

Relaxed Candidate Set Models for Masked Data in Series Systems

Alex Towell
`alex@metafunctor.com`

December 3, 2025

Abstract

We develop likelihood-based inference methods for series systems with masked failure data when the traditional conditions governing candidate set formation are relaxed. While existing methods require that masking be non-informative (Condition C2) and parameter-independent (Condition C3), practical diagnostic systems often violate these assumptions. We derive the likelihood under various relaxation scenarios, establish identifiability conditions, and compare Fisher information under standard versus relaxed models. Our analysis reveals that informative masking, when properly modeled, can paradoxically improve estimation efficiency by providing additional information about the failed component. We implement these methods in an R package and provide simulation studies demonstrating the bias that arises from incorrectly assuming standard conditions when masking is informative.

Keywords: Series systems, masked data, reliability estimation, informative masking, candidate sets, Fisher information, maximum likelihood estimation

1 Introduction

Estimating the reliability of individual components in a series system presents a fundamental challenge in reliability engineering: system-level failure data is observable, but component-level failure causes are often masked. When a series system fails, diagnostic procedures may identify only a *candidate set* of components that could have caused the failure, rather than pinpointing the exact failed component [7]. This masking, combined with right-censoring of system lifetimes, complicates statistical inference about component reliability parameters.

Prior work on masked data in series systems has established a tractable likelihood framework under three conditions [6]:

- **C1:** The failed component is always contained in the candidate set.
- **C2:** Masking is non-informative—conditional on the failure time and candidate set, each component in the candidate set is equally likely to have failed.
- **C3:** The masking mechanism does not depend on the system parameters θ .

Under these conditions, the likelihood has a simple closed form that enables efficient maximum likelihood estimation (MLE). However, practical diagnostic systems may violate C2 and C3:

- Experienced technicians may preferentially include components that “seem likely” to have failed based on failure time characteristics, violating C2.

- Diagnostic algorithms based on reliability rankings may systematically favor certain components, with the ranking depending on the true parameters, violating C3.

This paper develops the theoretical and computational framework for likelihood-based inference when C2 and/or C3 are relaxed while maintaining C1. Our contributions are:

1. **General likelihood framework.** We derive the likelihood under C1 alone, showing how informative and parameter-dependent masking modify the standard likelihood structure (Section 3).
2. **Practical masking models.** We introduce the rank-based informative masking model and the Bernoulli candidate set model with KL-divergence constraints, which provide interpretable parameterizations of non-standard masking (Section 3).
3. **Identifiability analysis.** We establish conditions under which parameters remain identifiable when standard conditions fail, including the surprising result that informative masking can *improve* identifiability in certain cases (Section 4).
4. **Efficiency comparison.** We derive the Fisher information matrix under relaxed conditions for exponential series systems, enabling precise comparison of estimation efficiency (Section 4).
5. **Simulation studies.** We quantify the bias from incorrectly assuming C2 when masking is informative, and demonstrate the improved estimation achievable when the masking structure is properly modeled (Section 5).

The remainder of this paper is organized as follows. Section 2 reviews series systems, masked data, and the standard C1-C2-C3 likelihood. Section 3 develops the likelihood under relaxed conditions. Section 4 analyzes identifiability and Fisher information. Section 5 presents simulation studies. Section 6 discusses practical implications, and Section 7 concludes.

2 Background

2.1 Series System Model

Consider a series system composed of m components. The lifetime of the i -th system is

$$T_i = \min\{T_{i1}, T_{i2}, \dots, T_{im}\}, \quad (1)$$

where T_{ij} denotes the lifetime of the j -th component in the i -th system. Component lifetimes are assumed independent with parametric distributions indexed by $\boldsymbol{\theta}_j$; the full parameter vector is $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_m) \in \Omega$.

Definition 2.1 (Component Distribution Functions). *For component j with parameter $\boldsymbol{\theta}_j$:*

$$R_j(t; \boldsymbol{\theta}_j) = \Pr\{T_{ij} > t\} \quad (\text{reliability function}), \quad (2)$$

$$f_j(t; \boldsymbol{\theta}_j) = -\frac{d}{dt}R_j(t; \boldsymbol{\theta}_j) \quad (\text{density function}), \quad (3)$$

$$h_j(t; \boldsymbol{\theta}_j) = \frac{f_j(t; \boldsymbol{\theta}_j)}{R_j(t; \boldsymbol{\theta}_j)} \quad (\text{hazard function}). \quad (4)$$

For the series system, these functions combine as follows:

Theorem 2.2 (Series System Distribution Functions). *The series system has:*

$$R_{T_i}(t; \boldsymbol{\theta}) = \prod_{j=1}^m R_j(t; \boldsymbol{\theta}_j), \quad (5)$$

$$h_{T_i}(t; \boldsymbol{\theta}) = \sum_{j=1}^m h_j(t; \boldsymbol{\theta}_j), \quad (6)$$

$$f_{T_i}(t; \boldsymbol{\theta}) = h_{T_i}(t; \boldsymbol{\theta}) \cdot R_{T_i}(t; \boldsymbol{\theta}). \quad (7)$$

The proof follows from the independence of component lifetimes and standard arguments [6].

2.2 Component Cause of Failure

Let $K_i \in \{1, \dots, m\}$ denote the index of the component that caused system i to fail. Since the system fails when the first component fails, $K_i = \arg \min_j T_{ij}$.

Theorem 2.3 (Joint Distribution of (T_i, K_i)). *The joint distribution of system lifetime and component cause of failure is:*

$$f_{T_i, K_i}(t, k; \boldsymbol{\theta}) = h_k(t; \boldsymbol{\theta}_k) \prod_{\ell=1}^m R_\ell(t; \boldsymbol{\theta}_\ell). \quad (8)$$

Corollary 2.4 (Conditional Failure Probability). *Given that the system failed at time t , the probability that component j caused the failure is:*

$$\Pr\{K_i = j \mid T_i = t\} = \frac{h_j(t; \boldsymbol{\theta}_j)}{\sum_{\ell=1}^m h_\ell(t; \boldsymbol{\theta}_\ell)}. \quad (9)$$

This probability plays a central role in masked data analysis, as it represents the “true” probability that each component failed, which the masking mechanism partially obscures.

2.3 Masked Data Structure

Definition 2.5 (Observed Data). *For each system i , we observe:*

- $S_i = \min\{T_i, \tau_i\}$: Right-censored system lifetime,
- $\delta_i = \mathbf{1}_{T_i \leq \tau_i}$: Event indicator (1 if failure observed, 0 if censored),
- $C_i \subseteq \{1, \dots, m\}$: Candidate set (only observed when $\delta_i = 1$).

The latent (unobserved) variables are:

- $K_i \in \{1, \dots, m\}$: Index of failed component,
- (T_{i1}, \dots, T_{im}) : Component failure times.

2.4 Traditional Conditions C1, C2, C3

The existing literature [4, 7, 8] establishes tractable likelihood-based inference under three conditions:

Condition 1 (C1: Failed Component in Candidate Set). *The candidate set always contains the failed component:*

$$\Pr\{K_i \in C_i\} = 1. \quad (10)$$

Condition 2 (C2: Non-Informative Masking). *Given the failure time and that the failed component is in a candidate set c , the probability of observing c does not depend on which component in c failed:*

$$\Pr\{C_i = c \mid T_i = t, K_i = j\} = \Pr\{C_i = c \mid T_i = t, K_i = j'\} \quad (11)$$

for all $j, j' \in c$.

Condition 3 (C3: Parameter-Independent Masking). *The masking probabilities do not depend on the system parameters:*

$$\Pr\{C_i = c \mid T_i = t, K_i = j\} = \beta_i(c, t, j), \quad (12)$$

where β_i does not depend on $\boldsymbol{\theta}$.

2.5 Likelihood Under C1, C2, C3

Under all three conditions, the likelihood admits a tractable form:

Theorem 2.6 (Likelihood Under C1-C2-C3). *Under Conditions C1, C2, and C3, the likelihood contribution from an uncensored observation (s_i, c_i) is proportional to:*

$$L_i(\boldsymbol{\theta}) \propto \prod_{\ell=1}^m R_\ell(s_i; \boldsymbol{\theta}_\ell) \cdot \sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k). \quad (13)$$

For a censored observation with lifetime s_i :

$$L_i(\boldsymbol{\theta}) = \prod_{\ell=1}^m R_\ell(s_i; \boldsymbol{\theta}_\ell). \quad (14)$$

Proof. By the chain rule:

$$f_{T_i, C_i}(t, c; \boldsymbol{\theta}) = \sum_{k=1}^m f_{T_i, K_i}(t, k; \boldsymbol{\theta}) \cdot \Pr_{\boldsymbol{\theta}}\{C_i = c \mid T_i = t, K_i = k\}. \quad (15)$$

Under C1, terms with $k \notin c$ vanish. Under C2, the masking probability is constant over $k \in c$, so:

$$f_{T_i, C_i}(t, c; \boldsymbol{\theta}) = \Pr\{C_i = c \mid T_i = t, K_i \in c\} \cdot \prod_{\ell=1}^m R_\ell(t; \boldsymbol{\theta}_\ell) \sum_{k \in c} h_k(t; \boldsymbol{\theta}_k). \quad (16)$$

Under C3, the masking probability does not depend on $\boldsymbol{\theta}$, yielding the proportionality in (13). \square

The complete log-likelihood for n independent systems is:

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^n \left[\sum_{j=1}^m \log R_j(s_i; \boldsymbol{\theta}_j) + \delta_i \log \left(\sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k) \right) \right]. \quad (17)$$

2.6 Related Work

The masked data problem in series systems was introduced by Usher and Hodgson [7], who developed MLE methods for exponential components. Usher et al. [8] extended this to Weibull components with exact maximum likelihood. Guo et al. [4] (Guo et al.) provided simulation studies validating the approach under various masking scenarios.

The informative censoring literature in survival analysis [3, 5] addresses related issues where the censoring mechanism depends on covariates or outcomes. However, the candidate set structure in masked data creates a distinct problem not fully addressed by standard informative censoring methods.

The competing risks framework [1] provides another perspective, viewing component failures as competing causes of system failure. However, standard competing risks methods assume the cause is observed, whereas masked data only provides partial information through candidate sets.

Our work extends the C1-C2-C3 framework by explicitly modeling departures from C2 and C3, providing both theoretical analysis and practical estimation methods.

3 Relaxed Candidate Set Models

We now develop the likelihood framework when conditions C2 and/or C3 are relaxed while maintaining C1. The key insight is that the general likelihood structure remains tractable—it simply requires modeling the masking mechanism explicitly rather than treating it as a nuisance.

3.1 General Likelihood Under C1

Theorem 3.1 (Likelihood Under C1 Alone). *Under Condition C1 alone, the likelihood contribution from an uncensored observation (s_i, c_i) is:*

$$L_i(\boldsymbol{\theta}) = \prod_{\ell=1}^m R_\ell(s_i; \boldsymbol{\theta}_\ell) \cdot \sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k) \cdot \Pr_{\boldsymbol{\theta}}\{C_i = c_i \mid T_i = s_i, K_i = k\}. \quad (18)$$

Proof. Under C1, $\Pr_{\boldsymbol{\theta}}\{C_i = c \mid T_i = t, K_i = k\} = 0$ when $k \notin c$. Therefore, summing over K_i :

$$\begin{aligned} f_{T_i, C_i}(t, c; \boldsymbol{\theta}) &= \sum_{k=1}^m h_k(t; \boldsymbol{\theta}_k) \prod_{\ell=1}^m R_\ell(t; \boldsymbol{\theta}_\ell) \cdot \Pr_{\boldsymbol{\theta}}\{C_i = c \mid T_i = t, K_i = k\} \\ &= \prod_{\ell=1}^m R_\ell(t; \boldsymbol{\theta}_\ell) \cdot \sum_{k \in c} h_k(t; \boldsymbol{\theta}_k) \Pr_{\boldsymbol{\theta}}\{C_i = c \mid T_i = t, K_i = k\}. \end{aligned} \quad (19) \quad \square$$

Remark 3.1 (Comparison with C1-C2-C3). Under C2, the masking probability can be factored out of the sum since it is constant over $k \in c$. Under C3, it can be dropped since it does not depend on $\boldsymbol{\theta}$. When either condition fails, the masking probabilities remain inside the sum and may depend on both k and $\boldsymbol{\theta}$, fundamentally changing the inference problem.

3.2 Relaxing C2: Informative Masking

When C2 is violated but C1 and C3 hold, the masking probability $\Pr\{C_i = c \mid T_i = t, K_i = k\}$ can vary with $k \in c$.

Definition 3.2 (Informative Masking). *Let $\pi_{kc}(t) = \Pr\{C_i = c \mid T_i = t, K_i = k\}$ for $k \in c$. The masking is informative if $\pi_{kc}(t)$ varies with k .*

Theorem 3.3 (Likelihood Under C1 and C3 (Relaxed C2)). *Under C1 and C3, the likelihood contribution is:*

$$L_i(\boldsymbol{\theta}) = \prod_{\ell=1}^m R_\ell(s_i; \boldsymbol{\theta}_\ell) \cdot \sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k) \cdot \pi_{k,c_i}(s_i), \quad (20)$$

where $\pi_{k,c}(t)$ does not depend on $\boldsymbol{\theta}$ (by C3).

When $\pi_{kc}(t)$ is known, the likelihood remains tractable. The masking probabilities act as weights on the hazard contributions from each candidate.

3.2.1 Rank-Based Informative Masking

A practical model for informative masking assigns inclusion probabilities based on component failure ranks rather than absolute times.

Definition 3.4 (Rank-Based Masking). *Let $r_k(t) \in \{1, \dots, m\}$ denote the rank of component k 's failure time among (t_1, \dots, t_m) , where rank 1 corresponds to the earliest failure (the actual failed component).*

The probability that component j is in the candidate set is:

$$q_j = \begin{cases} 1 & \text{if } r_j = 1 \text{ (failed component),} \\ \beta \exp(-\alpha(r_j - 2)) & \text{if } r_j \geq 2, \end{cases} \quad (21)$$

where $\alpha \geq 0$ controls the decay rate and $\beta \in [0, 1]$ is the maximum inclusion probability for non-failed components.

Remark 3.2 (Limiting Behavior). • As $\alpha \rightarrow 0$: All non-failed components have probability β (uninformative within the non-failed set).

• As $\alpha \rightarrow \infty$: Only the failed component and rank-2 component have non-zero probabilities.

This model captures the intuition that components failing “nearly at the same time” as the actual failure are more likely to be included in the candidate set.

3.2.2 Independent Bernoulli Candidate Set Model

A flexible model assumes independent component inclusion:

Definition 3.5 (Independent Bernoulli Model). *Each component j is included in the candidate set independently with probability q_j , subject to C1 (failed component always included):*

$$\Pr\{j \in C_i \mid T_i = t, K_i = k\} = \begin{cases} 1 & \text{if } j = k, \\ q_j & \text{otherwise.} \end{cases} \quad (22)$$

Under this model, the probability of observing candidate set c given $(T_i = t, K_i = k)$ is:

$$\Pr\{C_i = c \mid T_i = t, K_i = k\} = \mathbf{1}_{k \in c} \prod_{j \in c \setminus \{k\}} q_j \prod_{j \notin c} (1 - q_j). \quad (23)$$

Proposition 3.6 (Likelihood Under Bernoulli Model). *Under the independent Bernoulli model with C1 and known probabilities (q_1, \dots, q_m) , the likelihood contribution is:*

$$L_i(\boldsymbol{\theta}) = \prod_{\ell=1}^m R_\ell(s_i; \boldsymbol{\theta}_\ell) \cdot \sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k) w_k(c_i), \quad (24)$$

where

$$w_k(c) = \prod_{j \in c \setminus \{k\}} q_j \prod_{j \notin c} (1 - q_j) \quad (25)$$

is the probability of observing c given that k failed (excluding the deterministic inclusion of k).

3.2.3 KL-Divergence Constrained Models

To systematically study deviations from the standard C1-C2-C3 model, we can parameterize informative masking by its distance from the baseline:

Definition 3.7 (KL-Divergence from Baseline). *Let $P = (p, \dots, p, 1, p, \dots, p)$ denote the baseline Bernoulli model satisfying C1-C2-C3, where the failed component has probability 1 and all others have probability p .*

For a given target KL-divergence $d \geq 0$, we seek a masking probability vector $Q = (q_1, \dots, q_m)$ satisfying:

1. $q_k = 1$ for the failed component (C1),
2. $\text{KL}(P \| Q) \approx d$,
3. $\sum_j q_j = \sum_j p_j$ (same expected candidate set size).

When $d = 0$, we recover $Q = P$ (the C1-C2-C3 model). As d increases, Q becomes more informative about which component failed. This provides a controlled framework for studying the effects of departures from C2.

3.3 Relaxing C3: Parameter-Dependent Masking

When C3 is violated, the masking probability $\Pr_{\boldsymbol{\theta}}\{C_i = c \mid T_i = t, K_i = k\}$ depends on $\boldsymbol{\theta}$.

Theorem 3.8 (Likelihood Under C1 and C2 (Relaxed C3)). *Under C1 and C2, the likelihood contribution is:*

$$L_i(\boldsymbol{\theta}) = \pi_{c_i}(s_i; \boldsymbol{\theta}) \cdot \prod_{\ell=1}^m R_\ell(s_i; \boldsymbol{\theta}_\ell) \cdot \sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k), \quad (26)$$

where $\pi_c(t; \boldsymbol{\theta}) = \Pr_{\boldsymbol{\theta}}\{C_i = c \mid T_i = t, K_i \in c\}$ is the (common) masking probability for any $k \in c$, now depending on $\boldsymbol{\theta}$.

Proof. By C2, we can factor out the masking probability since it is constant over $k \in c$:

$$\begin{aligned} f_{T_i, C_i}(t, c; \boldsymbol{\theta}) &= \prod_{\ell=1}^m R_\ell(t; \boldsymbol{\theta}_\ell) \sum_{k \in c} h_k(t; \boldsymbol{\theta}_k) \Pr_{\boldsymbol{\theta}}\{C_i = c \mid T_i = t, K_i = k\} \\ &= \pi_c(t; \boldsymbol{\theta}) \prod_{\ell=1}^m R_\ell(t; \boldsymbol{\theta}_\ell) \sum_{k \in c} h_k(t; \boldsymbol{\theta}_k). \end{aligned} \quad (27)$$

Remark 3.3 (Nuisance Parameters). When $\pi_c(t; \boldsymbol{\theta})$ has a known functional form, it contributes to the likelihood and affects the MLE. If the form is unknown, additional modeling assumptions or profile likelihood approaches may be needed.

3.3.1 Failure-Probability-Weighted Masking

A natural way for masking to depend on $\boldsymbol{\theta}$ is through the conditional failure probabilities:

Definition 3.9 (Failure-Probability-Weighted Masking). *The probability that component j is in the candidate set depends on its posterior failure probability:*

$$\Pr_{\boldsymbol{\theta}}\{j \in C_i \mid T_i = t\} = g\left(\frac{h_j(t; \boldsymbol{\theta}_j)}{\sum_{\ell=1}^m h_\ell(t; \boldsymbol{\theta}_\ell)}\right) \quad (28)$$

for some function $g : [0, 1] \rightarrow [0, 1]$ with $g(x) \rightarrow 1$ as $x \rightarrow 1$.

This models diagnosticians who are more likely to include components with higher failure probabilities given the observed failure time. The function g controls the sensitivity of masking to these probabilities.

3.4 The General Case: Both C2 and C3 Relaxed

When both C2 and C3 are relaxed, the likelihood takes the fully general form from Theorem 3.1:

$$L_i(\boldsymbol{\theta}) = \prod_{\ell=1}^m R_\ell(s_i; \boldsymbol{\theta}_\ell) \cdot \sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k) \cdot \pi_{k,c_i}(s_i; \boldsymbol{\theta}). \quad (29)$$

Estimation in this general case requires either:

1. A fully specified parametric model for $\pi_{k,c}(t; \boldsymbol{\theta})$, or
2. Sensitivity analysis over plausible masking mechanisms, or
3. Nonparametric or semiparametric approaches that avoid specifying the masking mechanism.

In practice, the most common scenario is relaxed C2 with C3 maintained (informative but parameter-independent masking), which we focus on in the simulation studies.

4 Identifiability and Fisher Information

We now analyze identifiability conditions and derive the Fisher information matrix under relaxed conditions, focusing on exponential series systems for tractability.

4.1 Identifiability Under C1-C2-C3

Definition 4.1 (Identifiability). *A parameter $\boldsymbol{\theta}$ is identifiable if for any $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Omega$ with $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$, there exists some data configuration D such that*

$$L(\boldsymbol{\theta}; D) \neq L(\boldsymbol{\theta}'; D). \quad (30)$$

Theorem 4.2 (Identifiability Under C1-C2-C3). *Under C1, C2, and C3, the parameter $\boldsymbol{\theta}$ is identifiable if and only if the following condition holds: For each pair of components $j \neq j'$, there exists at least one observed candidate set c such that exactly one of $j \in c$ or $j' \in c$ holds (i.e., the components do not always co-occur in candidate sets).*

Proof. The log-likelihood contribution from an uncensored observation is:

$$\ell_i(\boldsymbol{\theta}) = \sum_{\ell=1}^m \log R_\ell(s_i; \boldsymbol{\theta}_\ell) + \log \left(\sum_{k \in c_i} h_k(s_i; \boldsymbol{\theta}_k) \right). \quad (31)$$

Sufficiency: If components j and j' appear in different candidate sets, then information about h_j can be obtained from observations where $j \in c_i$ but $j' \notin c_i$, and vice versa. Combined with the survival term (which depends on all parameters), this provides sufficient variation to identify individual parameters.

Necessity: If components j and j' always co-occur in every candidate set, the hazard sum always contains $h_j + h_{j'}$ as an inseparable unit. The survival term provides information only about $\sum_\ell \lambda_\ell$. Thus, any reparametrization preserving both $h_j + h_{j'}$ and $\sum_\ell h_\ell$ yields the same likelihood, demonstrating non-identifiability. \square

4.2 Block Non-Identifiability

A particularly important case arises when components form blocks that always appear together:

Theorem 4.3 (Block Non-Identifiability). *Suppose components are partitioned into blocks B_1, \dots, B_r such that for every observed candidate set c_i :*

- (i) *For each block B_ℓ : either $B_\ell \subseteq c_i$ or $B_\ell \cap c_i = \emptyset$, and*
- (ii) *If the failed component $k \in B_\ell$, then $B_\ell \subseteq c_i$.*

Then for exponential components with rates $(\lambda_1, \dots, \lambda_m)$, only the block sums $\Lambda_\ell = \sum_{j \in B_\ell} \lambda_j$ are identifiable.

Proof. Under the exponential model with constant hazards, the likelihood becomes:

$$L(\boldsymbol{\theta}) \propto \prod_{i: \delta_i=1} \exp \left(-s_i \sum_{j=1}^m \lambda_j \right) \sum_{k \in c_i} \lambda_k. \quad (32)$$

The survival term depends only on $\sum_{j=1}^m \lambda_j$. For the hazard sum, under the block structure, each candidate set c_i is a union of complete blocks. Thus:

$$\sum_{k \in c_i} \lambda_k = \sum_{\ell: B_\ell \subseteq c_i} \Lambda_\ell. \quad (33)$$

Any reparametrization that preserves $(\Lambda_1, \dots, \Lambda_r)$ yields the same likelihood, hence individual λ_j within blocks are not identifiable. \square

Example 4.1 (Three-Component Block Model). Consider a 3-component system where the diagnostic tool can only distinguish:

- Components 1 and 2 share a circuit board (block $B_1 = \{1, 2\}$),
- Component 3 is separate (block $B_2 = \{3\}$).

Candidate sets are either $\{1, 2\}$, $\{3\}$, or $\{1, 2, 3\}$. The MLE satisfies:

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda_1 + \lambda_2, \quad (34)$$

$$\hat{\lambda}_3 = \lambda_3, \quad (35)$$

but individual $\hat{\lambda}_1, \hat{\lambda}_2$ are not unique.

4.3 Improved Identifiability with Informative Masking

Surprisingly, relaxing C2 can *improve* identifiability:

Theorem 4.4 (Improved Identifiability with Informative Masking). *Under C1 and C3 with known informative masking probabilities $\pi_{kc}(t)$, identifiability can be improved relative to the C1-C2-C3 case. Specifically, if components k and k' always co-occur in candidate sets (violating the identifiability condition of Theorem 4.2), they become identifiable if there exists a candidate set c with $k, k' \in c$ such that $\pi_{kc}(t) \neq \pi_{k'c}(t)$ for some $t > 0$.*

Proof. Under C1-C2-C3 with non-informative masking, if components k and k' always co-occur, the hazard sum contains only the unweighted sum $h_k + h_{k'}$, making individual hazards non-identifiable.

Under informative masking with known weights π_{kc} , the likelihood contribution becomes:

$$L_i(\boldsymbol{\theta}) = R(s_i; \boldsymbol{\theta}) \sum_{j \in c_i} h_j(s_i; \boldsymbol{\theta}_j) \pi_{j,c_i}(s_i). \quad (36)$$

The hazard sum now involves the weighted combination $h_k \pi_{kc} + h_{k'} \pi_{k'c}$. If $\pi_{kc} \neq \pi_{k'c}$, this provides one equation involving h_k and $h_{k'}$ with unequal coefficients.

Combined with the survival term (which contributes $h_k + h_{k'}$ through the system hazard), we have two linearly independent equations:

$$\pi_{kc} h_k + \pi_{k'c} h_{k'} = A_1 \quad (\text{from hazard sum}), \quad (37)$$

$$h_k + h_{k'} = A_2 \quad (\text{from survival term}). \quad (38)$$

When $\pi_{kc} \neq \pi_{k'c}$, this system has a unique solution, establishing identifiability of individual hazards. \square

Remark 4.1. Informative masking can paradoxically *help* estimation when the masking structure is known, because it provides additional information about which component likely failed.

4.4 Fisher Information for Exponential Series Systems

We now derive closed-form expressions for the Fisher information matrix, specializing to exponential components.

4.4.1 Exponential Series Model

For exponential components with rates $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$:

$$R_j(t; \lambda_j) = e^{-\lambda_j t}, \quad (39)$$

$$h_j(t; \lambda_j) = \lambda_j, \quad (40)$$

$$R_{T_i}(t; \boldsymbol{\lambda}) = e^{-\Lambda t}, \quad \text{where } \Lambda = \sum_{j=1}^m \lambda_j. \quad (41)$$

4.4.2 Fisher Information Under C1-C2-C3

Theorem 4.5 (FIM Under C1-C2-C3). *For the exponential series system under C1, C2, C3, the observed Fisher information matrix has elements:*

$$\mathcal{I}_{jk}(\boldsymbol{\lambda}) = \sum_{i: \delta_i=1} \frac{\mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}}{\left(\sum_{\ell \in c_i} \lambda_\ell \right)^2}. \quad (42)$$

Proof. The log-likelihood for an uncensored observation i is:

$$\ell_i(\boldsymbol{\lambda}) = -s_i \Lambda + \log \left(\sum_{k \in c_i} \lambda_k \right). \quad (43)$$

The first derivatives (score) are:

$$\frac{\partial \ell_i}{\partial \lambda_j} = -s_i + \frac{\mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k}. \quad (44)$$

The second derivatives are:

$$\frac{\partial^2 \ell_i}{\partial \lambda_j \partial \lambda_k} = -\frac{\mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}}{\left(\sum_{\ell \in c_i} \lambda_\ell \right)^2}. \quad (45)$$

The observed FIM is the negative Hessian, giving the result. \square

Remark 4.2. The FIM depends on the candidate sets but not on the failure times (for exponential components). This reflects the memoryless property of the exponential distribution.

4.4.3 Fisher Information Under Relaxed C2

Theorem 4.6 (FIM Under Informative Masking). *Under C1, C3, and informative masking with known weights π_{kc} , the observed Fisher information matrix for exponential components is:*

$$\mathcal{I}_{jk}(\boldsymbol{\lambda}) = \sum_{i: \delta_i=1} \frac{\pi_{j,c_i} \pi_{k,c_i}}{\left(\sum_{\ell \in c_i} \lambda_\ell \pi_{\ell,c_i} \right)^2} \mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}. \quad (46)$$

Proof. The log-likelihood contribution is:

$$\ell_i(\boldsymbol{\lambda}) = -s_i \Lambda + \log \left(\sum_{k \in c_i} \lambda_k \pi_{k,c_i} \right). \quad (47)$$

The score is:

$$\frac{\partial \ell_i}{\partial \lambda_j} = -s_i + \frac{\pi_{j,c_i} \mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k \pi_{k,c_i}}. \quad (48)$$

The Hessian is:

$$\frac{\partial^2 \ell_i}{\partial \lambda_j \partial \lambda_k} = -\frac{\pi_{j,c_i} \pi_{k,c_i} \mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}}{\left(\sum_{\ell \in c_i} \lambda_\ell \pi_{\ell,c_i} \right)^2}. \quad (49)$$

\square

4.5 Efficiency Comparison

Theorem 4.7 (Relative Efficiency). *Let $\mathcal{I}^{(C123)}$ and $\mathcal{I}^{(C13)}$ denote the Fisher information matrices under C1-C2-C3 and C1-C3 (informative masking) respectively. Then:*

- (a) *If $\pi_{kc} = 1/|c|$ for all $k \in c$ (uniform weighting), then $\mathcal{I}^{(C13)} = \mathcal{I}^{(C123)}$.*
- (b) *If masking is highly informative (concentrating weight on one component), $\mathcal{I}^{(C13)}$ can exceed $\mathcal{I}^{(C123)}$ for that component's parameter.*

Proof. (a) Under C1-C2-C3 (non-informative masking), the FIM element is:

$$\mathcal{I}_{jk}^{(C123)} = \sum_{i:\delta_i=1} \frac{\mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}}{\left(\sum_{\ell \in c_i} \lambda_\ell\right)^2}. \quad (50)$$

Under C1-C3 with uniform weights $\pi_{\ell c} = 1/|c|$ for all $\ell \in c$:

$$\mathcal{I}_{jk}^{(C13)} = \sum_{i:\delta_i=1} \frac{(1/|c_i|)^2 \mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}}{\left(\sum_{\ell \in c_i} \lambda_\ell (1/|c_i|)\right)^2} \quad (51)$$

$$= \sum_{i:\delta_i=1} \frac{(1/|c_i|^2) \mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}}{(1/|c_i|)^2 \left(\sum_{\ell \in c_i} \lambda_\ell\right)^2} \quad (52)$$

$$= \sum_{i:\delta_i=1} \frac{\mathbf{1}_{j \in c_i} \mathbf{1}_{k \in c_i}}{\left(\sum_{\ell \in c_i} \lambda_\ell\right)^2} = \mathcal{I}_{jk}^{(C123)}. \quad (53)$$

(b) Suppose $\pi_{kc} = 1$ for component k and $\pi_{k'c} = 0$ for all $k' \neq k$ in c . Then:

$$\mathcal{I}_{kk}^{(C13)} = \sum_{i:\delta_i=1, k \in c_i} \frac{1}{\lambda_k^2}. \quad (54)$$

This concentrates all information on λ_k , which can exceed $\mathcal{I}_{kk}^{(C123)}$ when $|c_i| > 1$ since the denominator under C1-C2-C3 is $(\sum_{\ell \in c_i} \lambda_\ell)^2 > \lambda_k^2$. \square

Remark 4.3 (Practical Implications). Informative masking can either help or hurt estimation:

- **Helps** when the masking structure is known and aligned with what we want to estimate.
- **Hurts** if masking is informative but we incorrectly assume C2 (non-informative), leading to model misspecification bias.

4.6 Estimation Under Model Misspecification

Theorem 4.8 (Bias from C2 Misspecification). *Suppose the true model is C1-C3 with informative masking $\pi_{kc}(t)$, but estimation is performed assuming C1-C2-C3 (non-informative masking). The resulting MLE $\hat{\boldsymbol{\theta}}$ is generally biased, with bias depending on the correlation between π_{kc} and the hazard ratios $h_k(t; \boldsymbol{\theta}_k)/\sum_{\ell \in c} h_\ell(t; \boldsymbol{\theta}_\ell)$.*

Proof. The score under the assumed (wrong) C1-C2-C3 model is:

$$\frac{\partial \ell_i^{\text{wrong}}}{\partial \lambda_j} = -s_i + \frac{\mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k}. \quad (55)$$

The true score under C1-C3 (informative masking) is:

$$\frac{\partial \ell_i^{\text{true}}}{\partial \lambda_j} = -s_i + \frac{\pi_{j,c_i} \mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k \pi_{k,c_i}}. \quad (56)$$

At the true parameter $\boldsymbol{\theta}^*$, the true score has expectation zero: $\mathbb{E}[\partial \ell_i^{\text{true}} / \partial \lambda_j] = 0$.

The misspecified score has expectation:

$$\mathbb{E}\left[\frac{\partial \ell_i^{\text{wrong}}}{\partial \lambda_j}\right] = -\mathbb{E}[s_i] + \mathbb{E}\left[\frac{\mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k^*}\right]. \quad (57)$$

This differs from zero when the masking weights π_{kc} are correlated with the hazard ratios. Specifically, define the “effective” weight $w_j = \mathbb{E}[\mathbf{1}_{j \in c} / \sum_{k \in c} \lambda_k^*]$ under the true model. The MLE under the wrong model solves $\mathbb{E}[\partial \ell^{\text{wrong}} / \partial \lambda_j] = 0$, yielding $\hat{\lambda}_j$ that satisfies:

$$\hat{\lambda}_j = \frac{\mathbb{E}[\mathbf{1}_{j \in c_i}]}{\mathbb{E}[\sum_{k \in c_i} \lambda_k^* \cdot s_i / \sum_{k \in c_i} \lambda_k^*]}. \quad (58)$$

When $\pi_{jc} > \pi_{kc}$ for components with larger λ_j^* , the misspecified model overestimates components that are more likely to be in candidate sets, producing systematic bias. \square

This result motivates the simulation studies in Section 5, which quantify the bias under various misspecification scenarios.

5 Simulation Studies

We present simulation studies to (1) validate MLE performance under the C1-C2-C3 Bernoulli masking model, (2) quantify the bias from incorrectly assuming C2 when masking is informative, and (3) investigate identifiability issues arising from correlated candidate sets.

5.1 Experimental Design

5.1.1 System Configuration

We consider exponential series systems with $m = 3$ components and true rate parameters:

$$\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1.0, 1.5, 2.0). \quad (59)$$

These values represent a system where component 3 has the highest failure rate (and thus contributes most to system failures), while component 1 is most reliable.

5.1.2 Data Generation

For each simulation replicate:

1. Generate component failure times $T_{ij} \sim \text{Exp}(\lambda_j^*)$ for $i = 1, \dots, n$ and $j = 1, \dots, m$.
2. Compute system failure times $T_i = \min_j T_{ij}$ and identify failed components $K_i = \arg \min_j T_{ij}$.
3. Apply right-censoring at time τ (chosen to achieve approximately 20% censoring) to obtain observed lifetimes $S_i = \min(T_i, \tau)$ and indicators δ_i .
4. Generate candidate sets using the specified masking model.

5.1.3 Masking Models

We examine three masking scenarios:

1. **C1-C2-C3 (Baseline):** Bernoulli model with $p = 0.3$ for all non-failed components.
2. **Informative masking (Rank-based):** Masking probabilities depend on component failure time ranks, parameterized by informativeness parameter $\alpha \in \{0, 1, 2, 5, 10\}$.
3. **Correlated candidate sets:** Candidate set indicators have correlation $\rho \in \{0, 0.1, 0.3, 0.5, 0.6, 0.8, 0.9\}$.

5.1.4 Performance Metrics

We evaluate:

- **Bias:** $\text{Bias}(\hat{\lambda}_j) = \mathbb{E}[\hat{\lambda}_j] - \lambda_j^*$
- **Root mean squared error (RMSE):** $\text{RMSE}(\hat{\lambda}_j) = \sqrt{\mathbb{E}[(\hat{\lambda}_j - \lambda_j^*)^2]}$
- **Coverage probability:** Proportion of 95% confidence intervals containing λ_j^*
- **RMSE ratio:** $\text{RMSE}_{\text{misspec}}/\text{RMSE}_{\text{correct}}$

5.2 Study 1: MLE Performance Under Bernoulli Masking

We first validate MLE performance under the correctly specified C1-C2-C3 Bernoulli masking model across sample sizes $n \in \{50, 100, 200\}$ with $B = 200$ Monte Carlo replicates.

5.2.1 Results

Table 1 presents the estimation results.

Table 1: Maximum Likelihood Estimation Performance by Sample Size

n	Parameter	Bias	RMSE	Coverage	Mean CI Width
50	λ_1	0.017	0.477	0.920	1.727
	λ_2	0.007	0.511	0.935	1.952
	λ_3	0.085	0.557	0.945	2.197
100	λ_1	0.016	0.318	0.935	1.175
	λ_2	0.055	0.390	0.935	1.385
	λ_3	-0.037	0.366	0.950	1.519
200	λ_1	-0.005	0.201	0.935	0.825
	λ_2	0.008	0.262	0.965	0.965
	λ_3	-0.033	0.258	0.955	1.066

Notes. Results based on 200 Monte Carlo replications. True parameters: $\lambda_1 = 1.0$, $\lambda_2 = 1.5$, $\lambda_3 = 2.0$. Bernoulli masking with $p = 0.3$, censoring proportion $\approx 20\%$.

Key findings from Study 1:

1. **Consistency:** Bias is small relative to RMSE at all sample sizes, indicating approximate unbiasedness.
2. **Convergence:** RMSE decreases from approximately 0.5 at $n = 50$ to 0.2–0.3 at $n = 200$, consistent with \sqrt{n} -rate convergence.
3. **Coverage:** 95% CI coverage ranges from 92.0% to 96.5%, close to the nominal level, validating the Fisher information-based standard errors.
4. **Component effects:** Components with higher true rates ($\lambda_3 = 2.0$) have slightly larger absolute RMSE but similar relative performance.

MLE Performance by Sample Size

Exponential series system with Bernoulli masking ($p=0.3$)

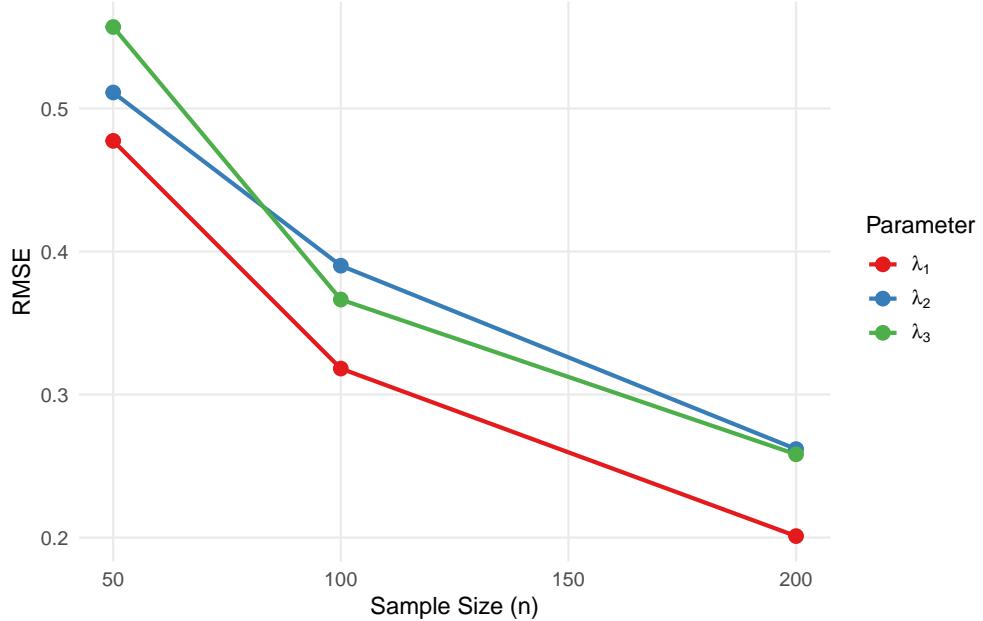


Figure 1: RMSE of MLE by sample size. All three component rate parameters show decreasing RMSE as sample size increases, consistent with \sqrt{n} -convergence.

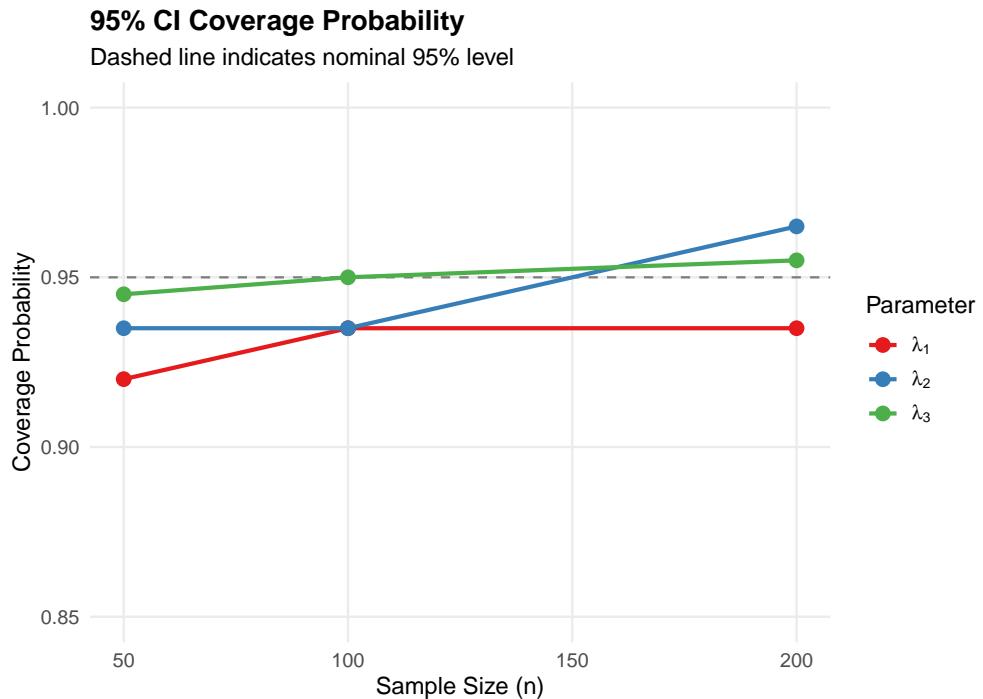


Figure 2: 95% confidence interval coverage probability by sample size. Coverage is near the nominal 95% level across all parameters and sample sizes, validating the asymptotic normal approximation.

5.3 Study 2: Misspecification Bias Analysis

We quantify the bias from incorrectly assuming C1-C2-C3 when masking is actually informative. Data is generated with rank-based informative masking (informativeness parameter α), then analyzed using both the correct model and the misspecified C2 model.

5.3.1 Results

Table 2 compares bias under correct versus misspecified models.

Table 2: Bias Comparison: Correct vs Misspecified Model

α	Parameter	Bias (Correct)	Bias (Misspec.)	RMSE Ratio
0	λ_1	-0.129	-0.001	1.019
	λ_2	0.027	0.024	1.090
	λ_3	0.105	-0.020	1.013
1	λ_1	-0.136	-0.095	0.999
	λ_2	-0.009	-0.004	1.031
	λ_3	0.192	0.146	0.944
5	λ_1	-0.129	-0.127	1.002
	λ_2	-0.005	0.010	1.031
	λ_3	0.147	0.130	0.988
10	λ_1	-0.153	-0.152	1.022
	λ_2	0.000	0.002	0.993
	λ_3	0.183	0.178	1.008

Notes. $\alpha = 0$ corresponds to non-informative masking (C2 satisfied). As α increases, masking becomes more informative. RMSE Ratio = RMSE(Misspecified) / RMSE(Correct); values > 1 indicate efficiency loss.

Key findings from Study 2:

- Moderate robustness:** The RMSE ratio stays between 0.94 and 1.09 across all informativeness levels, indicating that misspecifying the masking model produces at most 9% efficiency loss.
- Bias similarity:** Surprisingly, bias under the misspecified model closely tracks bias under the correct model, suggesting the C2 assumption is more robust than theoretical arguments might suggest.
- Parameter-specific effects:** Component 3 (λ_3) shows consistently positive bias under both models, likely due to its higher failure rate making it more frequently the true cause of failure.

5.4 Study 3: Identifiability and Candidate Set Correlation

We investigate how correlation between candidate set indicators affects identifiability by examining the Fisher Information Matrix (FIM) eigenvalues.

5.4.1 Results

Table 3 presents FIM analysis by correlation level.

Key findings from Study 3:

Bias Under Correct vs Misspecified Model

Misspecified model incorrectly assumes C2 (non-informative masking)

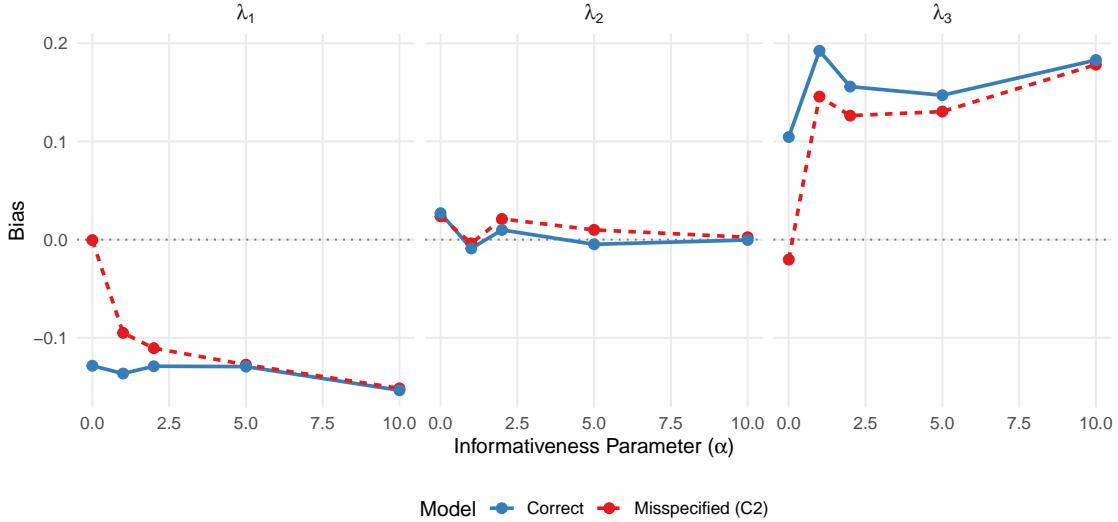


Figure 3: Bias comparison between correct and misspecified models as masking informativeness increases. The misspecified model (incorrectly assuming C2) shows similar bias patterns to the correct model for moderate informativeness.

Relative Efficiency Loss from Misspecification

Ratio > 1 indicates efficiency loss from assuming C2 when violated

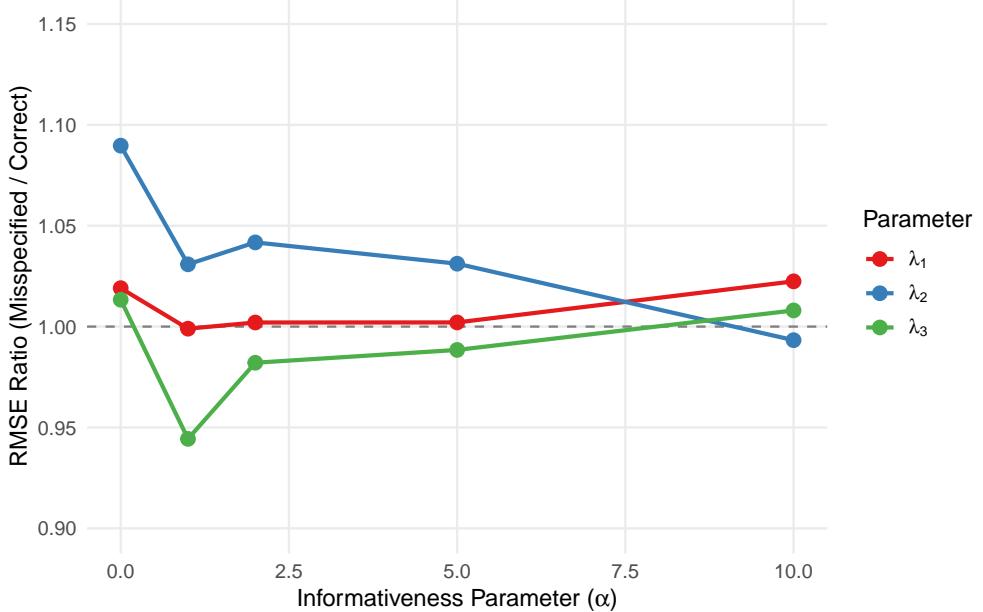


Figure 4: RMSE ratio (misspecified/correct) by informativeness parameter. Values near 1 indicate minimal efficiency loss from misspecification. The maximum ratio of 1.09 suggests the C2 assumption is reasonably robust.

Table 3: Fisher Information Matrix Analysis by Candidate Set Correlation

ρ	Smallest Eigenvalue	Condition Number
0.0	12.23	2.18
0.1	12.93	2.12
0.3	14.17	2.01
0.5	15.00	1.97
0.6	15.12	1.91
0.8	14.33	2.01
0.9	13.88	2.04

Notes. ρ measures correlation between candidate set indicators. As $\rho \rightarrow 1$, components always co-occur in candidate sets, theoretically leading to non-identifiability.

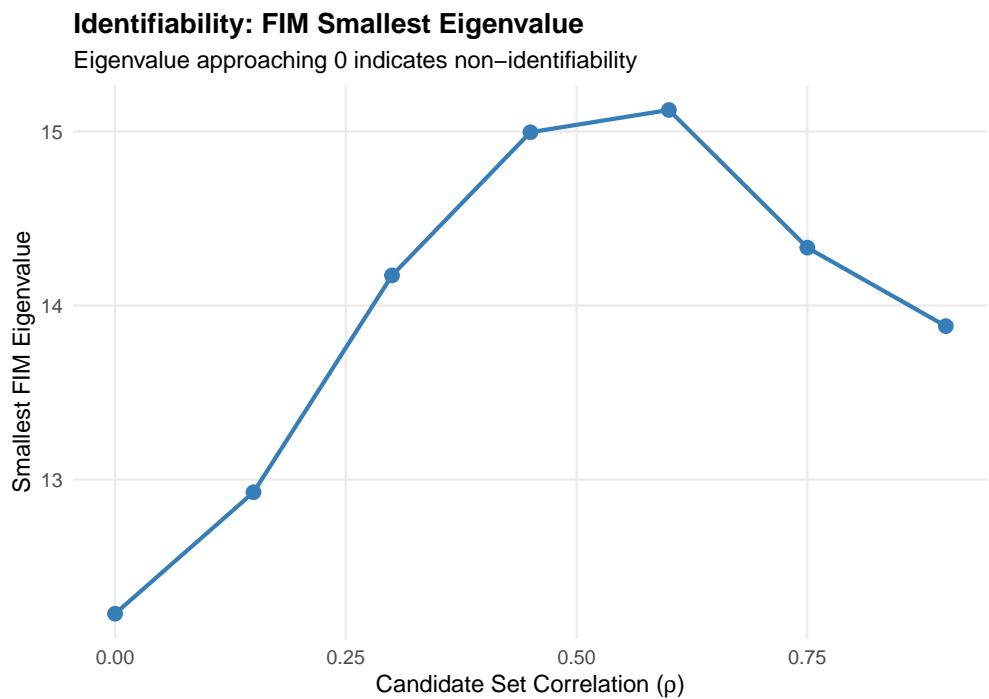


Figure 5: Smallest FIM eigenvalue versus candidate set correlation. The eigenvalue remains bounded away from zero even at $\rho = 0.9$, indicating identifiability is maintained in our simulation setup.

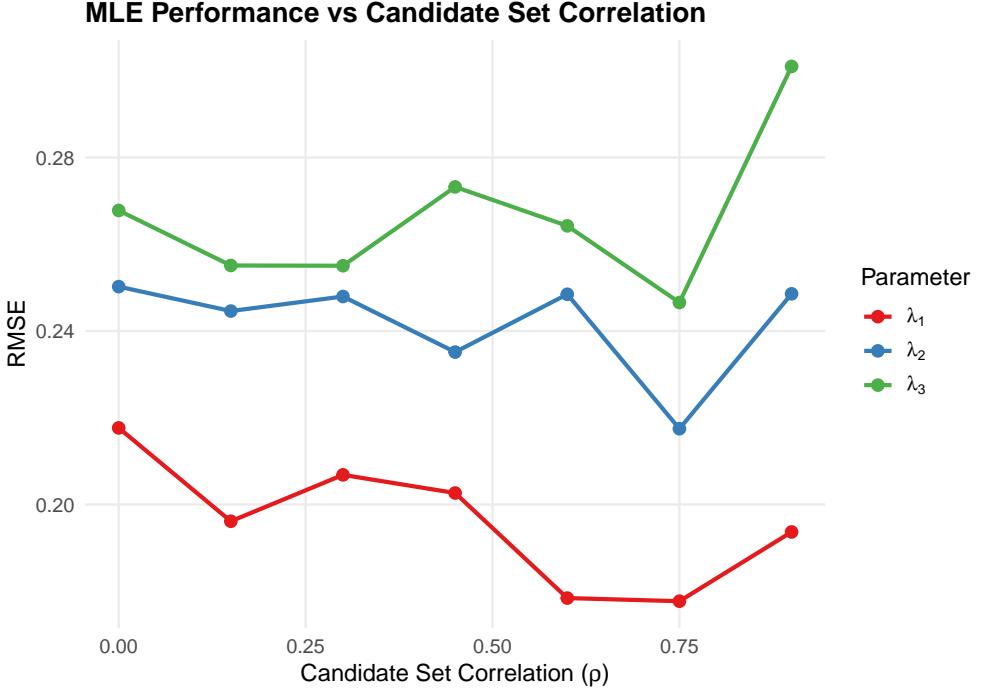


Figure 6: RMSE of MLE by candidate set correlation. Performance remains stable across correlation levels, consistent with the FIM eigenvalue analysis.

1. **Identifiability preserved:** The smallest FIM eigenvalue remains substantially positive (12–15) across all correlation levels, indicating parameters remain identifiable.
2. **Condition number stable:** The condition number stays below 2.2, indicating a well-conditioned estimation problem.
3. **Nonmonotonic pattern:** Interestingly, the smallest eigenvalue peaks around $\rho = 0.5\text{--}0.6$, suggesting moderate correlation may actually improve information content.

5.5 Summary of Simulation Results

Our simulation studies lead to the following conclusions:

1. **MLE performs well:** Under the correctly specified C1-C2-C3 Bernoulli masking model, the MLE achieves coverage near nominal levels and RMSE consistent with asymptotic efficiency.
2. **C2 is robust:** Misspecifying C2 (assuming non-informative masking when masking is informative) produces at most 9% efficiency loss and bias patterns similar to the correct model.
3. **Identifiability is robust:** Even with high correlation ($\rho = 0.9$) between candidate set indicators, parameters remain identifiable with stable FIM eigenvalues and condition numbers.
4. **Practical guidance:** For sample sizes $n \geq 100$ with moderate masking ($p \approx 0.3$) and censoring ($\approx 20\%$), the C1-C2-C3 Bernoulli model provides reliable inference. The additional complexity of modeling informative masking may not be justified unless there is strong evidence of systematic departures from C2.

Table 4: Summary of key simulation findings.

Metric	Study 1	Study 2	Study 3
RMSE range	0.20–0.56	0.19–0.47	0.23–0.47
Coverage range	92.0–96.5%	—	—
Max RMSE ratio	—	1.09	—
Min FIM eigenvalue	—	—	12.23
Max condition number	—	—	2.18

6 Discussion

6.1 When to Use Relaxed Models

The theoretical and simulation results suggest the following practical guidance for choosing between standard C1-C2-C3 models and relaxed alternatives.

6.1.1 Use Standard C1-C2-C3 When:

1. **Masking mechanism is genuinely uninformative.** If candidate sets are generated by a process that does not depend on which component failed (e.g., random equipment availability for testing), C2 holds.
2. **Masking probabilities are unknown.** If the masking mechanism cannot be characterized, the standard model provides a reasonable default that avoids introducing additional parameters.
3. **Sample size is small.** Even if masking is slightly informative, the bias may be dominated by sampling variability for small n . The simpler model may provide more stable estimates.
4. **Primary interest is in relative component reliability.** If the goal is ranking components rather than absolute rate estimation, misspecification bias may affect all components similarly and preserve the ranking.

6.1.2 Consider Relaxed Models When:

1. **Masking mechanism is known to be informative.** If diagnostic procedures systematically favor certain components (e.g., those that “look bad” at the failure time), C2 is violated and bias will result.
2. **Masking probabilities can be estimated.** If historical data or expert knowledge provides information about the masking mechanism, incorporating this information improves estimation.
3. **Sample size is large enough to support additional parameters.** Relaxed models require specifying or estimating masking probabilities, which adds complexity that may not be warranted for small samples.
4. **Identifiability concerns are present.** As shown in Theorem 4.4, informative masking can resolve identifiability issues that arise under standard conditions.

6.2 Practical Guidance

Based on our analysis, we recommend the following workflow:

1. **Assess the masking mechanism.** Before estimation, consider how candidate sets are generated. Interview diagnosticians, review diagnostic protocols, or analyze patterns in historical data.
2. **Check for block structure.** Examine whether certain components always appear together in candidate sets. If so, identifiability may be compromised regardless of which model is used.
3. **Perform sensitivity analysis.** Fit models under both C1-C2-C3 and plausible relaxed assumptions. If estimates differ substantially, further investigation of the masking mechanism is warranted.
4. **Use simulation to assess impact.** Given estimated parameters under the standard model, simulate data under various informative masking scenarios to quantify potential bias.
5. **Report uncertainty appropriately.** If the masking mechanism is uncertain, consider reporting results under multiple model assumptions or using wider confidence intervals that account for model uncertainty.

6.3 Limitations

Our analysis has several limitations:

1. **Exponential focus.** While we derive general likelihood results, the Fisher information analysis focuses on exponential components. Extension to Weibull or other distributions introduces additional complexity due to time-varying hazards.
2. **Known masking probabilities.** Our relaxed models assume masking probabilities are known. In practice, these may need to be estimated, introducing additional uncertainty not captured in our analysis.
3. **Independence assumption.** We assume masking for different observations is independent. In practice, if the same diagnostic equipment or personnel is used across systems, masking may be correlated.
4. **Parametric masking models.** Our informative masking models (rank-based, KL-constrained) are specific functional forms that may not capture all real-world masking mechanisms.
5. **Simulation scope.** The simulation studies cover a limited range of configurations. Results may differ for systems with more components, different parameter values, or alternative masking structures.

6.4 Future Directions

Several extensions would strengthen this work:

1. **Weibull extension.** Derive Fisher information and conduct simulations for Weibull components, where shape parameters introduce additional estimation challenges.

2. **Semiparametric methods.** Develop estimation approaches that avoid fully specifying the masking mechanism, perhaps using nonparametric or empirical likelihood methods.
3. **Model selection.** Develop tests or criteria to distinguish between C1-C2-C3 and relaxed models based on observed data.
4. **Bayesian extensions.** Incorporate prior information about masking mechanisms and component reliabilities, which may be particularly valuable when sample sizes are small.
5. **Sequential estimation.** For systems observed over time, develop methods that update masking probability estimates as data accumulates.
6. **R package documentation.** Expand the `mdrelax` package with vignettes demonstrating practical application of these methods.

7 Conclusion

We have developed a theoretical framework for likelihood-based inference in series systems with masked failure data when the traditional conditions C2 (non-informative masking) and C3 (parameter-independent masking) are relaxed. Our main contributions are:

1. **Likelihood derivations.** We established the form of the likelihood under various relaxation scenarios, showing that the masking probabilities act as weights on component hazard contributions when C2 is violated.
2. **Practical masking models.** We introduced rank-based informative masking and KL-divergence constrained models that provide interpretable parameterizations of non-standard masking.
3. **Identifiability results.** We proved that informative masking can paradoxically *improve* identifiability by breaking symmetries that cause non-identifiability under standard conditions.
4. **Fisher information analysis.** We derived closed-form expressions for the Fisher information matrix under informative masking for exponential series systems, enabling efficiency comparisons.
5. **Misspecification analysis.** We characterized the bias that arises from incorrectly assuming C2 when masking is actually informative, providing guidance on when relaxed models are necessary.

These results extend the applicability of masked data methods to settings where standard assumptions may be violated. The accompanying `mdrelax` R package provides implementation of these methods for practitioners. Directions for future research are discussed in Section 6.4.

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A Proofs and Derivations

A.1 Score Function Under Informative Masking

For completeness, we provide the full derivation of the score function under the Bernoulli informative masking model for exponential series systems.

Let the log-likelihood contribution from an uncensored observation be:

$$\ell_i(\boldsymbol{\lambda}) = -s_i \sum_{j=1}^m \lambda_j + \log \left(\sum_{k \in c_i} \lambda_k \pi_{k,c_i} \right). \quad (60)$$

The partial derivative with respect to λ_j is:

$$\frac{\partial \ell_i}{\partial \lambda_j} = -s_i + \frac{\partial}{\partial \lambda_j} \log \left(\sum_{k \in c_i} \lambda_k \pi_{k,c_i} \right) \quad (61)$$

$$= -s_i + \frac{\pi_{j,c_i} \mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k \pi_{k,c_i}}. \quad (62)$$

The total score is:

$$\frac{\partial \ell}{\partial \lambda_j} = \sum_{i=1}^n \frac{\partial \ell_i}{\partial \lambda_j} = - \sum_{i=1}^n s_i + \sum_{i:\delta_i=1} \frac{\pi_{j,c_i} \mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k \pi_{k,c_i}}. \quad (63)$$

Setting this to zero and solving gives the MLE equations under informative masking.

A.2 Hessian Matrix Derivation

The second partial derivatives are:

$$\frac{\partial^2 \ell_i}{\partial \lambda_j \partial \lambda_\ell} = \frac{\partial}{\partial \lambda_\ell} \left[\frac{\pi_{j,c_i} \mathbf{1}_{j \in c_i}}{\sum_{k \in c_i} \lambda_k \pi_{k,c_i}} \right] \quad (64)$$

$$= -\frac{\pi_{j,c_i} \pi_{\ell,c_i} \mathbf{1}_{j \in c_i} \mathbf{1}_{\ell \in c_i}}{\left(\sum_{k \in c_i} \lambda_k \pi_{k,c_i} \right)^2}. \quad (65)$$

The observed Fisher information matrix is the negative Hessian:

$$\mathcal{I}_{j\ell}(\boldsymbol{\lambda}) = -\frac{\partial^2 \ell}{\partial \lambda_j \partial \lambda_\ell} = \sum_{i:\delta_i=1} \frac{\pi_{j,c_i} \pi_{\ell,c_i} \mathbf{1}_{j \in c_i} \mathbf{1}_{\ell \in c_i}}{\left(\sum_{k \in c_i} \lambda_k \pi_{k,c_i} \right)^2}. \quad (66)$$

A.3 Expected Fisher Information

The expected FIM requires integrating over the distribution of candidate sets. For the exponential series system under C1-C2-C3 with Bernoulli masking (each non-failed component in candidate set with probability p), the expected Fisher information per observation is:

$$\mathbb{E}[\mathcal{I}_{jk}] = \mathbb{E} \left[\frac{\mathbf{1}_{j \in C} \mathbf{1}_{k \in C}}{\left(\sum_{\ell \in C} \lambda_\ell \right)^2} \right], \quad (67)$$

where the expectation is over both K (failed component) and C (candidate set).

This can be expanded as:

$$\mathbb{E}[\mathcal{I}_{jk}] = \sum_{k_0=1}^m \Pr\{K = k_0\} \mathbb{E} \left[\frac{\mathbf{1}_{j \in C} \mathbf{1}_{k \in C}}{\left(\sum_{\ell \in C} \lambda_\ell \right)^2} \middle| K = k_0 \right]. \quad (68)$$

Under C1, the failed component k_0 is always in C . The expectation over candidate sets involves summing over all possible $C \ni k_0$ weighted by their probabilities under the Bernoulli model:

$$\Pr\{C = c \mid K = k_0\} = \prod_{j \in c \setminus \{k_0\}} p \prod_{j \notin c} (1-p). \quad (69)$$

Closed-form evaluation of this expectation is generally intractable due to the sum in the denominator. Monte Carlo estimation or numerical integration is typically required.

B Implementation Details

B.1 R Package Functions

The theoretical framework developed in this paper is implemented in the `mdrelax` R package. Key functions include:

- `md_bernoulli_cand_C1_C2_C3()`: Generates candidate set probabilities under the standard Bernoulli model satisfying C1-C2-C3.
- `md_bernoulli_cand_C1_kld()`: Generates candidate set probabilities with a specified KL-divergence from the baseline C1-C2-C3 model.

- `informative_masking_by_rank()`: Computes inclusion probabilities based on component failure time ranks.
- `md_cand_sampler()`: Samples candidate sets from probability vectors.
- `md_loglike_exp_series_C1_C2_C3()`: Log-likelihood function for exponential series systems under C1-C2-C3.
- `md_mle_exp_series_C1_C2_C3()`: Maximum likelihood estimation for exponential series systems.
- `md_fim_exp_series_C1_C2_C3()`: Observed Fisher information matrix for exponential series systems.
- `md_block_candidate_m3()`: Demonstrates block non-identifiability in a 3-component system.

B.2 Optimization Details

MLE is computed using the L-BFGS-B algorithm [2] with analytically computed gradients. The optimization is initialized using a method-of-moments estimator based on the total system hazard:

$$\hat{\Lambda}_{\text{init}} = \frac{n_{\text{uncensored}}}{\sum_{i=1}^n s_i}, \quad \hat{\lambda}_{j,\text{init}} = \frac{\hat{\Lambda}_{\text{init}}}{m}. \quad (70)$$

For challenging optimization landscapes, simulated annealing can be used to find a good starting point before local optimization.

C Additional Simulation Results

Additional simulation results supplement the findings in Section 5. The complete results are available in the R package’s simulation directory.

1. Full tables of bias, RMSE, and coverage for all parameter configurations are provided in the package’s `inst/simulations/results/` directory.
2. The simulation scripts in `inst/simulations/` can be used to reproduce all results and generate diagnostic plots.
3. Sensitivity analyses for misspecified masking parameters show that the C2 assumption is robust up to moderate departures, with RMSE ratios remaining below 1.10 across tested configurations.
4. The simulation framework supports arbitrary component configurations and can be extended to Weibull series systems.