1. Y_t's uncorrelated. $E(Y_t) = \mu$ $Var(Y_t) = \sigma^2$

a. $M_t = \left(\sum_{j=1}^{N} Y_{t+1-j}\right)/N$

$$Var(M_t) = \frac{1}{N^2} Var\left(\sum_{j=1}^{N} Y_{t+1-j}\right) = \frac{1}{N^2}\sum_{j=1}^{N} Var(Y_{t+1-j}) = \frac{1}{N^2}\cdot N\sigma^2 = \sigma^2/N$$

b. $Cov(M_t, M_{t+k})$

$$= Cov\left(\frac{1}{N}\sum_{j=1}^{N} Y_{t+1-j}, \frac{1}{N}\sum_{j=1-k}^{N-k} Y_{t+1-j}\right)$$

$$= \frac{1}{N^2} Cov\left(\sum_{j=N-k+1}^{N} Y_{t+1-j} + \sum_{j=1}^{N-k} Y_{t+1-j}, \sum_{j=1-k}^{0} Y_{t+1-j} + \sum_{j=1}^{N-k} Y_{t+1-j}\right)$$

$\sum_{j=N-k+1}^{N} Y_{t+1-j}$ and $\sum_{j=1-k}^{0} Y_{t+1-j}$ are indep.

$$= \frac{1}{N^2} Var\left(\sum_{j=1}^{N-k} Y_{t+1-j}\right) = \frac{1}{N^2}\sum_{j=1}^{N-k}\sigma^2 = \sigma^2 \sum_{j=1}^{N-k}\frac{1}{N^2} = \sigma^2\frac{N-k}{N^2}$$

when $k < N$

c. ACF. Assume $k > 0$

$$\rho_k = \frac{Cov(M_t, M_{t+k})}{\sqrt{Var(M_t)\cdot Var(M_{t+k})}}$$

$$= \frac{\sigma^2\frac{N-k}{N^2}}{\sqrt{\sigma^2/N \cdot \sigma^2/N}} = \frac{\frac{N-k}{N^2}}{1/N} = \frac{N-k}{N} \quad k<N.$$

Symmetric for $k<0$, so $\rho_k = 1 - \frac{|k|}{N}$ $k<N$

2. a. $E(Y_t) = E(Z_1\cos(wt) + Z_2\sin(wt) + e_t)$

$= E(Z_1)\cos(wt) + E(Z_2)\sin(wt) + E(e_t) = 0$

$Var(Y_t) = Var(Z_1\cos(wt) + Z_2\sin(wt) + e_t)$

$= \cos^2(wt) + \sin^2(wt) + \sigma^2 = 1+\sigma^2$

$Cov(Y_t, Y_{t+k}) = E(Y_t Y_{t+k}) - E(Y_t)E(Y_{t+k})$

$= E(Y_t Y_{t+k}) = E((Y_t - e_t)(Y_{t+k} - e_{t+k}))$

$= E\left[\left(Z_1\cos(wt) + Z_2\sin(wt)\right)\left(Z_1\cos(w(t+k)) + Z_2\sin(w(t+k))\right)\right]$

$= E\left[Z_1^2\cos(wt)\cos(w(t+k)) + Z_2^2\sin(wt)\sin(w(t+k))\right]$

both 0 $+ E(Z_1Z_2\cos(wt)\sin(w(t+k)))$

$+ E(Z_1Z_2\sin(wt)\cos(w(t+k)))$

$= \cos(wt)\cos(w(t+k))EZ_1^2 + \sin(wt)\sin(w(t+k))EZ_2^2$

$(EZ_1^2 = EZ_2^2 = 1)$ $= \cos(wt)\cos(w(t+k)) + \sin(wt)\sin(w(t+k))$

$= \cos(w(t+k) - wt) = \cos(wk)$

ACF. $\rho_k = \dfrac{\cos(wk)}{1+\sigma^2}$ not depend on t. $\Rightarrow$ stationary.

b. See R code

c. $E(X_t) = \beta_0 + \beta_1 t$ not constant over t.

So $\{X_t\}$ is not stationary. See R code for simulation.

d. $\nabla X_t = \beta_1 + Z_1[\cos(wt) - \cos(wt-w)] + Z_2[\sin(wt) - \sin(wt-w)]$

$+ (e_t - e_{t-1})$

$E(\nabla X_t) = \beta_1$ constant over t.

let $s=t+k$ $Cov(\nabla X_t, \nabla X_s) = E\left[(Z_1[\cos(wt) - \cos(wt-w)] + Z_2[\sin wt - \sin(wt-w)])\right.$

$\left.\cdot (Z_1[\cos(ws) - \cos(ws-w)] + Z_2[\sin(ws) - \sin(ws-w)])\right]$

Inside expectation $= Z_1^2[\cos wt - \cos(wt-w)][\cos(ws) - \cos(ws-w)]$

$+ Z_2^2[\sin wt - \sin(wt-w)][\sin(ws) - \sin(ws-w)]$

$+ \underbrace{Z_1Z_2[\cdots]}$ expectation $= 0$.

So $\operatorname{cov}(\nabla X_t, \nabla X_s) = [\cos \omega t - \cos(\omega t - \omega)][\cos \omega s - \cos(\omega s - \omega)]$

$+ [\sin \omega t - \sin(\omega t - \omega)][\sin \omega s - \sin(\omega s - \omega)]$

$= \cos(\omega t)\cos(\omega s) + \sin(\omega t)\sin(\omega s)$

$+ \cos(\omega t - \omega)\cos(\omega s - \omega) + \sin(\omega t - \omega)\sin(\omega s - \omega)$

$- [\cos(\omega t)\cos(\omega s - \omega) + \sin(\omega t)\sin(\omega s - \omega)]$

$- [\cos(\omega s)\cos(\omega t - \omega) + \sin(\omega s)\sin(\omega t - \omega)]$

$= \cos(\omega(s-t)) + \cos(\omega(s-t))$

$- \cos(\omega(s-t+1)) - \cos(\omega(s-t-1))$

$s-t=k$ $\quad = 2\cos(\omega k) - \cos(\omega(k+1)) - \cos(\omega(k-1))$

not depend on t.

$\Rightarrow$ stationary

See R-code for simulation.

3. See R code.