1. (a) $E(Z_1-3Z_2-2Z_3) = E(Z_1)-3E(Z_2)-2E(Z_3)$
$= 0-3-2(-1) = -1$

(b) $Var(2Z_1+Z_3) = Var(2Z_1)+Var(Z_3)+2cov(2Z_1, Z_3)$
$= 4\,Var(Z_1)+Var(Z_3)+4\,cov(Z_1,Z_3)$
$= 4(1)+3+0 = 7$

(c). $cov(3Z_1-Z_2, Z_2+2Z_3) = cov(3Z_1, Z_2)+cov(3Z_1, 2Z_3)$
$+ cov(-Z_2, Z_2)+cov(-Z_2, 2Z_3)$
$= 3cov(Z_1, Z_2)+6cov(Z_1, Z_3)$
$- cov(Z_2, Z_2) - 2cov(Z_2, Z_3)$
$= 3(-0.5)+0-2-2(1.5)$
$= -1.5-2-3 = -6.5$

2. (a) $E(Y_t) = E(e_t e_{t-1})$
$= E(e_t)E(e_{t-1})$ since e_t are indep.
$= 0$

$Var(Y_t) = Var(e_t e_{t-1})$
$= E(e_t^2 e_{t-1}^2) - [E(e_t e_{t-1})]^2$
$= E(e_t^2)E(e_{t-1}^2)$

$E(e_t^2) = E(e_{t-1}^2) = Var(e_t)+[E(e_t)]^2 = Var(e_t) = \sigma^2$

So $Var(Y_t) = \sigma^4$

(b) ACF: $cov(Y_t, Y_{t+k}) = cov(e_t e_{t-1}, e_{t+k}e_{t+k-1})$
$= E(e_t e_{t-1} e_{t+k} e_{t+k-1}) - E(e_t e_{t-1})E(e_{t+k}e_{t+k-1})$
$= E(e_t e_{t-1} e_{t+k} e_{t+k-1})$

when $k=0$, $cov(Y_t, Y_{t+k}) = Var(Y_t) = \sigma^4$
$k=1$, $cov(Y_t, Y_{t+k}) = E(e_t e_{t-1} e_{t+1} e_t)$
$= E(e_t^2)E(e_{t-1})E(e_{t+1}) = 0$
$k>1$, $cov(Y_t, Y_{t+k}) = 0$.

So $\rho_k = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases}$

(c) $Var(Y_t)$, $E(Y_t)$ constant over t. ρ_k (ACF) doesn't depend on t. Stationary!

Write you answers clearly on separate sheets of paper. Show all your steps. You may use notes, your textbook, etc. You are to work completely independently on this exam. Submit your solutions on Blackboard or through email, by 11:59pm Friday March 5th.

1.(15pt) Suppose that $Z = (Z_1, Z_2, Z_3)$ is a random vector with

$$\mathrm{E}(Z) = (0, 1, -1)$$

$$\mathrm{Var}(Z) = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 2 & 1.5 \\ 0 & 1.5 & 3 \end{bmatrix}$$

Calculate each of the following:

(a) $\mathrm{E}(Z_1 - 3Z_2 - 2Z_3)$

(b) $\mathrm{Var}(2Z_1 + Z_3)$

(c) $\mathrm{Cov}(3Z_1 - Z_2, Z_2 + 2Z_3)$

2. (15pt) Let $\{e_t\}$ be a normal white noise process with mean zero and variance σ^2. Consider the process

$$Y_t = e_t e_{t-1}.$$

(a) Calculate $\mathrm{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.

(b) Calculate the ACF (Autocorrelation Function).

(c) Is the process weakly stationary? Why?

3. (10 pt) Suppose that we have fit the straight-line regression without intercept $\hat{y} = \hat{\beta}_1 x_1$. However, the response y is in fact affected by a second variable x_2. So the true regression function is

$$y = \beta_1 x_1 + \beta_2 x_2 + \epsilon.$$

Assume ϵ's are i.i.d. with mean 0 and variance σ^2. Calculate the bias of $\hat{\beta}_1$ in the original simple linear regression, i.e. calculate $\mathrm{E}(\hat{\beta}_1 - \beta_1)$.

4. (10 pt) Consider the simple linear regression model $y = \beta_0 + \beta_1 x + \epsilon$, where β_0 is known. ϵ's are i.i.d. with mean 0 and variance σ^2.

(a) Find the least square estimator of β_1 in this model.

(b) Construct a $100(1-\alpha)\%$ confidence interval for β_1. Compare the interval with the one when β_0 is also unknown. Is it narrower?

3. $\hat{\beta}_1 = (X_1^T X_1)^{-1} X_1^T Y$

where $X_1 = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}$ $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

so $E(\hat{\beta}_1) = E((X_1^T X_1)^{-1} X_1^T Y)$

$= (X_1^T X_1)^{-1} X_1^T E(Y)$

$E(Y) = E(\beta_1 X_1 + \beta_2 X_2 + \varepsilon) = \beta_1 X_1 + \beta_2 X_2$. $E(Y) = X\beta$

so $E(\hat{\beta}_1) = (X_1^T X_1)^{-1} X_1^T \begin{pmatrix} \beta_1 x_{11} + \beta_2 x_{12} \\ \vdots \\ \beta_1 x_{n1} + \beta_2 x_{n2} \end{pmatrix}$

$= (X_1^T X_1)^{-1} X_1^T X \beta$.

where $X = \begin{pmatrix} x_{11} & x_{12} \\ \vdots & \vdots \\ x_{n1} & x_{n2} \end{pmatrix}$ $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

$E(\hat{\beta}_1) = (\sum x_{i1}^2)^{-1} (x_{11} \cdots x_{n1}) \begin{pmatrix} x_{11} & x_{12} \\ \vdots & \vdots \\ x_{n1} & x_{n2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

$= (\sum x_{i1}^2)^{-1} (\sum x_{i1}^2 \quad \sum x_{i1} x_{i2}) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$

$= (\sum x_{i1}^2)^{-1} (\beta_1 \sum x_{i1}^2 + \beta_2 \sum x_{i1} x_{i2})$

$= \beta_1 + \beta_2 \dfrac{\sum x_{i1} x_{i2}}{\sum x_{i1}^2}$

$E(\hat{\beta}_1 - \beta_1) = \beta_2 \dfrac{\sum x_{i1} x_{i2}}{\sum x_{i1}^2}$

$\hat{\beta}_1$ no longer unbiased!

4. (a) $y_i - \beta_0 = \beta_1 x_i + \varepsilon_i$, let $z_i = y_i - \beta_0$

so $\hat{\beta}_1 = (X^T X)^{-1} X^T Z$, where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $Z = \begin{pmatrix} y_1 - \beta_0 \\ \vdots \\ y_n - \beta_0 \end{pmatrix}$

$$= \frac{\sum x_i (y_i - \beta_0)}{\sum x_i^2}$$

$$= \frac{\sum x_i y_i}{\sum x_i^2} - \beta_0 \frac{\sum x_i}{\sum x_i^2}$$

(b) $Var(\hat{\beta}_1) = Var\left(\frac{\sum x_i y_i}{\sum x_i^2} - \beta_0 \frac{\sum x_i}{\sum x_i^2} \right)$

$$= Var\left(\frac{\sum x_i y_i}{\sum x_i^2} \right)$$

$$= \frac{1}{(\sum x_i^2)^2} \sum x_i^2 Var(y_i) = \frac{1}{(\sum x_i^2)^2} \sum x_i^2 \cdot \sigma^2 = \frac{\sigma^2}{\sum x_i^2}$$

CI: $\hat{\beta}_1 \pm t_{\alpha/2, n-1} \sqrt{\frac{1}{\sum x_i^2}}\, \hat{\sigma}$

when β_0 is unknown

recall $Var(\hat{\beta}_1) = \hat{\sigma}^2 C_{jj}$

where C_{jj} is the ith diagnal of $(X^T X)^{-1}$

$X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$ $X^T X = \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}$ $(X^T X)^{-1} = \begin{pmatrix} \frac{\sum x_i^2}{n\sum x_i^2 - (\sum x_i)^2} & \frac{-\sum x_i}{n\sum x_i^2 - (\sum x_i)^2} \\ \frac{-\sum x_i}{n\sum x_i^2 - (\sum x_i)^2} & \frac{n}{n\sum x_i^2 - (\sum x_i)^2} \end{pmatrix}$

so $Var(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{\sum x_i^2 - (\sum x_i)^2/n}$

$Var(\hat{\beta}_1)$ is larger if β_0 is unknown, so CI will be wider.