6.1

Multi-Factor ANOVA (Sec. 5.4), 2^k Design (Secs. 6.2, 6.3)

model: 3-factor ANOVA

$$Y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijkl}$$

lth response, $(i,j,k)^{th}$ treatment combination

$i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, c$; $l = 1, \ldots, n$

example 6.1 : Quality control department, fabric finishing shop

see R output

response = quality score (exp. unit = cloth specimen)

$a=3$, $b=2$, $c=3$, $(n=3)$, $abc=18$

factors = cycle time (40, 50, 60), temperature (300°, 350°), operator (1, 2, 3)

three.mod = aov(score ~ time * temp * operator)

$(a-1)(b-1)(c-1) = 4$

$abc(n-1) = 36$

$F^*_{ABC} = 3.523$, dfs = (4, 36), p-value = .016

There is evidence of a three factor interaction.

(i.e., two-factor interaction depends on the level of the third factor)

Let's look at interaction plots.

(*) If an interaction plot is parallel, or nearly so, then there is no need to include the interaction term.

(6.2) see R output, create an interaction plot for time, temperature for each operator.

summary: op. 1, time 50 is optimal, negligible temp effect
op. 2, temp 350 is best at time 40, no other temp effects
op. 3, temp effect at time 40, decreasing effect of time

comment: High order interactions are difficult to interpret and difficult to estimate

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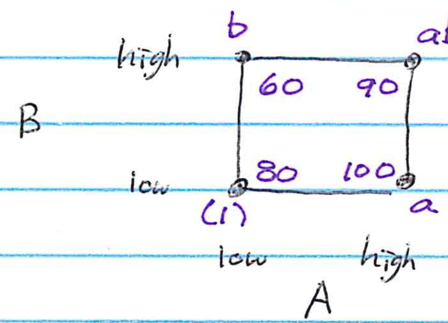
2^k design → k factors, tested at 2 levels each (low, high)

example: A = reactant concentration (15%, 25%)
B = catalyst amount (1 lb., 2 lbs.)
response = yield from a chemical process, $n=3$ ($N=12$)

see Figure 6.1

graphical display:



We use the notation (1), a, b, ab to denote the treatment combination, and the sum of responses at that treatment combination.

We use the notation A, B to represent the factors and the estimated main effects.

6.3

Define $A = \left[\frac{a-(1)}{n} + \frac{ab-b}{n} \right] \left(\frac{1}{2} \right)$

$$B = \left[\frac{b-(1)}{n} + \frac{ab-a}{n} \right] \left(\frac{1}{2} \right)$$

Note:

main effects are estimated by the change in mean response, averaged over levels of the other factor.

Define

$$AB = \left[\frac{ab-b}{n} - \frac{a-(1)}{n} \right] \left(\frac{1}{2} \right) = \frac{1}{2n} \left[ab - b - a + (1) \right]$$

Note: interaction effect

difference between effect of A (B) at high B (A) and effect of A (B) at low B (A).

Table of Contrasts

treatment	I	A	B	AB	sum
		effect			
(1)	+	−	−	+	80
a	+	+	−	−	100
b	+	−	+	−	60
ab	+	+	+	+	90

$$A = \frac{1}{2(3)} [50] = 8.33$$

$$B = \frac{1}{2(3)} [-30] = -5.00$$

$$AB = \frac{1}{2(3)} [-10] = -1.67$$

Recall: For orthogonal contrasts,

$$SS_{tr} = SS_A + SS_B + SS_{AB}$$

6.4

$$SS_A = nA^2, \quad SS_B = nB^2, \quad SS_{AB} = n(AB)^2$$

$$F = \frac{SS_c/1}{MS_E}$$

2^3 Design → Factors A, B, C each at 2 levels (low, high)

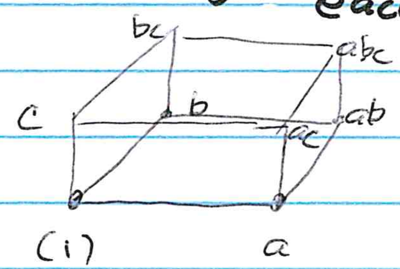


Table of Contrasts

treatment	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

Note: A three factor interaction occurs when a two factor interaction depends on the level of the remaining factor.

Comment: High order interactions may be difficult to interpret and difficult to estimate

(think about the factors that influence success ~~of a~~ in a professional baseball player)

6.5

see R handout

Example 6.2

A = time, B = concentration, C = pressure, D = temperature

Y = yield (2^4 Design, $n=1$)

Fit a model with two-factor interactions

```
two.mod = aov(Yield ~ (A+B+C+D)^2 )
```

select factors : A, C, D, A:C, A:D

```
reduced.mod = aov(Yield ~ A+C+D+A:C +A:D)
```

see Minitab handout

examine interaction plots, and main effect plots

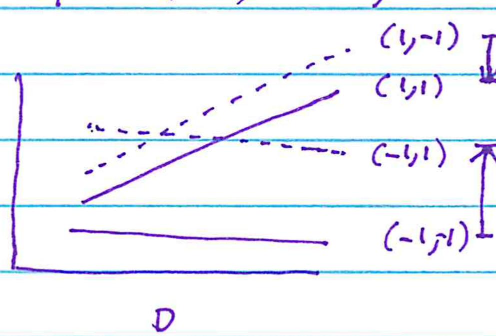
A:C] time ~~pressure~~ has a positive ~~negative~~ effect on yield when pressure is low.

A:D] time has a positive effect on yield when temperature is high

A, C, D] time, pressure, temperature each have a positive effect on yield

```
fits.r = predict(reduced.mod) ; A.C
```

```
interaction.plot( D, A.C , fits.r )
```



we estimate that the optimal yield is attained when

time = +1 (high)

pressure = -1 (low)

Effect Estimates, 2^k design

$k=2$:

$$A = \frac{1}{2n}\left[a-(1)+ab-b\right]$$

$$B = \frac{1}{2n}\left[b-(1)+ab-a\right]$$

$$AB = \frac{1}{2n}\left[ab-b-a+(1)\right]$$

$k=3$:

$$A = \frac{1}{4n}\left[a-(1)+ab-b+ac-c+abc-bc\right]$$

$$\vdots$$

$$ABC = \frac{1}{4n}\left[abc-bc-ac+c-ab+b+a-(1)\right]$$

$$= \frac{1}{4n}\left[\underbrace{\left((abc-bc)-(ac-c)\right)}_{AB,\ \text{high } C}-\underbrace{\left((ab-b)-(a-(1))\right)}_{AB,\ \text{low } C}\right]$$

In general,

$$Var(\hat{\text{effect}}) = \left(\frac{1}{2^{k-1}n}\right)^2 \cdot \left[2^k \cdot n\sigma^2\right] = \frac{\sigma^2}{n2^{k-2}}$$

$$SE(\hat{\text{effect}}) = \sqrt{\frac{MSE}{n2^{k-2}}}, \quad \boxed{CI = \hat{\text{effect}} \pm t_{\frac{\alpha}{2}, 2^k(n-1)} \cdot SE}$$

$$SS_{contrast} = n2^{k-2} \cdot (\hat{\text{effect}})^2$$

example:

A = concentration of reactant

B = catalyst amount

y = yield of chemical process

data:

A	B	treatment combination	data (n=3)	summary statistics
−	−	A low, B low	28, 25, 27	$(1) = 80,\ S^2_{(1)} = 2.333$
+	−	A high, B low	36, 32, 32	$a = 100,\ S^2_a = 5.333$
−	+	A low, B high	18, 19, 23	$b = 60,\ S^2_b = 7.000$
+	+	A high, B high	31, 30, 29	$ab = 90,\ S^2_{ab} = 1.000$

$MSE = \text{mean}\{S^2\} = 3.9167$, $SE(\widehat{\text{effect}}) = \sqrt{\frac{MSE}{n}}$

$= 1.143$

$t_{.025,8} = 2.365$, margin of error $= (2.365)(1.143) = 2.7$

$\widehat{A\text{effect}} = 8.33 \quad [5.63, 11.03]$

$\widehat{B\text{effect}} = -5.00 \quad [-7.70, -2.30]$

$\widehat{AB\text{effect}} = 1.67 \quad [-1.03, 4.37]$