HW#3

1\. (a) $Q(\beta) = \sum (Y_i - \beta X_i)^2$

$$\frac{\partial Q(\beta)}{\partial \beta} = \sum 2(Y_i - \beta X_i) X_i \overset{set}{=} 0$$

so. $\sum X_i Y_i - \beta \sum X_i^2 = 0$

$$\hat{\beta} = \frac{\sum X_i Y_i}{\sum X_i^2}$$

(b) $E(\hat{\beta}) = E\left(\frac{\sum X_i Y_i}{\sum X_i^2}\right) = \frac{\sum X_i E(Y_i)}{\sum X_i^2}$

$$E(Y_i) = E(\beta X_i + \varepsilon_i) = \beta X_i + 0 = \beta X_i$$

so $E(\hat{\beta}) = \frac{\sum X_i \beta X_i}{\sum X_i^2} = \beta$

(c) $Var(\hat{\beta}) = Var\left(\frac{\sum X_i Y_i}{\sum X_i^2}\right)$

$$= \frac{\sum X_i^2 Var(Y_i)}{\left(\sum X_i^2\right)^2} = \frac{Var(Y_i)}{\sum X_i^2}$$

$Var(Y_i) = Var(\beta X_i + \varepsilon_i) = Var(\varepsilon_i) = \sigma^2$

so $Var(\hat{\beta}) = \frac{\sigma^2}{\sum X_i^2}$

(d) $Y = X\beta + \varepsilon$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \quad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \quad \beta = [\beta], \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

(e) $\hat{\beta} = (X^T X)^{-1} X^T Y$

$$= \left([X_1 \ X_2 \ \cdots \ X_n] \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \right)^{-1} [X_1 \cdots X_n] \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

$$= (\Sigma X_i^2)^{-1} \ \Sigma X_i Y_i$$

$$= \frac{\Sigma X_i Y_i}{(\Sigma X_i^2)^{-1}}$$

(f) $Var(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$

$$= \sigma^2 \left([X_1 \cdots X_n] \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \right)^{-1}$$

$$= \sigma^2 (\Sigma X_i^2)^{-1}$$

$$= \frac{\sigma^2}{\Sigma X_i^2}$$

2. (a)

	Df	Sum Sq.	Mean Sq
Regression	2	81117	40558.5
Error	234	23565	100.7051
Total	236	104682	

(b) $F = \frac{MSR}{MSE} = 402.7451 \sim F(2, 234)$

$P(F > 402.7451) \approx 0 = P\text{-value}$

Reject H_0.

We have sufficient evidence to conclude at least one of the β_1 & β_2 is not 0.

(c) $H_0: \beta_1 = 0$

$H_a: \beta_1 \neq 0$

$$t = \frac{\hat{\beta}_1 - 0}{SE(\hat{\beta}_1)} = \frac{8.56069}{0.31150} = 27.482 \sim t(234)$$

$P\text{-value} = 2P(t > 27.482)$

$= 2P(t < -27.482)$

$= 3.325 \times 10^{-75}$

Reject H_0

$\beta_1 \neq 0$

Can't be dropped

$H_0: \beta_2 = 0$

$H_a: \beta_2 \neq 0$

$$t = \frac{\hat{\beta}_2}{SE(\hat{\beta}_2)} = \frac{0.28060}{0.04621} = 6.072 \sim t(234)$$

$P\text{-value} = 2P(t < -6.072)$

$= 5.062 \times 10^{-9}$

Reject H_0

$\beta_2 \neq 0$.

Can't be dropped

(d) For every one month increase in age, the learning time increases by 8.56 min, on average, holding GMS constant.
For every one unit increase in GMS, the learning time increases by 0.28 min, on average, holding age constant.

(e) $$R^2 = \frac{SSR}{SSTo} = \frac{81117}{104682} = 0.7749$$

The linear regression model (learning time $= \beta_0 + \beta_1 \text{Age} + \beta_2 \text{GMS}$) explains 77.49% of the total variation in learning time.

3. See R-code