Regression Analysis - STAT 482 - Exam 2: Due Dec 14, 2021

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Problem 1

A beer distributor is interested in the amount of time to service its retail outlets. Two factors are thought to influence the delivery time (y) in minutes: the number of cases delivered (x_1) and the distance traveled (x_2) in miles. A random sample of delivery time data has been collected. The data is available on Blackboard as a csv file.

Preliminary analysis

$$\begin{aligned} x_1 &= \text{number of cases delivered} \\ x_2 &= \text{distance traveled (miles)} \\ y &= \text{delivery times (minutes)} \end{aligned}$$

where

$$\operatorname{cor}(x_1, x_2) < 0.$$

(a)

Provide an interpretation of a regression coefficient in a multiple regression model.

Reproduce

β_ℓ is the difference in mean response from a 1 unit increase in x_ℓ, with all other input levels held fixed.

(b)

Compute b_1, b_2, the estimated the regression coefficients for the delivery time data.

```
data1 = read.csv('exam2-1.csv')
head(data1)
```

```
##   cases distance time
## 1    10       30   24
## 2    15       25   27
## 3    10       40   28
## 4    20       18   33
## 5    25       22   35
## 6    18       31   32
```

```
data1.m12 = lm(time ~ cases + distance, data=data1)
data1.m12
```

```
##
## Call:
## lm(formula = time ~ cases + distance, data = data1)
##
## Coefficients:
## (Intercept)        cases     distance
##      5.8933       0.9206       0.3286
```

Reproduce

$b_1 = .921$ $b_2 = .329$.

(c)

Compute t statistics for testing the effect of each input variable. Explain what type of effect is being tested here.

```
coef(summary(data1.m12))
```

```
##              Estimate Std. Error   t value     Pr(>|t|)
## (Intercept) 5.8932559 2.62973524  2.241007 4.471588e-02
## cases       0.9206362 0.06870747 13.399360 1.402168e-08
## distance    0.3286414 0.06589144  4.987620 3.157935e-04
```

Reproduce

parameter	t-statistic	p-value
cases	$t_1^* = 13.399$	$p_1 = .000$
distance	$t_2^* = 4.988$	$p_2 = .000$

The t_ℓ-statistic is testing the *partial* effect of input x_ℓ, accounting for the effects of all other inputs.

(d)

Compute $\mathrm{SS_R}(X_1)$ and $\mathrm{SS_R}(X_2|X_1)$. Explain what each sum of squares represents.

```
anova(data1.m12)
```

```
## Analysis of Variance Table
##
## Response: time
##           Df  Sum Sq Mean Sq F value    Pr(>F)
## cases      1 308.008 308.008 154.901 3.217e-08 ***
## distance   1  49.464  49.464  24.876 0.0003158 ***
## Residuals 12  23.861   1.988
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Reproduce

$\mathrm{SS_R}(x_1) = 308.008$ and $\mathrm{SS_R}(x_2|x_1) = 49.464$.

$\mathrm{SS_R}(x_1)$ measures the variation in delivery time (y) explained by number of cases (x_1). $\mathrm{SS_R}(x_2|x1)$ measures the variation in delivery time (y) explained by distance (x_2), beyond that explained by number of cases (x_1).

(e)

Test for a marginal effect of x_2 against a model which includes no other input variables. (Compute the test statistic and p-value.) Provide an interpretation of the result, stated in the context of the problem.

```
data1.m0 = lm(time ~ 1, data=data1)
data1.m2 = lm(time ~ distance, data=data1)
anova(data1.m2)
```

```
## Analysis of Variance Table
##
## Response: time
##           Df Sum Sq Mean Sq F value Pr(>F)
## distance   1   0.47  0.4672  0.0159 0.9014
## Residuals 13 380.87 29.2974
```

```
anova(data1.m0,data1.m2)  # equivalent to anova(data1.m2)
```

```
## Analysis of Variance Table
##
## Model 1: time ~ 1
## Model 2: time ~ distance
```

```
##   Res.Df    RSS Df Sum of Sq      F Pr(>F)
## 1     14 381.33
## 2     13 380.87  1   0.46715 0.0159 0.9014
```

Reproduce

$F_2^* = .016$ (p-value $= .901$).

The observed data is compatible with the no effects (reduced) model. It is not necessary to add distance (x_2) to the no effects model for predicting delivery time (y).

(f)

Test for a partial effect of x_2 against a model which includes x_1. (Compute the test statistic and p-value.) Provide an interpretation of the result, stated in the context of the problem.

```r
data1.m1 = lm(time ~ cases, data=data1)
anova(data1.m1,data1.m12)
```

```
## Analysis of Variance Table
##
## Model 1: time ~ cases
## Model 2: time ~ cases + distance
##   Res.Df    RSS Df Sum of Sq      F    Pr(>F)
## 1     13 73.325
## 2     12 23.861  1    49.464 24.876 0.0003158 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Reproduce

$F_{2|1} = 24.876$ (p-value $= .000$).

The observed data is *not* compatible with the reduced model. We accept the addition of distance (x_2) to a model which already includes number of cases (x_1).

(g)

Compute the correlation matrix. What feature of multidimensional modeling is illustrated in this problem?

```r
cor(data1)
```

```
##               cases    distance        time
## cases     1.0000000 -0.40529762  0.89872861
## distance -0.4052976  1.00000000 -0.03500075
## time      0.8987286 -0.03500075  1.00000000
```

Reproduce

Multicollinearity. We see that $r_{12} = -.405$, thus the inputs are highly (negative) correlated.

Investigating relationships in higher dimensions requires higher-level statistical methods, such as regression analysis. Two-dimensional methods and graphs are insufficient.

Problem 2

A bakery is interested in the best formulation for a new product. A small-scale experiment is conducted to investigate the relationship between the product satisfaction (y), and the moisture content (input 1) and sweetness (input 2) of the product. The input variables have been coded (x_1, x_2) for ease of calculation. The data is available on Blackboard as a csv file.

(a)

Provide a definition for an orthogonal design. Discuss an advantage to using an orthogonal design.

Reproduce

A design is orthogonal if $X'X$ is diagonal, where X is the design matrix. For orthogonal designs, regression coefficient estimates and variance explained by the inputs do not depend on which other inputs are included in the model.

(b)

Provide a definition for an interaction effect.

Reproduce

An interaction effect occurs when the effect of an input effect depends on the levels of the other inputs.

(c)

Fit an interaction model using the coded variables. Compute the regression coefficient estimates and their standard errors.

```
data2 = read.csv('exam2-2.csv')
data2
```

```
##    moisture x1 sweetness x2 satisfaction
## 1         4 -3         2 -1           65
## 2         4 -3         4  1           75
## 3         4 -3         2 -1           61
## 4         4 -3         4  1           76
## 5         6 -1         2 -1           75
## 6         6 -1         4  1           86
## 7         6 -1         2 -1           72
## 8         6 -1         4  1           85
## 9         8  1         2 -1           87
## 10        8  1         4  1           87
## 11        8  1         2 -1           80
## 12        8  1         4  1           88
## 13       10  3         2 -1           91
## 14       10  3         4  1           96
## 15       10  3         2 -1           89
## 16       10  3         4  1           99
```

```
data2.xmod = lm(satisfaction ~ x1*x2, data=data2)
summary(data2.xmod)
```

```
##
## Call:
## lm(formula = satisfaction ~ x1 * x2, data = data2)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -2.900 -1.938 -0.500  1.938  4.950
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  82.0000     0.6435 127.433  < 2e-16 ***
## x1            3.9750     0.2878  13.813 9.94e-09 ***
## x2            4.5000     0.6435   6.993 1.45e-05 ***
## x1:x2        -0.5750     0.2878  -1.998   0.0689 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.574 on 12 degrees of freedom
## Multiple R-squared:  0.9531, Adjusted R-squared:  0.9413
## F-statistic: 81.23 on 3 and 12 DF,  p-value: 3.069e-08
```

Reproduce

coefficient	estimate	std. error
β_0	$b_0 = 82.000$	$\text{se}(b_0) = .644$
β_1	$b_1 = 3.975$	$\text{se}(b_1) = .288$
β_2	$b_2 = 4.500,$	$\text{se}(b_2) = .644$
β_{12}	$b_{12} = 0.575$	$\text{se}(b_{12} = .288$

(d)

Write the estimated regression as a function of x_1 for $x_2 = 1, 0, -1$.

```
b0 = coef(data2.xmod)[1]
b1 = coef(data2.xmod)[2]
b2 = coef(data2.xmod)[3]
b12 = coef(data2.xmod)[4]
reg.est.x1 = matrix(c(b0+b2,b1+b12,  # x2=+1
                      b0,b1,         # x2=0
                      b0-b2,b1-b12), # x2=-1
                    nrow=3,byrow=T)

dimnames(reg.est.x1) = list(c("x2=+1","x2=0","x2=-1"),c("intercept","slope"))
reg.est.x1
```

```
##       intercept slope
## x2=+1      86.5 3.400
## x2=0       82.0 3.975
## x2=-1      77.5 4.550
```

Reproduce

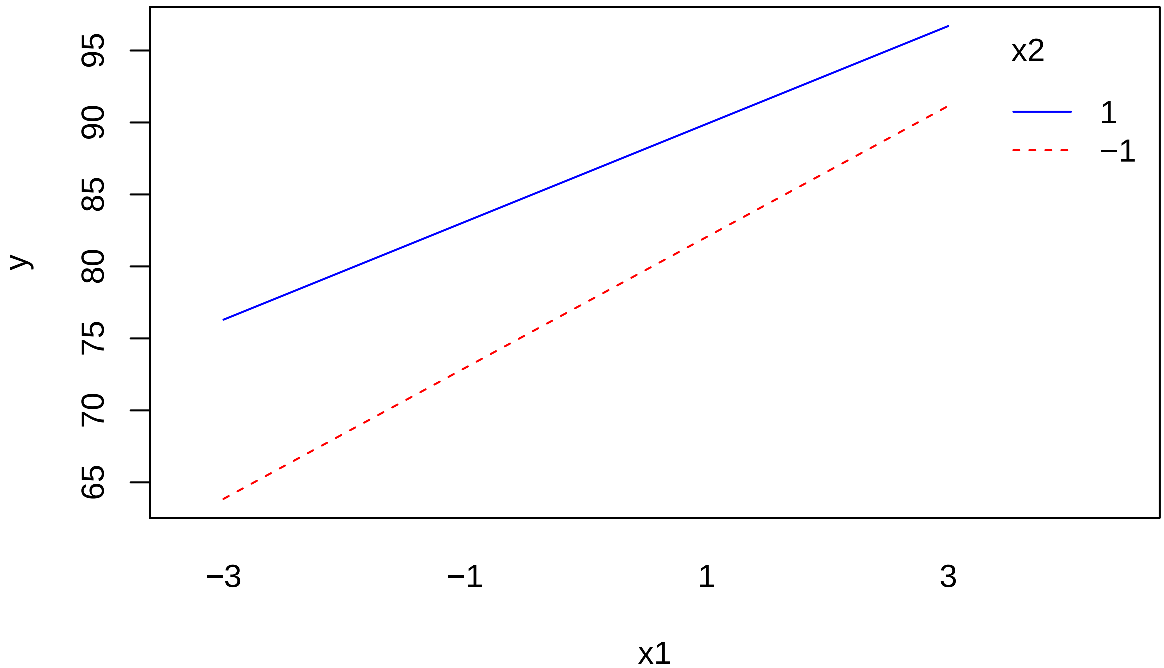
We have the regression function $\hat{E}(Y|x) = b_0 + b_1x_1 + b_2x_2 + b_{12}x_1x_2$. The regression function for $x_2 = 1, 0, -1$ is thus:

- $\hat{E}(Y|x_2 = +1) = (b_0 + b_2) + (b_1 + b_{12})x_1 = 86.500 + 3.400x_1$
- $\hat{E}(Y|x_2 = 0) = b_0 + b_1x_1 = 82.000 + 3.975x_1$
- $\hat{E}(Y|x_2 = -1) = (b_0 - b_2) + (b_1 - b_{12})x_1 = 77.500 + 4.550x_1$

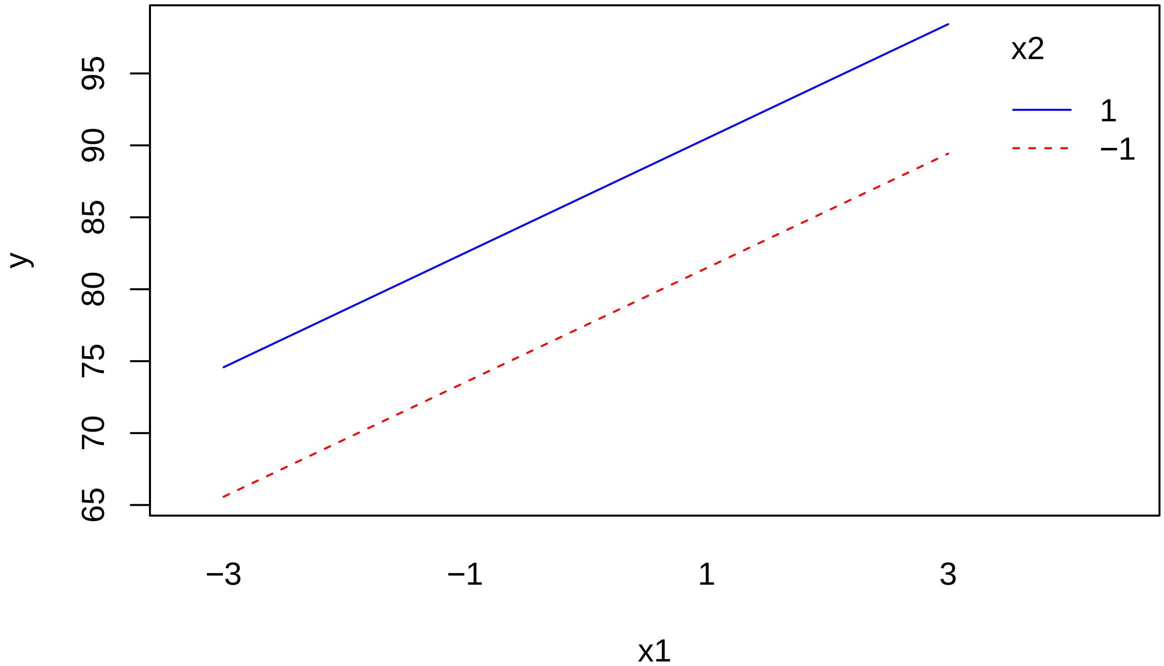
(e)

Create interaction plots for both the interaction model and the additive effects model.

```
# interaction model:
data2.xmod.fits = predict(data2.xmod)
interaction.plot(data2$x1,data2$x2,data2.xmod.fits,
            col=c("red","blue"),
            trace.label="x2",
            xlab="x1",
            ylab="y")
```




```r
# additive model
data2.amod = lm(satisfaction ~ x1+x2,data=data2)
data2.amod.fits = predict(data2.amod)
interaction.plot(data2$x1,data2$x2,data2.amod.fits,
            col=c("red","blue"),
            trace.label="x2",
            xlab="x1",
            ylab="y")
```



(f)

Test for an interaction effect. (Compute the test statistic and p-value.)

```r
anova(data2.amod, data2.xmod)
```

```
## Analysis of Variance Table
##
## Model 1: satisfaction ~ x1 + x2
## Model 2: satisfaction ~ x1 * x2
```

```
##   Res.Df    RSS Df Sum of Sq      F  Pr(>F)
## 1     13 105.95
## 2     12  79.50  1     26.45 3.9925 0.06888 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Reproduce

$F^* = 3.99$ (p-value $= .07$).

Since the p-value is so close to .05, the experimental result is hard to interpret. However, we can see from the interaction plots that they are pretty similiar, so it would seem like we do not lose much information with the additive model.

If we wish to stick to the hard cut-off of $\alpha = .05$, then we would say that the experiment finds the data to be compatible with the additive (reduced) model.

Problem 3

An engineer is interested in comparing three chemical processes (categorical input with groups A,B,C) for manufacturing a compound. It is suspected that the impurity (continuous input x) of the raw material will affect the yield (response variable y) of the product. The data is available on Blackboard as a csv file.

Preliminary

$$\begin{aligned} \text{categorical process} &= \{A, B, C\} \\ x &= \text{impurity} \\ y &= yield. \end{aligned}$$

(a)

Define indicator variables I_1 and I_2 using chemical process C as the baseline level.

Reproduce

$$I_1 \begin{cases} 1, & \text{if } \texttt{process} = A, \\ 0, & \text{otherwise} \end{cases}$$

$$I_2 \begin{cases} 1, & \text{if } \texttt{process} = B, \\ 0, & \text{otherwise.} \end{cases}$$

(b)

Write an additive model for response y using continuous input variable x and indicator variables I_1, I_2.

Reproduce

$$E(Y|\cdot) = \beta_0 + \beta_1 x + \beta_2 I_1 + \beta_3 I_2.$$

(c)

Write a regression function for each of the chemical processes.

Reproduce

$$E(Y|\cdot) = \begin{cases} (\beta_0 + \beta_2) + \beta_1 x & \text{if } \texttt{process} = A, \\ (\beta_0 + \beta_3) + \beta_1 x & \text{if } \texttt{process} = B, \\ \beta_0 + \beta_1 x & \text{if } \texttt{process} = C. \end{cases}$$

(d)

Provide an interpretation for each effect parameter, stated in the context of the problem.

Reproduce

β_1 is the difference in mean `yield` from a 1 unit increase in `impurity` (x), with the `process` held constant.

β_2 is the difference in mean `yield` between `process` A and `process` C, with `impurity` (x) held constant.

β_3 is the difference in mean `yield` between `process` B and `process` C, with `impurity` (x) held constant.

$\beta_2 - \beta_3$ is the difference in mean `yield` between `process` A and `process` B, with `impurity` (x) held constant.

(e)

Compute interval estimates for each of the effect parameters.

```
library(matlib)
data3 = read.csv('exam2-3.csv')
data3$process = as.factor(data3$process)

contrasts(data3$process) = contr.treatment(3,base=3)
data3.amod = lm(yield ~ impurity+process,data=data3)

b.hat = coef(data3.amod)
dfe = nrow(model.matrix(data3.amod)) - ncol(model.matrix(data3.amod))
V = vcov(data3.amod)
a = c(0,0,1,-1)

b.hat.12 = a %*% b.hat
se.12 = sqrt(a %*% V %*% a)

b.hat.12.lower = b.hat.12 - qt(.975,dfe) * se.12
b.hat.12.upper = b.hat.12 + qt(.975,dfe) * se.12

confint(data3.amod)
```

```
##                 2.5 %    97.5 %
## (Intercept) 1.5446801 5.7006292
## impurity    0.1806097 0.9967766
## process1    4.3264225 7.3832047
## process2    2.2017852 5.0353307
```

```
cat("beta2-beta1",c(b.hat.12.lower,b.hat.12.upper))
```

```
## beta2-beta1 0.5974664 3.875045
```

Reproduce

$$\begin{aligned} \mathrm{CI}(\beta_1) &= [.181, .997], \\ \mathrm{CI}(\beta_2) &= [4.326, 7.383], \\ \mathrm{CI}(\beta_3) &= [2.202, 5.035], \\ \mathrm{CI}(\beta_2 - \beta_3) &= [.597, 3.875]. \end{aligned}$$

(f)

Create a scatterplot of the data with the estimated regression lines.

```
intercept.3 = b.hat[1]
intercept.1 = b.hat[1]+b.hat[3]
intercept.2 = b.hat[1]+b.hat[4]
slope = b.hat[2]

cat(slope, intercept.1, intercept.2, intercept.3)
```

```
## 0.5886932 9.477468 7.241213 3.622655
```

```r
attach(data3)

plot(impurity[process == "A"],
     yield[process == "A"],
     xlab='impurity',
     ylab='yield',
     pch=1,
     col='blue',
     xlim=c(min(impurity)-1,max(impurity)+1),
     ylim=c(min(yield)-1,max(yield)+1))

points(impurity[process=="B"], yield[process=="B"], pch=2, col='red')
points(impurity[process=="C"], yield[process=="C"], pch=15, col='green')

abline(intercept.1,slope,col='blue')
abline(intercept.2,slope,col='red')
abline(intercept.3,slope,col='green')
```