Homework #2

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Question 1

Suppose n_1 is observed from a $\mathrm{BIN}(n; \pi)$ distribution. Derive the maximum likelihood estimate $\hat{\pi}$.

Answer. Let $X \sim \mathrm{BIN}(n; \pi)$. Then, $\mathrm{f_X}(k|n, \pi) = \binom{n}{k}\pi^k(1-\pi)^{n-k}$ and the likelihood is

$$\ell(\pi; n_1) = \pi^{n_1}(1-\pi)^{n-n_1}\,.$$

which has a log-likelihood

$$\mathrm{L}(\pi; n_1) = n_1 \log \pi + (n - n_1)\log(1-\pi)\,. \tag{1}$$

Assuming $\pi \in (0, 1)$, to find the maximum likelihood of π given that we observe $X = n_1$, we find

$$\hat{\pi} = \arg\max_{\pi} \mathrm{L}(\pi; n_1)$$

by solving for the zeros of the derivative of L,

$$\begin{aligned}
\left.\frac{\mathrm{d\,L}}{\mathrm{d}\pi}\right|_{\hat{\pi}} &= 0\\
\frac{n_1}{\hat{\pi}} - \frac{n-n_1}{1-\hat{\pi}} &= 0\\
\frac{n_1(1-\hat{\pi}) - (n-n_1)\hat{\pi}}{\hat{\pi}(1-\hat{\pi})} &= 0\\
n_1(1-\hat{\pi}) - (n-n_1)\hat{\pi} &= 0\\
n_1 - n_1\hat{\pi} - n\hat{\pi} + n_1\hat{\pi} &= 0\\
n_1 - n\hat{\pi} &= 0\\
n\hat{\pi} &= n_1\,,
\end{aligned}$$

and thus the MLE of π given an observation $X = n_1$ is

$$\hat{\pi} = \frac{n_1}{n}\,. \tag{2}$$

Question 2

Prove that if $\mathrm{P}(A|B) > \mathrm{P}(A|B')$ then $\mathrm{P}(B|A) > \mathrm{P}(B|A')$.

Proof. Starting at the proposition $\mathrm{P}(A|B) > \mathrm{P}(A|B')$, we apply rewrite rules that preserve the relation, eventually reaching the goal state $\mathrm{P}(B|A) > \mathrm{P}(B|A')$.

If $\mathrm{P}(A|B) > \mathrm{P}(A|B')$ then $\mathrm{odds}(A|B) > \mathrm{odds}(A|B')$ where $\mathrm{odds}(X) := \mathrm{P}(X)/\,\mathrm{P}(X')$. Making this substitution, we have

$$\frac{\mathrm{P}(A|B)}{\mathrm{P}(A'|B)} > \frac{\mathrm{P}(A|B')}{\mathrm{P}(A'|B')}\,.$$

Multiplying the LHS and RHS by convenient expressions for 1, respectively $\mathrm{P}(B)/\,\mathrm{P}(B)$ and $\mathrm{P}(B')/\,\mathrm{P}(B')$, we rewrite the above as

$$\frac{\mathrm{P}(A|B)\,\mathrm{P}(B)}{\mathrm{P}(A'|B)\,\mathrm{P}(B)} > \frac{\mathrm{P}(A|B')\,\mathrm{P}(B')}{\mathrm{P}(A'|B')\,\mathrm{P}(B')}\,.$$

Both sides are now a ratio of joint distributions. We rewrite them as

$$\frac{\mathrm{P}(B|A)\,\mathrm{P}(A)}{\mathrm{P}(B|A')\,\mathrm{P}(A')} > \frac{\mathrm{P}(B'|A)\,\mathrm{P}(A)}{\mathrm{P}(B'|A')\,\mathrm{P}(A')}\,.$$

Discarding the common factor $\mathrm{P}(A)/\,\mathrm{P}(A')$ from both sides, we rewrite the above as

$$\frac{\mathrm{P}(B|A)}{\mathrm{P}(B|A')} > \frac{\mathrm{P}(B'|A)}{\mathrm{P}(B'|A')}\,.$$

Multiplying both sides by $\mathrm{P}(B|A')/\,\mathrm{P}(B'|A)$, we may rewrite the above as

$$\frac{\mathrm{P}(B|A)}{\mathrm{P}(B'|A)} > \frac{\mathrm{P}(B|A')}{\mathrm{P}(B'|A')}$$

which may be rewritten as $\mathrm{odds}(B|A) > \mathrm{odds}(B|A')$. Finally, observe that

$$\mathrm{odds}(B|A) > \mathrm{odds}(B|A') \implies \mathrm{P}(B|A) > \mathrm{P}(B|A')\,.$$

$\square$