Context & Motivation

Reliability in **series systems** is like a chain's strength – determined by its weakest link.

- Essential for system design and maintenance.

Main Goal: Estimate individual component reliability from *failure data*.

Challenges:

- *Masked* component-level failure data.
- *Right-censoring* in system-level failure data.

Our Response:

- Derive techniques to interpret such ambiguous data.
- Aim for precise and accurate reliability estimates for individual components using maximum likelihood estimation (MLE) and bootstrapped (BCa) confidence intervals.

- **Chain Analogy**: Think of a series system as a chain. Its reliability, just like a chain's strength, is determined by its weakest link or component. When any component fails, the whole system does.
- **Reliability Importance**: Understanding the reliability of each component is essential for the design and maintenance of these systems.
- **Data Challenge**: The data we rely on can come with its own challenges. We sometimes encounter ambiguous data like right-censored information or masked component-level failures, where we don't know precisely which component failed.
- **Aim**: Our goal is to interpret such ambiguous data and provide accurate reliability estimates for each component, which includes providing correctly specified 9595using the BCa method.

Core Contributions

Likelihood Model for **series systems**.

- Accounts for *right-censoring* and *masked component failure*.
- Can easily incorporate additional failure data.

Specifications of Conditions:

- Assumptions about the masking of component failures.
- Simplifies and makes the model more tractable.

Simulation studies:

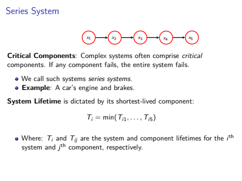
- Components with *Weibull* lifetimes.
- Evaluate MLE and confidence intervals under different scenarios.

R Library: Methods available on GitHub.

- github.com/queelius/wei.series.md.c1.c2.c3

Our core contributions can be broken down into several parts:

- **Likelihood model**: We've derived a likelihood model for series systems that accounts for the ambiguous data.
- **Explain conditions**: We've clarified the conditions this model assumes about the masking of component failures. These conditions simplify the model and make it more tractable.
- **Validated with simulation study**: We've validated our model with extensive simulations using Weibull distributions to gauge its performance under various scenarios.
- **R Library**: For those interested, we made our methods available in an R Library hosted on GitHub.


Series System

Critical Components: Complex systems often comprise *critical* components. If any component fails, the entire system fails.

- We call such systems *series systems*.
- **Example**: A car's engine and brakes.

System Lifetime is dictated by its shortest-lived component:

$$T_i = \min(T_{i1}, \ldots, T_{i5})$$

- Where: T_i and T_{ij} are the system and component lifetimes for the i^{th} system and j^{th} component, respectively.

- **Critical Components**: Many complex systems have components that are essential to their operation.
- **Series System**: If any of these components fail, the entire system fails. We call these series systems.
- **Car**: Think of a car - if the engine or brakes fail, the car can't be operated.
- **Lifetime**: Its lifetime is the lifetime of its shortest-lived component.
- **Notation**: For reference, we show the math notation we'll use throughout the talk.

Reliability Function

Reliability Function represents the probability that a system or component functions beyond a specified time.

- Essential for understanding longevity and dependability.

Series System Reliability: Product of the reliability of its components:

$$R_{T_i}(t;\boldsymbol{\theta}) = \prod_{j=1}^{m} R_j(t;\boldsymbol{\theta_j}).$$

- Here, $R_{T_i}(t;\boldsymbol{\theta})$ and $R_j(t;\boldsymbol{\theta_j})$ are the reliability functions for the system i and component j, respectively.

Relevance:

- Forms the foundation for most reliability studies.
- Integral to our likelihood model, e.g., right-censoring events.

- **Reliability Function** The reliability function tells us the chance a component or system functions past a specific time. It's our key metric for longevity.
- **Product of Component Reliability**: In a series system, the overall reliability is the product of its component reliabilities. So, if even one component has a low reliability, it can impact the whole system.
- **Relevance**: Why does this matter to us? This concept is foundational to our studies, especially when we're handling right-censored data.

Hazard Function: Understanding Risks

Hazard Function: Measures the immediate risk of failure at a given time, assuming survival up to that moment.

- Reveals how the risk of failure evolves over time.
- Guides maintenance schedules and interventions.

Series System Hazard Function is the sum of the hazard functions of its components:

$$h_{T_i}(t;\boldsymbol{\theta}) = \sum_{j=1}^{m} h_j(t;\boldsymbol{\theta_j}).$$

- Components' risks are additive.

Hazard Function: Let's shift focus to the hazard function. Essentially, it's a way to gauge the immediate risk of failure, especially if it's been functioning up until that point.

Series Hazard Function Lastly, the hazard function for a series system is just the sum of the hazard functions of its components.

Additive: We see that the component risks are additive.

Joint Distribution of Component Failure and System Lifetime

Our likelihood model depends on the **joint distribution** of the system lifetime and the component that caused the failure.

- **Formula**: Product of the failing component's hazard function and the system reliability function:

$$f_{K_i,T_i}(j,t;\boldsymbol{\theta}) = h_j(t;\boldsymbol{\theta_j})R_{T_i}(t;\boldsymbol{\theta}).$$

- **Single Point of Failure**: A series system fails due to one component's malfunction.
- **Representation**:
  - K_i: Component causing the i^{th} system's failure.
  - $h_j(t;\boldsymbol{\theta_j})$: Hazard function for the j^{th} component.

- **Joint Distribution** In our likelihood model, understanding the joint distribution of a system's lifetime and the component that led to its failure is fundamental.
- **Formula**: It is the product of the failing component's hazard function and the system reliability function.
- **Unique Cause**: Which emphasizes that in a series system, failure can be attributed to a single component's malfunction.
- **Notation**: Here, K_i denotes the component responsible for the failure.

Component Failure & Well-Designed Series Systems

The **marginal probability** of component failure helps predict the cause of failure.

- **Derivation**: Marginalize the joint distribution over the system lifetime:

$$\Pr\{K_i = j\} = E_{\boldsymbol{\theta}}\left[\frac{h_j(T_i; \boldsymbol{\theta_j})}{h_{T_i}(T_i; \boldsymbol{\theta_l})}\right].$$

Well-Designed Series System: Components exhibit comparable chances of causing system failures.

- **Relevance**: Our simulation study employs a (reasonably) well-designed series system.

- **Marginal**: We can use this joint distribution to calculate the marginal probability of component failure.
- **Expected Value**: When we do so, we find that it is the expected value of the ratio of component and system hazard functions.
- **Well-Designed**: We say that a series system is *well-designed* if each components has a comparable chance of failing.
- **Relevance**: Our simulation study is based on a reasonably well-designed series system.

Likelihood Model

Likelihood measures how well our model parameters ($\boldsymbol{\theta}$) explain the data. Each system contributes to the **total likelihood** via its *likelihood contribution*:

$$L(\boldsymbol{\theta}|\text{data}) = \prod_{i=1}^{n} L_i(\boldsymbol{\theta}|\text{data}_i).$$

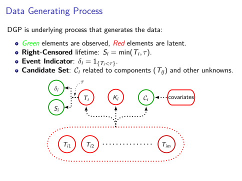
where $\textbf{data}_i$ is the data for the i^{th} system and L_i is its contribution.

Our model handles the following data: **Right-Censored**: Experiment ends before failure (Event Indicator: $\delta_i = 0$). - Contribution is system reliability: $L_i(\boldsymbol{\theta}) = R_{T_i}(\tau; \boldsymbol{\theta})$. **Masked Failure**: Failure observed, but the failed component is masked by a *candidate set*. More on its contribution later.

System	Right-Censored Lifetime	Event Indicator	Candidate Set
1	1.1	1	$\{1, 2\}$
2	5	0	$\emptyset$

Let's talk about the likelihood model, which is a way of measuring how well our model explains the data.

- **Total likelihood** is the product of the likelihood contributions of each system.
- **Contributions**: Our likelihood model deals with right-censoring and masked cause of failure.
- **Right-Censoring** occurs when the experiment ends before the system fails. Its contribution is just the system reliability, since the event indicator tells us if the system was right-censored.
- **Masking** occurs when we observe a failure but we don't know the precise component cause. Instead, we observe a candidate set of components that could have failure. More on this later.
- Here's an example of observed data.
- **System 1**: We see that the system failed at 1.1. We don't know which component failed, but we know it was either component 1 or 2.



- **DGP**: Let's discuss the data generating process to motivate our model.
- **Graph**: Here's the graph: green is observed and red is latent.
- **Infer**: We don't get to see the red elements, but we can infer most of them from the green elements.
- **Green**: So, let's focus on the green elements.
- **Right-censoring** time is the minimum of the system lifetime and the right-censoring time.
- **Event** indicator is 1 if the system fails before the right-censoring time, 0 otherwise.
- The candidate sets are related to the component lifetimes and many other factors.
- **Difficult** to model. Seek a simple model that is valid under certain assumptions, which we discuss next.

Likelihood Contribution: Masked Failure Conditions

The candidate sets that **mask** the failed component are assumed to satisfy the following conditions:

Candidate Set Contains Failed Component: The candidate set includes the failed component.

Equal Probabilities Across Candidate Sets: The probability of of the candidate set is constant across different components within it.

Masking Probabilities Independent of Parameters: The masking probabilities when conditioned on the system lifetime and the failed component aren't functions of the system parameter.

- Reasonable conditions in many industrial settings.

- The right-censoring contribution is straightforward. But the masked failure contribution is a bit more complicated.
- **Masking** occurs when a system fails but the precise failed component is masked by a candidate set.
- **Tractable**: To make problem more tractable, we introduce certain conditions.
- **Condition 1**: The candidate set always includes the failed component.
- **Condition 2**: The probability of the candidate set is constant across different components within it.
- **Condition 3**: The masking probabilities when conditioned on the system lifetime and the failed component aren't functions of the system parameter.
- **Reasonable**: These conditions are often reasonable.

Likelihood Contribution: Derivation for Masked Failures

Take the **joint distribution** of T_i, K_i, and $\mathcal{C}_i$ and marginalize over K_i:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \sum_{j=1}^{m} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 1** to get a sum over candidate set:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 2** to move probability outside the sum:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j'\} \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}).$$

Apply **Condition 3** to remove the probability's dependence on $\boldsymbol{\theta}$:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \beta_i \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}).$$

Result: $L_i(\boldsymbol{\theta}) \propto \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) = R_{T_i}(t_i; \boldsymbol{\theta}) \sum_{j \in c_i} h_j(t_i; \boldsymbol{\theta_j})$.

- **Derive**: Here, we derive the likelihood contribution for masked failures.
- **Joint**: To start, we use the joint distribution of the system lifetime, the failed component, and the candidate set.
- **Marginalize**: We marginalize over the failed component, since we don't know which component failed.
- Apply Condition 1 to get a sum over the candidate set instead.
- Apply Condition 2 to move the probability outside the sum.
- Apply condition 3 to remove the probability's dependence on $\boldsymbol{\theta}$.
- The result: the likelihood contribution is proportional to the product of the system reliability and the sum of the component hazards in the masking set.

Methodology: Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE): Maximize the likelihood function:

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} L(\boldsymbol{\theta}).$$

Solution: Numerically solved system of equations for $\hat{\boldsymbol{\theta}}$:

$$\nabla_{\boldsymbol{\theta}} \log L(\hat{\boldsymbol{\theta}}) = \mathbf{0}.$$

- **MLE**: We use the standard MLE approach.
- **ArgMax**: We find the parameter values that maximize the log-likelihood function.
- **Solution**: Since there is no closed-form solution, we numerically solve it.

Bootstrap Confidence Intervals (CIs)

Confidence Intervals (CI) quantify the uncertainty in our estimate.

Asymptotic Sampling Distribution of MLE is a popular choice for constructing CIs.

- **Challenge**: Asymptotic distribution may not be accurate for small sample sizes.
  - Particularly since we're dealing with right-censoring and masking.

Bootstrapped CIs: Resample data and obtain MLE for each.

- Use **percentiles** of bootstrapped MLEs for CIs.

Correctly Specified CIs:

- Desired: Coverage probability near 95%. (> 90% acceptable.)
- **Challenge**: Actual coverage may deviate.

BCa adjustments counteracts bias and skewness in estimates.

- **Goal**: Need a way to measure the uncertainty in our estimate.
- **CIs** are a popular; they help us pin down the likely range of values for our parameters.
- **Bootstrap** the CIs, since there is a lot of bias and variability in our estimate due to the masking and censoring in our small data sets and the asymptotic distribution is not likely to be accurate.
- **Specified**: We want our CIs to be correctly specified, meaning they cover the true parameter value around 95
- **BCa**: But they may be too low or too high; we use the BCa method to adjust for bias and skewness in the estimate. A coverage probability above 90% is acceptable.

Challenges with MLE on Masked Data

We discovered some challenges with the MLE.

Convergence Issues: Flat likelihood regions were observed due to the ambiguity in the masked data and small sample sizes.

Bootstrap Issues: Bootstrap relies on the Law of Large Numbers.

- It might not represent the true variability for small samples.
- Due to censoring and masking, the effective sample size is reduced.

Mitigation: In simulation study, we discard non-convergent samples for the MLE on original data, but keep all resamples for the bootstrap.

- Helps ensure robustness of the results, while acknowledging the inherent complexities introduced by masking and censoring.
- We report convergence rates in our simulation study.

Like any model, ours has its challenges:

- **Masking**: Masking and censoring, combined with small sample sizes, can cause flat likelihood regions, which can lead to convergence issues.
- **Small**: For small samples, bootstrapping may not always capture the true variability in the data **Approach**: We take the following approach in our simulation study. **Discard**: We discard non-convergent samples for the MLE on original data but retain all MLEs for the resampled data. **Robustness**: This helps ensure the robustness of our results while acknowledging the inherent complexities introduced by masking and censoring. **Convergence Rate**: We report the convergence rate in our simulation study.

Series System: Weibull Components

The lifetime of the j^{th} component in the i^{th} system:

$$T_{ij} \sim \text{Weibull}(k_j, \lambda_j)$$

- λ_j is the **scale** parameter
- k_j is the **shape** parameter:
  - $k_j < 1$: Indicates infant mortality.
  - $k_j = 1$: Indicates random failures.
  - $k_j > 1$: Indicates wear-out failures.

Recall that for a series system:

- **Series Reliability** is the product of the component reliabilities.
- **Hazard** is the sum of the component hazard functions.
- **Likelihood**: $L(\boldsymbol{\theta}) \propto \prod_{i=1}^{n} R_{T_i}(t_i; \boldsymbol{\theta}) \left[\sum_{j \in c_i} h_j(t_i; \boldsymbol{\theta_j})\right]^{\delta_i}$.

- **Weibull**: We model a series system with Weibull components.
- **Component Functions**: Hazard and reliability functions are well-known for Weibull.
- **Shape** parameter tells us a lot about the failure characteristics.
- **Increasing**: When the function is increasing, think of it as wearing-out over time.
- **Decreasing**: If it's decreasing, it usually signals some early-life challenges.
- **Series System**: Recall that for a series system, the reliability is the product of the component reliabilities and the hazard function is the sum of the component hazard functions.
- **Likelihood**: The likelihood is the same as before, we've just reproduced it here.

Well-Designed Series System

Simulation study centered around series system with Weibull components:

Component	Shape	Scale	$\Pr\{K_i\}$
1	1.26	994.37	0.17
2	1.16	908.95	0.21
3	1.13	840.11	0.23
4	1.18	940.13	0.20
5	1.20	923.16	0.20

- Based on (Guo, Niu, and Szidarovszky 2013) which studies a 3-component series system.
  - We add components 4 and 5 to make the system more complex.
- **Probabilities** are comparable: it is *reasonably well-designed*.
  - Component 1 is most reliable, component 3 is least reliable.
- **Shape** parameters are greater than 1, indicating wear-out failures.

- **Centered**: This study is centered around a series system with Weibull components.
- **Based**: It's based on a paper that studies a 3-component series system.
- **Added**: We added components 4 and 5 to make it more complex.
- **Probability**: We show the probability of each component being the cause of failure.
- **Well-Designed**: The probabilities are comparable, so no weak links. It's reasonably well-designed. Component 1 is most reliable, component 3 is least.
- **Parameters**: We show the shape and scale parameters for each component.
- **Wear-Out**: The shape parameters are greater than 1, indicating components are likely to fail due to wear-out.

Data Generation

Latent Component Lifetimes are generated for each system in the study.

Right-censoring: In our simulation study, we independently control the probability q (quantile) of right-censoring by finding the value τ that satisfies $\Pr\{T_i < \tau\} = q$.

- $S_i = \min(T_i, \tau)$ and $\delta_i = 1_{\{T_i < \tau\}}$.

Masking Component Failures: The *Bernoulli Masking Model* is used to mask component cause of failure, parameterized by masking probability p.

- p chosen independently: at the extremes, if $p = 0$ there is no masking, and if $p = 1$, there is total masking.
- We describe the process and how it satisfies the masking conditions next.

- **Data Generation**: We generate the latent component lifetimes for the series system we just discussed.
- **Observed Data**: Then, we generate the data we actually see, which is based on the component data.
- **Right-Censoring**: We control the probability of right-censoring by finding the value of τ that satisfies the quantile q. Then, we set the right-censoring time to be the minimum of the system lifetime and τ. The event indicator is 1 if the system fails before τ, 0 otherwise.
- **Masking**: We use a Bernoulli masking model to mask the component cause of failure. We parameterize the level of masking by the masking probability, p.
- We parameterize the level of masking by the masking probability, p, which specifies that each non-failed component has a p probability of masking the failed component by including it in the candidate set.

Data Generation: Satisfying Masking Conditions

We generate the candidate sets for each system in the study.

Satisfying Masking Conditions:

- **Condition 1**: The failed component deterministically placed in candidate set.
- **Condition 2**: By using a Bernoulli distribution with a constant probability p for all components, probability of a candidate set is constant as we vary which component failed within set.
- **Condition 3**: Masking only depends on the fixed parameter p and doesn't interact with the system parameter θ.

- **Masking**: We use a Bernoulli masking model for masking the failed component.
- This satisifes the masking failure conditions in the following ways:
- **Condition 1**: The failed component is deterministically placed in the candidate set.
- **Condition 2**: The probability of masking is the same for all components, so the probability of the candidate set is constant across components.
- **Condition 3**: The masking probability is independent of the parameters.

Performance Metrics

Objective: Evaluate the MLE and BCa confidence intervals' performance across various scenarios.

- **MLE Evaluation**:
  - **Accuracy**: Proximity of the MLE's expected value to the actual value.
  - **Precision**: Consistency of the MLE across samples.
- **BCa Confidence Intervals Evaluation**:
  - **Accuracy**: Confidence intervals (CIs) should cover true parameters around 95% of the time.
    - Coverage probability (CP)
  - **Precision**: Assessed by the width of the CIs.

Both accuracy and precision are crucial for confidence in the analysis.

- **Objective**: We want to evaluate the performance of the MLE and BCa confidence intervals across various scenarios.
- **MLE Evaluation**: We evaluate the MLE in terms of accuracy and precision.
- **Accuracy**: Accuracy is the proximity of the MLE's expected value to the actual value.
- **Precision**: Precision is the consistency of the MLE across samples.
- **BCa Confidence Intervals Evaluation**: We evaluate the BCa confidence intervals in terms of accuracy and precision.
- **Accuracy**: Accuracy is measured by the coverage probability, which is the proportion of times the confidence interval covers the true parameter.
- **Precision**: Precision is assessed by the width of the confidence interval.