otimes \hat{q}_i \tag{56}$$

describes the bilinear coupling of the system's position coordinate X to the positions q_i of the environmental oscillators and λ denote coupling strength. This interaction represents the continuous environmental monitoring of the position coordinate of the system[1].

Now our primary focus is to derive the dynamics of the reduced density matrix under this Total Hamiltonian. We now consider the dynamics of system $\mathcal{S}$ interacting with a heat bath. The density matrix ρ evolves in time under the unitary operator:

$$\rho(t) = \exp\left[-i\frac{H_{\text{tot}}t}{\hbar}\right]\rho(0)\exp\left[i\frac{H_{\text{tot}}t}{\hbar}\right]. \tag{57}$$

If we are interested in the detailed dynamics of the system but only the coarse-grained effect of the bath we can work with the reduced density matrix obtained by tracing the density matrix of the total system described by ρ_{tot}, over the bath variables

$$\rho_r = \text{Tr}_{\text{bath}}\rho(t). \tag{58}$$

This traced out density matrix is referred as reduced Density matrix and the dynamics of the ρ_r has been solved exactly [1] using the path integral techniques[6].

The Born–Markov master equation describing the evolution of the density matrix $\rho_S(t)$ of the system is given by[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\big[H_S + \tfrac{1}{2}M\widetilde{\Omega}^2 X^2, \rho_S(t)\big] - i\gamma\big[X, \{P, \rho_S(t)\}\big]$$
$$- D\big[X, [X, \rho_S(t)]\big] - f\big[X, [P, \rho_S(t)]\big]. \quad (59)$$

The coefficients $\widetilde{\Omega}^2$, γ, D, and f are defined as

$$\widetilde{\Omega}^2 \equiv -\frac{2}{M}\int_0^\infty \mathrm{d}\tau\, \eta(\tau)\cos(\Omega\tau), \qquad (60\mathrm{a})$$

$$\gamma \equiv \frac{1}{M\Omega}\int_0^\infty \mathrm{d}\tau\, \eta(\tau)\sin(\Omega\tau), \qquad (60\mathrm{b})$$

$$D \equiv \int_0^\infty \mathrm{d}\tau\, \nu(\tau)\cos(\Omega\tau), \qquad (60\mathrm{c})$$

$$f \equiv -\frac{1}{M\Omega}\int_0^\infty \mathrm{d}\tau\, \nu(\tau)\sin(\Omega\tau). \qquad (60\mathrm{d})$$

By the master equation above one can determine the dynamics of the reduced density matrix and check the diagonalization of reduced density matrix in both position and momentum basis.

From[1] the interpretation of the different terms appearing in the master equation is given below. The first term in Eqn(59)

$$-i\big[H_S + \tfrac{1}{2}M\widetilde{\Omega}^2 X^2, \rho_S(t)\big] \qquad (61)$$

describes the usual unitary dynamics of a with frequency shifted harmonic oscillator.

The next term,

$$-i\gamma\big[X, \{P, \rho_S(t)\}\big] \qquad (62)$$

describes momentum damping—and thus dissipation—due to the interaction with the environment[1]. The term describes momentum damping at a rate proportional to γ.

The third term,

$$-D\big[X, \big[X, \rho_S(t)\big]\big] \tag{63}$$

describes the environmental monitoring of the position coordinate $\hat{X}$ of the system and thus decoherence in the position basis. In position representation we have,

$$\frac{\partial}{\partial t}\rho(X, X', t) = -D(X - X')^2 \rho(X, X', t) \tag{64}$$

$$\rho(X, X', t) = \rho(X, X', 0)e^{-D(X-X')^2 t} \tag{65}$$

implies that the off-diagonal terms vanishes rapidly, causing decoherence in the position basis. From the results quoted from [1] we conclude that it has decohered in the position basis as it has become diagonal in that basis for this type of interaction.

Under the limit of a high-temperature environment, the master equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\big[H'_S, \rho_S(t)\big] - i\gamma\big[X, \{P, \rho_S(t)\}\big]$$
$$- 2M\gamma k_b T\big[X, \big[X, \rho_S(t)\big]\big]. \tag{66}$$

Under the limit of a high-temperature environment, Caldeira–Leggett model Master Equation for ρ_S in position basis is

$$\frac{\partial}{\partial t}\rho(X, X', t) = \left[-i\left\{\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial X'^2} - \Omega^2(X^2 - X'^2)\right\}\right.$$
$$\left. -i\gamma(X - X')(\frac{\partial}{\partial X} - \frac{\partial}{\partial X'}) - 2M\gamma k_b T(X - X')^2\right]\rho(X, X', t) \tag{67}$$

3.6 Quantum Brownian Motion with momentum and position coupling

In the usual Quantum Brownian motion model we modified the interaction terms to include the momentum and position coupling of the system to the environment of the form,

$$H_{\text{int}} = (\lambda \hat{X} - \mu \hat{P}) \sum_i \hat{q}_i \tag{68}$$

In terms of these new variables the Hamiltonian H_{tot} takes the following form:

$$H_{\text{tot}} = \frac{\hat{P}^2}{2M} + \frac{1}{2} M \Omega_S^2 \hat{X}^2 + \sum_i \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega_i^2 \hat{q}_i^2$$
$$+ (\hat{X} - \mu \hat{P}) \sum_i \lambda \hat{q}_i \tag{69}$$

Lets go back to Hamiltonian of the form given by Quantum Brownian motion with linear interaction which is already solved in the literature [6].

$$H_{\text{tot}} = \frac{\hat{P}'^2}{2M'} + \frac{1}{2} M' \tilde{\omega}^2 \hat{X}'^2 + \sum_i \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m \omega_i^2 \hat{q}_i^2$$
$$+ \kappa \sum_i \hat{X}' \otimes \hat{q}_i \tag{70}$$

Under the canonical transformation given by

$$\hat{X}' \mapsto \hat{X} - \mu \hat{P} \tag{71}$$

$$\hat{P}' \mapsto \alpha \hat{X} + \beta \hat{P} \tag{72}$$

the H_{tot} in Eqn(70) transform to the form in Eqn(69) by preserving canonical commutations.

Tuning different coupling constants with environment will fix the coefficients λ, μ and the canonical commutation fixing the values of coefficients α, β is given by,

$$[X, P] = i\hbar, \qquad [x_i, p_j] = i\hbar \delta_{\text{ij}} \tag{73}$$

$$[P, x_i] = [p_i, X] = [X, x_i] = [P, p_i] = 0. \tag{74}$$

Similarly under same canonical transformation of QBM master equation (Eqn.66) would result in modified Born–Markov master equation (high temperature limit) with position and momentum coupling,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S(t) = -i\big[H_S' + \frac{1}{2}M'\widetilde{\Omega}^2 X^2, \rho_S(t)\big] - i\gamma\big(\beta\big[X, \{P, \rho_S(t)\}\big]$$
$$- \mu\alpha\big[P, \{X, \rho_S(t)\}\big] + \alpha\big[X^2, \rho_S(t)\big] - \mu\beta\big[P^2, \rho_S(t)\big]\big)$$
$$- D\big(\big[X, [X, \rho_S(t)]\big] + \mu^2\big[P, [P, \rho_S(t)]\big]\big)$$
$$+ D\big(\mu\big[X, [P, \rho_S(t)]\big] - \mu\big[P, [X, \rho_S(t)]\big]\big) \quad (75)$$

where $D = 2M\gamma k_b T$.

Lets interpret the different terms in Master equation(75). We can observe that the equation is symmetric in X and P (i.e treating X and P on equal footing) compared to Eqn(66). The first term describes the usual unitary dynamics of a with frequency shifted harmonic oscillator. the next term contains,

$$-i\gamma\big(\beta\big[X, \{P, \rho_S(t)\}\big] - \mu\alpha\big[P, \{X, \rho_S(t)\}\big]\big) \quad (76)$$

which describes both momentum damping and position damping—and thus dissipation—due to both interaction with the environment. The decoherence term,

$$-D\big(\big[X, [X, \rho_S(t)]\big] + \mu^2\big[P, [P, \rho_S(t)]\big]\big) \quad (77)$$

In position basis the term,

$$-D\big[X, [X, \rho_S(t)]\big] \longrightarrow -D(X - X')^2 \rho(X, X', t) \quad (78)$$

and other term will contain derivatives in X. So term in Eqn(78) causes decoherence in the position basis. Similarly in momentum basis,

$$-\mu^2 D\big[P, [P, \rho_S(t)]\big] \longrightarrow -\mu^2 D(P - P')^2 \rho(P, P', t) \quad (79)$$

thus causing decoherence in momentum basis.

Remark: If we take $\mu = 1$ then both coupling of position and momentum with environment are equal which implies that Decoherence rates to be same and it can be observed from Eqn(78,79).

4 Conclusion

Decoherence is a fundamental process that reveals the limits of quantum mechanics. Understanding open quantum systems is currently of great interest due to its implications for emerging fields such as quantum computing, quantum optics etc. Having made the efforts to study the foundation of decoherence through literature reviews led me to realize its importance in answering the basics of quantum mechanics. By having these foundations, I have analysed the Quantum Brownian motion(QBM) model and its interpretation of the master equation. Further we modified the QBM model that incorporate momentum coupling as well and solved for the master equation. The resulting master equation is symmetric in position(X) and momentum(P) (i.e treating X and P on equal footing). From the analysis on Eqn(78) and Eqn(79) it seems to have decohered (at different rates for different coupling constants) in both X and P basis. However, it is worth noting that we have yet to examine the Decoherence effects from the remaining terms. Exploring these effects may provide valuable insights and could be a potential area of focus for future research endeavors.

4.1 Future directions

- **Master Equation:** The detailed analysis of the various terms in the master equation contributing to decoherence and other effects is potential area to focus on.

- **Study on emergence of classical trajectory:** After deriving the exact dynamics of the reduced Density matrix, we can investigate how the classical trajectories would emerge after decoherence.

- **Study on Wigner function:** The emergence of classicality can be well understood by studying the Wigner function and its dynamics.

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