1\. (a) $E(\hat{y}_T) = (1-\theta)\sum_{t=0}^{\infty} E(\theta^t y_{T-t})$

$$= (1-\theta)\sum_{t=0}^{\infty} \theta^t E(y_{T-t})$$

$E(y_{T-t}) = E(\mu + e_t) = \mu$ when $T-t \neq t^*$

$E(y_{T-t}) = E(\mu+\delta+e_t) = \mu+\delta$ when $T-t = t^*$, i.e. $t = T-t^*$

So $E(\hat{y}_T) = (1-\theta)\sum_{t=0}^{\infty}\theta^t \mu = \mu$ when $T < t^*$

$$E(\hat{y}_T) = (1-\theta)\left[\sum_{t=0}^{\infty}\theta^t\mu + \delta\theta^{T-t^*}\right]$$

$$= \mu + \theta^{T-t^*}\delta(1-\theta)$$

(b) To make $\theta^{T-t^*}\delta(1-\theta) < 0.1\delta$

$$\theta^{T-t^*}(1-\theta) < 0.1$$

$$0.5^{T-t^*+1} < 0.1$$

$$T-t^*+1 = \log 0.1 / \log 0.5 = 3.32$$

After **3** periods, the expected value returns to within 0.1δ of μ.

2\. (a). $E(Y_t) = \phi E(Y_t) + 0$. so $E(Y_t) = 0$.

$E(W_t) = E(Y_t - Y_{t-1}) = 0$

$Cov(W_t, W_{t-k}) = E(W_t W_{t-k})$

$$= E[(Y_t - Y_{t-1})(Y_{t-k} - Y_{t-k-1})]$$

$$= E(Y_t Y_{t-k}) - E(Y_{t-k}Y_{t-1}) - E(Y_t Y_{t-k-1}) + E(Y_{t-1}Y_{t-k-1})$$

$$= \gamma_k - \gamma_{k-1} - \gamma_{k+1} + \gamma_k$$

$corr(W_t, W_{t-k}) = \frac{1}{2(1-\phi)}(2\rho_k - \rho_{k-1} - \rho_{k+1}) = \frac{-\phi^{k-1}(\phi-1)^2}{2(1-\phi)} = \frac{(\phi-1)\phi^{k-1}}{2}$

$corr(W_t, W_{t-k})$
$= \gamma_0(2\phi^k - \phi^{k-1} - \phi^{k+1}) / \frac{2\sigma^2}{1+\phi}$
$= \frac{\sigma^2}{1-\phi^2} \times \frac{1+\phi}{2\sigma^2}(2\phi^k - \phi^{k-1} - \phi^{k+1})$
$= \frac{1}{2(1-\phi)} \cdot (-\phi^{k-1})(\phi-1)^2$

Note: $\gamma_0 = Var(Y_t) = \frac{\sigma^2}{1-\phi^2}$

$\gamma_k = \phi^k \gamma_0$

(b) $Var(W_t) = Var(Y_t - Y_{t-1}) = E(Y_t - Y_{t-1})^2$

$$= E(Y_t^2) + E(Y_{t-1}^2) - 2E(Y_t Y_{t-1})$$

$$= 2\gamma_0 - 2\gamma_1 = \frac{\sigma^2}{1-\phi^2}2(1-\phi) = \frac{2\sigma^2}{1+\phi}$$

$$= 2(1-\phi)\gamma_0$$

3. (a) $E(Y_t) = E(X_t) = 0$

$$\begin{aligned}
cov(Y_t, Y_{t-k}) &= E(Y_t Y_{t-k}) \\
&= E((X_t + e_t)(X_{t-k} + e_{t-k})) \\
&= E(X_t X_{t-k}) + E(e_t X_{t-k}) + E(X_t e_{t-k}) \\
&\quad + E(e_t e_{t-k}) \\
&= E(X_t X_{t-k}) = \phi^k \cdot \frac{\sigma_z^2}{1-\phi^2}
\end{aligned}$$

$$\begin{aligned}
Var(Y_t, Y_t) &= Var(X_t) + \sigma_e^2 \\
&= \frac{\sigma_z^2}{1-\phi^2} + \sigma_e^2
\end{aligned}$$

(b) $\cancel{k} = 1$, $cov(U_t, U_{t-k})$

$$\begin{aligned}
&= cov(Y_t - \phi Y_{t-1}, Y_{t-1} - \phi Y_{t-2}) \\
&= cov(Y_t, Y_{t-1}) + cov(Y_t, -\phi Y_{t-2}) \\
&\quad + cov(-\phi Y_{t-1}, Y_{t-1}) + \phi^2 cov(Y_{t-1}, Y_{t-2}) \\
&= \gamma_1 - \phi \gamma_2 - \phi \gamma_0 + \phi^2 \gamma_1 \neq 0.
\end{aligned}$$

for $k > 1$,

$U_t = Y_t - \phi Y_{t-1}$ $\quad E(U_t) = 0$

$$\begin{aligned}
&cov(U_t, U_{t-k}) \\
&= cov(Y_t - \phi Y_{t-1}, Y_{t-k} - \phi Y_{t-k-1}) \\
&= E(Y_t Y_{t-k}) - \phi E(Y_{t-1} Y_{t-k}) - \phi E(Y_t Y_{t-k-1}) \\
&\quad + \phi^2 E(Y_{t-1} Y_{t-k-1}) \\
&= \phi^k \cdot \frac{\sigma_z^2}{1-\phi^2} - \phi \cdot \phi^{k-1} \frac{\sigma_z^2}{1-\phi^2} - \phi \phi^{k+1} \frac{\sigma_z^2}{1-\phi^2} \\
&\quad + \phi^2 \phi^k \frac{\sigma_z^2}{1-\phi^2} = 0
\end{aligned}$$

4. (i) $(1-0.8B+0.15B^2)y_t = (1-0.3B)e_t$

$(1-\cancel{0.3B})(1-0.5B)y_t = (1-\cancel{0.3B})e_t$

$(1-0.5B)y_t = e_t$

ARMA (1, 0). or AR (1)

root : 2 >1 $\Rightarrow$ stationary

AR process always invertible.

(ii) $(1-B+0.5B^2)y_t = (1-1.2B)e_t$

ARMA(2,1)

MA root : $\frac{1}{1.2} = 0.833 < 1$ not invertible.

AR root : $\frac{1\pm\sqrt{-1}}{1} = 1\pm\sqrt{-1} = 1\pm i$

modulus $= \sqrt{1+1} = \sqrt{2} > 1$. stationary.