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Parameter estimations in linear failure rate model using masked data

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Abstract

Estimations of the parameters included in the lifetime distributions of the individual components in a series system with independent and nonidentical components are introduced in this paper by using masked system life data. Particularly, we deduce the maximum likelihood and Bayes estimates of these parameters when the failure rates of the system components are assumed to be linear with different parameters. In Bayes approach, it is assumed that the unknown parameters to be estimated behave as independent random variables having symmetrical triangular prior distributions. The problem is illustrated on a two-component series system. To show how one can apply the theoretical results obtained here, numerical simulation study is introduced. Also in such simulation study, a comparison between the procedures used is discussed.

1. Introduction

In reliability analysis, it is interesting to estimate the unknown parameters included in the lifetime distributions of the individual components in a multi-component systems. Such estimates may be useful since they reflect, in some ways, the component reliabilities after assembly into an operational system (see [10]). Under some appropriate conditions, the estimations obtained can be used to predict the reliability of new configurations of the system components. The

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analysis of system life data is often used to obtain the estimations of these parameters. Under ideal circumstances, this system life data contains the time to failure along with information on the exact component causing the system failure. However, in several cases such as a circuit card containing many individual components, the exact component responsible for the system failure cannot be identified. Instead, it is assumed that the component causes the system failure belongs to some subset of the components which considered potentially responsible for the failure. In this case, the cause of failure is masked (see [10]). Masking can occur in such type of data due to variety of reasons, for example, the cost, time constraints associated with failure analysis, recording errors, lack of proper diagnostic component, and the destructive nature of certain component failures that makes exact diagnosis impossible.

Under a simplifying assumption that components have constant failure rates, Miyakawa [5] considered a two-component series system and he derived closed-form expressions for the maximum likelihood estimate. Usher and Hodgson [10] extended Miyakawa's results to a three-component series system, under the same constant failure rates assumption. Under auxiliary conditions, they derived closed-form expressions for the MLEs. Guess et al. [1] extended and clarified the derivation of the likelihood function in the masked life data under the assumption that masking is independent of the exact failure cause. Lin et al. [2] derived exact MLE using masked system data under the same assumption that components have constant failure rates. Usher [9] used iterative maximum likelihood procedure in the case of two-component series system under the assumption that components have Weibull lifetime distributions. He illustrated the approach with a simple numerical example. Sarhan [7] obtained maximum likelihood and Bayes estimates of the reliability of system components in the case of n component series system when the components have constant failure rates. Sarhan [6] derived the maximum likelihood estimates of the parameters included in the cases of two-component and three-component series systems under assumption that the lifetime distributions of the components are Weibull. He derived closed-form expressions for MLEs in some particular cases, which generalize the results obtained by Usher and Hodgson [10].

In this paper we use the masked life data to derive the maximum likelihood and Bayes estimates of the included parameters in the cases of two-component series systems under assumption that the components have linear failure rates with different parameters. We derive closed-form expressions for Bayes estimations. The MLEs have no closed forms. Their values can be obtained by using a numerical technique method. Numerical simulation study is introduced to illustrate how one can apply the theoretical results obtained and compare the estimations obtained by using maximum likelihood and Bayes approaches.

This paper is divided into five sections. Section 2 gives the model. The maximum likelihood approach is discussed in Section 3. Section 4 introduces

Bayes estimators for the unknown parameters and the values of system components. Numerical results are given in Section 5.

2. The model

In the model considered here, it is assumed that:

Assumption 1. The system consists of m independent and nonidentical components connected in series.

Assumption 2. n of such system are placed on the life test. The test is terminated when all systems have failed. For each system, i, $i = 1, 2, \ldots, n$, the observable quantities are the system lifetime, T_i, and the set S_i of system components that may cause the system i failure.

Assumption 3. Masking is s-independent of the failure cause.

Assumption 4. The hazard rate function of component j, $j = 1, 2, \ldots, m$, in system i, $i = 1, 2, \ldots, n$, is linear with slope β_j and reciprocal α_j. That is, the hazard rate function of component j, $j = 1, 2, \ldots, m$, in system i, $i = 1, 2, \ldots, n$, is

$$h_j(t) = \alpha_j + \beta_j t, \quad t \geqslant 0, \ \ \alpha_j, \beta_j > 0. \tag{1}$$

Assumption 5. The parameters α_j and β_j are independent random variables with prior symmetrical triangular distributions on nonnegative intervals $\mathbf{A}_j = (a_j, b_j)$ and $\mathbf{B}_j = (c_j, d_j)$, respectively. That is, the prior pdf of α_j is

$$g_j(u) = \begin{cases} \dfrac{1}{\epsilon_{1j}^2}(\epsilon_{1j} - |u - \mu_{1j}|), & u \in \mathbf{A}_j, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $\mu_{1j} = (b_j + a_j)/2$ and $\epsilon_{1j} = (b_j - a_j)/2$. Also, the prior pdf of β_j is

$$\pi_j(u) = \begin{cases} \dfrac{1}{\epsilon_{2j}^2}(\epsilon_{2j} - |u - \mu_{2j}|), & u \in \mathbf{B}_j, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $\mu_{2j} = (d_j + c_j)/2$ and $\epsilon_{2j} = (d_j - c_j)/2$.

Assumption 6. The loss function associated with the estimate $\hat{\theta}$ of the vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$, $k \geqslant 1$, is the quadratic loss given by

$$L(\hat{\theta}, \theta) = k(\hat{\theta} - \theta)^2. \tag{4}$$

The probability density function, the cumulative distribution function and reliability function of the linear failure rate model can be obtained as follows:

Using (1) the conditional lifetime probability density function $f_j(t|\alpha_j, \beta_j)$ can be obtained as

$$f_j(t|\alpha_j, \beta_j) = (\alpha_j + \beta_j t)\exp\left[-\left(\alpha_j t + \tfrac{1}{2}\beta_j t^2\right)\right], \quad t \geqslant 0, \ \alpha_j, \beta_j > 0. \tag{5}$$

The cumulative distribution $F_j(t|\alpha_j, \beta_j)$ becomes

$$F_j(t|\alpha_j, \beta_j) = 1 - \exp\left[-\left(\alpha_j t + \tfrac{1}{2}\beta_j t^2\right)\right], \quad t \geqslant 0, \ \alpha_j, \beta_j > 0. \tag{6}$$

The reliability function $\overline{F}_j(t|\alpha_j, \beta_j)$ is

$$\overline{F}_j(t|\alpha_j, \beta_j) = \exp\left[-\left(\alpha_j t + \tfrac{1}{2}\beta_j t^2\right)\right], \quad t \geqslant 0, \ \alpha_j, \beta_j > 0. \tag{7}$$

From now and henceforth we denote by θ to the vector of unknown parameters $\alpha_1, \alpha_2, \ldots, \alpha_m, \ \beta_1, \beta_2, \ldots, \beta_m$.

3. Maximum likelihood estimates

Under the Assumptions 1–3, the likelihood function is given by (see [1]):

$$L(\text{data}, \theta) = \prod_{i=1}^{n}\left(\sum_{j\in S_i}\left\{f_j(t_i)\prod_{l\in M_j}\overline{F}_l(t_i)\right\}\right), \quad M_j = \{1, 2, \ldots, m\} \setminus \{j\}. \tag{8}$$

The probability density function $f_j(t)$ is related with the hazard rate function $h_j(t)$ and the reliability function $\overline{F}_j(t)$ according to the following relation:

$$f_j(t) = h_j(t)\overline{F}_j(t). \tag{9}$$

Substituting from (9) into (8), the likelihood function $L(\text{data}, \theta)$ becomes

$$L(\text{data}, \theta) = \prod_{i=1}^{n}\left(\sum_{j\in S_i} h_j(t_i) \cdot \prod_{l=1}^{m}\overline{F}_l(t_i)\right). \tag{10}$$

Substituting from (1) and (7) into (10), one can get the likelihood function for the data collected from the linear failure rate model as follows:

$$L(\text{data}, \theta) = \prod_{i=1}^{n}\left(\sum_{j\in S_i}(\alpha_j + \beta_j t_i) \cdot \prod_{l=1}^{m}\exp\left[-\left(\alpha_l t_i + \frac{1}{2}\beta_l t_i^2\right)\right]\right) \tag{11}$$

which can be written as

$$L(\text{data}, \theta) = \left\{ \prod_{i=1}^{n} \sum_{j \in S_i} (\alpha_j + \beta_j t_i) \right\} \exp\left[-T_1 \sum_{j=1}^{m} \alpha_j - \frac{1}{2} T_2 \sum_{j=1}^{m} \beta_j \right], \tag{12}$$

where $T_1 = \sum_{i=1}^{n} t_i$ and $T_2 = \sum_{i=1}^{n} t_i^2$.

Therefore, the log-likelihood function becomes

$$\ln L = \sum_{i=1}^{n} \ln\left(\sum_{j \in S_i} (\alpha_j + \beta_j t_i) \right) - T_1 \sum_{j=1}^{m} \alpha_j - \frac{1}{2} T_2 \sum_{j=1}^{m} \beta_j. \tag{13}$$

The MLE of the parameters α_j and β_j ($j = 1, 2, \ldots, m$) can be obtained by solving the set of likelihood equations given by

$$\frac{\partial \ln L}{\partial \alpha_j} = 0, \quad \frac{\partial \ln L}{\partial \beta_j} = 0, \qquad j = 1, 2, \ldots, m. \tag{14}$$

Using (13) and (14), one can deduce the set of likelihood equations as in the following form:

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\sum_{j \in S_i} \delta_{lj}}{\sum_{j \in S_i} (\alpha_j + \beta_j t_i)} - T_1 = 0, \quad l = 1, 2, \ldots, m, \\ &\sum_{i=1}^{n} \frac{t_i \sum_{j \in S_i} \delta_{lj}}{\sum_{j \in S_i} (\alpha_j + \beta_j t_i)} - \frac{1}{2} T_2 = 0, \quad l = 1, 2, \ldots, m, \end{aligned} \tag{15}$$

where δ_{lj} is the Kroneker delta defined by

$$\delta_{lj} = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

The MLE of the unknown parameters α_j, β_j ($j = 1, 2, \ldots, m$) can be obtained by solving the set of equations given by (15) with respect to α_j, β_j ($j = 1, 2, \ldots, m$). As it seems such set of equations has no closed form solution. Therefore a numerical technique is required to obtain the MLE. In what follows, we study this problem on two-component series system.

3.1. Two-component system

Let us study the problem when the system consists of two-components ($m = 2$) where the cause of system failure may be unknown. In this case we need the following supplementary assumptions. Assume that n of such systems are put on the life test. Let $(t_1, S_1), (t_2, S_2), \ldots, (t_n, S_n)$ be the observations that are obtained from the life test. Let $x_1, x_2, \ldots, x_{n_1}$ denote the observed system times to failures when component 1 causes the system failure. It means that n_1 is the number of failed systems when $S_i = \{1\}$. Let $y_1, y_2, \ldots, y_{n_2}$ be the observed system time to failure when component 2 causes the system failure. It means that n_2 is the number of failed systems when $S_i = \{2\}$. Also, let $z_1, z_2, \ldots, z_{n_{12}}$ be the observed system times to failures when the causing system failure is masked. That is, n_{12} is the number of observations when $S_i = \{1, 2\}$. Note that, $n = n_1 + n_2 + n_{12}$. The likelihood function in this case becomes

$$L = \exp\left\{ -(\alpha_1 + \alpha_2)T_1 - \frac{1}{2}(\beta_1 + \beta_2)T_2 \right\} \times \prod_{i=1}^{n_1}(\alpha_1 + \beta_1 x_i) \prod_{j=1}^{n_2}(\alpha_2 + \beta_2 y_j) \prod_{k=1}^{n_{12}}((\alpha_1 + \alpha_2) + (\beta_1 + \beta_2) z_k). \tag{16}$$

Then the log-likelihood function becomes

$$\ln L = -(\alpha_1 + \alpha_2)T_1 - \frac{1}{2}(\beta_1 + \beta_2)T_2 + \sum_{i=1}^{n_1} \ln(\alpha_1 + \beta_1 x_i) + \sum_{j=1}^{n_2} \ln(\alpha_2 + \beta_2 y_j) + \sum_{k=1}^{n_{12}} \ln(\alpha_1 + \alpha_2 + (\beta_1 + \beta_2) z_k). \tag{17}$$

Using Eqs. (14) and (17), one can derive the likelihood equations to be, $l = 1, 2$,

$$\begin{gathered} \delta_{1l} \sum_{i=1}^{n_1} \frac{1}{\alpha_1 + \beta_1 x_i} + \delta_{2l} \sum_{i=1}^{n_2} \frac{1}{\alpha_2 + \beta_2 y_i} + \sum_{i=1}^{n_{12}} \frac{1}{\alpha_1 + \alpha_2 + (\beta_1 + \beta_2) z_i} - T_1 = 0, \\ \delta_{1l} \sum_{i=1}^{n_1} \frac{x_i}{\alpha_1 + \beta_1 x_i} + \delta_{2l} \sum_{i=1}^{n_2} \frac{y_i}{\alpha_2 + \beta_2 y_i} + \sum_{i=1}^{n_{12}} \frac{z_i}{\alpha_1 + \alpha_2 + (\beta_1 + \beta_2) z_i} - \frac{1}{2} T_2 = 0. \end{gathered} \tag{18}$$

To obtain the MLEs $\hat{\alpha}_j$, $\hat{\beta}_j$ $(j = 1, 2)$, we have to solve the above system of nonlinear equations with respect to α_j, β_j. The solution of such system has no closed form and a numerical technique is required.

The MLE of the value of reliability function of component j at a specified mission time t_0, say $\hat{\overline{F}}_j(t_0)$, can be obtained by replacing the unknown parameters α_j and β_j with their MLE in Eq. (7) when $t = t_0$. That is, the MLE of $\overline{F}_j(t_0)$ is given by

$$\hat{\overline{F}}_j(t_0) = \exp\left[-\left(\hat{\alpha}_j t_0 + \tfrac{1}{2} \hat{\beta}_j t^2 \right) \right].$$

4. Bayes approach

As it was noted in the previous section the MLE of the parameters α_j, β_j $(j = 1, 2)$ have no closed forms and a numerical technique method is required to obtain such estimates. For this reason we look for another kind of estimates of the parameters α_j, β_j $(j = 1, 2)$. In this section we use Bayes procedure to estimate the unknown parameters α_j, β_j, $(j = 1, 2)$ and reliability of the system components at a specified mission time, say t_0, in the case of two-component system. Also, we consider here the same supplementary assumptions assumed for the two-component system given in Section 3.1. As we shall see, the Bayes approach gives not only estimates of the unknown parameters in closed forms but also estimates better than MLE. For the motivation of assuming symmetrical triangular prior distributions, we refer to Sarhan [8]. To obtain Bayes estimates we need the following corollary and lemmas.

Lemma 1. *The following relation is fulfilled, for nonnegative integers* $n = 1, 2, \ldots$

$$\prod_{i=1}^{n}(1 + at_i) = \sum_{l=0}^{n} \tau_l a^l, \tag{19}$$

where

$$\tau_l = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n} t_{i_1} t_{i_2} \cdots t_{i_l}, \quad l = 1, 2, \ldots, n \ \text{and} \ \tau_0 = 1.$$

Proof. Let $\phi(n) = \prod_{i=1}^{n}(1 + at_i)$. Using the following recurrence relation and making some arrangements on can reach the proof

$$\phi(n) = (1 + at_n)\phi(n-1), \quad n = 1, 2, \ldots, \quad \phi(0) = 1. \qquad \square$$

Corollary 1. *In the case of two-component system, the likelihood function becomes*

$$L = \exp\left\{ -(\alpha_1 + \alpha_2)T_1 - \frac{1}{2}(\beta_1 + \beta_2)T_2 \right\} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_{12}} \sum_{l=0}^{k} \sum_{m=0}^{n_{12}-k} \times \binom{k}{l} \binom{n_{12}-k}{m} \mathscr{X}_i \mathscr{Y}_j \mathscr{Z}_k \alpha_1^{n_1+m-i} \alpha_2^{n_2+n_{12}-j-k-m} \beta_1^{i+l} \beta_2^{j+k-l}, \tag{20}$$

where

$$\begin{aligned} T_1 &= \sum_{i=1}^{n} t_i, \qquad T_2 = \sum_{i=1}^{n} t_i^2, \\ \mathscr{X}_i &= \sum_{1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant n_1} x_{j_1} x_{j_2} \cdots x_{j_i}, \quad i = 1, 2, \ldots, n_1, \\ \mathscr{Y}_j &= \sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n_2} y_{i_1} y_{i_2} \cdots y_{i_j}, \quad j = 1, 2, \ldots, n_2, \\ \mathscr{Z}_k &= \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n_{12}} z_{i_1} z_{i_2} \cdots z_{i_k}, \quad k = 1, 2, \ldots, n_{12}. \end{aligned} \tag{21}$$

Proof. Using Eq. (19), one can obtain the following relations:

$$\prod_{i=1}^{n_1}(\alpha_1+\beta_1 x_i)=\sum_{j=0}^{n_1}\beta_1^j\alpha_1^{n_1-j}\mathscr{X}_j, \tag{22}$$

$$\prod_{i=1}^{n_2}(\alpha_2+\beta_2 y_i)=\sum_{j=0}^{n_2}\beta_2^j\alpha_2^{n_2-j}\mathscr{Y}_j, \tag{23}$$

and

$$\prod_{i=1}^{n_{12}}(\alpha_1+\alpha_2+(\beta_1+\beta_2)z_i)=\sum_{k=0}^{n_{12}}(\beta_1+\beta_2)^k(\alpha_1+\alpha_2)^{n_{12}-k}\mathscr{Z}_{l_3},$$

using binomial expansions for $(\beta_1+\beta_2)^k$ and $(\alpha_1+\alpha_2)^{n_{12}-k}$ we get

$$\prod_{i=1}^{n_{12}}(\alpha_1+\alpha_2+(\beta_1+\beta_2)z_i)$$
$$=\sum_{k=0}^{n_{12}}\sum_{l=0}^{k}\sum_{m=0}^{n_{12}-k}\binom{k}{l}\binom{n_{12}-k}{m}\beta_1^l\alpha_1^m\beta_2^{k-l}\alpha_2^{n_{12}-k-m}\mathscr{Z}_k. \tag{24}$$

Substituting from Eqs. (22)–(24) into Eq. (16) we can get Eq. (20), which completes the proof. □

The following lemma can be proved easily by using integration by parts.

Lemma 2. *Let* $\Gamma(m,\phi)=\int_0^\phi x^{m-1}\,\mathrm{e}^{-x}\,\mathrm{d}x$, *then for* $m>0$ *the following statement is fulfilled*:

$$\Gamma(m,\phi)=\begin{cases}(m-1)\Gamma(m-1,\phi)-\phi^{m-1}\,\mathrm{e}^{-\phi} & \text{if } m>1,\\ 1-\mathrm{e}^{-\phi} & \text{if } m=1.\end{cases} \tag{25}$$

Lemma 3. *Let*

$$I(a,b,n,\tau)=\epsilon^2\int_a^b g(u)u^n\,\mathrm{e}^{-\tau u}\,\mathrm{d}u, \tag{26}$$

where

$$g(u)=\begin{cases}\frac{1}{\epsilon^2}[\epsilon-|u-\mu|], & u\in(a,b),\\ 0, & \text{otherwise},\end{cases}$$

$\mu=(b+a)/2$, $\epsilon=(b-a)/2$. *Then*

$$I(a,b,n,\tau) = \frac{1}{\tau^{n+2}}\Big\{(a\tau)^{n+1}\,\mathrm{e}^{-a\tau} + (b\tau)^{n+1}\,\mathrm{e}^{-b\tau} - 2(\mu\tau)^{n+1}\,\mathrm{e}^{-\mu\tau} + 2(n+1-\mu\tau)\Gamma(n+1,\mu\tau) - (n+1-a\tau)\Gamma(n+1,a\tau) - (n+1-b\tau)\Gamma(n+1,b\tau)\Big\}. \tag{27}$$

Proof. The function $g(u)$ can be written as in the following form:

$$g(u) = \frac{1}{\epsilon^2}\begin{cases} u-a, & a \leqslant u < \mu, \\ b-u, & \mu \leqslant u \leqslant b, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

Substituting from Eq. (28) into Eq. (26), $I(a,b,n,\tau)$ becomes

$$I(a,b,n,\tau) = \int_a^\mu (u-a)u^n\,\mathrm{e}^{-u\tau}\,\mathrm{d}u + \int_\mu^b (b-u)u^n\,\mathrm{e}^{-u\tau}\,\mathrm{d}u. \tag{29}$$

Thus,

$$I(a,b,n,\tau) = \Gamma(n+2,a,\tau,\mu) - a\Gamma(n+1,a,\tau,\mu) - \Gamma(n+2,\tau,\mu,b) + \Gamma(n+1,\tau,\mu,b), \tag{30}$$

where $\Gamma(m+1,T,x,y) = \int_x^y u^m\,\mathrm{e}^{-Tu}\,\mathrm{d}u$. One can easily verify that

$$\Gamma(m+1,T,x,y) = \frac{1}{T^{m+1}}[\Gamma(m+1,bT) - \Gamma(m+1,aT)]. \tag{31}$$

Using Eqs. (30) and (31) together with Eq. (25) one can deduce Eq. (27) which completes the proof. □

Now we are ready to introduce the following theorem that gives the joint posterior pdf of $(\alpha_1,\alpha_2,\beta_1,\beta_2)$.

Theorem 1. *Under Assumptions* 1–5, *when* $m = 2$, *the joint posterior pdf of* $\underline{\theta} = (\alpha_1,\alpha_2,\beta_1,\beta_2)$ *is given by*

$$g(\underline{\theta}|\text{data}) = \frac{1}{I_0}\left\{\prod_{j=1}^{2}(\epsilon_{1j} - |\alpha_j - \mu_{1j}|)(\epsilon_{2j} - |\beta_j - \mu_{2j}|)\right\} \times \exp\left\{-(\alpha_1+\alpha_2)T_1 - \frac{1}{2}(\beta_1+\beta_2)T_2\right\}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_{12}}\sum_{l=0}^{k}\sum_{m=0}^{n_{12}-k} \times \binom{k}{l}\binom{n_{12}-k}{m}\alpha_1^{n_1+m-i}\beta_1^{i+l}\alpha_2^{n_2+n_{12}-j-k-m}\beta_2^{j+k-l}\mathscr{X}_i\mathscr{Y}_j\mathscr{Z}_k, \tag{32}$$

where $\alpha_j \in \mathbf{A}_j$, $\beta_j \in \mathbf{B}_j$,

$$
\begin{aligned}
I_0 = {} & \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_{12}}\sum_{l=0}^{k}\sum_{m=0}^{n_{12}-k}\binom{k}{l}\binom{n_{12}-k}{m}\mathscr{X}_i\mathscr{Y}_j\mathscr{Z}_k \\
& \times I(a_2,b_2,n_2+n_{12}-j-k-m,T_1)I(a_1,b_1,n_1+m-i,T_1) \\
& \times I\left(c_1,d_1,i+l,\frac{1}{2}T_2\right)I\left(c_2,d_2,j+k-l,\frac{1}{2}T_2\right). \qquad (33)
\end{aligned}
$$

Proof. The joint posterior pdf of $\underline{\theta}$ is related with the joint prior pdf of $\underline{\theta}$ and the likelihood function $L(\text{data},\underline{\theta})$ according to the Bayes theorem (see [4]),

$$
g(\underline{\theta}|\text{data}) = \frac{g(\underline{\theta})L(\text{data},\underline{\theta})}{\int_{\underline{\theta}} g(\underline{\theta})L(\text{data},\underline{\theta})\,\mathrm{d}\underline{\theta}} \qquad (34)
$$

using Assumption 5, the joint prior pdf of $\underline{\theta}$ becomes

$$
\begin{aligned}
g(\underline{\theta}) = {} & \prod_{j=1}^{2}\left\{\frac{1}{\epsilon_{1j}^2\epsilon_{2j}^2}\left(\epsilon_{1j}-|\alpha_j-\mu_{1j}|\right)\left(\epsilon_{2j}-|\beta_j-\mu_{2j}|\right)\right\}, \\
& \alpha_j\in\mathbf{A}_j,\ \ \beta_j\in\mathbf{B}_j,\ \ j=1,2. \qquad (35)
\end{aligned}
$$

Substituting from Eqs. (20) and (35) into Eq. (34) and using some calculus, one can reach the proof. □

Theorem 2. *The posterior* pth *moment of* α_j *and* β_j $(j=1,2)$ *are given respectively by*

$$
M_{\alpha_j}^{(p)} = \frac{I_{\alpha_j}^{(p)}}{I_0},\quad M_{\beta_j}^{(p)} = \frac{I_{\beta_j}^{(p)}}{I_0},\qquad p=1,2,\ldots, \qquad (36)
$$

where

$$
\begin{aligned}
I_{\alpha_1}^{(p)} = {} & \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_{12}}\sum_{l=0}^{k}\sum_{m=0}^{n_{12}-k}\binom{k}{l}\binom{n_{12}-k}{m}\mathscr{X}_i\mathscr{Y}_j\mathscr{Z}_k \\
& \times I(a_2,b_2,n_2+n_{12}-j-k-m,T_1)I(a_1,b_1,n_1+m-i+p,T_1) \\
& \times I\left(c_1,d_1,i+l,\frac{1}{2}T_2\right)I\left(c_2,d_2,j+k-l,\frac{1}{2}T_2\right), \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
I_{\alpha_2}^{(p)} = {} & \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_{12}}\sum_{l=0}^{k}\sum_{m=0}^{n_{12}-k}\binom{k}{l}\binom{n_{12}-k}{m}\mathscr{X}_i\mathscr{Y}_j\mathscr{Z}_k \\
& \times I(a_2,b_2,n_2+n_{12}-j-k-m+p,T_1)I(a_1,b_1,n_1+m-i,T_1) \\
& \times I\left(c_1,d_1,i+l,\frac{1}{2}T_2\right)I\left(c_2,d_2,j-l,\frac{1}{2}T_2\right), \qquad (38)
\end{aligned}
$$

$$
\begin{aligned}
I_{\beta_1}^{(p)} = & \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_{12}}\sum_{l=0}^{k}\sum_{m=0}^{n_{12}-k}\binom{k}{l}\binom{n_{12}-k}{m}\mathscr{X}_i\mathscr{Y}_j\mathscr{Z}_k \\
& \times I(a_2,b_2,n_2+n_{12}-j-k-m,T_1)I(a_1,b_1,n_1+m-i,T_1) \\
& \times I\left(c_1,d_1,i+l+k+p,\frac{1}{2}T_2\right)I\left(c_2,d_2,j-l,\frac{1}{2}T_2\right),
\end{aligned} \tag{39}
$$

and

$$
\begin{aligned}
I_{\beta_2}^{(p)} = & \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\sum_{k=0}^{n_{12}}\sum_{l=0}^{k}\sum_{m=0}^{n_{12}-k}\binom{k}{l}\binom{n_{12}-k}{m}\mathscr{X}_i\mathscr{Y}_j\mathscr{Z}_k \\
& \times I(a_2,b_2,n_2+n_{12}-j-k-m,T_1)I(a_1,b_1,n_1+m-i,T_1) \\
& \times I\left(c_1,d_1,i+l+k,\frac{1}{2}T_2\right)I\left(c_2,d_2,j-l+p,\frac{1}{2}T_2\right),
\end{aligned} \tag{40}
$$

Proof. The posterior pth moment of α_j and β_j $(j=1,2)$ are defined respectively as the posterior expectation of α_j^p and β_j^p. That is,

$$
M_{\alpha_j}^{(p)} = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{c_1}^{d_1}\int_{c_2}^{d_2}\alpha_j^p g(\alpha_1,\alpha_2,\beta_1,\beta_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2 \tag{41}
$$

and

$$
M_{\beta_j}^{(p)} = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{c_1}^{d_1}\int_{c_2}^{d_2}\beta_j^p g(\alpha_1,\alpha_2,\beta_1,\beta_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2. \tag{42}
$$

Substituting from Eq. (32) into Eqs. (41) and (42) and using Lemma 3, one can reach the proof. $\square$

Theorem 3. *Under Assumptions* 1–6, *the Bayes estimates of α_j and β_j, say $\tilde{\alpha}_j$ and $\tilde{\beta}_j$, and their associated minimum posterior risks, say $R\alpha_j$ and R_{β_j}, are*

$$
\tilde{\alpha}_j = \frac{I_{\alpha_j}^{(1)}}{I_0}, \quad \tilde{\beta}_j = \frac{I_{\beta_j}^{(1)}}{I_0}, \qquad j=1,2, \tag{43}
$$

$$
R_{\alpha_j} = \frac{I_{\alpha_j}^{(2)}}{I_0} - \left[\frac{I_{\alpha_j}^{(1)}}{I_0}\right]^2, \quad R_{\beta_j} = \frac{I_{\beta_j}^{(2)}}{I_0} - \left[\frac{I_{\beta_j}^{(1)}}{I_0}\right]^2 \tag{44}
$$

where $I_{\alpha_j}^{(p)}$, $I_{\beta_j}^{(p)}$ $(p=1,2)$ are given by Eqs. (39) *and* (40).

Proof. Under the squared error loss function, the Bayes estimate of the unknown parameter and its associated minimum posterior risk are defined respectively as the posterior mean and variance (see [4]). That is,

$$\tilde{\alpha}_j = M_{\alpha_j}^{(1)}, \quad \tilde{\beta}_j = M_{\beta_j}^{(1)}, \qquad j = 1, 2, \tag{45}$$

$$R_{\alpha_j} = M_{\alpha_j}^{(2)} - \left[M_{\alpha_j}^{(1)}\right]^2, \qquad R_{\beta_j} = M_{\beta_j}^{(2)} - \left[M_{\beta_j}^{(1)}\right]^2 \tag{46}$$

Substituting from Eq. (36) into Eqs. (45) and (46) we can reach the proof. □

Theorem 4. *Under Assumptions* 1–6, *the Bayes estimates of the value of reliability function of system components* $\overline{F}_j(t_0)$ *at a specified mission time* t_0, *say* $\tilde{\tilde{\overline{F}}}_j$, *and their associated minimum posterior risks*, *say* $R_{\overline{F}_j}$, *are*

$$\tilde{\tilde{\overline{F}}}_j = \frac{I_{\overline{F}_j}^{(1)}}{I_0}, \quad R_{\overline{F}_j} = \frac{I_{\overline{F}_j}^{(2)}}{I_0} - \left[\frac{I_{\overline{F}_j}^{(1)}}{I_0}\right]^2, \qquad j = 1, 2, \tag{47}$$

where, *for* $p = 1, 2$

$$\begin{aligned} I_{\overline{F}_1}^{(p)} = {} & \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_{12}} \sum_{l=0}^{k} \sum_{m=0}^{n_{12}-k} \binom{k}{l} \binom{n_{12}-k}{m} \mathscr{X}_i \mathscr{Y}_j \mathscr{Z}_k \\ & \times I(a_2, b_2, n_2 + n_{12} - j - k - m, T_1) I(a_1, b_1, n_1 + m - i, T_1 + pt_0) \\ & \times I\left(c_1, d_1, i + l, \frac{1}{2}(T_2 + pt_0^2)\right) I\left(c_2, d_2, j - l + p, \frac{1}{2} T_2\right), \end{aligned} \tag{48}$$

and

$$\begin{aligned} I_{\overline{F}_2}^{(p)} = {} & \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_{12}} \sum_{l=0}^{k} \sum_{m=0}^{n_{12}-k} \binom{k}{l} \binom{n_{12}-k}{m} \mathscr{X}_i \mathscr{Y}_j \mathscr{Z}_k \\ & \times I(a_2, b_2, n_2 + n_{12} - j - k - m, T_1 + pt_0) I(a_1, b_1, n_1 + m - i, T_1) \\ & \times I\left(c_1, d_1, i + l, \frac{1}{2} T_2\right) I\left(c_2, d_2, j - l + p, \frac{1}{2}(T_2 + pt_0^2)\right), \end{aligned} \tag{49}$$

Proof. Under the squared error loss function, the Bayes estimate of a function of an unknown parameter and the associated minimum posterior risk are defined respectively as the expectation and variance of that function with respect to posterior pdf of that unknown parameter (as a random variable). That is, for $j = 1, 2$,

$$\tilde{\tilde{\overline{F}}}_j = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{c_1}^{d_1} \int_{c_2}^{d_2} \overline{F}_j(t_0 | \alpha_j, \beta_j) g(\alpha_1, \alpha_2, \beta_1, \beta_2) \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\beta_1 \, \mathrm{d}\beta_2, \tag{50}$$

$$R_{\overline{F}_j} = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{c_1}^{d_1}\int_{c_2}^{d_2} \left\{\overline{F}_j(t_0|\alpha_j,\beta_j)\right\}^2 g(\alpha_1,\alpha_2,\beta_1,\beta_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\beta_1\,\mathrm{d}\beta_2 - \left[\tilde{r}_j(t_0)\right]^2. \tag{51}$$

Substituting from Eqs. (7) and (32) into Eqs. (50) and (51) and using Lemma 3, we can reach the proof of the theorem. □

5. Simulation study and conclusion

We show in this section how one can apply the previous theoretical results obtained to the lifetime data. This section is devoted to present numerical results based on a large simulation study. This simulation has been done by writing a computer program. It is assumed in the simulation that $\alpha_1 = 2$, $\beta_1 = 1.5$, $\alpha_2 = 1$ and $\beta_2 = 2.5$. Tables 1–4 show four samples obtained from the simulation when $n = 30$. Each of these tables contains the number of system failed, its time to failure and the set containing the components that may cause the system failure.

Based on the simulated data given in Tables 1–4, the values of T_1, T_2, n_1, n_2 and n_{12} are determined and given in Table 5.

Also, using the data presented in the above Tables 1–5, the theoretical results previously obtained in Eqs. (33)–(40), (48), (49) and assuming that $\mathbf{A}_1 = (0.5, 3.5)$, $\mathbf{A}_2 = (0.2, 1.8)$, $\mathbf{B}_1 = (0.3, 2.7)$ and $\mathbf{B}_2 = (0.8, 4.2)$ we have computed the integrals $I_0, I_{\alpha_1}, I_{\alpha_2}, I_{\beta_1}, I_{\beta_2}, I_{\overline{F}_1}$ and $I_{\overline{F}_2}$. Table 6 gives values of the integrals I_0, $I^{(1)}_{\alpha_1}$, $I^{(1)}_{\alpha_2}$, $I^{(1)}_{\beta_1}$, $I^{(1)}_{\beta_2}$, $I^{(1)}_{\overline{F}_1}$ and $I^{(1)}_{\overline{F}_2}$ obtained based on the simulated data given in samples 1–4. The integrals $I_{\overline{F}_1}$ and $I_{\overline{F}_2}$ are obtained when $t_0 = 0.8$.

Using the data summarized in Tables 1–5 we solved the system of nonlinear equations (18) and obtained the MLEs of α_j, β_j and $\overline{F}_j(0.8)$ $(j = 1, 2)$. The

Table 1
Simulated system life data for the first sample

i	t_i	S_i	i	t_i	S_i	i	t_i	S_i
1	0.008	{1}	11	0.684	{1}	21	0.233	{1}
2	0.157	{1}	12	0.443	{1}	22	0.033	{2}
3	0.298	{2}	13	0.344	{1}	23	0.123	{2}
4	0.584	{2}	14	0.520	{1}	24	0.301	{1}
5	0.050	{1}	15	0.137	{1}	25	0.122	{1}
6	0.305	{2}	16	0.017	{2}	26	0.177	{2}
7	0.003	{1}	17	0.308	{1}	27	0.042	{1}
8	0.013	{2}	18	0.377	{2}	28	0.029	{2}
9	0.070	{1}	19	0.441	{1}	29	0.462	{2}
10	0.762	{2}	20	0.159	{1}	30	0.169	{1}

Table 2
Simulated system life data for the second sample

i	t_i	S_i	i	t_i	S_i	i	t_i	S_i
1	0.449	$\{1,2\}$	11	0.101	$\{1\}$	21	0.347	$\{2\}$
2	0.150	$\{1,2\}$	12	0.353	$\{2\}$	22	0.297	$\{2\}$
3	0.022	$\{2\}$	13	0.026	$\{1\}$	23	0.602	$\{2\}$
4	0.174	$\{1\}$	14	0.145	$\{2\}$	24	0.663	$\{2\}$
5	0.499	$\{1\}$	15	0.328	$\{1\}$	25	0.178	$\{2\}$
6	0.081	$\{1\}$	16	0.207	$\{1\}$	26	0.588	$\{1\}$
7	0.032	$\{2\}$	17	0.092	$\{1\}$	27	0.085	$\{1\}$
8	0.040	$\{1\}$	18	0.110	$\{2\}$	28	0.383	$\{2\}$
9	0.427	$\{1\}$	19	0.092	$\{2\}$	29	0.090	$\{1\}$
10	0.180	$\{1\}$	20	0.254	$\{1\}$	30	0.535	$\{2\}$

Table 3
Simulated system life data for the third sample

i	t_i	S_i	i	t_i	S_i	i	t_i	S_i
1	0.154	$\{1,2\}$	11	0.041	$\{2\}$	21	0.095	$\{1\}$
2	0.579	$\{1,2\}$	12	0.494	$\{1\}$	22	0.565	$\{1\}$
3	0.506	$\{2\}$	13	0.538	$\{2\}$	23	0.428	$\{1\}$
4	0.216	$\{2\}$	14	0.050	$\{1\}$	24	0.976	$\{2\}$
5	0.402	$\{2\}$	15	0.270	$\{1\}$	25	0.110	$\{2\}$
6	0.177	$\{1\}$	16	0.100	$\{2\}$	26	0.205	$\{1\}$
7	0.022	$\{1\}$	17	0.390	$\{1\}$	27	0.324	$\{1\}$
8	0.308	$\{1\}$	18	0.150	$\{1\}$	28	0.279	$\{1\}$
9	0.133	$\{1\}$	19	0.346	$\{1\}$	29	0.011	$\{2\}$
10	0.262	$\{1\}$	20	0.037	$\{2\}$	30	0.362	$\{2\}$

Table 4
Simulated system life data for the fourth sample

i	t_i	S_i	i	t_i	S_i	i	t_i	S_i
1	0.030	$\{1,2\}$	11	0.146	$\{1\}$	21	0.172	$\{1\}$
2	0.209	$\{1,2\}$	12	0.172	$\{2\}$	22	0.607	$\{2\}$
3	0.458	$\{1,2\}$	13	1.063	$\{2\}$	23	0.725	$\{1\}$
4	0.259	$\{2\}$	14	0.123	$\{2\}$	24	0.175	$\{1\}$
5	0.104	$\{1\}$	15	0.554	$\{2\}$	25	0.109	$\{2\}$
6	0.381	$\{1\}$	16	0.782	$\{1\}$	26	0.265	$\{1\}$
7	0.044	$\{1\}$	17	0.003	$\{1\}$	27	0.202	$\{2\}$
8	0.342	$\{2\}$	18	0.086	$\{2\}$	28	0.362	$\{1\}$
9	0.060	$\{1\}$	19	0.023	$\{1\}$	29	0.594	$\{2\}$
10	0.045	$\{2\}$	20	0.085	$\{1\}$	30	0.161	$\{1\}$

Turbo Pascal procedure "newtsys.pas" in [3] is used to solve the system of nonlinear equations (18). These MLEs are given in Tables 7–10. Further, The results presented in Table 6 are used to obtain the Bayes estimates of α_j, β_j and

Table 5
Some quantities required to obtain the estimations

Sample	T_1	T_2	n_1	n_2	n_{12}
1	7.372	3.117	18	12	0
2	7.531	2.965	15	13	2
3	8.530	3.786	17	11	2
4	8.341	4.319	15	12	3

Table 6
The integrals required for Bayes estimations

Integral	Sample			
	1	2	3	4
I_0	2.162×10^{-4}	6.893×10^{-4}	2.109×10^{-5}	1.831×10^{-5}
I_{α_1}	4.576×10^{-4}	1.347×10^{-3}	3.967×10^{-5}	3.442×10^{-5}
I_{α_2}	2.329×10^{-4}	7.539×10^{-4}	2.046×10^{-5}	1.843×10^{-5}
I_{β_1}	3.274×10^{-4}	1.033×10^{-3}	3.266×10^{-5}	2.516×10^{-5}
I_{β_2}	5.390×10^{-4}	1.874×10^{-3}	3.146×10^{-5}	4.547×10^{-5}
$I_{\overline{F}_1}$	2.443×10^{-5}	8.691×10^{-5}	2.847×10^{-6}	2.270×10^{-6}
$I_{\overline{F}_2}$	4.108×10^{-5}	1.250×10^{-4}	4.450×10^{-6}	3.699×10^{-6}

Table 7
The ML, Bayes estimates and their respective percentage errors for sample 1

Parameter	Method			
	Maximum likelihood		Bayes	
	Estimate	PE	Estimate	PE
α_1	2.096	4.8	2.117	5.85
α_2	0.882	11.8	1.077	7.7
β_1	1.636	9.07	1.514	0.93
β_2	3.594	43.76	2.493	0.32
$\overline{F}_1$	0.111	11.34	0.113	0.934
$\overline{F}_2$	0.156	22.56	0.190	5.73

$\overline{F}_j(0.8)$ $(j = 1, 2)$, according to relations (43) and (47). Tables 7–10 give these estimations.

In order to compare the accuracy of the maximum likelihood and Bayes procedure, we calculate the percentage error associated with the estimate obtained using each procedure. The percentage error associated with the estimate $\hat{\theta}$ of θ, say $\mathrm{PE}_{\hat{\theta}}$, is defined by

$$\mathrm{PE}_{\hat{\theta}} = \frac{|\hat{\theta} - \text{exact value of } \theta|}{\text{exact value of } \theta} \times 100\%.$$

Table 8
The ML, Bayes estimates and their respective percentage errors for sample 2

Parameter	Method			
	Maximum likelihood		Bayes	
	Estimate	PE	Estimate	PE
α_1	1.817	9.15	1.955	2.25
α_2	0.345	65.5	1.094	9.40
β_1	1.590	6.00	1.499	0.07
β_2	6.371	154.8	2.719	8.76
$\overline{F}_1$	0.141	12.48	0.130	10.35
$\overline{F}_2$	0.099	51.07	0.181	3.70

Table 9
The ML, Bayes estimates and their respective percentage errors for sample 3

Parameter	Method			
	Maximum likelihood		Bayes	
	Estimate	PE	Estimate	PE
α_1	1.218	39.1	1.881	5.85
α_2	0.056	94.4	0.970	7.7
β_1	3.392	126.1	1.549	0.93
β_2	15.21	568.4	2.440	0.32
$\overline{F}_1$	0.127	11.34	0.135	8.28
$\overline{F}_2$	0.007	96.36	0.211	4.42

Table 10
The ML, Bayes estimates and their respective percentage errors for sample 4

Parameter	Method			
	Maximum likelihood		Bayes	
	Estimate	PE	Estimate	PE
α_1	2.272	13.6	1.880	6.0
α_2	0.755	24.5	1.007	0.7
β_1	−0.992	166.1	1.374	8.4
β_2	2.569	2.76	2.483	0.7
$\overline{F}_1$	0.223	78.58	0.124	0.8
$\overline{F}_2$	0.240	18.70	0.202	0.0

The percentage errors obtained are also given in Tables 7–10.

Based on the results obtained and shown in Tables 7–10, one can conclude that follows:

1. The maximum likelihood procedure fails in sample 4 to give a reasonable estimate of β_1, it gives a negative value.
2. In spite of the maximum likelihood procedure fails in sample 4, the Bayes procedure gives not only reasonable estimates of all parameters but also it gives better estimates in sense of having smaller percentage errors.
3. The percentage errors associated with the Bayes estimates obtained are smaller than that obtained by using maximum likelihood procedure.

From the above one can say that for the studied cases, the Bayes procedure has two advantages than the maximum likelihood procedure. The first is it gives reasonable estimates from all samples given. The second is it gives estimates with percentage errors smaller than that associated with MLEs.

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