Southern Illinois University-Edwardsville

Homework #1

Student name: *Alex Towell* (*atowell@siue.edu*)

Course: *STAT 579 - Discrete Multivariate Analysis* – Professor: *Dr. Andrew Neath*
Due date: *01/28/2021*

Question 1

Consider repeated, independent rolls of a fair die.

(a) Let Y_1 be the number of ones in 50 rolls. Completely specify the probability function for Y_1.

(b) Let $(Y_1, Y_2, \ldots, Y_6)$ be the number of ones, twos, etc. in 50 rolls. Completely specify the probability function for $(Y_1, Y_2, \ldots, Y_6)$.

Answer.

(a) Each roll is an independent trial. We model each roll as a discrete uniform distribution, $X_j \sim \text{DU}(1, 6)$ with probability distribution function (PDF)

$$\mathrm{f}_{X_i}(k) = \frac{1}{6}, k \in \{1, \ldots, 6\} \tag{1}$$

for $i = 1, \ldots, 50$.

The probability that $X_i = 1$ is $\pi = \mathrm{f}_{X_i}(1) = \frac{1}{6}$ and the probability that $X_i \neq 1$ is $1 - \pi = \frac{5}{6}$. Y_j is the number of outcomes where $X_i = j$, i.e., $Y_j = \sum_{i=1}^{50} [X_i = j]$. Since each X_i is i.i.d., Y_j is binomially distributed as

$$Y_j \sim \text{BIN}\left(n = 50, \pi = \frac{1}{6}\right), \tag{2}$$

with the PDF

$$\mathrm{f}_{Y_j}(k \,|\, n = 50, \pi = 1/6) = \binom{50}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{50-k}, \tag{3}$$

Thus, $Y_1 \sim \mathrm{f}_{Y_1}(k \,|\, n = 50, \pi = 1/6)$.

(b) A single observation of X_i may be considered as a random Boolean vector of dimension 6 whose j-th component realizes 1 if $X_i = j$ and otherwise 0, e.g., $X_i = 3$ maps to $(0, 0, 1, 0, 0, 0)$. Then, $\sum_{i=1}^{50} X_i$ maps to a random vector $(Y_1, \ldots, Y_6)$ where $Y_j = k_j$ if k_j of $\{X_i\}$ realizes j.

This joint distribution $(Y_1, \ldots, Y_6)$ models the multinomial distribution,

$$(Y_1, \ldots, Y_6) \sim \text{MULT}(n = 50, \{\pi_j\}) \tag{4}$$

where $\pi_j = 1/6$ for $j = 1, \ldots, 6$, which has the PDF

$$\mathrm{f}(k_1, \ldots, k_6) = \frac{50!}{\prod_{j=1}^{6} k_j!} \prod_{j=1}^{6} \left(\frac{1}{6}\right)^{k_j} \tag{5}$$

with the constraint that $\sum_{j=1}^{6} k_j = n = 50$. We may rewrite the above equation as

$$\mathrm{f}(k_1, \ldots, k_6) = \frac{50!}{\prod_{j=1}^{6} k_j!} \left(\frac{1}{6}\right)^{\sum_{j=1}^{6} k_j} . \tag{6}$$

Since $k_1 + \ldots + k_6 = 50$, we may rewrite the above equation as

$$\mathrm{f}(k_1, \ldots, k_6) = \frac{50!}{6^{50} \prod_{j=1}^{6} k_j!} . \tag{7}$$

Also, observe that $k_6 = 50 - k_1 - \ldots - k_5$, and thus we may reparamerterize as

$$\mathrm{f}(k_1, \ldots, k_5) = \frac{50!}{6^{50}(50 - \sum_{j=1}^{5} k_j) \prod_{j=1}^{5} k_j!} . \tag{8}$$

Question 2

Let $Y_1 \sim \mathrm{POI}(\lambda_1 = 1)$, $Y_2 \sim \mathrm{POI}(\lambda_2 = 2)$, and $Y_3 \sim \mathrm{POI}(\lambda_3 = 3)$ be independent random variables.

(a) Completely specify the probability function for (Y_1, Y_2, Y_3).

(b) Completely specify the probability function for $Y_+ = \sum_{i=1}^{3} Y_i$.

(c) Completely specify the conditional probability function for (Y_1, Y_2, Y_3) given $Y_+ = n$.

Answer.

(a) By independence,

$$\mathrm{f}_{\mathrm{Y}_1,\mathrm{Y}_2,\mathrm{Y}_3}(k_1, k_2, k_3 \,|\, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3) = \prod_{i=1}^{3} \mathrm{f}_{\mathrm{Y}_i}(k_i \,|\, \lambda_i) . \tag{9}$$

Dropping the subscripts on f for notational simplicity and substituting the *Poisson* probability distribution functions into the above equation, we rewrite the joint distribution function as

$$\mathrm{f}(k_1, k_2, k_3 \,|\, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3) = \frac{\lambda_1^{k_1} \mathrm{e}^{-k_1}}{k_1!} \frac{\lambda_2^{k_2} \mathrm{e}^{-k_2}}{k_2!} \frac{\lambda_3^{k_3} \mathrm{e}^{-k_3}}{k_3!} \tag{10}$$

which may be rewritten as

$$\mathrm{f}(k_1, k_2, k_3 \,|\, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3) = \frac{\lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3} \mathrm{e}^{-(k_1 + k_2 + k_3)}}{k_1! k_2! k_3!} . \tag{11}$$

We may rewrite the above by substituting the given lambda values into right-hand-side,

$$\mathrm{f}(k_1, k_2, k_3) = \frac{2^{k_2} 3^{k_3} \mathrm{e}^{-(k_1 + k_2 + k_3)}}{k_1! k_2! k_3!} \tag{12}$$

(*b*) The sum of independent poisson random variables is poisson with a failure rate given by the sum of the poisson failure rates, thus

$$Y_+ \sim \mathrm{POI}(\lambda_+ = \lambda_1 + \lambda_2 + \lambda_3 = 6)\,, \tag{13}$$

which has the PDF

$$\mathrm{f}_{Y_+}(n \,|\, \lambda = 6) = \frac{6^n \mathrm{e}^{-n}}{n!} \tag{14}$$

where $k \in \{0, 1, 2, \ldots\}$.

(*c*) By the laws of probability, the conditional distribution of (Y_1, Y_2, Y_3) given $Y_+ = n$ is

$$\mathrm{P}(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 \,|\, Y_+ = n) = \tag{15}$$

$$\frac{\mathrm{P}(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3, Y_+ = n)}{\mathrm{P}(Y_+ = n)}\,. \tag{16}$$

It is given that $k_1 + k_2 + k + k_3 = n$, and therefore the only value that Y_+ can realize is n with probability 1, therefore

$$\mathrm{P}(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 \,|\, Y_+ = n) = \tag{17}$$

$$\frac{\mathrm{P}(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3)}{\mathrm{P}(Y_+ = n)}\,. \tag{18}$$

Plugging in the joint distribution function for (Y_1, Y_2, Y_3) and the distribution function for Y_+ results in

$$\mathrm{P}(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 \,|\, Y_+ = n) = \frac{\mathrm{f}(k_1, k_2, k_3 \,|\, \lambda_1, \lambda_2, \lambda_3)}{\mathrm{f}_{\mathrm{Y}_+}(n \,|\, \lambda_+)}\,. \tag{19}$$

We may rewrite the above by plugging in the derivations of these distribution functions,

$$\mathrm{P}(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 \,|\, Y_+ = n) = \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3} \mathrm{e}^{-(k_1 + k_2 + k_3)}}{6^n \mathrm{e}^{-n} k_1! k_2! k_3!} \tag{20}$$

Since it is given that $k + 1 + k_2 + k_3 = n$, we may perform these substitutions as desired, resulting in sequence of transformations given by

$$\begin{aligned}
\mathrm{P}(Y_1 = k_1, Y_2 = k_2, Y_3 = k_3 \,|\, Y_+ = n) &= \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3} \mathrm{e}^{-n}}{6^n \mathrm{e}^{-n} k_1! k_2! k_3!} && (21)\\
&= \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3}}{6^n k_1! k_2! k_3!} && (22)\\
&= \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3}}{6^{k_1 + k_2 + k_3} k_1! k_2! k_3!} && (23)\\
&= \frac{n! \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_3}}{6^{k_1} 6^{k_2} 6^{k_3} k_1! k_2! k_3!} && (24)\\
&= \frac{n!}{k_1! k_2! k_3!} \left(\frac{\lambda_1}{6}\right)^{k_1} \left(\frac{\lambda_2}{6}\right)^{k_2} \left(\frac{\lambda_3}{6}\right)^{k_3}\,, && (25)
\end{aligned}$$

which is the distribution function of the multinomial.

If we let W denote the conditional distribution of (Y_1, Y_2, Y_3) given $Y_+ = n = 50$, then

$$W \sim \text{MULT}(n = 50, \{\pi_j\}) \tag{26}$$

where $\pi_j = \frac{\lambda_j}{\lambda_+} = \frac{j}{6}$. This may finally be rewritten to

$$W \sim \text{MULT}\left(n = 50, \pi_1 = \frac{1}{6}, \pi_2 = \frac{2}{6}, \pi_3 = \frac{3}{6}\right) . \tag{27}$$