Shape and scale of Weibull distribution

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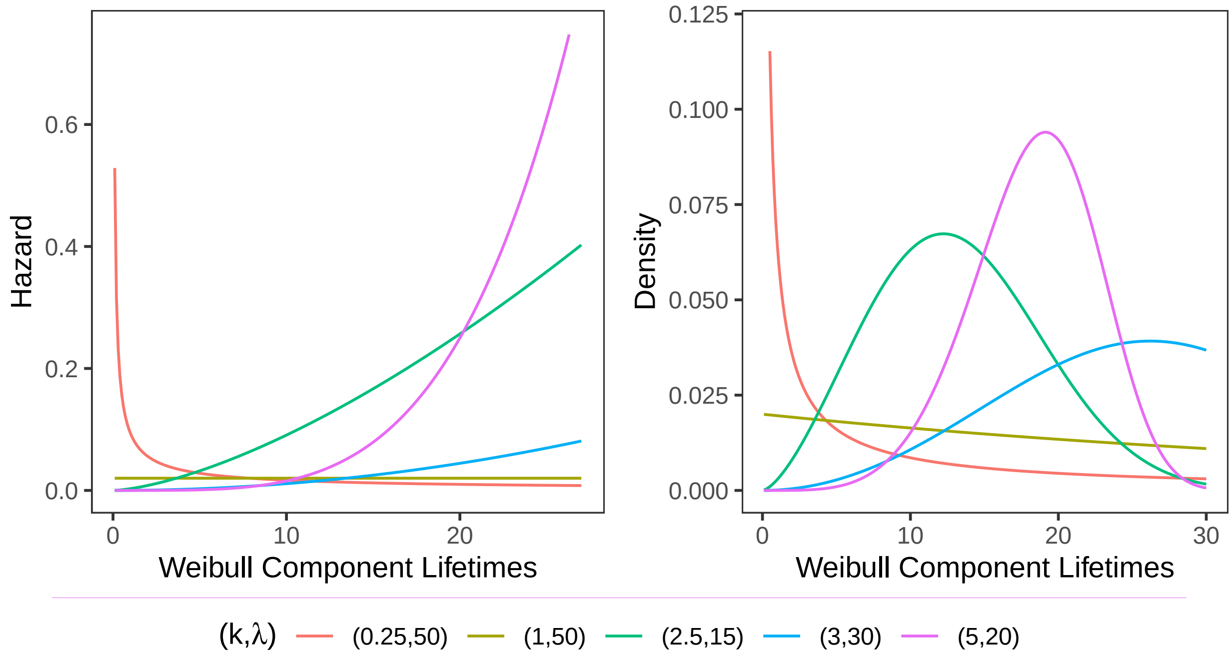


Plots of five different components with Weibull distributed lifetimes. Key observations:
(1) Components with a shape < 1 have decreasing hazards, e.g., component 1.
(2) Components with shapes > 1 have increasing hazards, e.g., components 3, 4, and 5.
(3) Components with shape = 1 have constant hazards, e.g., component 2.

Figure 1: Component lifetime plots

The shape parameter k may be understood in the following way:

- If $k < 1$, then the hazard function decreases with respect to system lifetime, which may occur if defective items fail early and are weeded out.
- If $k > 1$, then the hazard function is increases with respect to time, which may occur as a result of an aging process.
- If $k = 1$, then the failure rate is constant, which means it is exponentially distributed.

See Figure 1 for plots of the hazard and pdf functions of five different Weibull distributed components. We will use these plots to illustrate the different shapes of the hazard and pdf functions. The first component has a shape parameter $k = 0.25$, which is less than 1, and so the hazard function decreases with respect to time. The second component has a shape parameter $k = 1$, and so the hazard function is constant. The third, fourth, and fifth components have shape parameters $k = 2.5$, $k = 3$, and $k = 5$, respectively, and so the hazard functions increase with respect to time.

The lifetime of the series system composed of m Weibull components has a reliability function given by

$$R(t;\boldsymbol{\theta}) = \exp\left\{-\sum_{j=1}^{m}\left(\frac{t}{\lambda_j}\right)^{k_j}\right\}. \tag{1.1}$$

Proof. By Theorem **??**,

$$R(t;\boldsymbol{\theta}) = \prod_{j=1}^{m} R_j(t;\lambda_j,k_j).$$

Plugging in the Weibull component reliability functions obtains the result

$$\begin{aligned} R(t;\boldsymbol{\theta}) &= \prod_{j=1}^{m} \exp\left\{-\left(\frac{t}{\lambda_j}\right)^{k_j}\right\} \\ &= \exp\left\{-\sum_{j=1}^{m}\left(\frac{t}{\lambda_j}\right)^{k_j}\right\}. \end{aligned}$$

□

The Weibull series system's hazard function is given by

$$h(t;\boldsymbol{\theta}) = \sum_{j=1}^{m} \frac{k_j}{\lambda_j}\left(\frac{t}{\lambda_j}\right)^{k_j-1}, \tag{1.2}$$

whose proof follows from Theorem **??**.

In Figure 2, we plot the hazard function and the pdf of the Weibull series system with the component lifetime parameters considered earlier,

$$\begin{aligned} T_{i1} &\sim \mathrm{WEI}(3, 30) \\ T_{i2} &\sim \mathrm{WEI}(2, 50) \\ T_{i3} &\sim \mathrm{WEI}(0.5, 15) \\ T_{i4} &\sim \mathrm{WEI}(5, 20) \\ T_{i5} &\sim \mathrm{WEI}(0.25, 50) \end{aligned}$$

for the i-th series system where $i = 1, \ldots, n$. By Theorem **??**, the series system has a random lifetime given by

$$T_i = \min\{T_{i1}, \ldots, T_{i5}\}.$$

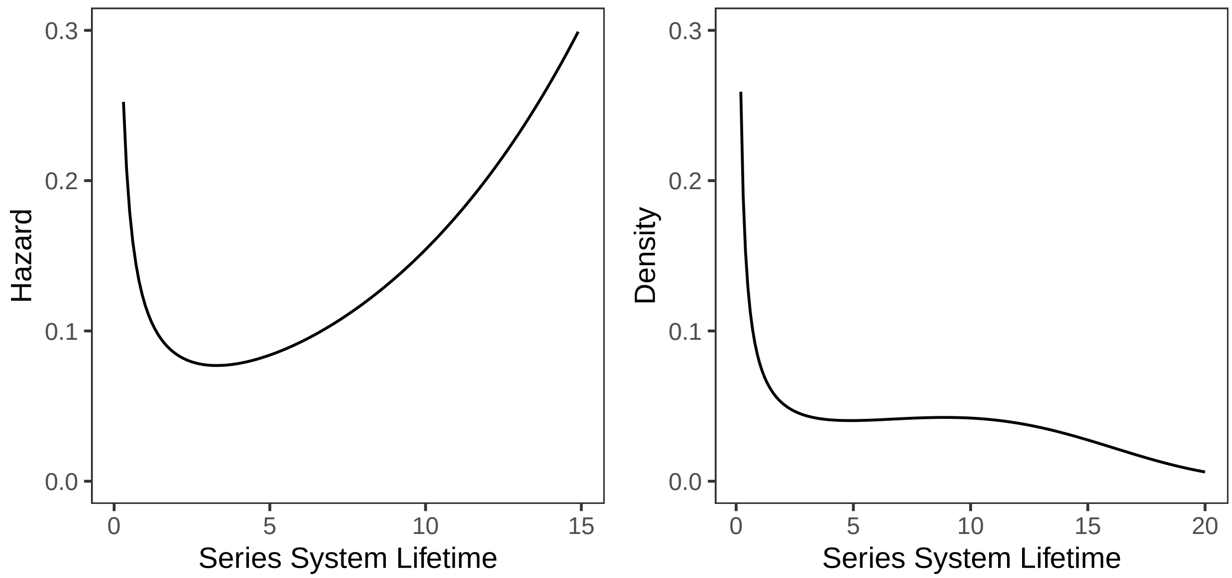
where $\boldsymbol{\theta} = (k_1, \lambda_1, \ldots, k_5, \lambda_5)$.

The series system, due to being a mixture of Weibull components with different shapes, has both a high infant mortality rate and an aging process, which is reflected in the plot of the hazard function. The hazard is initially high then decreases to some minimum before increasing again. This is a pattern we see in nature, e.g., electronic appliances may fail early due to defects, but those that survive the initial period of high failure rate can be expected to last for a long time before finally wearing out due to an aging process.

The pdf of the series system also appears to be multimodal, where the modes correspond to the high infant mortality rate and the aging process.

The pdf of the series system is given by

$$f(t;\boldsymbol{\theta}) = \left\{\sum_{j=1}^{m} \frac{k_j}{\lambda_j}\left(\frac{t}{\lambda_j}\right)^{k_j-1}\right\} \exp\left\{-\sum_{j=1}^{m}\left(\frac{t}{\lambda_j}\right)^{k_j}\right\}. \tag{1.3}$$



Plots of the hazard function and the pdf of a series system with the previously discussed Weibull components. Key observations:
(1) The hazard is initially large but decreases to some minimum before increasing again, exhibiting both a high infant mortality rate and an aging process. This is a pattern we see in nature (e.g., humans).
(2) The pdf has a rather unusual form, a result of being a combination of Weibull distributions.

Figure 2: System lifetime plots

Proof. By definition,

$$f(t; \boldsymbol{\theta}) = h(t; \boldsymbol{\theta}) R(t; \boldsymbol{\theta}).$$

Plugging in the failure rate and reliability functions given respectively by Equations (1.1) and (1.2) completes the proof. □