

Homework #1

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Course: STAT 579 - Discrete Multivariate Analysis – Professor: Dr. Andrew Neath

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Question 1

Consider repeated, independent rolls of a fair die.

- (a) Let Y_1 be the number of ones in 50 rolls. Completely specify the probability function for Y_1 .
- (b) Let (Y_1, Y_2, \dots, Y_6) be the number of ones, twos, etc. in 50 rolls. Completely specify the probability function for (Y_1, Y_2, \dots, Y_6) .

Answer.

- (a) Each roll is an independent trial. We model each roll as a discrete uniform distribution, $X_j \sim \text{DU}(1, 6)$ with probability distribution function (PDF)

$$f_{X_i}(k) = \frac{1}{6}, k \in \{1, \dots, 6\} \quad (1)$$

for $i = 1, \dots, 50$.

The probability that $X_i = 1$ is $\pi = f_{X_i}(1) = \frac{1}{6}$ and the probability that $X_i \neq 1$ is $1 - \pi = \frac{5}{6}$. Y_j is the number of outcomes where $X_i = j$, i.e., $Y_j = \sum_{i=1}^{50} [X_i = j]$. Since each X_i is i.i.d., Y_j is binomially distributed as

$$Y_j \sim \text{BIN} \left(n = 50, \pi = \frac{1}{6} \right), \quad (2)$$

with the PDF

$$f_{Y_j}(k | n = 50, \pi = 1/6) = \binom{50}{k} \left(\frac{1}{6} \right)^k \left(\frac{5}{6} \right)^{50-k}, \quad (3)$$

Thus, $Y_1 \sim f_{Y_1}(k | n = 50, \pi = 1/6)$.

- (b) A single observation of X_i may be considered as a random Boolean vector of dimension 6 whose j -th component realizes 1 if $X_i = j$ and otherwise 0, e.g., $X_i = 3$ maps to $(0, 0, 1, 0, 0, 0)$. Then, $\sum_{i=1}^{50} X_i$ maps to a random vector (Y_1, \dots, Y_6) where $Y_j = k_j$ if k_j of $\{X_i\}$ realizes j .

This joint distribution (Y_1, \dots, Y_6) models the multinomial distribution,

$$(Y_1, \dots, Y_6) \sim \text{MULT}(n = 50, \{\pi_j\}) \quad (4)$$

where $\pi_j = 1/6$ for $j = 1, \dots, 6$, which has the PDF

$$f(k_1, \dots, k_6) = \frac{50!}{\prod_{j=1}^6 k_j!} \prod_{j=1}^6 \left(\frac{1}{6}\right)^{k_j} \quad (5)$$

with the constraint that $\sum_{j=1}^6 k_j = n = 50$. We may rewrite the above equation as

$$f(k_1, \dots, k_6) = \frac{50!}{\prod_{j=1}^6 k_j!} \left(\frac{1}{6}\right)^{\sum_{j=1}^6 k_j}. \quad (6)$$

Since $k_1 + \dots + k_6 = 50$, we may rewrite the above equation as

$$f(k_1, \dots, k_6) = \frac{50!}{6^{50} \prod_{j=1}^6 k_j!}. \quad (7)$$

Also, observe that $k_6 = 50 - k_1 - \dots - k_5$, and thus we may reparameterize as

$$f(k_1, \dots, k_5) = \frac{50!}{6^{50} (50 - \sum_{j=1}^5 k_j) \prod_{j=1}^5 k_j!}. \quad (8)$$

Question 2

Let $Y_1 \sim \text{POI}(\lambda_1 = 1)$, $Y_2 \sim \text{POI}(\lambda_2 = 2)$, and $Y_3 \sim \text{POI}(\lambda_3 = 3)$ be independent random variables.

- (a) Completely specify the probability function for (Y_1, Y_2, Y_3) .
- (b) Completely specify the probability function for $Y_+ = \sum_{i=1}^3 Y_i$.
- (c) Completely specify the conditional probability function for (Y_1, Y_2, Y_3) given $Y_+ = n$.

Answer.

- (a) By independence,

$$f_{Y_1, Y_2, Y_3}(k_1, k_2, k_3 \mid \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3) = \prod_{i=1}^3 f_{Y_i}(k_i \mid \lambda_i). \quad (9)$$

Dropping the subscripts on f for notational simplicity and substituting the *Poisson* probability distribution functions into the above equation, we rewrite the joint distribution function as

$$f(k_1, k_2, k_3 \mid \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3) = \frac{\lambda_1^{k_1} e^{-\lambda_1}}{k_1!} \frac{\lambda_2^{k_2} e^{-\lambda_2}}{k_2!} \frac{\lambda_3^{k_3} e^{-\lambda_3}}{k_3!} \quad (10)$$

which may be rewritten as

$$f(k_1, k_2, k_3 \mid \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3) = \frac{\lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3} e^{-(\lambda_1 + \lambda_2 + \lambda_3)}}{k_1! k_2! k_3!}. \quad (11)$$

We may rewrite the above by substituting the given lambda values into right-hand-side,

$$f(k_1, k_2, k_3) = \frac{2^{k_2} 3^{k_3} e^{-(k_1 + k_2 + k_3)}}{k_1! k_2! k_3!} \quad (12)$$

- (b) The sum of independent poisson random variables is poisson with a failure rate given by the sum of the poisson failure rates, thus

$$Y_+ \sim \text{POI}(\lambda_+ = \lambda_1 + \lambda_2 + \lambda_3 = 6), \quad (13)$$

which has the PDF

$$f_{Y_+}(n | \lambda = 6) = \frac{6^n e^{-n}}{n!} \quad (14)$$

where $k \in \{0, 1, 2, \dots\}$.

- (c) By the laws of probability, the conditional distribution of (Y_1, Y_2, Y_3) given $Y_+ = n$ is

$$P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 | Y_+ = n) = \quad (15)$$

$$\frac{P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3, Y_+ = n)}{P(Y_+ = n)}. \quad (16)$$

It is given that $k_1 + k_2 + k_3 = n$, and therefore the only value that Y_+ can realize is n with probability 1, therefore

$$P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 | Y_+ = n) = \quad (17)$$

$$\frac{P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3)}{P(Y_+ = n)}. \quad (18)$$

Plugging in the joint distribution function for (Y_1, Y_2, Y_3) and the distribution function for Y_+ results in

$$P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 | Y_+ = n) = \frac{f(k_1, k_2, k_3 | \lambda_1, \lambda_2, \lambda_3)}{f_{Y_+}(n | \lambda_+)} \quad (19)$$

We may rewrite the above by plugging in the derivations of these distribution functions,

$$P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 | Y_+ = n) = \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3} e^{-(k_1 + k_2 + k_3)}}{6^n e^{-n} k_1! k_2! k_3!} \quad (20)$$

Since it is given that $k_1 + k_2 + k_3 = n$, we may perform these substitutions as desired, resulting in sequence of transformations given by

$$P(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 | Y_+ = n) = \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3} e^{-n}}{6^n e^{-n} k_1! k_2! k_3!} \quad (21)$$

$$= \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3}}{6^n k_1! k_2! k_3!} \quad (22)$$

$$= \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3}}{6^{k_1 + k_2 + k_3} k_1! k_2! k_3!} \quad (23)$$

$$= \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3}}{6^{k_1} 6^{k_2} 6^{k_3} k_1! k_2! k_3!} \quad (24)$$

$$= \frac{n!}{k_1! k_2! k_3!} \left(\frac{\lambda_1}{6}\right)^{k_1} \left(\frac{\lambda_2}{6}\right)^{k_2} \left(\frac{\lambda_3}{6}\right)^{k_3}, \quad (25)$$

which is the distribution function of the multinomial.

If we let W denote the conditional distribution of (Y_1, Y_2, Y_3) given $Y_+ = n = 50$, then

$$W \sim \text{MULT}(n = 50, \{\pi_j\}) \quad (26)$$

where $\pi_j = \frac{\lambda_j}{\lambda_+} = \frac{j}{6}$. This may finally be rewritten to

$$W \sim \text{MULT}\left(n = 50, \pi_1 = \frac{1}{6}, \pi_2 = \frac{2}{6}, \pi_3 = \frac{3}{6}\right). \quad (27)$$