5.1

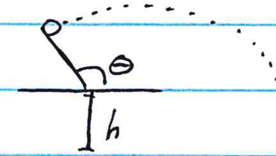
Two Factor Analysis of Variance
(Secs. 5.1, 5.2, 5.3)

previous: investigate the effect of (one) factor on response.

now: Define Factor A with a levels, and factor B with b levels. (There are ab treatment combinations)

examples:
- factor A = amount of Fertilizer (drug dose)
  factor B = soil quality (severity of illness)
  response = crop yield (recovery time)

- factor A = height of catapult
  factor B = angle
  response = distance traveled



Example (5.1) See Handout

(B) temperature (15°, 75°, 125° F)
(A) plate material (1, 2, 3)

response
battery lifetime

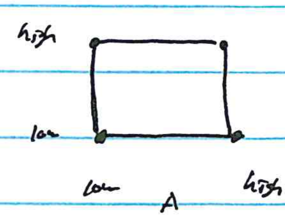
Data

Factor B

Factor A	1	2	...	b
1	$\bar{Y}_{11\cdot}$	$\bar{Y}_{12\cdot}$	...	$\bar{Y}_{1b\cdot}$
2	$\bar{Y}_{21\cdot}$	$\bar{Y}_{22\cdot}$	...	$\bar{Y}_{2b\cdot}$
⋮	⋮	⋮	...	⋮
a	$\bar{Y}_{a1\cdot}$	$\bar{Y}_{a2\cdot}$	...	$\bar{Y}_{ab\cdot}$

$\{Y_{ijk}\}$, $i = 1, \ldots, a$; $j = 1, \ldots, b$
$k = 1, \ldots, n$

kth response, ith level A, jth level B

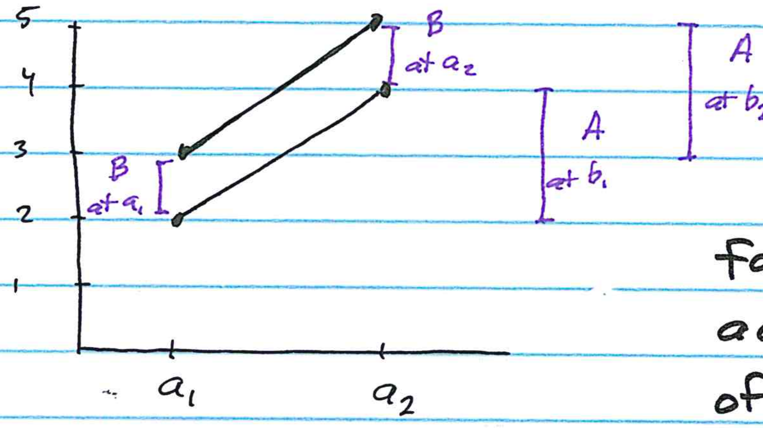


(5.2)

model: $Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$

overall mean — effect of i^{th} level A — effect of j^{th} level B — interaction effect

example: (underlying means, i.e., signal among noise)



The effect of one factor is the same across all levels of the other factor.

parallel interaction plot

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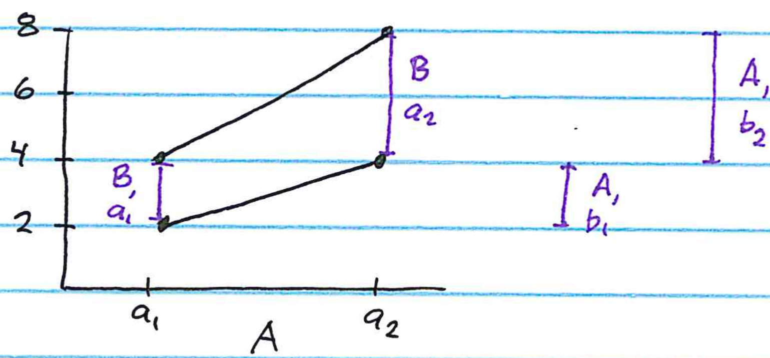
example: (Now suppose $\mu_{11} = 2$, $\mu_{21} = 4$, $\mu_{12} = 4$, $\mu_{22} = 8$)



see HW

(*) An interaction effect occurs when the effect of one factor depends on the level of the other factor.

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parameter estimates: $\hat{\mu}_{ij} = \bar{Y}_{ij\cdot}$

$$\hat{\mu} = \bar{Y}_{\cdots}, \quad \underbrace{\hat{\tau}_i = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdots}, \quad \hat{\beta}_j = \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdots}}_{\text{main effects}}$$

(5.3)

$$(\hat{\tau\beta})_{ij} = \hat{\mu}_{ij} - (\hat{\mu} + \hat{\tau}_i + \hat{\beta}_j) = \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdot\cdot\cdot}$$

interaction effect

Sum of squares:

$$SS_A, \; SS_B, \; SS_{AB}, \; SS_E$$

degrees of freedom:

$$a-1, \; b-1, \; (a-1)(b-1), \; ab(n-1) = N - ab$$

test statistics:

$$F_A = \frac{MS_A}{MS_E}, \quad F_B = \frac{MS_B}{MS_E}, \quad F_{AB} = \frac{MS_{AB}}{MS_E}$$

Note: Main effects are averaged over the levels of the other factor.

Back to the example: $a = 3, b = 3, n = 4$ (balanced)

See R output

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contr.sum  ← used to define parameter estimates
two.way.mod = aov(Y ~ A + B + A:B)
summary(two.way.mod)
```

ANOVA Table:

Source	F_0	dfs	p-value
Material	7.91	2,27	.002
Temperature	28.97	2,27	.0001
Interaction	3.56	4,27	.0186

see HW (*)

see HW (*)

interpretation: The experiment finds an interaction effect (between plate material and temperature) on battery lifetime.

(5.4)

next steps: (0) parameter estimates

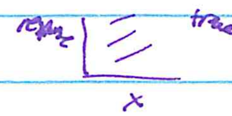
(1.) graphical displays (interaction plot, main effects plot)

(2.) pairwise comparisons (Fisher, Tukey)

see R output

dummy.coef — computes parameter estimates $\hat{\mu} = 105.5278$, $\hat{\tau}_1 = -22.36, \dots, (\hat{\tau\beta})_{33} = 1.78$

interaction.plot (X.factor, trace.factor, response)



Note the clear departure from parallel.

Because effects change, there is no simplification.

combined = interaction (material, temperature) — create one factor with ab = 9 levels

comparisons = glht (comb.mod , "Tukey")

contrasts( )

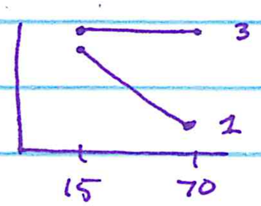
$\binom{9}{2} = 36$ comparisons

Minitab output may be helpful

temp = 15	2, 3, 1	plate = 1	15, 70, 125
temp = 70	3, 2 1	plate = 2	15, 70, 125
temp = 125	3, 1, 2	plate = 3	15, 70, 125

interpretation:

see HW (*)



The experiment finds that plate material has no effect on lifetime when temp is 15°, but that material 3 leads to greater lifetimes than material 1 when temp is 70°.

5.5

Example 5.2 → see R output

factor A = pressure (200, 215, 230)
factor B = temperature (150, 160, 170)
response = yield

see R output:

X.mod = aov(y ~ p*T) ← fits an interaction model

F_A, F_B, F_{AB}, p's = .0004, .0085, .4700

interpretation: The experiment finds that pressure and temperature both have an effect on yield. ~~However~~ Furthermore, the experiment finds that there is no interaction effect.

see HW (*)

a.mod = aov(y ~ p+T) ← fits an additive model
a.means = predict(a.mod)

additive model

HW(*): The model smooths over the randomness in the data, simplifying the analysis

see R output: grouping information, interval estimates, plots

t-statistics, p-values, grouping information, box plots, confidence intervals (Fisher, Tukey)

$\mu_3 > \mu_1 > \mu_2$
$\mu_3 > \mu_2 > \mu_1$
$\mu_2 > \mu_3 > \mu_2$

temperature: 170, 150, 160 (3, 1, 2) Tukey

pressure: 215, 200, 230 Tukey

Example 5.3: pour temp, titanium amt., strength

see interaction plot for an illustration of a large interaction effect.

<u>example</u>: Computing SSA, SSB, SSAB, SSE (MSE) in a two factor ANOVA. ($MSE = \text{mean}\{s_{ij}^2\}$)

$\boxed{a=2, b=2 \\ n=4}$

		B: $j=1$	$j=2$	
A	$i=1$	$\bar{Y}_{11.} = 90$	$\bar{Y}_{12.} = 78$	$\bar{Y}_{1..} = 84$ ($\hat{\tau}_1 = -2$)
	$i=2$	$\bar{Y}_{21.} = 92$	$\bar{Y}_{22.} = 84$	$\bar{Y}_{2..} = 88$ ($\hat{\tau}_2 = +2$)
		$\bar{Y}_{.1.} = 91$	$\bar{Y}_{.2.} = 81$	$\bar{Y}_{...} = 86$ ($\hat{\mu} = 86$)
		($\hat{\beta}_1 = +5$)	($\hat{\beta}_2 = -5$)	

$$SSA = \sum_{i,j,k} (\bar{Y}_{i..} - \bar{Y}_{...})^2 = bn \sum_i^a \hat{\tau}_i^2 = 2(4)\left((-2)^2 + 2^2\right) = 64$$

$$SSB = \sum_{i,j,k} (\bar{Y}_{.j.} - \bar{Y}_{...})^2 = an \sum_i^b \hat{\beta}_j^2 = 2(4)\left((5)^2 + (-5)^2\right) = 400$$

Fitted values for main effects model

$$\hat{Y}_{ij}^{(A)} = \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j = \bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}$$

	$j=1$	$j=2$
$i=1$	89	79
$i=2$	93	83

$(\widehat{\tau\beta})_{11} = +1 \quad (\widehat{\tau\beta})_{12} = -1$

$(\widehat{\tau\beta})_{21} = -1 \quad (\widehat{\tau\beta})_{22} = +1$

$$SSAB = \sum_{i,j,k} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 = n \sum_{i,j} (\widehat{\tau\beta})_{ij}^2 = 4(4) = 16$$

$$(\widehat{\tau\beta})_{ij} = \bar{Y}_{ij.} - \hat{Y}_{ij}^{(A)}$$

Example 5.2, some more details (study temperature effects)

R : compare.T = glht(a.mod, linfct = mcp(T = "tukey"))

summary(compare.T, test = univariate())

CI

$H_0: \mu_{160} = \mu_{150}$, $t_0 = -2.176$, $p = .049$ $[-.37, .001]$

$H_0: \mu_{170} = \mu_{150}$, $t_0 = 1.958$, $p = .072$ $[-.02, .32]$

$H_0: \mu_{170} = \mu_{160}$, $t_0 = 4.134$, $p = .001$ $[.15, .48]$

R : cld(   )

largest ⟶ smallest

$\underbrace{170 \quad 150}_{\text{"B"}} \qquad \underbrace{160}_{\text{"A"}}$

R: confint(   )

R: default is Tukey multiple pairwise comparisons

$H_0: \mu_{160} = \mu_{150}$, $t_0 = -2.176$, $p = .113$ $[-.37, .040]$

$H_0: \mu_{170} = \mu_{150}$, $t_0 = 1.958$, $p = .162$ $[-.05, .35]$

$H_0: \mu_{170} = \mu_{160}$, $t_0 = 4.134$, $p = .003$ $[.11, .52]$

3	1	2
170	150	160

Tukey orderings: $\mu_3 > \mu_1 > \mu_2$

$\mu_3 > \mu_2 > \mu_1$

$\mu_1 > \mu_3 > \mu_2$

Example 5.3

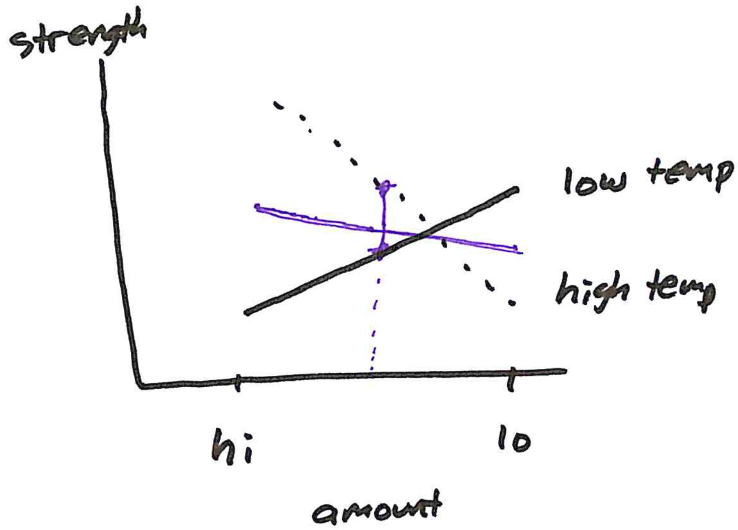
A = temparture
B = amount
y = strength of titanium rods

$F_A = 1.522$ $(P_A = .285)$

$F_B = 2.516$ $(P_B = .188)$

$F_{AB} = 14.759$ $(P_{AB} = .018)$

interaction plot:



large interaction effect,
main effects
average out