8.1

Mixed Models
(sec. 13.3)

idea: A, B factors $\Rightarrow$ two factor ANOVA (we saw this earlier in the semester.)

Now, A is a ~~fixed~~ (fixed) factor (levels $1, 2, \dots, a$)
B is a (random) block (randomly select levels $1, 2, \dots, b$)

motivating example: Consider an experiment to compare $a = 3$ drug protocols across a very large number of medical centers. A random sample of $b = 4$ is selected. Each drug produces a sample of size n from each of the selected medical centers.

A = drug protocal (fixed factor, $a = 3$ levels)
B = medical center (random factor / block, $b = 4$ randomly selected levels) 1.

<u>Data</u>:

		B: 1	2	$\cdots$	b
A	1	$Y_{111}, \dots, Y_{11n}$	$\cdots$	$\cdots$	$Y_{1b1}, \dots, Y_{1bn}$
	2	$\vdots$			$\vdots$
	$\vdots$				
	a	$Y_{a11}, \dots, Y_{a1n}$	$\cdots$	$\cdots$	$Y_{ab1}, \dots, Y_{abn}$

$Y_{ijk} \rightarrow$ kth response,
ith level of fixed factor A
jth selected level of random factor B.

(8.2) The data layout seems to be exactly like that of the ordinary two factor ANOVA. One of our goals in this section is to motivate why the analysis is different

Application: Handout 8, Example 1

A = operator (fixed factor) $a=3$
B = part (random factor / block) $b=20$

Each instrument is used to make $n=2$ measurements on each selected part.

Note: This application looks much like our problems modeled as a block design. However, we are taking multiple measurements for each fixed factor level, instead of a single measurement.

R → interaction.plot(part, operator, m)

model: $$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i=1,\dots,a \\ j=1,\dots,b \\ k=1,\dots,n \end{cases}$$

HW (*)

Fixed effect parameters: $\tau_1,\dots,\tau_a$ $\left(\sum_1^a \tau_i = 0\right)$

random effect parameters: σ^2_β, $\sigma^2_{\tau\beta}$, σ^2

mean squares:
$$MSA = \frac{bn\sum_{i=1}^a(\bar{y}_{i..}-\bar{y}_{...})^2}{a-1}, \quad MS_B = \frac{an\sum_{j=1}^b(\bar{y}_{.j.}-\bar{y}_{...})^2}{b-1}$$

$$MS_{AB} = \frac{n\sum_i\sum_j(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2}{(a-1)(b-1)}$$

$$MS_E = \sum_i\sum_j\sum_k(y_{ijk}-\bar{y}_{ij.})^2 / (N-ab)$$

(8.3) The sum of squares breakdown, and the mean squares, are the same for the mixed model as for ordinary two way ANOVA. But the test statistics are different!

Testing for a factor A effect
$H_0: \tau_1 = \tau_2 = \ldots = \tau_a = 0$ (null hypothesis: factor A has no effect on response)

HW [redacted]
(*)

$$E(MSA) = \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_1^a \tau_i^2}{a-1}$$

$$E(MSB) = \sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_\beta$$

$$E(MS_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta}, \quad E(MS_E) = \sigma^2$$ 2.

HW [redacted]
(*)

Under the null hypothesis ($H_0: \tau_1 = \ldots = \tau_a = 0$), $E(MSA) = \sigma^2 + n\sigma^2_{\tau\beta}$. Thus, the appropriate scaling requires a denominator with the same expected value. So, MS_{AB} is the appropriate error term.

HW [redacted]

$\cancel{F_0 = \frac{MSA}{MSAB}}$ $\left(F_A = \frac{MSA}{MS_{AB}} \right)$

Both notations for the F statistic testing factor A effects

Let F_0 denote the test statistic based on a mixed model likelihood approach $F_0 = 1.48$ dfs $= (2, 98)$ $P = .2324$

see R output

mixed.test (operator, part, m)

$MSA = 1.3083$

HW [redacted] $MS_{AB} = 0.712$

HW [redacted]

$F_A = 1.84$, $P = .173$, dfs $= (2, 38)$

interpretation: The experiment finds that operator does not have an effect on the measurement

Note: mixed.test is a user defined function in R. "lmerTest" does something similar, but not exactly the same.

3

8.4

Note: B as a random factor changes the statistic for testing the A effect.

Let's find more ways to understand this result.

data layout: (different approach)

$$
\begin{array}{cc|cccc}
 & & & B \text{ (think as ~~an~~ exp. units)} & & \\
 & & 1 & 2 & \cdots & b \\
 & 1 & \begin{bmatrix} y_{111} \\ \vdots \\ y_{11n} \end{bmatrix} & & \cdots & \begin{bmatrix} y_{1b1} \\ \vdots \\ y_{1bn} \end{bmatrix} \\
A & \vdots & \vdots & & & \\
 & a & \begin{bmatrix} y_{a11} \\ \vdots \\ y_{a1n} \end{bmatrix} & & \cdots & \begin{bmatrix} y_{ab1} \\ \vdots \\ y_{abn} \end{bmatrix}
\end{array}
$$

In example 1, random factor B is the part being measured, which is the experimental unit.

Because we have repeat measurements, it seems reasonable to use sample means as a summary.

data layout: (sample means)

$$
\begin{array}{cc|cccc}
 & & & B & & \\
 & & 1 & 2 & \cdots & b \\
\hline
 & 1 & \bar{y}_{11\cdot} & \bar{y}_{12\cdot} & \cdots & \bar{y}_{1b\cdot} \\
A & \vdots & \vdots & \vdots & & \vdots \\
 & a & \bar{y}_{a1\cdot} & \bar{y}_{a2\cdot} & \cdots & \bar{y}_{ab\cdot} \\
 & & \uparrow & \uparrow & & \uparrow
\end{array}
$$

Each factor level $(1, \ldots, a)$ provides a measurement on each block level / exp. unit $(1, \ldots, b)$.

This is the randomized complete block design from the previous problem set, 7.

HW 1(c)
(*)

(8.5) The algebraic formula for the test statistic from a block design on the sample means follows from Section 7.

$$F_0 = \frac{b \sum_{i=1}^{a} (\bar{Y}_{i..} - \bar{Y}_{...})^2 / (a-1)}{\sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 / (a-1)(b-1)} \qquad 4.$$

(we argued last section that the interaction between trtmnt and block measures how well we can generalize to the larger population.)

see R output

o = operator, p = part, ~~means = sample mean~~
means = sample mean of measurements

rcbd = lmer(means ~ o + (1|P))

random effects model

response is mean measurement

operator is a fixed effect

part is a random effect

$F_0 = 1.838$, $p = .173$ (same result as we got earlier) 4.

HW [illegible]
(*)

<u>comment</u>: Repeat measurements are summarized by a sample mean. The test statistic for a block design then leads to the interaction mean squares under a mixed model as the error term.

8.6

Example 8.2 (see R output)

A = temperature (fixed factor, levels = 800, 825, 850)
B = furnace position (random factor, $b = 2$)
response = baked density

		B: 1	B: 2
A	1	$Y_{111}, Y_{112}, Y_{113}$	
	2		
	3		

$n = 3$ measurements for each treatment combination.

But furnace position (random factor B) is the experimental unit. (i.e., $b = 2$ is the pertinent sample size, not $n = 3$.)

HW (*). Taking repeat measurements at each randomly selected level may serve to increase the measurement accuracy, but does not increase the pertinent sample size. 5.

mixed.test(temp, pos, density) $\rightarrow F_A = \frac{MS_A}{MS_{AB}}$

(correct test statistic for B random)

aov.mod = aov(density ~ temp*pos)

anova(aov.mod) $\rightarrow F_A = \frac{MS_A}{MS_E}$ (incorrect test for B random!)

Unbiased estimators for variance components:

$$\hat{\sigma}^2 = MS_E \,, \quad \hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} \,, \quad \hat{\sigma}^2_{\beta} = \frac{MS_B - MS_{AB}}{an}$$

Example 8.1 : $\hat{\sigma}^2 = 0.99$, $\hat{\sigma}^2_{\tau\beta} = -0.14$ (?) , $\hat{\sigma}^2_{\beta} = 10.28$

Block design on sample means: (Denote this as model (*))

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

$\Rightarrow$

$$(*) \quad \bar{Y}_{ij\cdot} = \mu + \tau_i + \beta_j + \varepsilon^{(*)}_{ij} \quad \begin{cases} i = 1, \dots, a \\ j = 1, \dots, b \end{cases}$$

where $\varepsilon^{(*)}_{ij} = (\tau\beta)_{ij} + \bar{\varepsilon}_{ij\cdot}$

($\varepsilon^{(*)}_{ij}$ includes both interaction variance and measurement variance)

$\left(Var(\varepsilon^{(*)}_{ij}) = \frac{\sigma^2_\varepsilon}{n} + \sigma^2_{\tau\beta} \right)$

$= (\sigma^2)^*$

$$SS^{(*)}_A = b \sum_{1}^{a} (\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2 = \frac{SS_A}{n}$$

Similarly, $SS^{(*)}_B = \frac{SS_B}{n}$, $SS^{(*)}_E = \frac{SS_{AB}}{n}$

and $$SS^{(*)}_{TO} = \frac{SS_A + SS_B + SS_{AB}}{n}$$

Thus,

$$SS_{TO} = SS_A + SS_B + SS_{AB} + SS_E$$

$$= n \cdot SS^{(*)}_{TO} + SS_E$$

(mixed model allows for decomposition of interaction variance and measurement variance)

$$MS_A^{(*)} = \frac{SS_A^{(*)}}{a-1} = MS_A/n$$

$$MS_B^{(*)} = \frac{SS_B^{(*)}}{b-1} = MS_B/n$$

$$MS_E^{(*)} = \frac{SS_E^{(*)}}{(a-1)(b-1)} = MS_{AB}/n$$

$$E(MS_A^{(*)}) = (\sigma^2)^* + \frac{b\sum_i^a \tau_i^2}{a-1} = \frac{\sigma_\varepsilon^2}{n} + \sigma_{\tau\beta}^2 + \frac{b\sum_i^a \tau_i^2}{a-1}$$

$$E(MS_A) = E(n \cdot MS_A^{(*)}) = \sigma_\varepsilon^2 + n\sigma_{\tau\beta}^2 + \frac{nb\sum_i^a \tau_i^2}{a-1}$$

Similarly,

$$E(MS_B) = E(n \cdot MS_B^{(*)}) = \sigma_\varepsilon^2 + n\sigma_{\tau\beta}^2 + na\sigma_\beta^2$$

$$E(MS_{AB}) = E(n \cdot MS_{AB}^{(*)}) = \sigma_\varepsilon^2 + n\sigma_{\tau\beta}^2$$

End Notes, #8

1. Because medical centers serve different sub groups (i.e., rural vs. city), a comparison between drug protocols should occur for each of the sampled medical centers. A generalization to the larger population is supported when the observed data shows a consistent effect across the different medical centers. That is, a small interaction supports the hypothesis of a treatment effect for the fixed factor in a Mixed Model.

2. Write the parameter estimates for the ANOVA model

$$\hat{\tau}_i = \bar{Y}_{i..} - \bar{Y}_{...}, \quad \hat{\beta}_j = \bar{Y}_{.j.} - \bar{Y}_{...}, \quad (\hat{\tau\beta})_{ij} = \bar{Y}_{ij.} - (\hat{\mu} + \hat{\tau}_i + \hat{\beta}_j)$$

$$\hat{\mu} = \bar{Y}_{...} \qquad\qquad = \bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}$$

From here, we derive the MS equations.

We are measuring different attributes, however, when one of the factors is random.

$$E(MS_A) = \underbrace{\frac{bn \sum \hat{\tau}_i}{a-1}}_{\text{Factor A effect}} + \underbrace{n\sigma^2_{\tau\beta} + \sigma^2}_{\text{random variance, depends on the interaction effect}}$$

3. Remember that "no effect" in our interpretation is not enough to claim that the measurement devices have no practical difference. An investigation into effect size would be necessary to make the stronger claim.

point estimates (Example 8.1) $\hat{\mu} = 22.39$, $\hat{\tau}_1 = -0.09$, $\hat{\tau}_2 = -0.12$, $\hat{\tau}_3 = 0.21$

4. The test statistic from a block design on the sample means is based on a model of the form:

$$\bar{Y}_{ij\cdot} = \mu + \tau_i + \beta_j + \varepsilon_{ij}^* \quad \begin{cases} i = 1, \dots, a \\ j = 1, \dots, b \end{cases}$$

Call this model (*). It can be shown that

$$SS_A^{(*)} = \frac{SS_A}{n}, \quad SS_B^{(*)} = \frac{SS_B}{n}, \quad SS_E^{(*)} = \frac{SS_{AB}}{n}.$$

That is, the sum of squares based on the sample means are smaller by the factor n, and the sum of squared errors is the treatment * block interaction.

5. Taking repeat measurements does allow us to learn more about the treatment effect at those selected levels. For example, you may have therapies that work well for you, but are not effective on average, across the population.