Southern Illinois University-Edwardsville

Homework #1

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Course: *STAT 478 - Time Series Analysis* – Professor: *Dr. Beidi Qiang*
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Question 1

Show $\operatorname{Var}(Y) = \operatorname{E}(Y^2) - \operatorname{E}^2(Y)$ starting from the definition $\operatorname{Var}(Y) \coloneqq \operatorname{E}(Y - \operatorname{E}(Y))^2$ by expanding and properties of expectation.

scale a

Answer. The computational variance is given by the following theorem.

Theorem 1. *The variance of Y is defined as*

$$\operatorname{Var}(Y) \coloneqq \operatorname{E}(Y - \operatorname{E}(Y))^2 . \tag{1}$$

The variance of Y has an equivalent expression known as the computational *variance and is given by*

$$\operatorname{Var}(Y) = \operatorname{E}(Y^2) - \operatorname{E}^2(Y) . \tag{2}$$

Proof. We may rewrite the definition of the variance by expanding the right-hand-side,

$$\operatorname{Var}(Y) = \operatorname{E}(Y^2 - \operatorname{E}(Y)Y - \operatorname{E}(Y)Y + \operatorname{E}^2(Y)) . \tag{3}$$

By the linearity of expectation, we may rewrite this as

$$\operatorname{Var}(Y) = \operatorname{E}(Y^2) - 2\operatorname{E}(\operatorname{E}(Y)Y) + \operatorname{E}(\operatorname{E}^2(Y)) \tag{4}$$

By the property that $\operatorname{E}(a) = a$, a is a constant, we may eliminate the outer expectations in the above, resulting in

$$\begin{aligned}\operatorname{Var}(Y) &= \operatorname{E}(Y^2) - 2\operatorname{E}^2(Y) + \operatorname{E}^2(Y) \\ &= \operatorname{E}(Y^2) - \operatorname{E}^2(Y) .\end{aligned}$$

$\square$

Question 2

Let (X, Y) have joint density $\mathrm{f}_{X,Y}(x, y) \coloneqq (x + y)$ over $R \coloneqq \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$, the unit square in the plane.

(*a*) Find $\operatorname{E}(X)$, $\operatorname{Var}(X)$, and $\operatorname{E}(XY)$.

(*b*) Find $\operatorname{Corr}(X, Y)$. Are X and Y independent?

(*c*) Find $\operatorname{Cov}(X, X + Y)$. Are X and Y independent?

Answer.

(*a*) The expectation and variance characteristics of X are given by the following theorem.

Theorem 2. *If* $(X, Y) \sim \mathrm{f}_{X,Y}(x, y) \coloneqq (x + y)$ *with a support over the unit square, then*

$$\mathrm{E}(X) = \frac{7}{12}\,, \tag{5}$$

$$\mathrm{Var}(X) = \frac{11}{144}\,, \tag{6}$$

$$\mathrm{E}(XY) = \frac{1}{3}\,. \tag{7}$$

Proof. To find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$, we first find the marginal distribution of X,

$$\begin{aligned}\mathrm{f}_X(x) &= \int_0^1 \mathrm{f}_{\mathrm{X,Y}}(x, y)\mathrm{d}y \\ &= \int_0^1 (x + y)\mathrm{d}y = x + \frac{1}{2}\,.\end{aligned}$$

The expectation of X is given by

$$\begin{aligned}\mathrm{E}(X) \coloneqq \int_0^1 x\,\mathrm{f}_{\mathrm{X}}(x)\mathrm{d}x\,. \coloneqq \int_0^1 x\left(x + \frac{1}{2}\right)\mathrm{d}x \\ = \left.\frac{x^3}{3}\right|_0^1 + \left.\frac{x^2}{4}\right|_0^1 = \frac{7}{12}\,.\end{aligned}$$

The variance of X is given by

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}^2(X)\,.$$

By theorem 2, $\mathrm{E}(X) = 7/12$. $\mathrm{E}(X^2)$ is given by

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^1 x^2\left(x + \frac{1}{2}\right)\mathrm{d}x \\ &= \left.\frac{x^4}{4}\right|_0^1 + \left.\frac{x^3}{6}\right|_0^1 = \frac{5}{12}\,.\end{aligned} \tag{8}$$

Thus,

$$\mathrm{Var}(X) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}\,.$$

The expectation of XY is given by

$$\begin{aligned}\mathrm{E}(XY) &\coloneqq \int_0^1\int_0^1 xy\,\mathrm{f}_{X,Y}(x, y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1\int_0^1 xy(x + y)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 y\int_0^1 x^2\mathrm{d}x\mathrm{d}y + \int_0^1 y^2\int_0^1 x\mathrm{d}x\mathrm{d}y = \frac{1}{3}\,.\end{aligned}$$

□

(*b*) The correlation of Y and Y is given by the following corollary.

Corollary 1. *If* $(X, Y) \sim \mathrm{f}_{X,Y}(x, y) \coloneqq (x + y)$ *with a support over the unit square, then*

$$\mathrm{Corr}(X, Y) = -\frac{1}{60}\,. \tag{9}$$

Proof. The correlation of X and Y, denoted by $\mathrm{Corr}(X, Y)$, is defined as

$$\mathrm{Corr}(X, Y) \coloneqq \frac{\mathrm{Cov}(X, Y)}{\sigma_X\sigma_Y} \tag{10}$$

where

$$\operatorname{Cov}(X,Y) = \operatorname{E}(XY) - \operatorname{E}(X)\operatorname{E}(Y)\,.$$

Without proof we claim the expectation of Y is the same as X by applying the same proof in part (a) for Y instead of X. We also found $\operatorname{E}(XY)$ in part (a). Plugging in these solutions to the above equation yields

$$\operatorname{Cov}(X,Y) = \frac{1}{3} - \frac{7}{12}\frac{7}{12} = -\frac{1}{144} \tag{11}$$

and thus the correlation is

$$\operatorname{Corr}(X,Y) = \frac{-\frac{1}{144}}{\sqrt{5/12}\sqrt{5/12}} = -\frac{1}{60}\,. \tag{12}$$

□

The random variables X and Y are dependent since they have a non-zero correlation.[1]

(c) The covariance of X and $X+Y$ is given by the following corollary.

Corollary 2. *If* $(X,Y) \sim \mathrm{f}_{X,Y}(x,y) \coloneqq (x+y)$ *with a support over the unit square, then*

$$\operatorname{Cov}(X, X+Y) = \frac{5}{72}\,. \tag{13}$$

Proof. The covariance X and $X+Y$ is given by

$$\begin{aligned}\operatorname{Cov}(X, X+Y) &= \operatorname{E}(X(X+Y)) - \operatorname{E}(X)\operatorname{E}(X+Y)\\ &= \left(\operatorname{E}(X^2) - \operatorname{E}^2(X)\right) + \left(\operatorname{E}(XY) - \operatorname{E}(X)\operatorname{E}(Y)\right)\\ &= \operatorname{Var}(X) + \operatorname{Cov}(X,Y)\,.\end{aligned} \tag{14}$$

By theorem 2, $\operatorname{Var}(X) = 11/144$ and by eq 11, $\operatorname{Cov}(X,Y) = -1/144$, and thus

$$\operatorname{Cov}(X, X+Y) = 11/144 - 1/144 = \frac{5}{72}\,, \tag{15}$$

□

By corollary 2, X and $X+Y$ have non-zero covariance and thus they are not independent.[2]

Question 3

The TSA library in R contains the data set *co2*, which lists monthly carbon dioxide (CO2) levels in northern Canada from 1/1994 to 12/2004.

(a) Construct a time series plot of the data. Print the plot and describe all systematic patterns you see in the plot.

(b) Apply a moving average filter of span 12 to the data. Plot the original data and overlay (superimpose) the moving average, and provide this plot. Discuss whether the moving average filter captures the overall trend in the time series.

Answer.

(a) In fig **??**, we see the time series plot of the *co2* data set. There are several regularities observable in the time series plot.

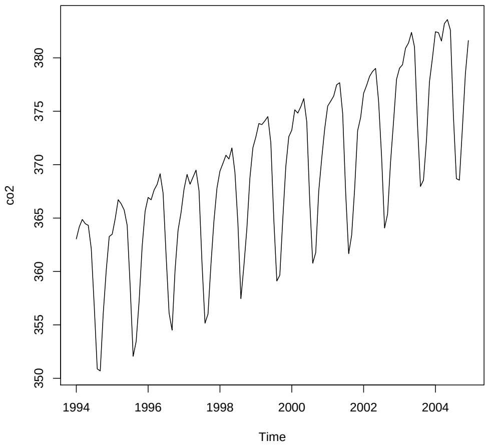
(a) The CO2 levels are increasing over time.

[1] We also knew they are dependent by the fact that their joint PDF cannot be factored into a product of two terms, one involving only x and the other only y.

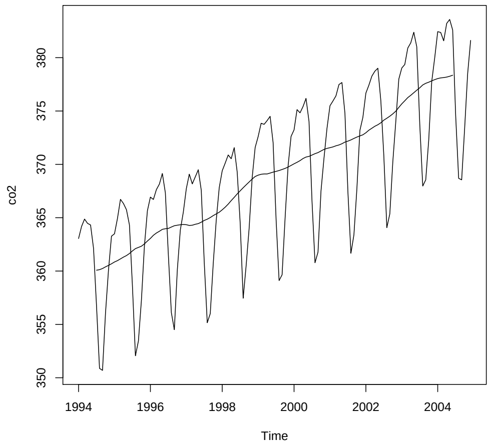
[2] Note that a zero covariance does not necessarily imply independence.

```r
# homework #1: problem 3
# requires:
#      - TSA library      : install.packages('TSA')
#      - forecast library : install.package('forecast')
library(TSA)
library(forecast)

# part (a)
data(co2)
pdf(file="plot3_a.pdf")
plot.ts(co2)

# part (b)
co2.ma12=ma(co2,order=12)
pdf(file="plot3_b.pdf")
plot.ts(co2)
lines(co2.ma12)
```

Listing 1: R script used to generate time series plots for problem 3.a and 3.b.



(a) Time series plot of *co2* data



(b) Time series plot of *co2* data with a 12th order moving average superimposed

Figure 1: Question 3 time series plots generated by the R script in listing 1.

(b) The CO2 levels have a non-linearity component. In particular, the CO2 levels, when we take out the increasing overall trend, seems to be sinusoidal (with a constant amplitude) with a constant period of around one year.

(c) The deviation from the pattern does not seem to be changing, i.e., the random deviation seems to be the result of i.i.d. noise.

(d) The deviation from the pattern seems pretty small, i.e., the i.i.d. noise seems to have a standard deviation that is small compared to the amplitude of the sinusoidal.

(*b*) In fig 1b, we see the time series plot of *co2* with the 12th order moving average applied to the co2 data superimposed over it. This moving averate filters out the seasonal pattern but makes it more apparent that the overall trend is one of increasing CO2 levels. Note that since the seasonal pattern has a period of a year, it is not surprising that the 12th order moving average filters it out since it takes the average of a one year interval.

Question 4

Suppose $\{e_t\}$ is a normal white noise process with mean 0 and variance σ^2. Let $\{Y_t\}$ be a process defined as (Y_t is a moving average of white noise process):

$$Y_t := \frac{1}{3}(e_t + e_{t-1} + e_{t-2})\,. \tag{16}$$

(*a*) Find the mean and variance function of $\{Y_t\}$.

(*b*) Find the autocovariance function and autocorrelation function of $\{Y_t\}$.

(*c*) Is the time series $\{Y_t\}$ stationary? Explain your answer.

(*d*) Simulate and plot the process in R. Provide your R code and print out the plot.

Answer.

(*a*) The expectation of Y_t is 0 as proven by

$$\begin{aligned} \mathrm{E}(Y_t) &= \mathrm{E}\left(\frac{1}{3}(e_t + e_{t-1} + e_{t-2})\right) \\ &= (\mathrm{E}(e_t) + \mathrm{E}(e_{t-1}) + \mathrm{E}(e_{t-2}))/3 \\ &= (0+0+0)/3 = 0 \end{aligned} \tag{17}$$

and the variance of Y_t is $\frac{\sigma^2}{3}$ as proven by

$$\begin{aligned} \mathrm{Var}(Y_t) &= \mathrm{Var}\left(\frac{1}{3}(e_t + e_{t-1} + e_{t-2})\right) \\ &= (\mathrm{Var}(e_t) + \mathrm{Var}(e_{t-1}) + \mathrm{Var}(e_{t-2}))/3^2 \\ &= (\sigma^2 + \sigma^2 + \sigma^2)/9 = \frac{\sigma^2}{3}\,. \end{aligned} \tag{18}$$

The marginal distribution, Y_t, as a function of independent normal random variables, is normally distributed with mean $\mathrm{E}(Y_t) = 0$ and variance $\mathrm{Var}(Y_t) = \sigma^2/3$ for all $t \geq 2$,

$$Y_t \sim \mathcal{N}(0, \sigma^2/3)\,. \tag{19}$$

In time series analysis we are primarily interested in the joint distribution of $\{Y_t\}$. If we were to repeatedly sample from $\{Y_t\}$ we would observe that the distribution of outcomes for Y_t for each $t \geq 2$ follows the above normal distribution. However, in a time series $\{Y_t\}$, we would see that $Y_2, Y_3, \ldots$ are correlated, as we show next.

(*b*) The autocovariance function of $\{Y_t\}$ is given by the following theorem.

Theorem 3. *The autocovariance of Y_t and Y_s is given by*

$$r_{t,s} = \begin{cases} \frac{3-\ell}{9}\sigma^2 & \ell \leq 2 \\ 0 & \ell > 2 \end{cases} \tag{20}$$

where $\ell = |s-t|$. Since the autocovariance function $r_{t,s}$ is only a function of ℓ, we may reparamterize it as r_ℓ.

Proof. The autocovariance function is defined as

$$r_{t,s} := \operatorname{Cov}(Y_t, Y_s)$$

which may be rewritten as

$$r_{t,s} = \operatorname{Cov}(e_t + e_{t-1} + e_{t-2}, e_s + e_{s-1} + e_{s-2})/9\,.$$

By an argument from symmetry, $r_{s,t} = r_{t,s}$, and so without loss of generality we assume $t \leq s$. We reparameterize the covariance in terms of a time lag ℓ by letting $t = s - \ell$. Then, we may rewrite the above as

$$r_{s-\ell,s} = \operatorname{Cov}(e_{s-\ell} + e_{s-\ell-1} + e_{s-\ell-2}, e_s + e_{s-1} + e_{s-2})/9$$

In what follows, we perform a case analysis for the lag time ℓ.

i If $\ell > 2$, then $Y_{s-\ell}$ and Y_s are independent since they are functions of different subsets of $\{e_t\}$ and therefore their covariance is 0.

ii If $\ell = 0$, then $r_{s,s} = \operatorname{Var}(Y_s) = \frac{\sigma^2}{3}$.

iii If $\ell = 1$, then

$$\begin{aligned} r_{s-1,s} &= \operatorname{Cov}(e_{s-1} + e_{s-2} + e_{s-3}, e_s + e_{s-1} + e_{s-2})/9 \\ &= \operatorname{Cov}(e_{s-1} + e_{s-2}, e_{s-1} + e_{s-2})/9 \\ &= \operatorname{Var}(e_{s-1} + e_{s-2})/9 = \frac{2\sigma^2}{9}\,. \end{aligned}$$

iv If $\ell = 2$, then

$$\begin{aligned} r_{s-2,s} &= \operatorname{Cov}(e_{s-2} + e_{s-3} + e_{s-4}, e_s + e_{s-1} + e_{s-2})/9 \\ &= \operatorname{Cov}(e_{s-2}, e_{s-2})/9 \\ &= \operatorname{Var}(e_{s-2})/9 = \frac{\sigma^2}{9}\,. \end{aligned}$$

These cases show that the autocovarince is strictly a function of ℓ. When we summarize these cases we get the desired result. ☐

Theorem 4. *The autocorrelation of $\{Y_t\}$ is given by*

$$\rho_{t,s} = \begin{cases} 1 - \frac{\ell}{3} & \ell \leq 2 \\ 0 & \ell > 2 \end{cases} \tag{21}$$

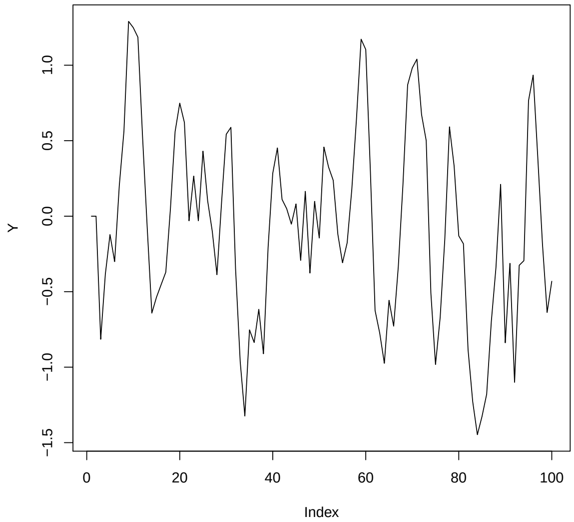
where $\ell = |t-s|$. Since $\rho_{t,s}$ is only a function of ℓ, we may reparameterize as ρ_ℓ.

Proof. The autocorrelation function of Y_t and Y_s is defined as

$$\rho_{t,s} := \frac{r_{t,s}}{\sqrt{\operatorname{Var}(Y_t)}\sqrt{\operatorname{Var}(Y_s)}}\,.$$

We consider the two cases of the autocovariance function, when $\ell = |s-t| > 2$ and $\ell \leq 2$.

i When ℓ is greater than 2, the autocovariance is 0 and therefore the autocorrelation is 0.



(a) Plot of $\{Y_t\}$.

```
# {e_t}, a white noise process
e <- rnorm(n=100,mean=0,sd=1)

# {Y_t} time series
Y <- vector(length=n)

# first two elements of {Y_t} are 0
Y[1]=0
Y[2]=0

# Y_t := 1/3 sum_{k=t-2}^t e_k
for (t in 3:n)
{
    Y[t]=(e[t]+e[t-1]+e[t-2])/3.0
}

pdf(file="plot4_d.pdf")
plot(Y,type="l")
```

(b) Generatative model of $\{Y_t\}$.

Figure 2: Question 4 plots and code

ii When $\ell \leq 2$, we rewrite the autocorrelation function by plugging in the autocovariance function of Y_t and Y_s, the variance of Y_t, and the variance of Y_s,

$$\begin{aligned}\rho_{t,s} &= \frac{(3-\ell)\sigma^2/9}{\sqrt{\sigma^2/3}\sqrt{\sigma^2/3}} \\ &= \frac{(3-\ell)\sigma^2/9}{\sigma^2/3} = \frac{3-\ell}{3} = 1 - \frac{\ell}{3}.\end{aligned}$$

□

Sensibly, the greater the lag between two random variables in the time series, the less they are correlated.

(*c*) The autocovariance between $Y_{t-\ell}$ and Y_t is given by $r_{t-\ell,t} = r_\ell$, which is zero if $\ell > 2$ and otherwise is $(3-\ell)\sigma^2/9$. We see that the autocovariance function only depends on the time lag ℓ and furthermore $\mathrm{E}(Y_t) = 0$ and $\mathrm{Var}(Y_t) = \frac{\sigma^2}{3}$ do not depend on time either. This is sufficient for weakly stationary. Furthermore, since Y_t is normally distributed, weakly stationary implies strictly stationary.

(*d*) See fig 2a for the plot of the time series $\{Y_t\}$ and fig 2b for the generative model.