1. Use Metropolis Hasting algorithm to generate $Y \sim$ Gamma $(\alpha, 1)$, where $\alpha > 1$. Note α need not to be an integer. Consider the proposal distribution g, which is the density of gamma(a, b), where a=$floor(\alpha)$ and $b = a/\alpha$.

(a) Implement your accept-reject algorithm to get a sample of 10000 from Gamma $(2.5, 1)$.
(b) Check on mixing and convergence using plots. Run multiple chains and compute the Gelman-Rubin Statistics. You may pick any reasonable burn-in
(c) Estimate $\mathrm{E}\left(Y^2\right)$ using the generated chain. Compare with the estimate you get with the Accept-Reject method (Exam I).

2. (Problem 7.1)Rework the textbook example. Consider the mixture normal $\delta N(7, 0.5^2) + (1 - \delta)N(10, 0.5^2)$.

(a) Simulate 200 realizations from the mixture distribution in Equation with $\delta = 0.7$. Draw a histogram of these data.
(b) Now assume δ is unknown. Implement an independence chain MCMC procedure to simulate from the posterior distribution of δ, using your data from part (a).
(c) Implement a random walk chain with $\delta^* = \delta^{(t)} + \epsilon$ with $\epsilon \sim Unif(-1, 1)$.
(d) Reparameterize the problem letting $U = log(\delta/(1-\delta))$ and $U^* = u(t) + \epsilon$. Implement a random walk chain with U as in Equation (7.8) page 208.
(e) Compare the estimates and convergence behavior of the three algorithms.

3. Consider a i.i.d sample $X_1, \cdots X_n$ from $N(\mu, \sigma^2)$. Consider the Baysian analysis to estimate μ and $\tau = (\sigma^2)^{-1}$. We put prior $\mu \sim N(m, p^{-1})$ and $\tau \sim Gamma(a, b)$.

(a) Write out the posterior distribution of $(\mu, \tau)|\mathbf{x}$. You may ignore the normalizing constant.
(b) Show the posterior conditional distribution of $\mu|(\tau, \mathbf{x})$ is

$$N(\frac{n\tau\bar{x} + pm}{n\tau + p}, (n\tau + p)^{-1}).$$

And the posterior conditional distribution of $\tau|(\mu, \mathbf{x})$ is

$$Gamma(a + n/2, b + n/2[s^2 + (\mu - \bar{x})^2]).$$

(c) First generate some "observed" sample data use $x = rnorm(200, mu = 5, sigma = 2)$. Hand-code Gibbs Sampler algorithm to sample (μ, τ) from the posterior using x. You may take prior parameters $a = 0.0001$; $b = 0.0001$; $p = 0.0001$; $m = 0$. Use the estimated posterior mean and compare your estimates with the true parameters $\mu = 5$ and $\tau = 0.25$.