Reliability Estimation in Series Systems

Maximum Likelihood Techniques for Right-Censored and Masked Failure Data

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Section 1

Introduction

Context & Motivation

- Quantifying reliability in series systems is essential.
- Real-world systems often only provide system-level failure data.
  - Masked and right-censored data obscure reliability metrics.
- Need robust techniques to decipher this data and make accurate estimations.

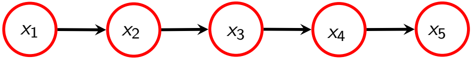
Core Contributions

- Derivation of likelihood model that accounts for right-censoring and masking.
  - Trivial to add more failure data via a likelihood contribution model.
  - R Library: github.com/queelius/wei.series.md.c1.c2.c3
- Clarification of the assumptions required for the likelihood model.
- Simulation studies with Weibull distributed component lifetimes.
  - Assess performance of MLE and BCa confidence intervals under various scenarios.

Section 2

Series System

Series System



A lot of complex systems have *critical* components $(x_1, \ldots, x_5)$.

- One component fails, the system fails.
- "A chain is only as strong as its weakest link." - Thomas Reid
- We call these *series systems*. Let $T_i = \min(T_{i1}, \ldots, T_{i5})$ be the lifetime of system i.
  - T_{ij} is the lifetime of component j in system i.

Reliability Function: System Longevity

Definition: Represents the probability that a system or component functions beyond a specified time t:

$$R_X(x) = \Pr\{X > x\}.$$

Interpretation:

- A high reliability value indicates a lower probability of failure.
- Essential for understanding the longevity and dependability of systems and components.

Series System Reliability: Product of the reliabilities of its components:

$$R_{T_i}(t; \boldsymbol{\theta}) = \prod_{j=1}^{m} R_j(t; \boldsymbol{\theta_j}).$$

Relevance: Forms the basis for most reliability analyses and helps in making informed decisions about system design and maintenance.

- Directly used in our likelihood model for right-censoring events.

Hazard Function: Failures Characteristics

Definition: Instantaneous failure rate at a specific time, given survival up to that point:

$$h_X(x) = \frac{f_X(t)}{R_X(t)}.$$

Interpretation

- A tool to understand how failure risk evolves over time.
- Guides maintenance schedules and interventions.
- Failure characteristics:
  - Rising: wear-out (aging).
  - Declining: infant mortality (defects).
  - Constant: random (accidents).

Series System Hazard Function: Sum of the hazard functions of its components:

$$h_{T_i}(t;\boldsymbol{\theta}) = \sum_{j=1}^{m} h_j(t;\boldsymbol{\theta_j}).$$

Component Cause of Failure

Let K_i denote component cause of failure of i^{th} system.

- The probability that the j^{th} component causes failure:

$$\Pr\{K_i = j\} = E_{\boldsymbol{\theta}}\left[\frac{h_j(T_i; \boldsymbol{\theta_j})}{h_{T_i}(T_i; \boldsymbol{\theta_I})}\right].$$

- The conditional probability that the j^{th} component causes a failure, given that the system failed at time t:

$$\Pr\{K_i = j | T_i = t\} = \frac{h_j(t; \boldsymbol{\theta_j})}{h_{T_i}(t; \boldsymbol{\theta_I})}.$$

- Joint Distribution of System Lifetime and Component Cause of Failure

$$f_{K_i, T_i}(j, t; \boldsymbol{\theta}) = h_j(t; \boldsymbol{\theta_j}) R_{T_i}(t; \boldsymbol{\theta}).$$

- **Relevance**: Important in our likelihood model for masked failures.

Reliability of Well-Designed Series Systems

- MTTF is a summary measure of reliability:
  - Equivalent to integrating its reliability function over its support.
  - MTTF can be misleading. We can't assume components with longer MTTFs are more reliable.
- A series system is only as strong as its weakest component.
- In a *well-designed series system*, components have similar failure characteristics:
  - Similar MTTFs and probabilities of being the cause of failure.
- **Relevance**: Our simulation study is based on a (reasonably) well-designed series system.

Section 3

Likelihood Model

Likelihood Model

We have our system model, but we don't observe component lifetimes. We observe data related to component lifetimes.

Observed Data

- Right censoring: No failure observed.
  - The experiment ended before the system failed.
    - τ is the right-censoring time.
    - $\delta_i = 0$ indicates right-censoring for system i.
- Masked causes
  - The system failed, but we don't know the component cause.
    - S_i is the observed time of system failure.
    - $\delta_i = 1$ indicates system failure for system i.
    - $\mathcal{C}_i$ are a subset of components that could have caused failure.

Observed Data Example

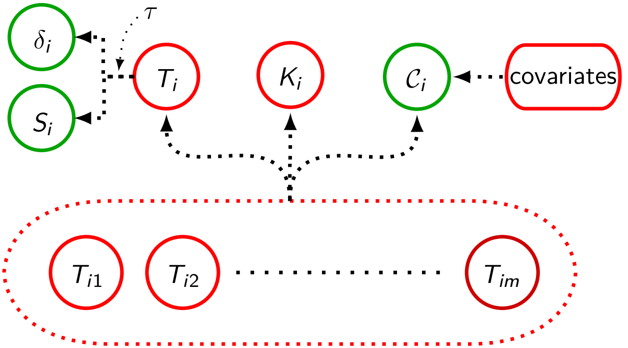
Observed data with a right-censoring time $\tau = 5$ for a series system with 3 components.

System	Right-censored lifetime	Event indicator	Candidate set
1	1.1	1	$\{1, 2\}$
2	1.3	1	$\{2\}$
4	2.6	1	$\{2, 3\}$
5	3.7	1	$\{1, 2, 3\}$
6	5	0	$\emptyset$
7	5	0	$\emptyset$

Data Generating Process

DGP is underlying process that generates observed data:

- Green elements are observed.
- Red elements are unobserved (latent).
- Candidate sets ($\mathcal{C}_i$) related to component lifetimes (T_{ij}) and other (unknown) covariates.
  - Distribution of candidate sets complex. Seek a simple model valid under certain assumptions.



Likelihood Function

Assumptions

- Right-censoring time τ independent of component lifetimes and parameters:

$$S_i = \min(\tau, T_i),$$
$$\delta_i = 1_{\{T_i < \tau\}}.$$

- Observed failure time with candidate sets. Candidate sets satisfy some conditions (discussed later).

Likelihood Contributions

$$L_i(\boldsymbol{\theta}) \propto \begin{cases} \prod_{l=1}^m R_l(s_i; \boldsymbol{\theta_l}) & \text{if } \delta_i = 0 \\ \prod_{l=1}^m R_l(s_i; \boldsymbol{\theta_l}) \sum_{j \in c_i} h_j(s_i; \boldsymbol{\theta_j}) & \text{if } \delta_i = 1. \end{cases}$$

Derivation: Likelihood Contribution for Masked Failures

Masking occurs when a system fails but the precise failed component is ambiguous. To make problem more tractable, we introduce certain conditions (which are reasonable for many real-world systems).

Conditions

1. **Candidate Set Contains Failed Component**: The candidate set, $\mathcal{C}_i$, always includes the failed component:
   - $\Pr_{\boldsymbol{\theta}}\{K_i \in \mathcal{C}_i\} = 1$.
2. **Equal Probabilities Across Candidate Sets**: For an observed system failure time $T_i = t_i$ and a candidate set $\mathcal{C}_i = c_i$, the candidate set probability is constant across different component failure causes within the set:
   - $\Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | K_i = j, T_i = t_i\} = \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | K_i = j', T_i = t_i\}$ for every $j, j' \in c_i$.
3. **Masking Probabilities Independent of θ**: Masking probabilities, when conditioned on T_i and failed component, aren't functions of $\boldsymbol{\theta}$.

Likelihood Contribution: Masked Component Cause of Failure

We construct the likelihood contribution for masked data like so:

- The joint distribution of T_i, K_i, and $\mathcal{C}_i$ is written as:

$$f_{T_i,K_i,\mathcal{C}_i}(t_i, j, c_i; \boldsymbol{\theta}) = f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\}.$$

- Marginalizing over K_i and applying Conditions 1, 2, and 3 yields:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \beta_i \prod_{l=1}^{m} R_l(t_i; \boldsymbol{\theta_l}) \sum_{j \in c_i} h_j(t_i; \boldsymbol{\theta_j}).$$

- The likelihood contribution: $L_i(\boldsymbol{\theta}) \propto f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta})$.
  - We do not need to model the distribution of the candidate sets.

Methodology: Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE): Maximize the likelihood function:

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} L(\boldsymbol{\theta}).$$

Log-likelihood: Easier to work with and numerically more stable:

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \ell_i(\boldsymbol{\theta}),$$

where ℓ_i is the log-likelihood contribution for the i^{th} observation:

$$\ell_i(\boldsymbol{\theta}) = \sum_{j=1}^{m} \log R_j(s_i; \boldsymbol{\theta_j}) + \delta_i \log\Bigg(\sum_{j \in c_i} h_j(s_i; \boldsymbol{\theta_j})\Bigg).$$

Solution: Numerically solve the following system of equations for $\hat{\boldsymbol{\theta}}$:

$$\nabla_{\boldsymbol{\theta}} \ell(\hat{\boldsymbol{\theta}}) = \mathbf{0}.$$

Bootstrap Method: Confidence Intervals

Sampling Distribution of MLE: Asymptotic normality is useful for constructing confidence intervals.

- **Issue**: May need large samples before asymptotic normality holds.

Bootstrapped CIs: Resample data and find an MLE for each. Use the distribution of the bootstrapped MLEs to construct CIs.

- **Percentile Method**: Simple and intuitive.
- **Coverage Probability**: Probability that the confidence interval contains the true parameter value $\boldsymbol{\theta}$.

Correctly Specified CIs: A coverage probability close to the nominal level of 95%.

- **Adjustments**: To improve coverage probability, we use the BCa method to adjust for bias (bias correction) and skewness (acceleration) in the estimate. Coverage probabilities above 90% acceptable.

Challenges with MLE on Masked Data

We discovered some challenges with the MLE on masked data.

Convergence Issues: Flat likelihood regions were observed due to the ambiguity in the masked data and small sample sizes.

Bootstrap Issues: Bootstrap relies on the Law of Large Numbers.

- Bootstrap might not represent the true variability, leading to inaccuracies.
- Due to right censoring and masking, the effective sample size is reduced.

Mitigation: We discard non-convergent samples for the MLE on original data, but keep all resamples for the bootstrap.

- This ensures that the bootstrap for "good" data is representative of the variability in the original data.
- We report convergence rates in our simulation study.

Section 4

Series System with Weibull Component Lifetimes

Series System with Weibull Component Lifetimes

The Weibull distribution has been crucial in reliability analysis due to its versatility. In our study, we model a system's components using Weibull distributed lifetimes.

- Introduced by Waloddi Weibull in 1937.
- Reflecting on its utility, Weibull modestly noted: "[...] may sometimes render good service."

Weibull Distribution Characteristics

The lifetime distribution for the j^{th} component of the i^{th} system is:

$$T_{ij} \sim \text{Weibull}(k_j, \lambda_j)$$

Where:

- $\lambda_j > 0$ is the scale parameter.
- $k_j > 0$ is the shape parameter.

Significance of the Shape Parameter:

- $k_j < 1$: Indicates infant mortality. E.g., defective components weeded out early.
- $k_j = 1$: Indicates random failures. E.g., result of random shocks.
- $k_j > 1$: Indicates wear-out failures. E.g., components wearing out with age.

Theoretical Results

Reliability and hazard functions of a series system with Weibull components:

$$R_{T_i}(t;\boldsymbol{\theta}) = \exp\left\{-\sum_{j=1}^{m}\left(\frac{t}{\lambda_j}\right)^{k_j}\right\},$$

$$h_{T_i}(t;\boldsymbol{\theta}) = \sum_{j=1}^{m}\frac{k_j}{\lambda_j}\left(\frac{t}{\lambda_j}\right)^{k_j-1},$$

where $\boldsymbol{\theta} = (k_1, \lambda_1, \ldots, k_m, \lambda_m)$ is the parameter vector of the series system.

Likelihood Model

We deal with right censoring and masked component cause of failure. The likelihood contribution of system i:

$$L_i(\boldsymbol{\theta}) \propto \begin{cases} R_{T_i}(t_i;\boldsymbol{\theta}) & \text{if } \delta_i = 0, \\ R_{T_i}(t_i;\boldsymbol{\theta})\sum_{j\in c_i} h_j(t_i;\boldsymbol{\theta_j}) & \text{if } \delta_i = 1. \end{cases}$$

Section 5

Simulation Study Overview

Simulation Study Overview

We conduct a simulation study based on a series system.

System Description

This study is centered around the following *well-designed series system*:

Component	Shape	Scale	MTTF	$\Pr\{K_i\}$
1	1.26	994.37	924.87	0.17
2	1.16	908.95	862.16	0.21
3	1.13	840.11	803.56	0.23
4	1.18	940.13	888.24	0.20
5	1.20	923.16	867.75	0.20
System	NA	NA	222.88	NA

Performance Metrics

Our main objective is to evaluate the MLE and BCa confidence intervals' performance across various scenarios.

- **MLE Evaluation**:
  - **Accuracy**: Proximity of the MLE's expected value to the actual value.
  - **Precision**: Consistency of the MLE across samples.
- **BCa Confidence Intervals Evaluation**:
  - **Accuracy**: Ideally, Confidence Intervals (CIs) should encompass true parameters around 95% of the time.
  - **Precision**: Assessed by the width of the CIs.

Both accuracy and precision are crucial for confidence in the analysis.

Data Generation

We generate data for n systems with 5 components each. We satisfy the assumptions of our likelihood model by generating data as follows:

- **Right-Censoring Model**: Right-censoring time set at a known value, parameterized by the quantile q.
  - Satisfies the assumption that the right-censoring time is independent of component lifetimes and parameters.
- **Masking Model**: Using a *Bernoulli masking model* for component cause of failure, parameterized by the probability p.
  - Satisfies masking Conditions 1, 2, and 3.

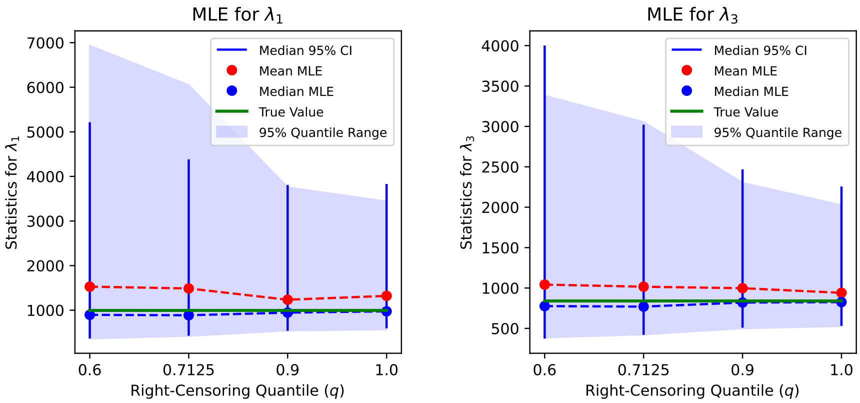
Scenario: Impact of Right-Censoring

Vary the right-censoring quantile (q): 60% to 100%. Fixed the parameters: $p = 21.5\%$ and $n = 90$.

Background

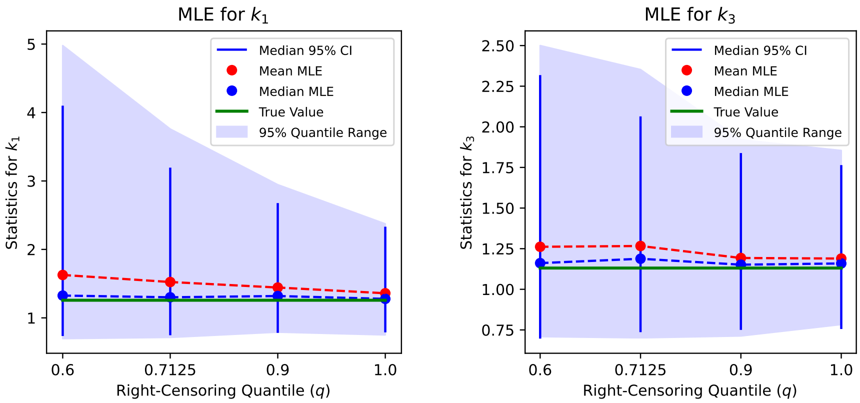
- **Right-Censoring**: No failure observed.
- **Impact**: Reduces the effective sample size.
- **MLE**: Bias and precision affected by censoring.

Scale Parameters



- **Dispersion**: Less censoring improves MLE precision.
- **Bias**: Both parameters are biased. Bias decreases with less censoring.
- **Median-Aggregated CIs**: Bootstrapped CIs become consistent with more data.

Shape Parameters



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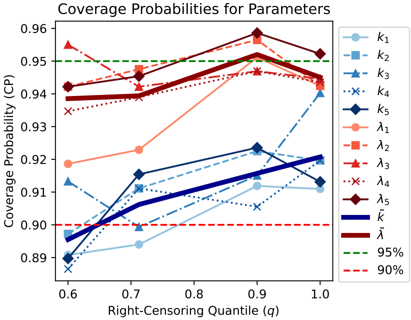
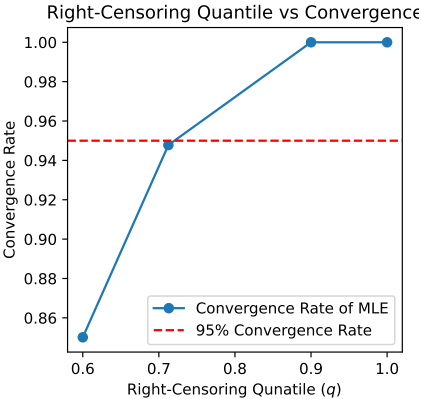
Coverage Probability and Convergence Rate





- **Calibration**: CIs converge to 95%. Scale parameters better calibrated.
- **Convergence Rate**: Increases as right-censoring reduces.

Conclusion

- MLE precision improves, bias drops with decreased right-censoring.
- BCa CIs perform well, particularly for scale parameters.
- MLE of most reliable component more affected by right-censoring.

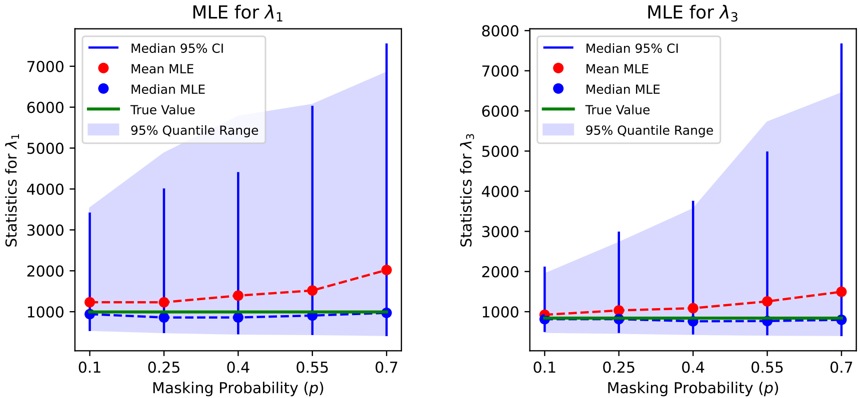
Impact of Masking Probability

Vary the masking probability p: 0.1 to 0.7. Fixed the parameters: $q = 0.825$ and $n = 90$.

Background

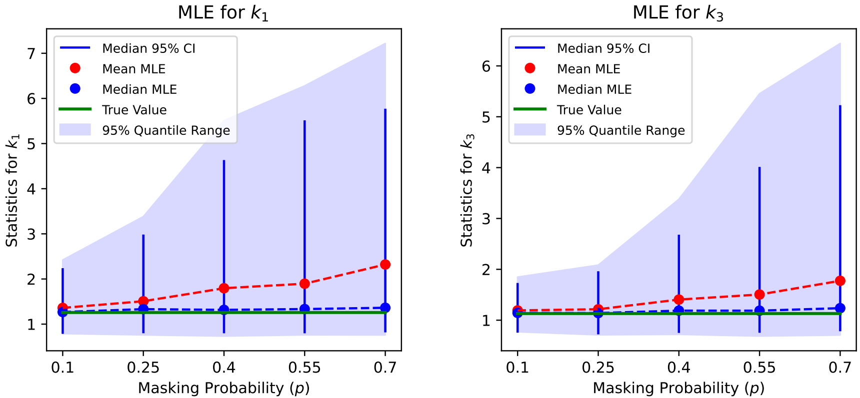
- **Masking** adds ambiguity in identifying the failed component.
- Impacts of masking on MLE:
  - **Ambiguity**: Higher p increases uncertainty in parameter adjustment.
  - **Bias**: Similar to right-censoring, but affected by both p and q.
  - **Precision**: Reduces as p increases.

Scale Parameters



- **Dispersion**: Increases with p, indicating reduced precision.
- **Bias**: Positive bias rises with p.
- **Median-Aggregated CIs**: Widen and show asymmetry as p grows.

Shape Parameters



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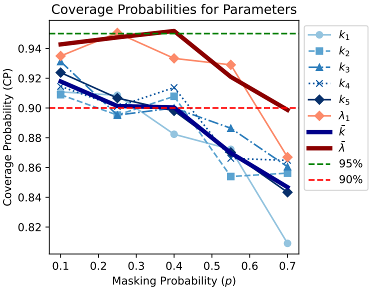
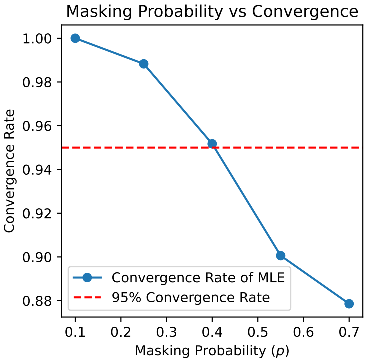
Coverage Probability and Convergence Rate





Calibration: Caution advised for severe masking with small samples.

- Scale parameters maintain coverage up to $p = 0.7$.
- Shape parameters drop below 90% after $p = 0.4$.

Convergence Rate: Reduces after $p > 0.4$, consistent with CP behavior.

Conclusion

- Masking influences MLE precision, coverage probability, and introduces bias.
- Despite significant masking, scale parameters have commendable CI coverage.

Impact of Sample Size

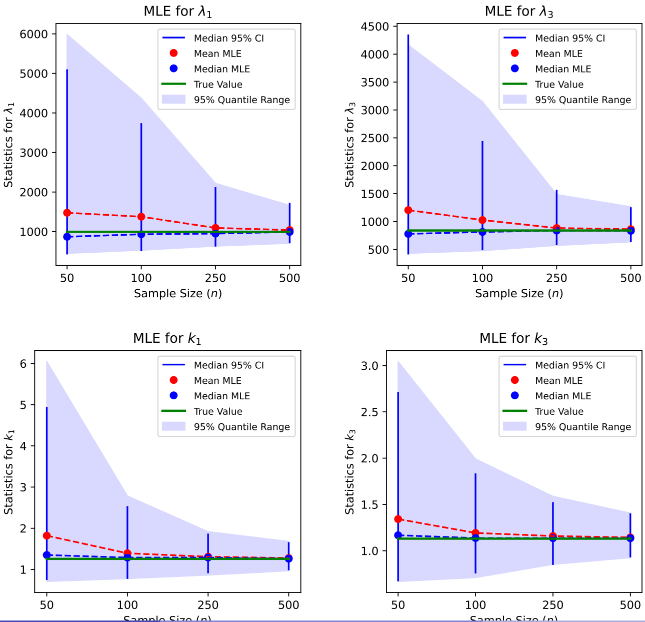
Assess the impact of sample size on MLEs and BCa CIs.

- Vary sample size n: 50 to 500
- Parameters: $p = 0.215$, $q = 0.825$

Background

- **Sample Size**: Number of systems observed.
- **Impact**: More data reduces uncertainty in parameter estimation.
- **MLE**: Mitigates biasing effects of right-censoring and masking.

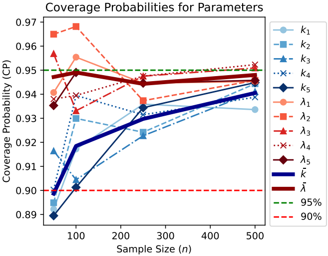
Both Scale and Shape Parameters

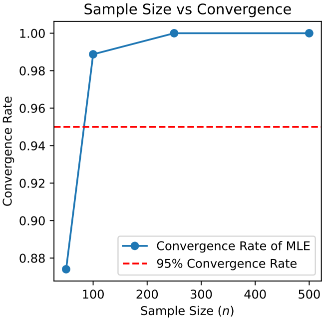


Parameters

- **Dispersion**:
  - Dispersion reduces with n—indicating improved precision.
  - Disparity observed between components k_1, λ_1 and k_3, λ_3.
- **Bias**:
  - High positive bias initially, but diminishes around $n = 250$.
  - Enough sample data can counteract right-censoring and masking effects.
- **Median-Aggregated CIs**:
  - CIs tighten as n grows—showing more consistency.
  - Upper bound more dispersed than lower, reflecting the MLE bias direction.

Coverage Probability and Convergence Rate





- **Calibration**:
  - CIs are mostly above 90% across sample sizes.
  - Converge to 95% as n grows.
  - Scale parameters have better coverage than shape.
- **Convergence Rate**:
  - Improves with n, surpassing 95% for $n \geq 100$.
  - Caution for estimates with $n < 100$ in specific setups.

Conclusion

- Sample size significantly mitigates challenges from right-censoring and masking.
- MLE precision and accuracy enhance notably with growing samples.
- BCa CIs become narrower and more reliable as sample size increases.

Section 6

Conclusion

Part 1

Key Findings

- Employed maximum likelihood techniques for component reliability estimation in series systems with masked failure data.
- Methods performed robustly despite masking and right-censoring challenges.

Simulation Insights

- Right-censoring and masking introduce positive bias; more reliable components are most affected.
- Shape parameters harder to estimate than scale.
- Large samples can counteract these challenges.

Part 2

Confidence Intervals

- Bootstrapped BCa CIs demonstrated commendable coverage probabilities, even in smaller sample sizes.

Takeaways

- Framework offers a rigorous method for latent component property estimation from limited observational data.
- Techniques validated to provide practical insights in diverse scenarios.
- Enhanced capability for learning from obscured system failure data.

Section 7

Discussion