

STAT 581 - Exam 2: Due Dec 14, 2021

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Problem 1

A randomized complete block design is used to study the effect of caliper on the measured diameters (in mm) of ball bearings. A sample of $b = 10$ ball bearings is randomly selected, and each of $a = 3$ calipers produces a measurement on each of the selected ball bearings. The data is provided as an attachment.

Preliminary

treatment = caliper($a = 3$)
block = ball bearing($b = 10$)
y = measured diameter

(a)

Write the model and distributional assumptions for a randomized complete block design.

Reproduce

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \begin{cases} i = 1, \dots, a = 3 \\ j = 1, \dots, b = 10 \end{cases}$$

where

$$\begin{aligned} \beta_j &\sim \mathcal{N}(0, \sigma_\beta^2), \\ \epsilon_{ij} &\sim \mathcal{N}(0, \sigma_\epsilon^2), \end{aligned}$$

and $\{\tau_i\}$ are fixed effects, $\{\beta_j\}$ are random effects, $\{\epsilon_{ij}\}$ are random errors, σ_β^2 is between block variance, and σ_ϵ^2 is within block variance (variance in the response, or measurement variance).

(b)

Provide the algebraic formulas for MS_{tr} , MS_{bl} , MS_E . Explain why it makes sense for an interaction effect to serve as a measure of error variance.

Reproduce

$$\begin{aligned}MS_{tr} &= \frac{b \sum_{i=1}^a (\bar{Y}_{i.} - \bar{Y}_{..})^2}{a - 1}, \\MS_{bl} &= \frac{a \sum_{j=1}^b (\bar{Y}_{.j} - \bar{Y}_{..})^2}{b - 1}, \\MS_E &= \frac{\sum_{i=1}^a \sum_{j=1}^b (Y_{ij} - \bar{Y}_{i.} - \bar{Y}_{.j} + \bar{Y}_{..})^2}{(a - 1)(b - 1)},\end{aligned}$$

We are testing whether or not an observed treatment effect is generalizable to a larger population. How the effect depends on the experimental units determines how well the effect can be generalized.

(c)

Test for systematic differences in the measurements provided by the calipers. Compute the F_0 statistic, and the p -value. Provide an interpretation, stated in the context of the problem.

```
library("readxl")
#library("lme4")
library("lmerTest")

data = read_excel("exam2data.xlsx")
A = as.factor(na.omit(data$caliper)) # fixed effect
B = as.factor(na.omit(data$ball.bearing)) # random effect
y = na.omit(data$diameter) # response
head(data.frame(caliper=A,ball.bearing=B,diameter=y))

##   caliper ball.bearing diameter
## 1      1           1    26.88
## 2      1           2    26.53
## 3      1           3    26.58
## 4      1           4    26.86
## 5      1           5    26.33
## 6      1           6    26.60

# fixed effect tau1 + tau2 + tau3 = 0 (for calipers)
contrasts(A)=contr.sum

random.mod = lmer(y ~ (1|B) + A)

# the anova command is used to compute the test for fixed effects.
anova(random.mod)

## Type III Analysis of Variance Table with Satterthwaite's method
##      Sum Sq Mean Sq NumDF DenDF F value Pr(>F)
## A 0.10085 0.050423      2     18  5.5157 0.01354 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Reproduce

$F_0 = 5.516$ (p -value = .014).

The experiment finds that caliper (factor A) has an effect on the measured diameter of ball bearings (response y).

(d)

Compute estimates of the variance components. Explain when a block design is better than a completely randomized design.

```
print(VarCorr(random.mod),comp="Variance")
```

```
## Groups   Name                Variance
## B        (Intercept) 0.0411089
## Residual                        0.0091419
```

Reproduce

$$\hat{\sigma}_\epsilon^2 = .0091,$$

$$\hat{\sigma}_{bl}^2 = .0441.$$

A block design is better than a CRD when between block variance is large relative to within block variance.

(e)

Compute estimates of the fixed effect parameters. Explain why block effects are modeled differently than treatment effects in this design. Explain what treatment effect is estimable in this design.

```
# since the fixed effect estimates must sum to 0, the estimate at level a=3 is
# the negative of the sum of the fixed effect estimates at levels 1 through a-1=2.
est = coef(summary(random.mod))[1:3,1]
est.tau.hat = c(est,0-sum(est[2:3]))
names(est.tau.hat) = c("mu","tau1","tau2","tau3")
round(est.tau.hat,digits=3)
```

```
##      mu   tau1   tau2   tau3
## 26.532  0.081 -0.032 -0.050
```

Reproduce

$$\hat{\mu} = 26.532,$$

$$\hat{\tau}_1 = .081,$$

$$\hat{\tau}_2 = -.032,$$

$$\hat{\tau}_3 = -.50.$$

Because ball bearings have no identifiable features, we model block level effects through a probability distribution. Since levels for a block are not identifiable, we can only estimate the main effect (average effect/aggregate effect) of the treatment variable.

Problem 2

Now, a mixed effects design is used to study the effect of caliper (fixed effect, factor A) on the measured diameters of ball bearings. There are $a = 2$ calipers under investigation. A random

sample of $b = 8$ ball bearings is selected (random effect, factor B), and each caliper produces $n = 3$ measurements on each of the selected ball bearings. The data is provided as an attachment.

(a)

Write the model for this mixed effects design, defining the fixed effect parameters, and the random effect parameters.

Reproduce

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \begin{cases} i = 1, \dots, a = 2, \\ j = 1, \dots, b = 8, \\ k = 1, \dots, n \end{cases}$$

Fixed effect parameters: τ_1, \dots, τ_a ($\sum_i \tau_i = 0$) Random effect parameters: $\sigma_\beta^2, \sigma_{\tau\beta}^2, \sigma^2$.

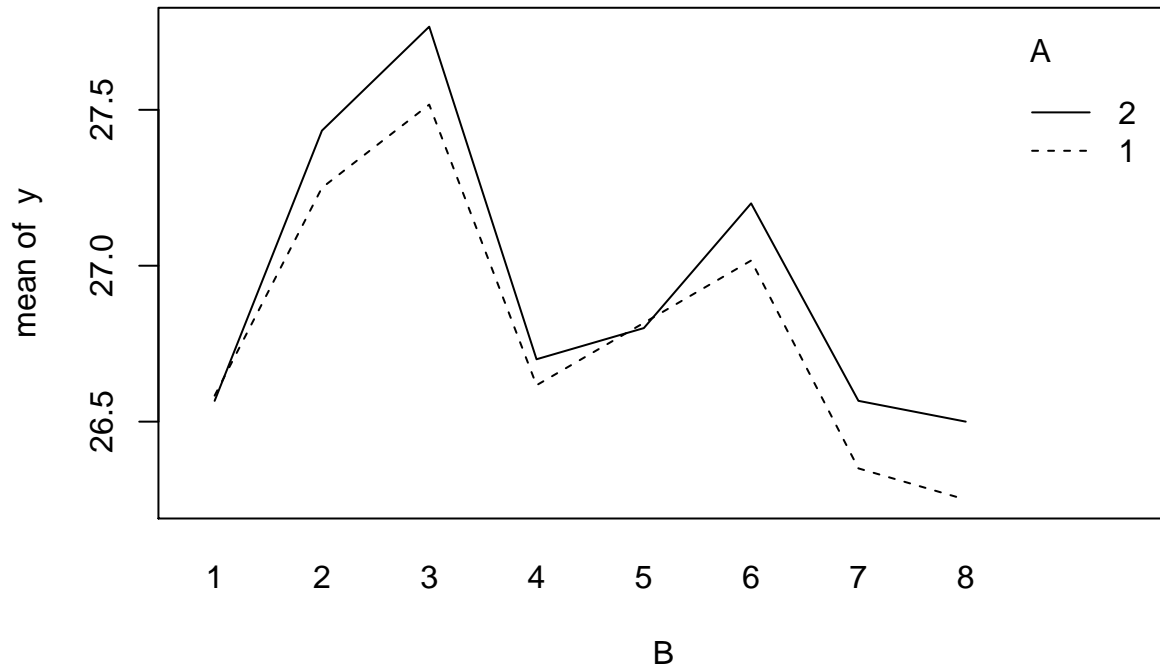
(b)

Create an interaction plot to display the caliper effect on measured diameter. Use a mixed model likelihood approach to test for a systematic difference in the measurements of the two calipers. Compute F_0 and the p -value.

```
A = as.factor(na.omit(data$device)) # fixed effect
B = as.factor(na.omit(data$ball)) # random effect
y = na.omit(data$measurement) # response
contrasts(A)=contr.sum
head(data.frame(caliper=A,ball=B,measurement=y))
```

```
##   caliper ball measurement
## 1      1     1      26.55
## 2      1     1      26.45
## 3      1     1      26.75
## 4      1     2      27.25
## 5      1     2      27.15
## 6      1     2      27.35
```

```
interaction.plot(B,A,y)
```



```
mixed.mod = lmer(y ~ A + (1|B) + (1|A:B))

## boundary (singular) fit: see ?isSingular

anova(mixed.mod)

## Type III Analysis of Variance Table with Satterthwaite's method
##      Sum Sq Mean Sq NumDF DenDF F value    Pr(>F)
## A  0.24083  0.24083      1     39   5.1916 0.02825 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Reproduce

$F_0 = 5.192$ (p -value = .0283).

(c)

State the expected value for each of MS_A , MS_B , MS_{AB} , and MS_E . Write the test statistic F_A as a ratio of mean squares. Use the expected values to argue why interaction mean squares is the appropriate error term.

Reproduce

$$E(MS_A) = \sigma^2 + n\sigma_{\tau\beta}^2 + \frac{bn \sum_{i=1}^a \tau_i^2}{a-1},$$

$$E(MS_B) = \sigma^2 + n\sigma_{\tau\beta}^2 + an\sigma_{\beta}^2,$$

$$E(MS_{AB}) = \sigma^2 + n\sigma_{\tau\beta}^2,$$

$$E(MS_E) = \sigma^2,$$

$$F_A = \frac{MS_A}{MS_{AB}}.$$

The appropriate scaling requires a denominator with the same expected value as the numerator, under $H_0 : \tau_1 = \dots = \tau_a = 0$.

(d)

Perform a test for caliper effects. Compute F_A and the p -value.

```
mixed.test(A,B,y)
```

```
##                SS df          MS
## Fixed Effect A  0.2408333  1 0.24083333
## Random Effect B 8.5491667  7 1.22130952
## Interaction AB  0.1291667  7 0.01845238
## Error          1.6800000 32 0.05250000
## F-test for fixed effect    p-value
##                13.05161 0.008593224
## error.var interaction.var block.var
##      0.0525      -0.01134921 0.2004762
```

Reproduce

$F_A = 13.042$ (p -value = .009).

(e)

Compute the unbiased estimates of the random effect parameters.

```
mixed.test(A,B,y)
```

```
##                SS df          MS
## Fixed Effect A  0.2408333  1 0.24083333
## Random Effect B 8.5491667  7 1.22130952
## Interaction AB  0.1291667  7 0.01845238
## Error          1.6800000 32 0.05250000
## F-test for fixed effect    p-value
##                13.05161 0.008593224
## error.var interaction.var block.var
##      0.0525      -0.01134921 0.2004762
```

Reproduce

$$\hat{\sigma}^2 = MS_E = .053$$

$$\hat{\sigma}_\beta^2 = .200$$

$$\hat{\sigma}_{\tau\beta}^2 = -.011.$$

Problem 3

A nested design is used to study the number of cases produced from three bottling machines (factor A , fixed effect). Four operators are randomly selected for each of the machines (nested factor B , random effect). Each operator makes $n = 2$ experimental runs. The data is provided as an attachment.

Preliminary

A = machine (fixed effect)
 $B(A)$ = operator (nested random effect)
 y = number of cases

(a)

Explain the difference between crossed factors and nested factors.

Reproduce

Factors A and B are *crossed* if the levels of B are the same at each level of A . Factor B is nested within A if the levels of B are different for each of the levels of A .

(b)

Write the model for this nested design, defining the fixed effect parameters, and the random effect parameters.

Reproduce

Model:

$$Y_{ijk} = \mu + \tau_i + \beta_{j(i)} + \epsilon_{ijk} \begin{cases} i = 1, \dots, a, \\ j = 1, \dots, b, \\ k = 1, \dots, n. \end{cases}$$

where $\epsilon_{ijk} \sim \mathcal{N}(0, \sigma^2)$.

Factor A fixed: $\{\tau_i : i = 1, \dots, a\}$ ($\sum_i \tau_i = 0$). Nested factor $B(A)$ random: σ_β^2 . Residual random: σ^2

(c)

Provide the algebraic formulas for the estimates $\hat{\tau}_i$, $\hat{\beta}_{j(i)}$, and \hat{y}_{ij} . State the expected value for each of MS_A , $MS_{B(A)}$, and MS_E .

Reproduce

$$\begin{aligned} \hat{\tau}_i &= \bar{Y}_{i..} - \bar{Y}_{...}, \\ \hat{\beta}_{j(i)} &= \bar{Y}_{ij.} - \bar{Y}_{i..}, \\ \hat{y}_{ij} &= \bar{Y}_{ij.}, \\ E(MS_A) &= \sigma^2 + n\sigma_\beta^2 + \frac{bn}{a-1} \sum_{i=1}^a \tau_i^2, \\ E(MS_{B(A)}) &= \sigma^2 + n\sigma_\beta^2, \\ E(MS_E) &= \sigma^2. \end{aligned}$$

(d)

Test for differences between bottling machines. Write the test statistic F_A statistic as a ratio of mean squares.

```
A = as.factor(na.omit(data$machine))
B = as.factor(na.omit(data$operator))
y = na.omit(data$cases)
```

```
nested.test(A,B,y)
```

```
##                SS df          MS
## Fixed Effect A    628.0833  2 314.041667
## Random Effect B(A) 885.7500  9  98.416667
## Error              80.0000 12   6.666667
## F-test for fixed effect    p-value
##                3.19094 0.08964972
## error.var  B.var
##    6.666667 45.875
```

Reproduce

Under the null model $H_0 : \tau_1 = \dots = \tau_a = 0$, MS_A and $MS_{B(A)}$ have the same expected value and thus an appropriate test statistic is

$$F_A = \frac{MS_A}{MS_{B(A)}}.$$

We see that $F_A = 3.19094$ ($p\text{-value} = .090$). The experiment finds that the machine does not have an effect on number of cases produced.

(e)

Explain why MS_E is the incorrect error term to use when the nested factor is random. In particular, comment on the pertinent sample size.

Reproduce

We think of operators as the experimental unit. The appropriate error term is then a measure of operator variance.

Taking repeat measurements from a selected operator does not increase the pertinent sample size.

Problem 4

A company wishes to study the effect of promotion type (1, 2, 3) on the sales of its crackers. A sample of $N = 15$ grocery stores is selected. Response variable y is the number of cases sold during the promotion period. Factor A is the promotion type. Covariate x is the same store sales prior to the promotion. The data is provided as an attachment.

Preliminary

factor = promotion type,
 x = pre-sales,
 y = sales.

(a)

Compute the estimated regression of presales on cases sold for each promotion type. Create a scatterplot of presales versus sales for each promotion type, including the estimated regression lines.

```
prom = as.factor(na.omit(data$promotion)) # factor A
presales = na.omit(data$presales)        # predictor x
sales = na.omit(data$sales)               # response

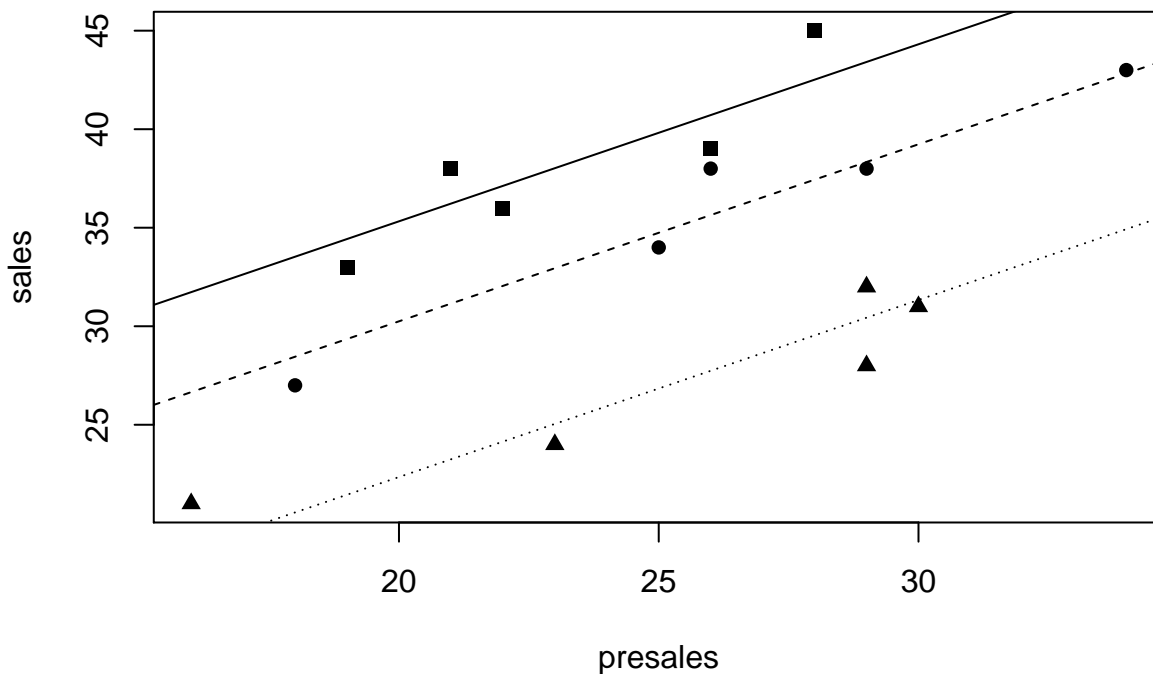
ancova.mod = lm(sales ~ presales + prom)
est = coef(ancova.mod)
b1 = est[2]
b0 = c(est[1], est[1]+est[3], est[1]+est[4])

reg.funcs = matrix(c(b0, rep(b1, 3)), nrow = 3)
dimnames(reg.funcs) = list(c("prom 1", "prom 2", "prom 3"), c("intercept", "slope"))
reg.funcs

##      intercept      slope
## prom 1 17.353421 0.8985594
## prom 2 12.278031 0.8985594
## prom 3  4.376591 0.8985594

# scatterplot
plot(presales[prom==1], sales[prom==1], xlab='presales',
     ylab='sales', pch=15,
     xlim=c(min(presales), max(presales)),
     ylim=c(min(sales), max(sales)))
points(presales[prom==2], sales[prom==2], pch=16)
points(presales[prom==3], sales[prom==3], pch=17)

abline(b0[1], b1, lty=1)
abline(b0[2], b1, lty=2)
abline(b0[3], b1, lty=3)
```



Reproduce

promotion type	estimated regression of pre-sales on sales
promotion 1	$17.353 + .898x$
promotion 2	$12.278 + .898x$
promotion 3	$4.377 + .898x$

(b)

Test for a promotion effect. Write the ANCOVA $F_{A|x}$ statistic using extra sum of squares notation. Compute $F_{A|x}$ and the p -value. Provide an interpretation, stated in the context of the problem. Note the role the covariate is playing in this analysis.

```
anova(ancova.mod)
```

```
## Analysis of Variance Table
##
## Response: sales
##           Df Sum Sq Mean Sq F value    Pr(>F)
## presales   1 190.68  190.678    54.379 1.405e-05 ***
## prom       2  417.15  208.575    59.483 1.264e-06 ***
## Residuals 11   38.57    3.506
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Reproduce

$$F_{A|x} = \frac{R(A|x)/(a-1)}{SS_E(A,x)/(N-a-1)} = 59.483,$$

which has a p -value = .000.

The experiment finds that the promotion type has an effect on sales, after adjusting for pre-sales.

(c)

Compute the sample mean sales and the sample mean presales for each promotion type. Compute the least squares means. Explain how the information from the covariate adjusts the determination of promotion effect.

```
library("lsmeans")
library("car")

xbar.prom = aggregate(presales, by=list(prom), FUN=mean)
ybar.prom = aggregate(sales, by=list(prom), FUN=mean)[2]
means.table = cbind(xbar.prom,ybar.prom)
colnames(means.table) = c("prom","presales.mean","sales.mean")
means.table
```

```
##   prom presales.mean sales.mean
## 1    1             23.2       38.2
## 2    2             26.4       36.0
## 3    3             25.4       27.2
```

```
# overall mean
```

```
mean(presales)
```

```
## [1] 25
```

```
# compute least squares means for adjusted means.
```

```
lsmeans(ancova.mod,pairwise ~ prom,adjust="none")$lsmeans
```

```
##   prom lsmean      SE df lower.CL upper.CL
## 1      39.8 0.858 11      37.9      41.7
## 2      34.7 0.850 11      32.9      36.6
## 3      26.8 0.838 11      25.0      28.7
##
## Confidence level used: 0.95
```

Reproduce

$$\bar{y}_{i,\text{adj}} = \bar{y}_i + \hat{\beta}(\bar{x}_{..} - \bar{x}_i)$$

promotion type	\bar{x}	\bar{y}		\bar{y}_{adj}
promotion 1	23.2	38.2	u	39.8
promotion 2	26.4	36.0	d	34.7
promotion 3	25.4	27.2	d	26.8

If $\bar{x}_i < \bar{x}_{..}$, then $\bar{y}_{i,\text{adj}} > \bar{y}_i$. (adjust upward for less favorable conditions). If $\bar{x}_i > \bar{x}_{..}$, then $\bar{y}_{i,\text{adj}} < \bar{y}_i$. (adjust downward for more favorable conditions).

Appendix: code