Reliability Estimation in Series Systems

Maximum Likelihood Techniques for Right-Censored and Masked Failure Data

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Context & Motivation

Reliability in **Series Systems** is like a chain's strength – determined by its weakest link.

- Essential for system design and maintenance.

Main Goal: Estimate individual component reliability from *failure data*.

Challenges:

- *Masked* component-level failure data.
- *Right-censoring* system-level failure data.

Our Response:

- Derive techniques to interpret such ambiguous data.
- Aim for precise and accurate reliability estimates for individual components using maximum likelihood estimation (MLE)
- Quantify uncertainty in estimates with bootstrap confidence intervals (CIs).

Core Contributions

Likelihood Model for **Series Systems**.

- Accounts for *right-censoring* and *masked component failure*.

Specifications of Conditions:

- Assumptions about the masking of component failures.
- Simplifies and makes the model more tractable.

Simulation Studies:

- Components with *Weibull* lifetimes.
- Evaluate MLE and confidence intervals under different scenarios.

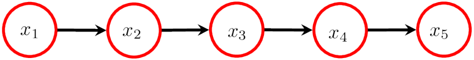
R Library: Methods available on GitHub.

- See: www.github.com/queelius/wei.series.md.c1.c2.c3

Section 1

Series System

What Is A Series System?



Critical Components: Complex systems often comprise *critical* components. If any component fails, the entire system fails.

- We call such systems *series systems*.
- **Example**: A car's engine and brakes.

System Lifetime is dictated by its shortest-lived component:

$$T_i = \min(T_{i1}, \ldots, T_{i5})$$

where:

- T_i is the lifetime of i^{th} system.
- T_{ij} is the j^{th} component of i^{th} system.

Reliability Function

Reliability Function represents the probability that a system or component functions beyond a specified time.

- Essential for understanding longevity and dependability.

Series System Reliability: Product of the reliability of its components:

$$R_{T_i}(t; \boldsymbol{\theta}) = \prod_{j=1}^{m} R_j(t; \boldsymbol{\theta_j}).$$

- If any component has low reliability, it can impact the whole system.
- Here, $R_{T_i}(t; \boldsymbol{\theta})$ and $R_j(t; \boldsymbol{\theta_j})$ are the reliability functions for the system i and component j, respectively.

Hazard Function: Understanding Risks

Hazard Function: Measures the immediate risk of failure at a given time, assuming survival up to that moment.

- Reveals how the risk of failure evolves over time.
- Guides maintenance schedules and interventions.

Series System Hazard Function: Sum of the component hazard functions:

$$h_{T_i}(t;\boldsymbol{\theta}) = \sum_{j=1}^{m} h_j(t;\boldsymbol{\theta_j}).$$

- Components' risks are additive.

Joint Distribution of Component Failure and System Lifetime

Our likelihood model depends on the **joint distribution** of the system lifetime and the component that caused the failure.

- **Formula**: Product of the failing component's hazard function and the system reliability function:

$$f_{K_i,T_i}(j, t; \boldsymbol{\theta}) = h_j(t; \boldsymbol{\theta_j}) R_{T_i}(t; \boldsymbol{\theta}).$$

- Here, K_i denotes component cause of i^{th} system's failure.

Component Cause of Failure

We can use the joint distribution to calculate the probability of component cause of failure.

- Helps predict the cause of failure.
- **Derivation**: Marginalize the joint distribution over the system lifetime:

$$\Pr\{K_i = j\} = E_{\boldsymbol{\theta}}\left[\frac{h_j(T_i; \boldsymbol{\theta_j})}{h_{T_i}(T_i; \boldsymbol{\theta_l})}\right].$$

- **Well-Designed Series System**: Components exhibit comparable chances of causing system failures.
- **Relevance**: Our simulation study employs a (reasonably) well-designed series system.

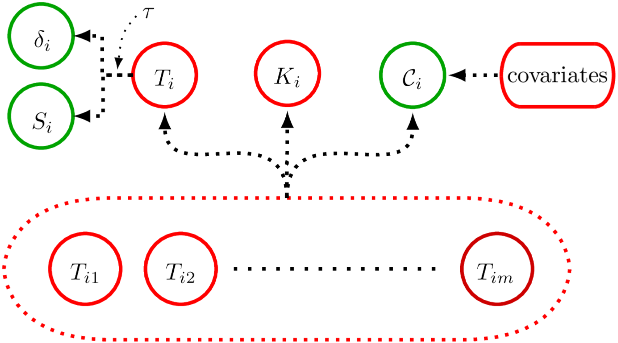
Section 2

Likelihood Model

Data Generating Process

The data generating process (DGP) is the underlying process that generates the data. *Green* elements are observed, *red* elements are latent:

- **Right-Censored** lifetime: $S_i = \min(T_i, \tau_i)$.
- **Event Indicator**: $\delta_i = 1_{\{T_i < \tau_i\}}$.
- **Candidate Set**: $\mathcal{C}_i$ related to components (T_{ij}) and other unknowns.



Likelihood Function

Likelihood Function measures how well model explains the data:

- **Right-Censored** data ($\delta_i = 0$).
- **Candidate Sets** or **Masked Failure** data ($\delta_i = 1$)

System	Right-Censored Lifetime (S_i)	Event Indicator (δ_i)	Candidate Set ($\mathcal{C}_i$)
1	1.1	1	$\{1, 2\}$
2	5	0	$\emptyset$

Each system contributes to *total likelihood* via its *likelihood contribution*:

$$L(\boldsymbol{\theta} | \text{data}) = \prod_{i=1}^{n} L_i(\boldsymbol{\theta} | \text{data}_i)$$

where $\textbf{data}_i$ is data for i^{th} system and L_i is its contribution.

Likelihood Contribution: Right-Censoring

Right-Censoring: For the i^{th} system, if right-censored ($\delta_i = 0$) at duration τ, its likelihood contribution is proportional to the system reliability function evaluated at τ:

$$L_i(\boldsymbol{\theta}) \propto R_{T_i}(\tau; \boldsymbol{\theta}).$$

- We only know that a failure occurred after the right-censoring time.
- This is captured by the system reliability function.

Key Assumptions:

- Censoring time (τ) independent of parameters.
- Event indicator (δ_i) is observed.
- **Reasonable** in many cases, e.g., right-censoring time τ predetermined by length of study.

Likelihood Contribution: Candidate Sets

Masking Component Failure: If the i^{th} system fails ($\delta_i = 1$), it is masked by a candidate set $\mathcal{C}_i$. Its likelihood contribution is complex and we use simplifying assumptions to make it tractable.

- **Condition 1**: The candidate set includes the failed component: $\Pr\{K_i \in \mathcal{C}_i\} = 1$.
- **Condition 2**: The condition probability of a candidate set given a cause of failure and a system lifetime is constant across conditioning on different failure causes within the candidate set: $\Pr\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\} = \Pr\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j'\}$ for $j, j' \in c_i$.
- **Condition 3**: The masking probabilities when conditioned on the system lifetime and the failed component aren't functions of the system parameter.

Likelihood Contribution: Derivation for Candidate Sets

Take the **joint distribution** of T_i, K_i, and $\mathcal{C}_i$ and marginalize over K_i:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \sum_{j=1}^{m} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 1** to get a sum over candidate set:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 2** to move probability outside the sum:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j'\} \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}).$$

Apply **Condition 3** to remove the probability's dependence on $\boldsymbol{\theta}$:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \beta_i \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}).$$

Result: $L_i(\boldsymbol{\theta}) \propto \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) = R_{T_i}(t_i; \boldsymbol{\theta}) \sum_{j \in c_i} h_j(t_i; \boldsymbol{\theta_j})$.

Bootstrap Confidence Intervals (CIs)

Confidence Intervals (CI) help capture the *uncertainty* in our estimate.

- **Normal** assumption for constructing CIs may not be accurate.
  - *Masking* and *censoring*.
- **Bootstrapped CIs**: Resample data and obtain MLE for each.
  - Use **percentiles** of bootstrapped MLEs for CIs.
- **Coverage Probability**: Probability the interval covers the true parameter value.
  - **Challenge**: Actual coverage may deviate to bias and skew in MLEs.
- **BCa** adjusts the CIs to counteract bias and skew in the MLEs.

Challenges with Masked Data

Like any model, ours has its challenges:

- **Convergence Issues**: Nearly flat likelihood regions can occur.
  - Ambiguity in masked, censored data
  - Complexities of estimating latent parameters.
- **Bootstrap Issues**: Relies on the empirical sampling distribution.
  - May not represent true variability for small samples.
  - *Censoring* and *masking* compound issue by reducing the **effective** sample size.
- **Mitigation**: In simulation, discard non-convergent samples for MLE on original data but retain all resamples for CIs.
  - More robust assessment at the cost of possible bias towards "well-behaved" data.
  - **Convergence Rates** reported to provide context.

Section 3

Simulation Study: Series System with Weibull Components

Series System Parameters

Component	Shape (k_j)	Scale (λ_j)	Failure Probability ($\Pr\{K_i\}$)
1	1.26	994.37	0.17
2	1.16	908.95	0.21
3	1.13	840.11	0.23
4	1.18	940.13	0.20
5	1.20	923.16	0.20

Lifetime of j^{th} component of i^{th} system: $T_{ij} \sim \text{Weibull}(k_j, \lambda_j)$.

- Based on (Guo, Niu, and Szidarovszky 2013)
- Extended to include components 4 and 5
  - Shapes greater than 1 indicates wear-outs.
  - Probabilities comparable: reasonably **well-designed**.
- Focus on Components 1 and 3 (most and least reliable) in study.

Synthetic Data and Simulation Values

How is the data generated in our simulation study?

- **Component Lifetimes** (latent $T_{i1}, \cdots, T_{im}$) generated for each system.
  - **Observed Data** is a function of latent components.
- **Right-Censoring** amount controlled with simulation value q.
  - Quantile q is probability system won't be right-censored.
  - Solve for right-censoring time τ in $\Pr\{T_i \leq \tau\} = q$.
  - $S_i = \min(T_i, \tau)$ and $\delta_i = 1_{\{T_i \leq \tau\}}$.
- **Candidate Sets** are generated using the *Bernoulli Masking Model*.
  - Masking level controlled with simulation value p.
  - Failed component (latent K_i) placed in candidate set (observed $\mathcal{C}_i$).
  - Each functioning component included with probability p.

Bernoulli Masking Model: Satisfying Masking Conditions

The Bernoulli Masking Model *satisfies* the masking conditions:

- **Condition 1**: The failed component deterministically placed in candidate set.
- **Condition 2** and **3**: Bernoulli probability p is same for all components and fixed by us.
  - Probability of candidate set is constant conditioned on component failure within set.
  - Probability of candidate set, conditioned on a component failure, only depends on the p.

Future Research: Realistically conditions may be violated.

- Explore sensitivity of likelihood model to violations.

Performance Metrics

Objective: Evaluate the MLE and BCa confidence intervals' performance across various scenarios.

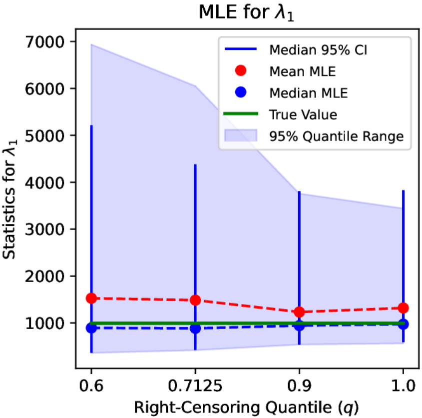
- Visualize the **simulated** sampling distribution of MLEs and 95% CIs.
- **MLE Evaluation**:
  - **Accuracy**: Bias
  - **Precision**: Dispersion of MLEs
    - 95% quantile range of MLEs.
- **95% CI Evaluation**:
  - **Accuracy**: Coverage probability (CP).
    - *Correctly Specified* CIs: CP near 95% ($>$ 90% acceptable).
  - **Precision**: Width of median CI.

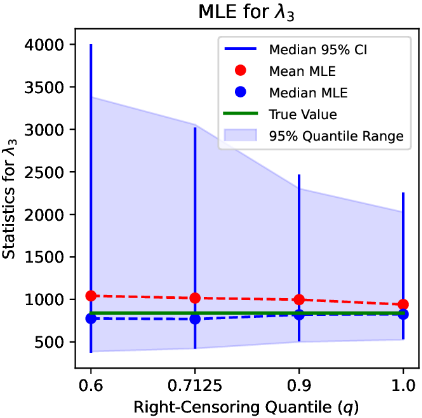
Scenario: Impact of Right-Censoring

Assess the impact of right-censoring on MLE and CIs.

- **Right-Censoring**: Failure observed with probability q: 60% to 100%.
  - Right censoring occurs with probability $1 - q$: 40% to 0%.
- **Bernoulli Masking Probability**: Each component is a candidate with probability p fixed at 21.5%.
  - Estimated from original study (Guo, Niu, and Szidarovszky 2013).
- **Sample Size**: n fixed at 90.
  - Small enough to show impact of right-censoring.

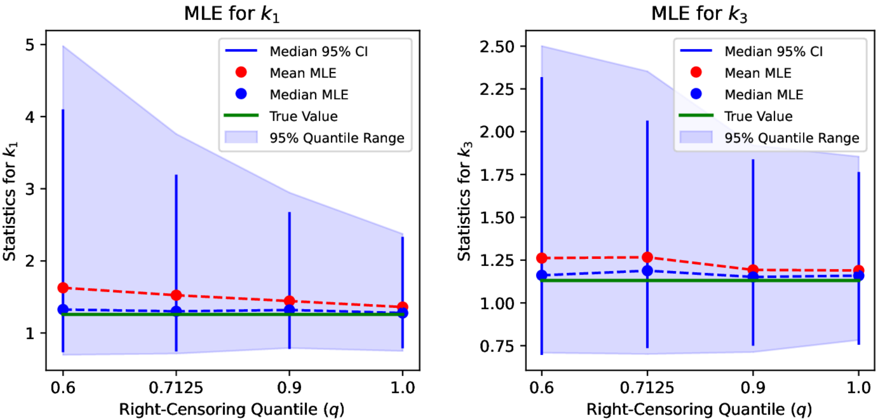
Scale Parameters





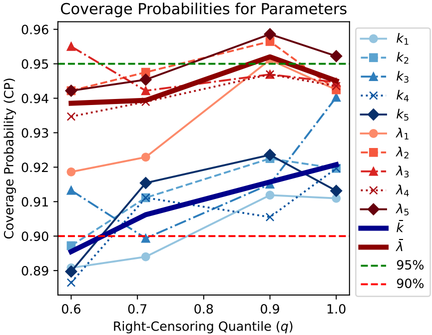
- **Dispersion**: Less censoring improves MLE precision.
  - Most reliable component more affected by censoring.
- **Bias**: MLE *positively* biased; decreases with less censoring.
- **Median CIs**: Tracks MLE dispersion.

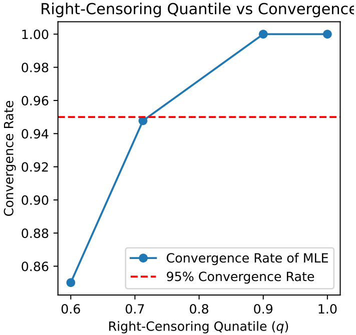
Shape Parameters



- Show a similar pattern as scale parameters.

Coverage Probability and Convergence Rate





- **Coverage** (left figure): CIs show good empirical coverage.
  - Scale parameters *correctly specified* (CP $\approx$ 95%)
  - Shape parameters *good enough* (CP $>$ 90%).
- **Convergence Rate** (right figure): Increases with less censoring.
  - **Caution**: Dips below 95% with more than 30% censoring.

Key Takeaways: Right-Censoring

Right-censoring has a notable impact on the MLE:

- **MLE Precision**:
  - Improves notably with reduced right-censoring levels.
  - More reliable components benefit more from reduced right-censoring.
- **Bias**:
  - MLEs show positive bias, but decreases with reduced right-censoring.
- **Convergence Rates**:
  - MLE convergence rate improves with reduced right-censoring.
  - Dips: $< 95\%$ at $> 30\%$ right-censoring.

BCa confidence intervals show good empirical coverage.

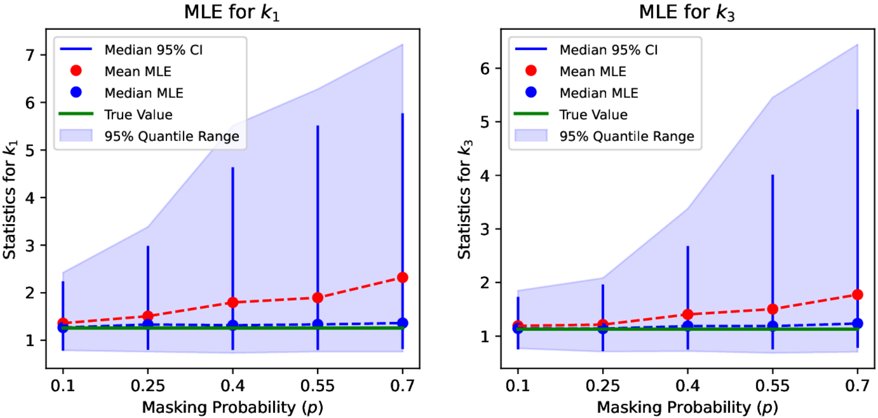
- CIs offer reliable *empirical coverage*.
- Scale parameters *correctly specified* across all right-censoring levels.

Scenario: Impact of Failure Masking

Assessing the impact of the failure masking level on MLE and CIs.

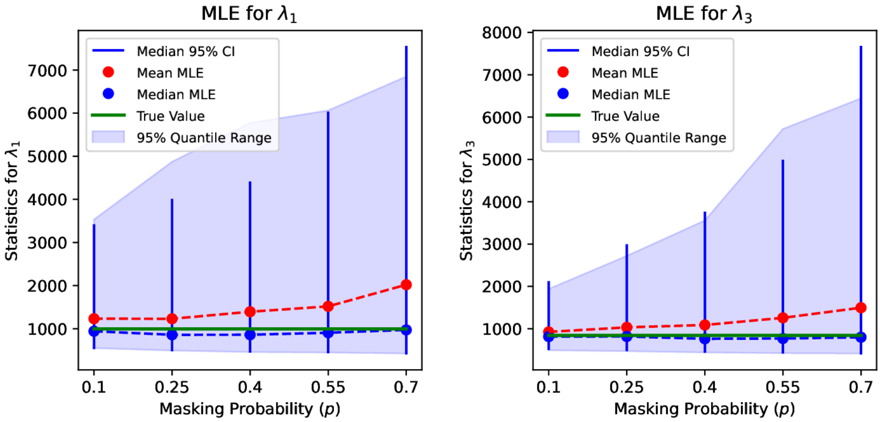
- **Bernoulli Masking Probability**: Vary Bernoulli probability p from 10% to 70%.
- **Right-Censoring**: q fixed at 82.5%.
  - Right-censoring occurs with probability $1 - q$: 17.5%.
  - Censoring less prevalent than masking.
- **Sample Size**: n fixed at 90.
  - Small enough to show impact of masking.

Shape Parameters



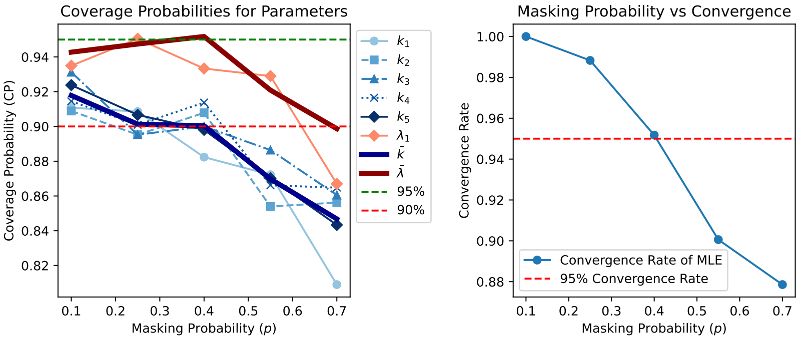
- **Dispersion**: Precision decreases with masking level (p).
- **Bias**: MLE *positively* biased and increases with masking level.
  - Applies a right-censoring like effect to the components.
- **Median CIs**: Tracks MLE dispersion.

Scale Parameters



- These graphs resemble the last ones for shape parameters.

Coverage Probability and Convergence Rate



- **Coverage**: Caution advised for severe masking with small samples.
  - Scale parameter CIs show acceptable coverage across all masking levels.
  - Shape parameter CIs dip below 90% when $p > 0.4$.
- **Convergence Rate**: Increases with less masking.
  - **Caution**: Dips under 95% when $p > 0.4$ (consistent with CP behavior).

Key Takeaways: Masking

The masking level of component failures profoundly affects the MLE:

- **MLE Precision**:
  - Decreases with more masking.
- **MLE Bias**:
  - Positive bias is amplified with increased masking.
  - Masking exhibits a right-censoring-like effect.
- **Convergence Rate**:
  - Commendable for Bernoulli masking levels $p \leq 0.4$.
    - *Extreme* masking: some masking occurs 90% of the time at $p = 0.4$.

The BCa confidence intervals show good coverage:

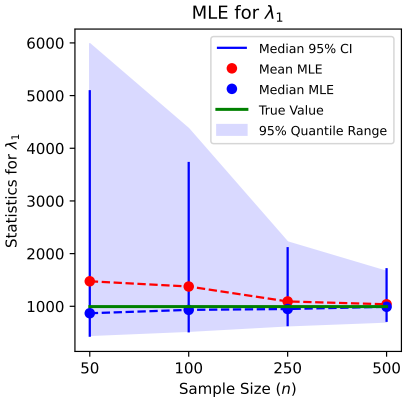
- **Scale** parameters maintain good coverage across all masking levels.
- **Shape** parameter coverage dip below 90% when $p > 0.4$.
  - Caution advised for severe masking with small samples.

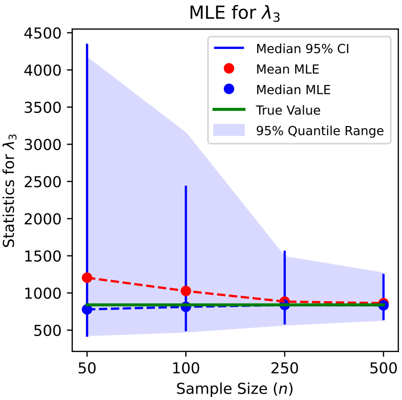
Scenario: Impact of Sample Size

Assess the mitigating affects of sample size on MLE and CIs.

- **Sample Size**: We vary the same size n from 50 to 500..
- **Right-Censoring**: q fixed at 82.5%
  - 17.5% chance of right-censoring.
- **Bernoulli Masking Probability**: p fixed at 21.5%
  - Some masking occurs 62% of the time.

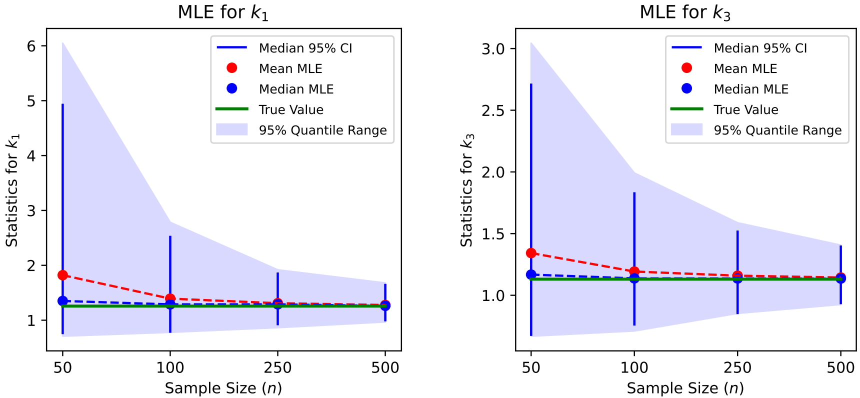
Scale Parameters





- **Dispersion**: Increasing sample size improves MLE precision.
  - Extremely precise for $n \geq 250$.
- **Bias**: Large *positive* bias initially, but diminishes to zero.
  - Large samples counteract right-censoring and masking effects.
- **Median CIs**: Track MLE dispersion. Very tight for $n \geq 250$.

Shape Parameters



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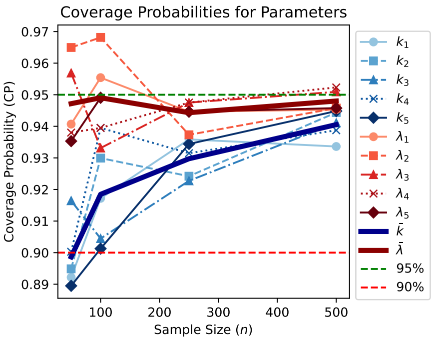
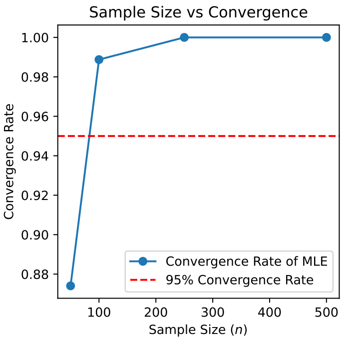
Coverage Probability and Convergence Rate





- **Coverage**: Good empirical coverage.
  - Correctly specified CIs for $n > 250$.
- **Convergence Rate**: Total convergence for $n \geq 250$.
  - Caution advised for estimates with $n < 100$ in specific setups.

Key Takeaways: Sample Size

Sample size has a notable impact on the MLE:

- **Precision**: Very precise for large samples ($n > 200$).
- **Bias**: Diminishes to near zero for large samples.
- **Coverage**: Correctly specified CIs for large samples.
- **Convergence Rate**: Total convergence for large samples.

Summary

Larger samples lead to more accurate, unbiased, and reliable estimations.

- Mitigates the effects of right-censoring and masking.

Section 4

Conclusion

Key Findings

MLE Performance:

- Right-censoring and masking introduce positive bias for our setup.
  - More reliable components are more affected.
- Shape parameters harder to estimate than scale parameters.
- Large samples can mitigate the affects of masking and right-censoring.

BCa Confidence Interval Performance:

- Width of CIs tracked MLE dispersion.
- Good empirical coverage in most scenarios.

Big Picture

MLE and CIs robust despite masking and right-censoring challenges.

Future Work

Directions to enhance learning from masked data:

- **Relax Masking Conditions**: Assess sensitivity to violations and and explore alternative likelihood models.
- **System Design Deviations**: Assess estimator sensitivity to deviations.
- **Homogenous Shape Parameter**: Analyze trade-offs with the full model.
- **Bootstrap Techniques**: Semi-parametric approaches and prediction intervals.
- **Regularization**: Data augmentation and penalized likelihood methods.
- **Additional Likelihood Contributions**: Predictors, etc.

Discussion