Time Series Analysis - STAT 478 - Final Exam - Part 1 Q2

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Part 1: Problem 2

Consider a linear trend process $Y_t = \beta_0 + \beta_1 t + e_t$, where $\{e_t\}$ is a 0 mean white noise process with variance σ^2. Let $\tilde{Y}_T$ be the simple exponential smoother, i.e.,

$$\tilde{Y}_T = (1-\theta)\sum_{t=0}^{\infty} \theta^t Y_{T-t}.$$

Show that the simple exponential smoother is a biased estimator for the linear trend process by calculating

$$\operatorname{Bias}(\tilde{Y}_T) = \operatorname{E}(Y_T) - \operatorname{E}(\tilde{Y}_T).$$

Proof. The estimator $\tilde{Y}_T$ is biased if $\operatorname{Bias}(\tilde{Y}_T) \neq 0$.

To solve $\operatorname{Bias}(\tilde{Y}_T)$, we must first solve $\operatorname{E}(Y_T)$ and $\operatorname{E}(\tilde{Y}_T)$. The expectation of Y_T is given by

$$\begin{aligned}
\operatorname{E}(Y_t) &= \operatorname{E}(\beta_0 + \beta_1 T + e_T)\\
&= \beta_0 + \beta_1 T + \operatorname{E}(e_T)\\
&= \beta_0 + \beta_1 T.
\end{aligned}$$

The expectation of $\tilde{Y}_T$ is given by

$$\begin{aligned}
\operatorname{E}(\tilde{Y}_T) &= \operatorname{E}\left((1-\theta)\sum_{t=0}^{\infty}\theta^t Y_{T-t}\right)\\
&= (1-\theta)\operatorname{E}\left(\sum_{t=0}^{\infty}\theta^t Y_{T-t}\right)\\
&= (1-\theta)\left(\sum_{t=0}^{\infty}\theta^t \operatorname{E}(Y_{T-t})\right).
\end{aligned}$$

By definition, $Y_{T-t} = \beta_0 + \beta_1(T-t) + e_{T-t}$, so we may perform that substitution, resulting in

$$\begin{aligned}
\operatorname{E}(\tilde{Y}_T) &= (1-\theta)\left(\sum_{t=0}^{\infty}\theta^t \operatorname{E}(\beta_0 + \beta_1(T-t) + e_{T-t})\right)\\
&= (1-\theta)\left(\sum_{t=0}^{\infty}\theta^t \left(\beta_0 + \beta_1(T-t) + \operatorname{E}(e_{T-t})\right)\right)\\
&= (1-\theta)\left(\sum_{t=0}^{\infty}\theta^t(\beta_0 + \beta_1 T) - \sum_{t=0}^{\infty}\theta^t\beta_1 t\right)\\
&= (1-\theta)\left((\beta_0 + \beta_1 T)\sum_{t=0}^{\infty}\theta^t - \beta_1\sum_{t=0}^{\infty} t\theta^t\right).
\end{aligned}$$

The only parts left to solve in the above are the infinite summations. Assuming $|\theta| < 1$, we observe that $\sum_{t=0}^{\infty} \theta^t$ is a geometric series that sums to $(1-\theta)^{-1}$ and $\sum_{t=0}^{\infty} t\theta^t$ is an infinite series that sums to $\theta(1-\theta)^{-2}$. Thus, we may make these substitutions, yielding

$$\begin{aligned} \mathrm{E}(\tilde{Y}_T) &= (1-\theta)\left((\beta_0 + \beta_1 T)(1-\theta)^{-1} - \beta_1\theta(1-\theta)^{-2}\right) \\ &= \beta_0 + \beta_1 T - \beta_1\theta(1-\theta)^{-1}. \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{Bias}(\tilde{Y}_T) &= \mathrm{E}(Y_T) - \mathrm{E}(\tilde{Y}_T) \\ &= (\beta_0 + \beta_1 T) - \beta_0 + \beta_1 T - \beta_1\theta(1-\theta)^{-1}, \end{aligned}$$

which simplifies to

$$\mathrm{Bias}(\tilde{Y}_T) = \frac{\theta}{1-\theta}\beta_1.$$

$\square$

There are two interesting special cases:

1. The bias is β_1 if $\theta = 0.5$.
2. The bias is 0 if $\theta = 0$.

Observe that the rate of change of the bias as a function of θ is given by

$$\frac{\partial\,\mathrm{Bias}}{\partial\theta} = \frac{1}{(1-\theta)^2}$$

and thus as θ moves away from 0.5 the bias increases without bound as it approaches 1 or 0, with the exception that when $\theta = 0$ the bias is 0.