Time Series Analysis - STAT 478 - Final Exam - Part 1 Q5

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Part 1: Problem 5

Suppose that $\{Y_t\}$ is a seasonal ARIMA process in the form

$$Y_t = (1 - \theta\,\mathrm{B})(1 - \Theta\,\mathrm{B}^4)e_t$$

where $\{e_t\}$ is a zero mean white noise process with variance σ^2.

Part (a)

$\boxed{\text{Derive expressions for } \mathrm{E}(Y_t) \text{ and } \mathrm{Var}(Y_t).}$

Note that

$$\begin{aligned} Y_t &= (1 - \theta\,\mathrm{B})(1 - \Theta\,\mathrm{B}^4)e_t \\ &= (1 - \Theta\,\mathrm{B}^4 - \theta\,\mathrm{B} + \theta\Theta\,\mathrm{B}^5)e_t \\ &= e_t - \theta e_{t-1} - \Theta e_{t-4} + \theta\Theta e_{t-5}. \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{E}(Y_t) &= \mathrm{E}(e_t) - \Theta\,\mathrm{E}(e_{t-4}) - \theta\,\mathrm{E}(e_{t-1}) + \theta\Theta\,\mathrm{E}(e_{t-5}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(Y_t) &= \mathrm{Var}(e_t) + \Theta^2\,\mathrm{Var}(e_{t-4}) + \theta^2\,\mathrm{Var}(e_{t-1}) + \theta^2\Theta^2\,\mathrm{Var}(e_{t-5}) \\ &= \sigma^2 + \Theta^2\sigma^2 + \theta^2\sigma^2 + \theta^2\Theta^2\sigma^2 \\ &= \sigma^2(1 + \Theta^2 + \theta^2 + \theta^2\Theta^2) \\ &= \sigma^2(1 + \Theta^2)(1 + \theta^2). \end{aligned}$$

Part (b)

$\boxed{\text{Derive the autocovariance function, that is, calculate } \mathrm{Cov}(Y_t, Y_{t-k}) \text{ for } k = 1, 2, \ldots}$

The autocovariance function of $\{Y_t\}$ is given by

$$\gamma_k = \begin{cases} \sigma^2(1 + \Theta^2)(1 + \theta^2) & k = 0 \\ -\theta\sigma^2(1 + \Theta^2) & k = 1 \\ 0 & k = 2 \\ \theta\Theta\sigma^2 & k = 3 \\ -\Theta\sigma^2(1 + \theta\Theta) & k = 4 \\ \theta\Theta\sigma^2 & k = 5 \\ 0 & k > 5. \end{cases}$$

Proof. We do a case analysis on values of $k = 1, 2, ...$, but first note that by symmetry

$$\text{Cov}(Y_{t-k}, Y_t) = \text{Cov}(Y_t, Y_{t-k})$$

and so we only consider non-negative values of k.

1. Let $k = 1$, then

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-1}) &= \text{Cov}(e_t - \theta e_{t-1} - \Theta e_{t-4} + \theta\Theta e_{t-5}, e_{t-1} - \theta e_{t-2} - \Theta e_{t-5} + \theta\Theta e_{t-6}) \\
&= \text{Cov}(-\theta e_{t-1}, e_{t-1}) + \text{Cov}(\theta\Theta e_{t-5}, -\Theta e_{t-5}) \\
&= -\theta\,\text{Cov}(e_{t-1}, e_{t-1}) - \theta\Theta^2\,\text{Cov}(e_{t-5}, e_{t-5}) \\
&= -\theta\sigma^2 - \theta\Theta^2\sigma^2 \\
&= -\theta\sigma^2(1 + \Theta^2).
\end{aligned}$$

2. Let $k = 2$, then

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-2}) &= \text{Cov}(e_t - \theta e_{t-1} - \Theta e_{t-4} + \theta\Theta e_{t-5}, e_{t-2} - \theta e_{t-3} - \Theta e_{t-6} + \theta\Theta e_{t-7}) \\
&= 0.
\end{aligned}$$

3. Let $k = 3$, then

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-3}) &= \text{Cov}(e_t - \theta e_{t-1} - \Theta e_{t-4} + \theta\Theta e_{t-5}, e_{t-3} - \theta e_{t-4} - \Theta e_{t-7} + \theta\Theta e_{t-8}) \\
&= \text{Cov}(-\Theta e_{t-4}, -\theta e_{t-4}) \\
&= \theta\Theta\,\text{Cov}(e_{t-4}, e_{t-4}) \\
&= \theta\Theta\sigma^2.
\end{aligned}$$

4. Let $k = 4$, then

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-4}) &= \text{Cov}(e_t - \theta e_{t-1} - \Theta e_{t-4} + \theta\Theta e_{t-5}, e_{t-4} - \theta e_{t-5} - \Theta e_{t-8} + \theta\Theta e_{t-9}) \\
&= \text{Cov}(-\Theta e_{t-4}, e_{t-4}) + \text{Cov}(\theta\Theta e_{t-5}, -\Theta e_{t-5}) \\
&= -\Theta\,\text{Cov}(e_{t-4}, e_{t-4}) - \theta\Theta^2\,\text{Cov}(e_{t-5}, e_{t-5}) \\
&= -\Theta\sigma^2 - \theta\Theta^2\sigma^2 \\
&= \sigma^2(-\Theta - \theta\Theta^2) \\
&= -\Theta\sigma^2(1 + \theta\Theta).
\end{aligned}$$

5. Let $k = 5$, then

$$\begin{aligned}
\text{Cov}(Y_t, Y_{t-5}) &= \text{Cov}(e_t - \theta e_{t-1} - \Theta e_{t-4} + \theta\Theta e_{t-5}, e_{t-5} - \theta e_{t-6} - \Theta e_{t-9} + \theta\Theta e_{t-10}) \\
&= \text{Cov}(\theta\Theta e_{t-1}, e_{t-5}) \\
&= \theta\Theta\,\text{Cov}(e_{t-1}, e_{t-5}) \\
&= \theta\Theta\sigma^2.
\end{aligned}$$

6. Let $k > 5$, then $\text{Cov}(Y_t, Y_{t-k})$, $k > 5$, is zero.

We see that the autocovariance function is independent of time t and only a function of lag k. We collect the cases and form the indicated piecewise function. □

Part (c)

$\boxed{\text{Characterize this models as SARIMA}(p, d, q) \times (P, D, Q)_s\text{, that is, identify } p, d, q, P, D, Q, s.}$

$\{Y_t\}$ is a zero mean process of type $\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_4$, i.e., $p = 0$, $d = 1$, $q = 1$, $P = 0$, $D = 1$, $Q = 1$, and $s = 4$.