Homework #2

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Course: *STAT 478 - Time Series Analysis* – Professor: *Dr. Beidi Qiang*
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Question 1

Consider the N-span moving average applied to data that is uncorrelated with mean μ and variance σ^2.

(a) Show that the variance of the moving average is $\mathrm{Var}(M_t) = \sigma^2/N$.

(b) Show that $\mathrm{Cov}(M_t, M_{t+k}) = \sigma^2 \sum_{j=1}^{N-k} (1/N)^2$.

(c) Show the ACF is

$$\rho_k = 1 - \frac{|k|}{N}, \text{for } k < N.$$

Answer.

(a) To simplify the presentation of the subsequent material, we define the following parameterized index set.

Definition 1. We define I_t to be the index set (of size N) given by

$$\mathcal{I}_t := \{t - N + 1, t - N, \ldots, t - 1, t\}\,. \tag{1}$$

M_t is defined as

$$M_t := \frac{1}{N} \sum_{i=t-N+1}^{t} Y_i = \frac{1}{N} \sum_{i \in \mathcal{I}_t} Y_i\,. \tag{2}$$

The variance of M_t is given by

$$\mathrm{Var}(M_t) = \mathrm{Var}\left(\frac{1}{N} \sum_{i \in \mathcal{I}_t} Y_i\right) = \frac{1}{N^2} \sum_{i \in \mathcal{I}_t} \mathrm{Var}(Y_i)\,. \tag{3}$$

It is given that the variance for the time series $\{Y_t\}$ is a constant denoted by σ^2, therefore

$$\mathrm{Var}(M_t) = \frac{1}{N^2} \sum_{i \in \mathcal{I}_t} \sigma^2 = \frac{1}{N^2} N\sigma^2 = \frac{\sigma^2}{N}\,. \tag{4}$$

(b) The covariance of M_t and M_{t+k} is given by

$$\begin{aligned}
\mathrm{Cov}(M_t, M_{t_k}) &= \mathrm{E}(M_t M_{t+k}) - \mathrm{E}(M_t)\,\mathrm{E}(M_{t+k}) && (5)\\
&= \mathrm{E}\left[\left(\frac{1}{N} \sum_{i \in \mathcal{I}_t} Y_i\right)\left(\frac{1}{N} \sum_{j \in \mathcal{I}_{t+k}} Y_j\right)\right] - \mu^2 && (6)\\
&= \frac{1}{N^2}\,\mathrm{E}\left[\left(\sum_{i \in \mathcal{I}_t} Y_i\right)\left(\sum_{j \in \mathcal{I}_{t+k}} Y_j\right)\right] - \mu^2\,. && (7)
\end{aligned}$$

We focus our attention on the following definition.

Definition 2. W is expected value given by the sum of products

$$W := \mathrm{E}\left[\left(\sum_{i \in \mathcal{I}_t} Y_i\right)\left(\sum_{j \in \mathcal{I}_{t+k}} Y_j\right)\right] \tag{8}$$

$$= \sum_{(i,j) \in S} \mathrm{E}(Y_i Y_j)\,, \tag{9}$$

where $S := \mathcal{I}_t \times \mathcal{I}_{t+k}$, which is a Cartesian product of cardinality $|S| = N^2$.

With this definition of W, the covariance of M_t and M_{t+k} may be rewritten as

$$\mathrm{Cov}(M_t, M_{t_k}) = \frac{1}{N^2} W - \mu^2\,. \tag{10}$$

By the assumption of independence, if $l \neq m$, then

$$\mathrm{E}(Y_l Y_m) = \mathrm{E}(Y_l)\,\mathrm{E}(Y_m) = \mu^2 \tag{11}$$

and since $\mathrm{Var}(Y_l) = \mathrm{E}(Y_l^2) - \mathrm{E}^2(Y_l)$,

$$\mathrm{E}(Y_l Y_l) = \mathrm{E}^2(Y_l) + \mathrm{Var}(Y_l) = \mu^2 + \sigma^2\,. \tag{12}$$

If $k \geq N$, then M_t and M_{t+k} have no components of $\{Y_t\}$ in common. By eq 11, this means that

$$W = N^2 \mu^2 \tag{13}$$

and therefore

$$\mathrm{Cov}(M_t, M_{t+k}) = \frac{1}{N^2} W - \mu^2 = \mu^2 - \mu^2 = 0\,. \tag{14}$$

If $k < N$, then M_t and M_{t+k} have some components of $\{Y_t\}$ in common. Therefore, W consists of $N - k$ expectations of the form $\mathrm{E}(Y_l Y_l)$ with the expected value $\mu^2 + \sigma^2$ and $N^2 - (N - k) = N^2 - N + k$ expectations of the form $\mathrm{E}(Y_l Y_m)$, $l \neq m$, with the expected value μ^2. Therefore, $W = (N-k)(\mu^2 + \sigma^2) + (N^2 - N + k)\mu^2 = (N-k)\sigma^2 + N^2\mu^2$. When we plug this W into the covariance equation, we get the result

$$\mathrm{Cov}(M_t, M_{t_k}) = \frac{1}{N^2}\left((N-k)\sigma^2 + N^2\mu^2\right) - \mu^2 = \frac{N-k}{N^2}\sigma^2 + \mu^2 - \mu^2 = \frac{N-k}{N^2}\sigma^2 \tag{15}$$

which may be rewritten as

$$\mathrm{Cov}(M_t, M_{t_k}) = \sigma^2 \sum_{j=1}^{N-k} (1/N)^2\,. \tag{16}$$

Observe that if $N - k < 1$, then $\sum_{j=1}^{N-k}(1/N)^2 = 0$, and thus this also covers the case when $k \geq N$ where we earlier proved the covariance is zero.

(c) In the last problem, we show that the autocovariance is strictly a function of lag k (with N constant). Thus, $r_k = \frac{N-k}{N^2}\sigma^2$.

The autocorrelation function is given by

$$\rho_k = \frac{r_k}{\mathrm{Var}(M_t)} = \frac{\frac{N-k}{N^2}\sigma^2}{\sigma^2/N} \tag{17}$$

$$= \frac{N-k}{N} = 1 - \frac{k}{N}\,. \tag{18}$$

Since the autocorrelation function is symmetric, the result is the same whether k is positive or negative, and thus

$$\rho_k = 1 - \frac{|k|}{N}\,. \tag{19}$$

Question 2

Suppose that Z_1 and Z_2 are uncorrelated random variables with $\mathrm{E}(Z_1) = \mathrm{E}(Z_2) = 0$ and $\mathrm{Var}(Z_1) = \mathrm{Var}(Z_2) = 1$. Consider the process defined by

$$Y_t := Z_1 \cos(\omega t) + Z_2 \sin(\omega t) + e_t \,, \tag{20}$$

where e_t's are iid and independent of both Z_1 and Z_2, $e_t \sim \mathcal{N}(0, \sigma^2)$.

(a) Prove that $\{Y_t\}$ is stationary. (Hint: $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.)

(b) Let Z_1 and Z_2 be independent $\mathcal{N}(0,1)$ random variables, and set $\sigma^2 = 1$ and $\omega = 0.5$. Use R to simulate $n = 250$ observations from the $\{Y_t\}$ process. Plot $\{Y_t\}$ and describe the appearance of your time series.

(c) Now consider the process

$$X_t := \beta_0 + \beta_1 t + Z_1 \cos\omega t + Z_2 \sin\omega t + e_t \,.$$

Show that the time series $\{X_t\}$ is not stationary. Then use R to simulate a realization of $\{X_t\}$. Plot $\{X_t\}$ and describe the appearance of your time series. Does your $\{X_t\}$ process appear to be stationary?

(d) Consider the differenced time series $\{\Delta X_t\}$, where $\Delta X_t := X_t - X_{t-1}$. Show that the first difference $\{\Delta X_t\}$ is actually stationary. Plot the first differences ΔX_t, you may use diff in R to get the difference. Describe the appearance of this difference process $\{\Delta X_t\}$. Does it appear to be stationary?

Answer.

(a) To prove that $\{Y_t\}$ is weakly stationary, it is sufficient to prove that it has a constant mean, a constant variance, and its ACF is strictly a function of lag. First, observe

$$\begin{aligned} \mathrm{E}(Y_t) &= \mathrm{E}(Z_1 \cos(\omega t) + Z_2 \sin(\omega t) + e_t) && (21)\\ &= \cos(\omega t)\,\mathrm{E}(Z_1) + \sin(\omega t)\,\mathrm{E}(Z_2) + \mathrm{E}(e_t) = 0 && (22) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(Y_t) &= \mathrm{Var}(Z_1 \cos(\omega t) + Z_2 \sin(\omega t) + e_t) && (23)\\ &= \cos^2(\omega t)\,\mathrm{Var}(Z_1) + \sin^2(\omega t)\,\mathrm{Var}(Z_2) + \mathrm{Var}(e_t) && (24)\\ &= \cos^2(\omega t) + \sin^2(\omega t) + \sigma^2 = 1 + \sigma^2 \,, && (25) \end{aligned}$$

so we see that the mean and variance are constant.

Next, the ACF is given by $\mathrm{Cov}(Y_t, Y_{t+k})$. By the properties of the covariance function,

$$\begin{aligned} \mathrm{Cov}(a_1 X_1 + a_2 X_2, a_3 X_3 + a_4 X_4) = a_1 a_3 \,\mathrm{Cov}(X_1, X_3) + a_1 a_4 \,\mathrm{Cov}(X_1, X_4) +{} \\ a_2 a_3 \,\mathrm{Cov}(X_2, X_3) + a_2 a_4 \,\mathrm{Cov}(X_2, X_4) \,. \end{aligned} \tag{26}$$

We let $Y_t = aZ_1 + bZ_2 + e_t$ and $Y_{t+k} = cZ_1 + dZ_2 + e_{t+k}$ where $a = \cos(\omega t)$, $b = \sin(\omega t)$, $c = \cos(\omega(t+k))$, and $d = \sin(\omega(t+k))$, and thus we are interested in

$$\mathrm{Cov}(aZ_1 + bZ_2 + e_t, cZ_1 + dZ_2 + e_{t+k}) \,. \tag{27}$$

We apply pattern matching to make the above equation match eq 26. We $X_2 = bZ_2 + e_t$ and $X_4 = dZ_2 + e_{t+k}$, and thus we wish to find

$$\mathrm{Cov}(aZ_1 + X_2, cZ_1 + X_4) = ac\,\mathrm{Cov}(Z_1, Z_1) + a\,\mathrm{Cov}(Z_1, X_4) + c\,\mathrm{Cov}(X_2, Z_1) + \mathrm{Cov}(X_2, X_4) \,. \tag{28}$$

Observe that $\mathrm{Cov}(Z_1, Z_1) = \mathrm{Var}(Z_1) = 1$ and Z_1 is independent of both X_4 and X_2, thus we may rewrite the above as

$$\mathrm{Cov}(aZ_1 + X_2, cZ_1 + X_4) = ac + \mathrm{Cov}(X_2, X_4) \,. \tag{29}$$

We now pattern match on $\mathrm{Cov}(X_2, X_4)$. Substituting the definitions of X_2 and X_4, we see that

$$\begin{aligned}\mathrm{Cov}(X_2, X_4) &= \mathrm{Cov}(bZ_2 + e_t, dZ_2 + e_{t+k}) \\ &= bd\,\mathrm{Cov}(Z_2, Z_2) + b\,\mathrm{Cov}(Z_2, e_{t+k}) + d\,\mathrm{Cov}(e_t, Z_2) + \mathrm{Cov}(e_t, e_{t+k})\,.\end{aligned}$$

Observe that $\mathrm{Cov}(Z_2, Z_2) = \mathrm{Var}(Z_2) = 1$ and Z_2 is independent of both e_t and e_{t+k}, thus we may rewrite the above as

$$\mathrm{Cov}(X_2, X_4) = bd + \mathrm{Cov}(e_t, e_{t+k})\,. \tag{30}$$

When we combine eq 30 with eq 29, we get the result

$$\mathrm{Cov}(aZ_1 + bZ_2 + e_t, cZ_1 + dZ_2 + e_{t+k}) = ac + bd + \mathrm{Cov}(e_t, e_{t+k})\,. \tag{31}$$

If $k = 0$, then $\mathrm{Cov}(e_t, e_t) = \mathrm{Var}(e_t) = \sigma^2$ and otherwise if $k \neq 0$ then $\mathrm{Cov}(e_t, e_{t+k}) = 0$ by independence.

By the trigometric idenity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, if we let $\alpha = \omega t$ and $\beta = \omega(t+k)$, we may rewrite $ac + bd$ as $\cos(-\omega k)$ since $\alpha - \beta = \omega t - \omega(t+k) = \omega(t - (t+k)) = -\omega k$. By the property that $\cos(-a) = \cos(a)$, we may finally rewrite $ac + bc$ to $\cos(\omega k)$.

Putting all of this together, we arrive at

$$r_k = \mathrm{Cov}(Y_t, Y_{t+k}) = \cos(\omega k) \tag{32}$$

if $k \neq 0$ and

$$r_0 = \mathrm{Var}(Y_t) = 1 + \sigma^2\,, \tag{33}$$

which is in agreement with our earlier more direct computation of the variance. By the fact that the mean is a constant 0, the variance $\mathrm{Var}(Y_t) = r_0$ is a constant $1 + \sigma^2$, and the ACF is strictly a function of lag, we may conclude that $\{Y_t\}$ satisfies the requirements of being weakly stationary.

Note to Dr. Q: I have another proof that directly uses $\mathrm{Cov}(Y_t, Y_{t+k}) = \mathrm{E}(Y_t Y_{t+k})$.

(b) Figure 1a is a plot of $\{Y_t\}$ with $n = 250$, $\sigma^2 = 1$, and $\omega = 0.5$ using R script in listing 1. The variance appears constant, but it does exhibit seasonality despite the fact that we proved it has a constant mean and autocovariance that is strictly a function of lag.

There is a white noise component e_t, but there is also a sinusoidal component. Each realization of a time series is sinusoidal with a random amplitude and random phase. Suppose $\{Y_t\}^{(i)}$ is defined as

$$\begin{aligned}Y_t^{(i)} &:= Z_{i,1}\cos(\omega t) + Z_{i,2}\sin(\omega t) + e_{i,t} &(34)\\ &= \sqrt{Z_{i,1}^2 + Z_{i,2}^2}\cos(\omega t - \arctan(Z_{i,2}/Z_{i,1})) + e_{i,t}\,, &(35)\end{aligned}$$

where $Z_{i,1}, Z_{i,2} \sim \mathcal{N}(0,1)$ for all i and $e_{i,t} \sim \mathcal{N}(0, \sigma^2)$. Then, $\mathrm{E}(Y_t^{(i)}) = 0$ for all i but any particular realization of $Z_{i,1}$ and $Z_{i,2}$ will be non-zero and thus will have a seasonable trend. For instance, given that $Z_{i,1} = a$ and $Z_{i,2} = b$, $Y_t^{(i)}$ has an expected value $a\cos(\omega t) + b\sin(\omega t)$ which is a function of time t.

We compare two different realizations of the time series in fig 1b.

It is interesting to note there is zero correlation between independent time series $\{Y_t\}^{(i)}$ and $\{Y_t\}^{(j)}$, $j \neq j$, since they have random phases.

(c) X_t is defined as

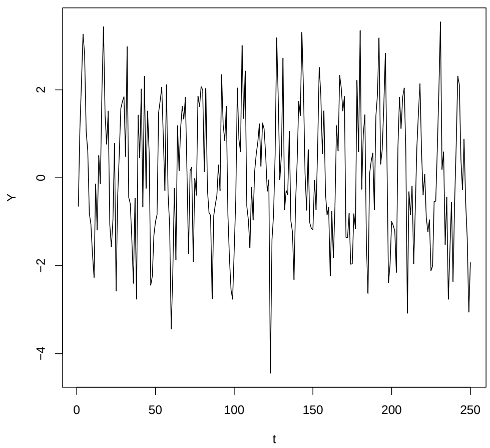
$$X_t := \beta_0 + \beta_1 t + Z_1\cos\omega t + Z_2\sin\omega t + e_t\,. \tag{36}$$

For $\{X_t\}$ to be stationary, one of the conditions is that it must have a constant mean. The expectation of X_t is given by

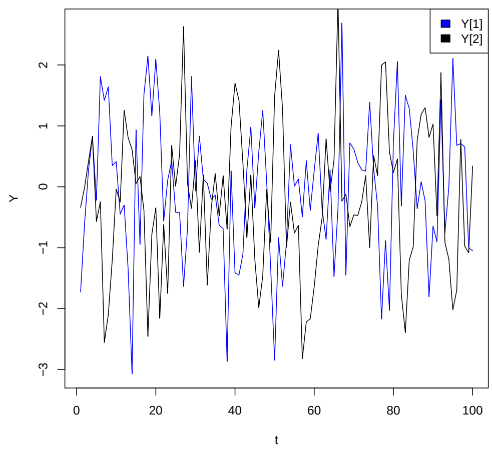
$$\begin{aligned}\mathrm{E}(X_t) &= \mathrm{E}(\beta_0 + \beta_1 t + Z_1\cos\omega t + Z_2\sin\omega t + e_t) &(37)\\ &= \beta_0 + \beta_1 t + \cos\omega t\,\mathrm{E}(Z_1) + \sin\omega t\,\mathrm{E}(Z_2) + \mathrm{E}(e_t) &(38)\\ &= \beta_0 + \beta_1 t\,, &(39)\end{aligned}$$

```r
# homework #2: problem 2.b

# n is size of time series
n <- 250

w <- 0.5

# $Z_1, Z_2 \sim \mathcal{N}(0,1)$
z <- rnorm(2, mean=0,sd=1)

# white noise $e_t \sim \mathcal{N}(0,1)$
e <- rnorm(n, mean=0,sd=1)

# $Y_t$ is the time series of interest
Y <- vector(length=n)

for (t in 1:n)
{
    Y[t] = z[1]*cos(w*t) + z[2]*sin(w*t) + e[t]
}

pdf(file="plot2_b_orig.pdf")
plot(Y,type="l", xlab="t", ylab="Y")
```

Listing 1: R script used to generate time series plots for Y_t.



(a) Time series plot of $\{Y_t\}$.



(b) Comparison of time series plot of $\{Y_t\}^{(1)}$ and $\{Y_t\}^{(2)}$.

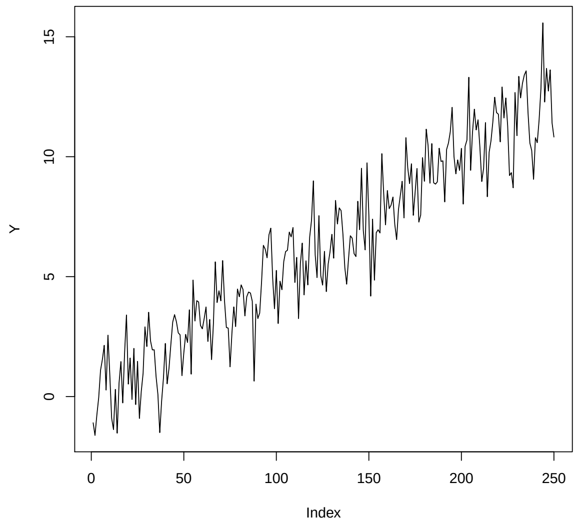


Figure 2: Time series plot of $\{X_t\}$.

which is a linear function with respect to time. Therefore, if $\beta_1 \neq 0$, the mean is a function of time and is non-constant, in which case $\{X_t\}$ is not stationary.

To illustrate, we plot $\{X_t\}$ with $\beta_0 = 0$ and $\beta_1 = 0.05$ in fig 2. Clearly, the time series is positively correlated with time t and thus is non-stationary. Otherwise, the variance seems to be constant. Moreover, $X_t = \beta_0 + \beta_1 t + Y_t$, and therefore $\{X_t - \beta_0 - \beta_1 t\} = \{Y_t\}$, so if we subtract the trendline out of the time series we have a stationary time series.

(d) The difference operator ∇ is defined as

$$\nabla A_t := A_t - A_{t-1}\,. \tag{40}$$

Therefore, substituting the definition of X_t into the ∇ function results in

$$\begin{aligned}\nabla X_t &:= X_t - X_{t-1}\\ &= (\beta_0 + \beta_1 t + Z_1 \cos\omega t + Z_2 \sin\omega t + e_t) -\\ &\quad\ (\beta_0 + \beta_1(t-1) + Z_1\cos(\omega(t-1)) + Z_2\sin(\omega(t-1)) + e_{t-1})\\ &= \beta_1 + Z_1(\cos\omega t - \cos(\omega(t-1))) + Z_2(\sin\omega t - \sin(\omega(t-1))) + (e_t - e_{t-1})\,.\end{aligned}$$

Letting $\mathrm{g}(t) := \cos\omega t - \cos(\omega(t-1))$ and $\mathrm{h}(t) := \sin\omega t - \sin(\omega(t-1))$, we rewrite the above as

$$\nabla X_t = \beta_1 + \mathrm{g}(t)Z_1 + \mathrm{h}(t)Z_2 + (e_t - e_{t-1})\,. \tag{41}$$

The expectation and variance of ∇X_t is given respectively by

$$\begin{aligned}\mathrm{E}(\nabla X_t) &= \beta_1 + \mathrm{g}(t)\,\mathrm{E}(Z_1) + \mathrm{h}(t)\,\mathrm{E}(Z_2) + \mathrm{E}(e_t) - \mathrm{E}(e_{t-1})\\ &= \beta_1 + \mathrm{g}(t)\cdot 0 + \mathrm{h}(t)\cdot 0 + 0 - 0\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(\nabla X_t) &= \mathrm{g}^2(t)\,\mathrm{Var}(Z_1) + \mathrm{h}^2(t)\,\mathrm{Var}(Z_2) + \mathrm{Var}(e_t) + \mathrm{Var}(e_{t-1}))\\ &= \mathrm{g}^2(t) + \mathrm{h}^2(t) + 2\sigma^2\\ &= 2(1 + \sigma^2 - \cos\omega)\,,\end{aligned}$$

which are both constants.

The covariance $\mathrm{Cov}(\nabla X_t, \nabla X_{t+k})$ is given by

$$\begin{aligned}\mathrm{Cov}(\nabla X_t, \nabla X_{t+k}) = \mathrm{E}[&(\mathrm{g}(t)Z_1 + \mathrm{h}(t)Z_2 + (e_t - e_{t-1}))\\ &(\mathrm{g}(t+k)Z_1 + \mathrm{h}(t+k)Z_2 + (e_{t+k} - e_{t+k-1}))]\,.\end{aligned} \tag{42}$$

Expanding the inner part of the expectation above results in an expectation of a sum of products. Taking advantage of the linearity of expectation, we rewrite this as a sum of expectation of products and we discard those expectations that are unconditionally zero, leaving us with

$$\begin{aligned}\operatorname{Cov}(\nabla X_t, \nabla X_{t+k}) =& \operatorname{g}(t)\operatorname{g}(t+k)\operatorname{E}(Z_1^2) + \operatorname{h}(t)\operatorname{h}(t+k)\operatorname{E}(Z_2^2) \\ &+ \operatorname{E}(e_t e_{t+k}) - \operatorname{E}(e_t e_{t+k-1}) - \operatorname{E}(e_{t-1}e_{t+k}) + \operatorname{E}(e_{t-1}e_{t+k-1})\,.\end{aligned} \tag{43}$$

Since $\operatorname{E}(Z_1^2) = \operatorname{E}(Z_2^2) = 1$, and assuming $k > 0$, then by the assumption of independence $\operatorname{E}(e_{t-1}e_{t+k}) = \operatorname{E}(e_{t-1})\operatorname{E}(e_{t+k}) = 0$ and we rewrite the above as

$$\begin{aligned}\operatorname{Cov}(\nabla X_t, \nabla X_{t+k}) =& \operatorname{g}(t)\operatorname{g}(t+k) + \operatorname{h}(t)\operatorname{h}(t+k) \\ &+ \operatorname{E}(e_t e_{t+k}) - \operatorname{E}(e_t e_{t+k-1}) + \operatorname{E}(e_{t-1}e_{t+k-1})\,,\end{aligned} \tag{44}$$

which simplies to

$$\operatorname{Cov}(\nabla X_t, \nabla X_{t+k}) = 2(1-\cos\omega)\cos(k\omega) + \operatorname{E}(e_t e_{t+k}) - \operatorname{E}(e_t e_{t+k-1}) + \operatorname{E}(e_{t-1}e_{t+k-1})\,. \tag{45}$$

The only point of consideration remaining are the expectations in the above equation. By independence, $\operatorname{E}(e_i e_j) = 0$, $i \neq j$, and $\operatorname{E}(e_j^2) = \sigma^2$. If $k = 0$, these expectations sum to $2\sigma^2$. If $k = 1$, these expectations sum to σ^2. Finally, if $k > 1$, these expectations sum to 0. Putting it all together, the autocovariance function is given by

$$r_k = -2(\cos\omega - 1)\cos(k\omega) + (2 - |k|)\sigma^2[|k| < 2]\,, \tag{46}$$

where $[p]$ is the Iverson bracket that is 1 if predicate p is true and otherwise 0.[1] As expected, $r_0 = \operatorname{Var}\Delta X_t$.

We have shown that $\{\Delta X_t\}$ has a constant expectation and variance and the autocovariance r_k is strictly a function of lag k. Thus, $\{\nabla X_t\}$ satisfies the requirements of a weakly stationary time series. In fig 3, we show a plot of $\{\Delta X_t\}$, which seems to be stationary as it jumps around 0 with a relatively constant variance and no obvious correlations.

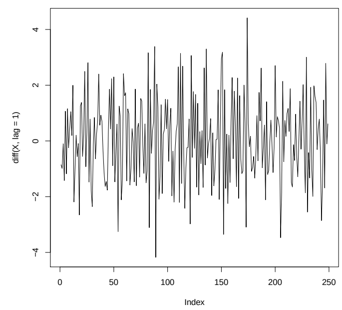


Figure 3: Time series plot of $\{\nabla X_t\}$.

[1] Also, observe that $\cos(-a) = \cos(a)$, so $r_k = r_{-k}$, as expected.

Question 3

The monthly values of the average hourly wages for U.S. apparel and textile workers for July 1981 to June 1987 are in the wages object in the TSA package. Type library(TSA); data(wages); print(wages) in R to see the data set.

(a) Plot the time series. What basic pattern do you see from the plot?

(b) Fit a linear time trend model using least squares. Give the plot of the linear trend overlain on the data, and give the estimated regression equation.

(c) Plot the standardized residuals from the linear regression over time. Comment on any notable pattern.

(d) Fit a quadratic time trend model using least squares. Give the plot of the quadratic trend overlain on the data, and give the estimated regression equation.

(e) Plot the standardized residuals from the quadratic regression over time. Comment on any notable pattern.

Answer.

(a) In fig 4a, we see a plot of wages with respect to year. The dominant feature of this time series is the increasing wages over this time period.

(b) The line of best fit to the *wages* time series dataset is given by

$$\hat{\mathrm{E}}(X_t) \coloneqq 7.93144 + 0.02342t\,. \tag{47}$$

We plot the wages time seres with this line of best fit in fig 4b.

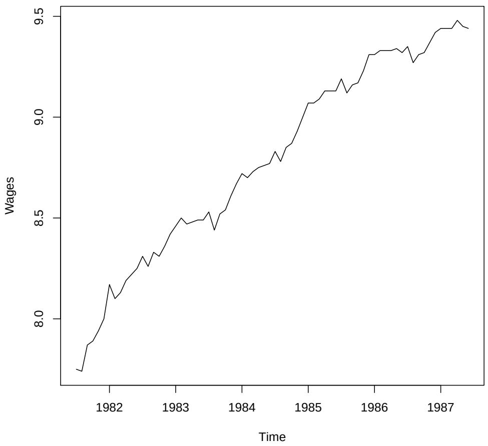
(c) The linear residuals are plotted in fig 4c. What stands out about this plot is the middle region has mostly positive residuals, meaning that most of these data points are below the line of best fit, and the end points have negative residuals, suggesting that most of these points are below the line of best fit. In other words, the residuals are correlated. Also, near the far left and far right of the plot, we see some outliers which take on relatively large negative values compared to the rest.

(d) The quadratic regression of best fit to the *wages* time series dataset is given by

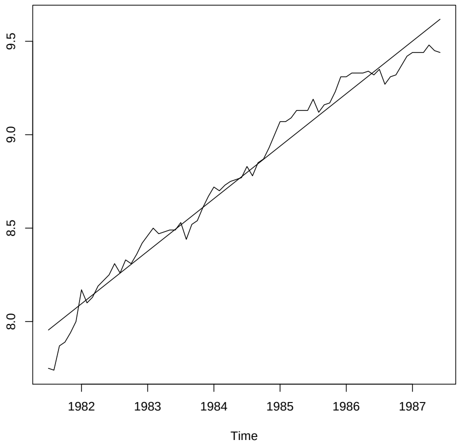
$$\hat{\mathrm{E}}(X_t) = 7.7974363 + 0.0342882t - 0.0001488t^2\,. \tag{48}$$

We plot the wages time seres with this line of best fit in fig 4d.

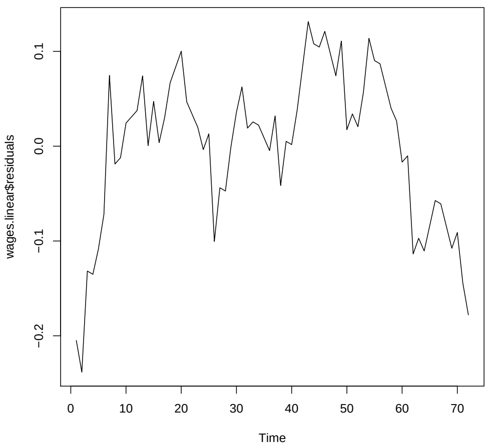
(e) The quadratic residuals are plotted in fig 4e. Unlike the linear regression line of best fit, the residuals are more uniformly centered around the mean and exhibit less correlation. These resdiuals seem to better model the concept of white noise.



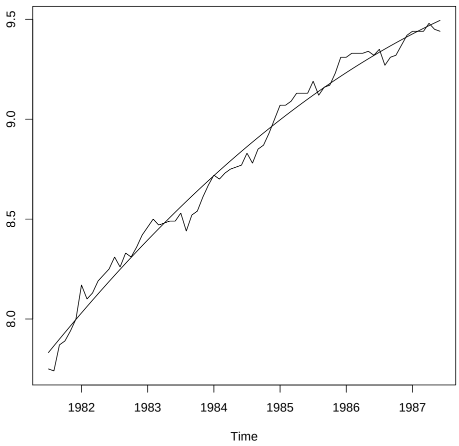
(a) Time series plot of wages.



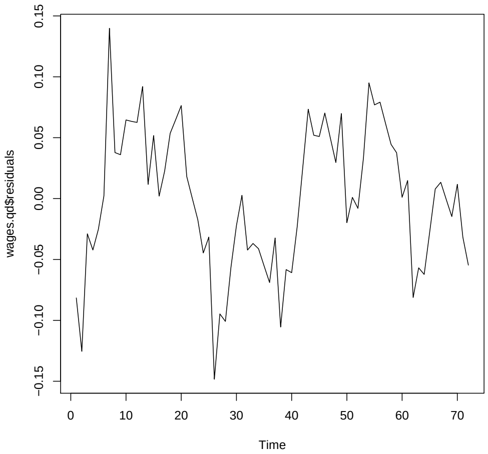
(b) Linear regression, $\hat{\mathrm{E}}(X_t) = 7.93144 + 0.02342t$.



(c) Linear regression residuals of the wages times series.



(d) Quadratic regression, $\hat{\mathrm{E}}(X_t) := 7.7974363 + 0.0342882t - 0.0001488t^2$



(e) Quadratic regression residuals of the wages times series.

Figure 4: Plots for problem 3