Reliability Estimation in Series Systems

Maximum Likelihood Techniques for Right-Censored and Masked Failure Data

Alex Towell

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Context & Motivation

Reliability in **Series Systems** is like a chain's strength – determined by its weakest link.

- Essential for system design and maintenance.

Main Goal: Estimate individual component reliability from *failure data*.

Challenges:

- *Masked* component-level failure data.
- *Right-censoring* system-level failure data.

Our Response:

- Derive techniques to interpret such ambiguous data.
- Aim for precise and accurate reliability estimates for individual components using maximum likelihood estimation (MLE) and BCa confidence intervals.

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- Derive techniques to interpret such ambiguous data.
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- Think of a series system as a **chain**: reliability is determined by its weakest component.
- When any component **fails**, the whole system does.
- So, understanding the **reliability** of each component is needed for the **design** and **maintenance** of these systems.
- So, our **main goal** is to estimate the reliability of each component from failure data.
- But the **data** can pose **challenges**, like right-censoring or masked failures where we don't know which component failed.
- Our **goal** is to use this data to provide accurate reliability estimates for each component, including accurate confidence intervals to quantify the uncertainty in our estimate.
- To obtain good **coverage**, we bootstrap the confidence intervals using the **BCa method**.

Core Contributions

Likelihood Model for **Series Systems**.

- Accounts for *right-censoring* and *masked component failure*.

Specifications of Conditions:

- Assumptions about the masking of component failures.
- Simplifies and makes the model more tractable.

Simulation Studies:

- Components with *Weibull* lifetimes.
- Evaluate MLE and confidence intervals under different scenarios.

R Library: Methods available on GitHub.

- See: www.github.com/queelius/wei.series.md.c1.c2.c3

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- Our **core contributions** can be broken down into several parts.
- We **derived** a **likelihood model** for series systems that accounts for **Right-censoring** and **masking** of **component failures**.
- We've **clarified** the conditions this model assumes about the masking of component failures. These conditions simplify the model and make it more tractable.
- We've **validated** our model with extensive simulations to gauge its performance under various simulation scenarios.
- The **simulation study** is based on components with **Weibull** lifetimes.
- For those interested, we made our methods available in an **R Library** hosted on GitHub.

Section 1

Series System

Section 1
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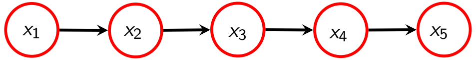
Series System



Critical Components: Complex systems often comprise *critical* components. If any component fails, the entire system fails.

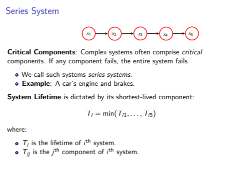
- We call such systems *series systems*.
- **Example**: A car's engine and brakes.

System Lifetime is dictated by its shortest-lived component:

$$T_i = \min(T_{i1}, \ldots, T_{i5})$$

where:

- T_i is the lifetime of i^{th} system.
- T_{ij} is the j^{th} component of i^{th} system.



- Many complex systems have **critical components** that are essential to their operation.
- If any of these components fail, the entire system fails. We call these **series systems**.
- Think of a **car** - if the engine or brakes fail, it can't be operated.
- Its **lifetime** is the lifetime of its **shortest-lived** component.
- For reference, we show the some notation we'll use throughout the talk.
- T_i is the system's lifetime and T_{ij} is its j^{th} component's lifetime.

Reliability Function

Reliability Function represents the probability that a system or component functions beyond a specified time.

- Essential for understanding longevity and dependability.

Series System Reliability: Product of the reliability of its components:

$$R_{T_i}(t;\boldsymbol{\theta}) = \prod_{j=1}^{m} R_j(t;\boldsymbol{\theta_j}).$$

- Here, $R_{T_i}(t;\boldsymbol{\theta})$ and $R_j(t;\boldsymbol{\theta_j})$ are the reliability functions for the system i and component j, respectively.

Relevance:

- Forms the foundation for most reliability studies.
- Integral to our likelihood model, e.g., right-censoring events.

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- The **reliability function** tells us the chance a component or system functions past a specific time.
- It's a **key metric** for longevity.
- In a series system, the overall reliability is the **product** of its component reliabilities.
- So, even if **one component** has a low reliability, it can impact the whole system.

Hazard Function: Understanding Risks

Hazard Function: Measures the immediate risk of failure at a given time, assuming survival up to that moment.

- Reveals how the risk of failure evolves over time.
- Guides maintenance schedules and interventions.

Series System Hazard Function is the sum of the hazard functions of its components:

$$h_{T_i}(t; \boldsymbol{\theta}) = \sum_{j=1}^{m} h_j(t; \boldsymbol{\theta_j}).$$

- Components' risks are additive.

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- Let's shift focus to the hazard function.
- Essentially, it's a way to gauge the immediate risk of failure, especially if it's been functioning up until that point.
- The hazard function for a series system is just the sum of the **component hazards**.
- We see that the component risks are **additive**.

Joint Distribution of Component Failure and System Lifetime

Our likelihood model depends on the **joint distribution** of the system lifetime and the component that caused the failure.

- **Formula**: Product of the failing component's hazard function and the system reliability function:

$$f_{K_i,T_i}(j,t;\boldsymbol{\theta}) = h_j(t;\boldsymbol{\theta_j})R_{T_i}(t;\boldsymbol{\theta}).$$

- **Single Point of Failure**: A series system fails due to one component's malfunction.
- **Representation**:
  - K_i: Component causing the i^{th} system's failure.
  - $h_j(t;\boldsymbol{\theta_j})$: Hazard function for the j^{th} component.

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- **Representation**:
  - K_i: Component causing the i^{th} system's failure.
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- In our likelihood model, understanding the **joint distribution** of a system's lifetime and the component that led to its failure is essential.
- It is the **product** of the failed component's hazard function and the system reliability function.
- Here, K_i denotes **failed component**.

Component Failure & Well-Designed Series Systems

The **marginal probability** of component failure helps predict the cause of failure.

- **Derivation**: Marginalize the joint distribution over the system lifetime:

$$\Pr\{K_i = j\} = E_{\boldsymbol{\theta}}\left[\frac{h_j(T_i;\boldsymbol{\theta_j})}{h_{T_i}(T_i;\boldsymbol{\theta_l})}\right].$$

Well-Designed Series System: Components exhibit comparable chances of causing system failures.

- **Relevance**: Our simulation study employs a (reasonably) well-designed series system.

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- We can use the **joint distribution** of the system lifetime and the failed component to calculate the **marginal** probability of component failure.
- When we do so, we find that it is the **expected value** of the ratio of component and system hazard functions.
- We say that a series system is **well-designed** if each components has a **comparable** chance of failing.
- Our simulation study is **based** on a reasonably well-designed series system.

Section 2

Likelihood Function

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Likelihood Function

Likelihood: Measures how well model explains the data. Each system contributes to *total likelihood* via its *likelihood contribution*:

$$L(\boldsymbol{\theta}|\text{data}) = \prod_{i=1}^{n} L_i(\boldsymbol{\theta}|\text{data}_i)$$

where $\textbf{data}_i$ = data for i^{th} system and L_i = its contribution.

Our model handles the following data:

- **Right-Censored**: Experiment ends before failure (Event Indicator: $\delta_i = 0$).
  - Contribution is system reliability: $L_i(\boldsymbol{\theta}) = R_{T_i}(\tau; \boldsymbol{\theta})$.
- **Masked Failure**: Failure observed, but the failed component is masked by a *candidate set*. More on its contribution later.

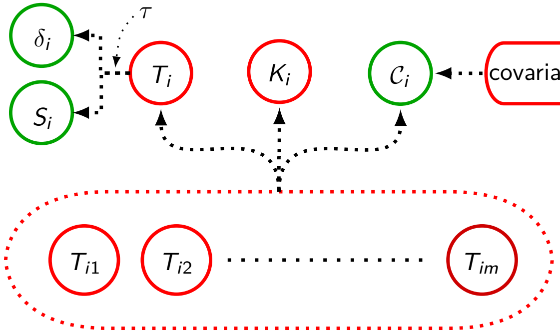
System	Right-Censored Lifetime	Event Indicator	Candidate Set
1	1.1	1	$\{1, 2\}$
2	5	0	$\emptyset$

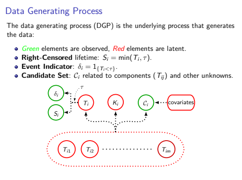
- Let's talk about the **likelihood function**, which is a way of **measuring** how well our model explains the data.
- The **total likelihood** is the product of the likelihood contributions of each system.
- Our model deals with right-censoring and masked failures.
- **Right-censoring** occurs when the experiment ends before the system fails. Its contribution is just the system reliability, since the event indicator tells us if the system was right-censored.
- **Masking** occurs when we observe a failure but we don't know which component failed. Instead, we see a set of components that mask the failure. More on this later.
- We show an **example** data set here. **System 1** failed and either component 1 or 2 caused it and **System 2** was right-censored.

Data Generating Process

The data generating process (DGP) is the underlying process that generates the data:

- *Green* elements are observed, *Red* elements are latent.
- **Right-Censored** lifetime: $S_i = \min(T_i, \tau)$.
- **Event Indicator**: $\delta_i = 1_{\{T_i < \tau\}}$.
- **Candidate Set**: $\mathcal{C}_i$ related to components (T_{ij}) and other unknowns.





- Let's discuss the **data generating process** to motivate our model.
- Here's the graph: **green** is observed and **red** is latent.
- We don't get to see the red elements, but we can **infer** most of them from the green elements.
- So, let's focus on the **green** elements, the **observed** data.
- The **right-censored** time is the minimum of the system lifetime and the right-censoring time τ.
- The **event** indicator is 1 if the system fails before the right-censoring time, 0 otherwise.
- The candidate set is related to the component lifetimes and many other factors.
- The candidate sets are difficult **difficult** to model, so we seek a simple model that is valid under certain assumptions, which we discuss next.

Likelihood Contribution: Masked Failure Conditions

The candidate sets that **mask** the failed component are assumed to satisfy the following conditions:

- **Candidate Set Contains Failed Component**: The candidate set includes the failed component.
- **Masking Probabilities Uniform Across Candidate Sets**: The probability of of the candidate set is constant across different components within it.
- **Masking Probabilities Independent of Parameters**: The masking probabilities when conditioned on the system lifetime and the failed component aren't functions of the system parameter.

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- The right-censoring contribution is straightforward. The **masked failure contribution** is more complex.
- **Masking** occurs when a system fails but the precise failed component is masked by a candidate set.
- To make the problem **tractable**, we introduce certain conditions or assumptions about the candidate sets.
- In **Condition 1**, the candidate set always includes the failed component.
- In **Condition 2**, the probability of the candidate set is constant across different components within it.
- In **Condition 3**, the masking probabilities when conditioned on the system lifetime and the failed component aren't functions of the system parameter.
- These conditions are often **reasonable** in industrial settings.

Likelihood Contribution: Derivation for Masked Failures

Take the **joint distribution** of T_i, K_i, and $\mathcal{C}_i$ and marginalize over K_i:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \sum_{j=1}^{m} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 1** to get a sum over candidate set:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 2** to move probability outside the sum:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \Pr_{\boldsymbol{\theta}}\{\mathcal{C}_i = c_i | T_i = t_i, K_i = j'\} \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}).$$

Apply **Condition 3** to remove the probability's dependence on $\boldsymbol{\theta}$:

$$f_{T_i,\mathcal{C}_i}(t_i, c_i; \boldsymbol{\theta}) = \beta_i \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}).$$

Result: $L_i(\boldsymbol{\theta}) \propto \sum_{j \in c_i} f_{T_i,K_i}(t_i, j; \boldsymbol{\theta}) = R_{T_i}(t_i; \boldsymbol{\theta}) \sum_{j \in c_i} h_j(t_i; \boldsymbol{\theta_j})$.

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- Here, we **derive** the likelihood contribution for masked failures.
- To start, we use the **joint distribution** of the system lifetime, the failed component, and the candidate set.
- Then, we **marginalize** over the failed component, since we don't know which component failed.
- We apply **condition 1** to get a **sum** over the **candidate set** instead.
- We apply **condition 2** to move the probability **outside** the sum.
- We apply **condition 3** to **remove** the probability's dependence on the system parameter.
- And we **end up** with a likelihood contribution proportional to the product of the system reliability and the sum of the component hazards in the candidate set.

Methodology: Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE): Maximize the log-likelihood function:

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} \log L(\boldsymbol{\theta}).$$

Solution: Numerically solved system of equations for $\hat{\boldsymbol{\theta}}$:

$$\nabla_{\boldsymbol{\theta}} \log L(\hat{\boldsymbol{\theta}}) = \mathbf{0}.$$

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- We use the standard **maximum likelihood** approach.
- We find a parameter value that **maximizes** the likelihood function using numerical methods.

Bootstrap Confidence Intervals (CIs)

Confidence Intervals (CI) help capture the *uncertainty* in our estimate.

- **Normal** assumption for constructing CIs may not be accurate.
  - *Masking* and *censoring*.
- **Bootstrapped CIs**: Resample data and obtain MLE for each.
  - Use **percentiles** of bootstrapped MLEs for CIs.
- **Coverage Probability**: Probability the interval covers the true parameter value.
  - **Challenge**: Actual coverage may deviate to bias and skew in MLEs.
- **BCa** adjusts the CIs to counteract bias and skew in the MLEs.

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- We need to measure the **uncertainty** in our estimate.
- **Confidence intervals** are a popular choice and help us pin down the likely range of values for our parameters.
- Due to masking and censoring, the **normal** approximation for constructing CIs may be inaccurate.
- So, we've chosen to **bootstrap** the intervals instead, which isn't as sensitive to these issues.
- **Coverage probability** is the probability the interval covers the true parameter value.
- Due to bias and skew in the MLE, the coverage probability may be too low/high, indicating over/under confidence.
- We use the **BCa method** to adjust the confidence intervals to counteract bias and skew.

Challenges with MLE on Masked Data

We discovered some challenges with the MLE.

- **Convergence Issues**: Flat likelihood regions observed.
  - Ambiguity in masked data with small samples.
- **Bootstrap Issues**: Bootstrap relies on the Law of Large Numbers.
  - It might not represent the true variability for small samples.
  - Censoring and masking reduces effective sample size.
- **Mitigation**: In simulation study, discard non-convergent samples for the MLE on original data, but keep all resamples for the BCa CIs.
  - Helps ensure robustness of the results, while acknowledging the inherent complexities introduced by masking and censoring.
  - We report convergence rates in our simulation study.

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- Like any model, ours has its challenges. **Masking** and **censoring**, combined with small sample sizes, can cause flat likelihood regions, which can lead to convergence issues.
- For **small samples**, bootstrapping may not always capture the true variability in the data.
- To **deal** with these issues in our simulation study, we **discard** samples that did not converge within 125 iterations for the MLE on original data.
- However, we **retain** all MLEs for the resampled data in the bootstrap for the confidence intervals.
- This helps ensure the **robustness** of our results while acknowledging the inherent complexities introduced by masking and censoring.
- We report the **convergence rate** in our simulation studies.

Section 3

Simulation Study

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Simulation Study

Series System: Weibull Components

The lifetime of the j^{th} component in the i^{th} system:

$$T_{ij} \sim \text{Weibull}(k_j, \lambda_j)$$

- λ_j is the **scale** parameter
- k_j is the **shape** parameter:
  - $k_j < 1$: Indicates infant mortality.
  - $k_j = 1$: Indicates random failures (exponential distribution).
  - $k_j > 1$: Indicates wear-out failures.

Weibull has well known *reliability* and *hazard* functions, so we won't reproduce them here. Recall that for a series system:

- **Series Reliability** is the product of the component reliabilities.
- **Hazard** is the sum of the component hazard functions.
- **Likelihood**: $L(\boldsymbol{\theta}) \propto \prod_{i=1}^{n} R_{T_i}(t_i; \boldsymbol{\theta}) \left[\sum_{j \in c_i} h_j(t_i; \boldsymbol{\theta_j})\right]^{\delta_i}$.

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- In our **simulation study**, we analyze a series system with **Weibull** components.
- The Weibull has two parameters: the **scale** and **shape**.
- **Shape** parameter tells us a lot about the failure characteristics.
  - When its **greater** than **one**, think of it as wearing-out over time.
  - If it's **less** than one, that usually signals some early-life challenges.
- For the Weibull, the **hazard** and **reliability** functions are well-known, so we won't reproduce them here.
- Recall that for a series system, the **reliability** is the product of the component reliabilities and the **hazard** function is the sum of the component hazard functions.
- The **formula** for the **likelihood function** for our data is the same as before, we've just reproduced it here as a reminder.

Well-Designed Series System

Simulation study centered around series system with Weibull components:

Component	Shape	Scale	$\Pr\{K_i\}$
1	1.26	994.37	0.17
2	1.16	908.95	0.21
3	1.13	840.11	0.23
4	1.18	940.13	0.20
5	1.20	923.16	0.20

- Based on (Guo, Niu, and Szidarovszky 2013) study of 3 components.
  - We added components 4 and 5.
- **Probabilities** are comparable: it is *reasonably well-designed*.
  - **Reliability**: Components 1 and 3 *most* and *least* reliable, respectively.
  - **Simulation Study**: Only show estimates for these two components.
- **Shapes** greater than 1, indicating wear-outs.

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Component Shape Scale Pr{K_i}
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3 1.13 840.11 0.23
4 1.18 940.13 0.20
5 1.20 923.16 0.20
Based on (Guo, Niu, and Szidarovszky 2013) study of 3 components.
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Shapes greater than 1, indicating wear-outs.

- This study is **centered** around a series system with **five** Weibull components.
- It's **based** on a paper that studies a 3-component series system.
- We **added** components 4 and 5 to make it more complex.
- We show the **probability** of each component being the cause of failure.
- Since the probabilities are comparable, no weak links, it's reasonably **well-designed**.
- Component 1 is the **most** reliable and component 3 is the **least**.
- In our simulation study, we only show the estimates for these two components, as they will show the biggest differences.
- We show the **shape** and **scale** parameters for each component.
- We see that the **shape** parameters are **greater** than 1, which indicates **wear-out** failures.

Data Generation and Simulation Values

How is the data generated in our simulation study?

- **Component Lifetimes** (latent $T_{i1}, \cdots, T_{im}$) generated for each system.
  - **Observed Data** is a function of latent components.
- **Right-Censoring** amount controlled with simulation value q.
  - Quantile q is probability system won't be right-censored.
  - Solve for right-censoring time τ in $\Pr\{T_i \leq \tau\} = q$.
  - $S_i = \min(T_i, \tau)$ and $\delta_i = 1_{\{T_i \leq \tau\}}$.
- **Candidate Sets** are generated using the *Bernoulli Masking Model*.
  - Masking level controlled with simulation value p.
  - Failed component (latent K_i) placed in candidate set (observed $\mathcal{C}_i$).
  - Each functioning component included with probability p.

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  - Each functioning component included with probability p.

- Let's talk about how we **generate** the data for our **simulation** study.
- First, we generate the latent **component lifetimes** for the system just discussed.
- Then, we generate the data we actually **see** based on these lifetimes.
- The **right-censored** lifetimes, the **censoring indicators**, and **candidate sets**.
- In the simulations, we **control** the amount of **right-censoring** with the value q, the probability the system won't be right-censored.
- We use the **Bernoulli Masking Model** to generate the candidate sets.
- We **control** the masking level with the value p, the **Bernoulli** probability.
- – Explain procedure for generating candidates –

Data Generation: Satisfying Masking Conditions

The Bernoulli Masking Model *satisfies* the masking conditions:

- **Condition 1**: The failed component deterministically placed in candidate set.
- **Condition 2** and **3**: Bernoulli probability p is same for all components and fixed by us.
  - Probability of candidate set is constant conditioned on component failure within set.
  - Probability of candidate set, conditioned on a component failure, only depends on the p.

Future Research: Realistically conditions may be violated.

- Explore sensitivity of likelihood model to violations.

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Future Research: Realistically conditions may be violated.

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- It's important to show how our **Bernoulli** masking model used in our simulation study **satisfies** these masking conditions.
- We obviously satisfy **Condition 1** because the failed component is always placed in the candidate set.
- We satisfy **Condition 2** because the Bernoulli probability is the same for all components. As we vary the component failure within the set, the probability of the set doesn't change.
- We satisfy **Condition 3** because, conditioned on a failed component, the probability of the candidate set only depends on the Bernoulli probability, which is fixed by us and doesn't interact with the the system parameters.
- In **real life**, these conditions may be violated. Future research could explore the **sensitivity** of our likelihood model to violations of these conditions.

Performance Metrics

Objective: Evaluate the MLE and BCa confidence intervals' performance across various scenarios.

- Visualize the **simulated** sampling distribution of MLEs and 95% CIs.
- **MLE Evaluation**:
  - **Accuracy**: Bias
  - **Precision**: Dispersion of MLEs
    - 95% quantile range of MLEs.
- **95% CI Evaluation**:
  - **Accuracy**: Coverage probability (CP).
    - *Correctly Specified* CIs: CP near 95% (> 90% acceptable).
  - **Precision**: Width of median CI.

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  - **Precision**: Width of median CI.

- We want to evaluate the accuracy and precision of our MLE and CIs under various conditions.
- For the **MLE**, we're looking at its **bias** and **spread**.
- A **tight spread** indicates **high precision**, but if it's biased, we can't trust it.
- For the **CIs**, when we talk about accuracy, we're looking at **coverage probability**.
- We want our intervals to be **correctly specified**, meaning they cover the true parameter value around 95% of the time.
- Our **goal** is to get close to the nominal 95% level, but we'll consider anything above 90
- As for **precision**, we use the width of these intervals.
- A **narrow width** points to a **higher precision**, but that's meaningless if the CP is too low.

Scenario: Impact of Right-Censoring

Assess the impact of right-censoring on MLE and CIs.

- **Right-Censoring**: Failure observed with probability q: 60% to 100%.
  - Right censoring occurs with probability $1 - q$: 40% to 0%.
- **Bernoulli Masking Probability**: Each component is a candidate with probability p fixed at 21.5%.
  - Estimated from original study (Guo, Niu, and Szidarovszky 2013).
  - Chance of **no** masking: $\Pr\{\text{only failed component in } \mathcal{C}_i\} \approx 62\%$.
- **Sample Size**: n fixed at 90.
  - Small enough to show impact of right-censoring.

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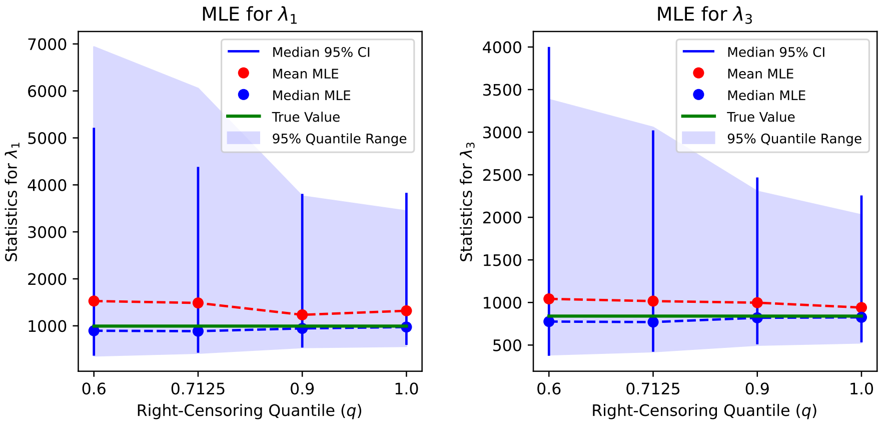
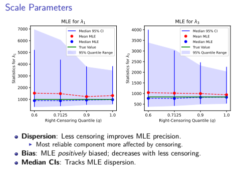
- We assess the impact of **right-censoring** on the MLE and confidence intervals.
- We **vary** the probability of observing a failure from 60% to 100%.
- We fix the masking probability at 21.5%, which is the probability that each component is a candidate.
- This masking probability is **based** on estimates from the original study.
- We fix the **sample size** at 90, which was small enough to show the impact of right-censoring on the MLE, but large enough so that the convergence rate was reasonable.

Scale Parameters

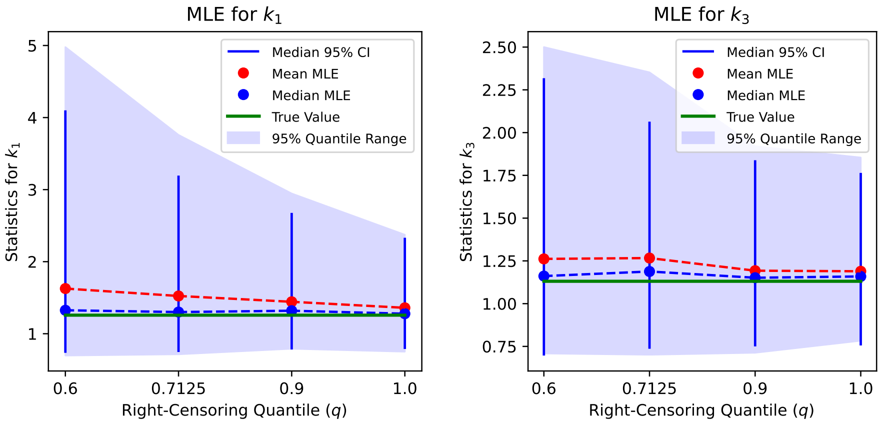


- **Dispersion**: Less censoring improves MLE precision.
  - Most reliable component more affected by censoring.
- **Bias**: MLE *positively* biased; decreases with less censoring.
- **Median CIs**: Tracks MLE dispersion.

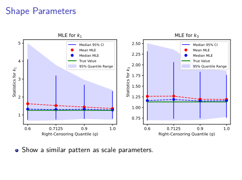


- Here, we show the impact of **right-censoring** on the MLE and confidence intervals for the **scale parameters**, the most reliable component on the **left** and the least reliable component on the **right**.
- In **light solid blue**, we show the dispersion of the MLE. We see that it improves with less censoring.
- We see that the **more reliable** component has more dispersion than the other component.
- This is **due** to more reliable components being more likely to be censored.
- In the **dashed red line**, we show the bias. The MLE is **positively** biased, but that bias decreases as censoring level is reduced.
- In the **dark blue** vertical lines, we show the median confidence intervals.
- We see they they **track** the MLE's dispersion, which is good.

Shape Parameters

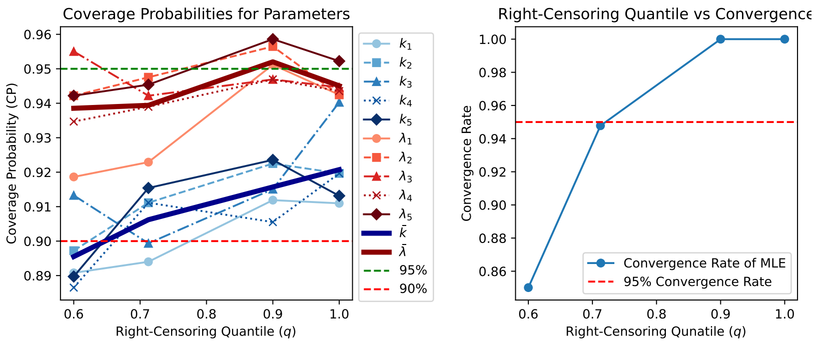


- Show a similar pattern as scale parameters.

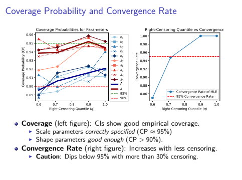


- We see similar results for the **shape parameters**.
- So, let's move on to evaluating the **accuracy** of the **confidence intervals**, where we do see some notable differences.

Coverage Probability and Convergence Rate



- **Coverage** (left figure): CIs show good empirical coverage.
  - Scale parameters *correctly specified* (CP $\approx$ 95%)
  - Shape parameters *good enough* (CP $>$ 90%).
- **Convergence Rate** (right figure): Increases with less censoring.
  - **Caution**: Dips below 95% with more than 30% censoring.



- On the **left** figure, we show the impact of **right-censoring** on the **coverage probability**.
- In the **bold** red line, we show the **mean** coverage for the scale parameters.
- It shows that the **coverage** is **correctly specified** across all censoring levels.
- In the **bold** blue line, we show the **mean** coverage for the shape parameters. They are **acceptable**, with coverage above 90%.
- In the **right** figure, we show the **convergence rate** for the MLE.
- At more than 30% censoring, the convergence rate dips below 95%.
- Combined with moderate failure masking and small samples, we suggest **caution** in interpreting the results.

Key Takeaways: Right-Censoring

Right-censoring has a notable impact on the MLE:

- **MLE Precision**:
  - Improves notably with reduced right-censoring levels.
  - More reliable components benefit more from reduced right-censoring.
- **Bias**:
  - MLEs show positive bias, but decreases with reduced right-censoring.
- **Convergence Rates**:
  - MLE convergence rate improves with reduced right-censoring.
  - Dips: $< 95\%$ at $> 30\%$ right-censoring.

BCa confidence intervals show good empirical coverage.

- CIs offer reliable *empirical coverage*.
- Scale parameters *correctly specified* across all right-censoring levels.

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Scenario: Impact of Failure Masking

Assessing the impact of the failure masking level on MLE and CIs.

- **Bernoulli Masking Probability**: Vary Bernoulli probability p from 10% to 70%.
- **Right-Censoring**: q fixed at 82.5%.
  - Right-censoring occurs with probability $1 - q$: 17.5%.
  - Censoring less prevalent than masking.
- **Sample Size**: n fixed at 90.
  - Small enough to show impact of masking.

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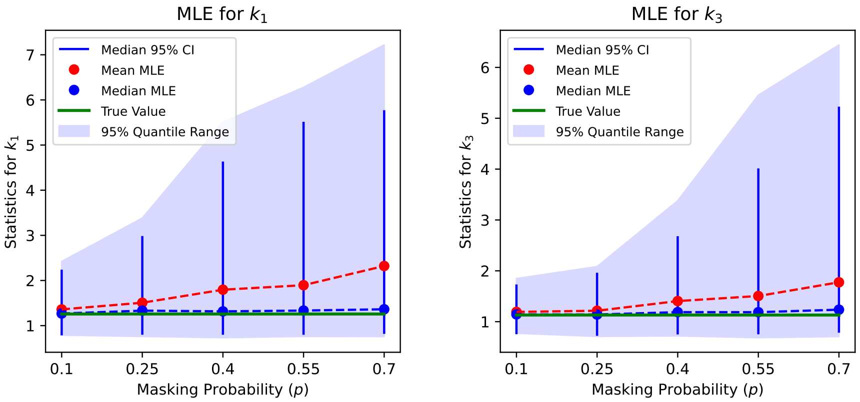
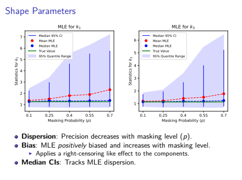
- Here, we assess the impact of **masking levels** on the MLE and confidence intervals.
- We **vary** the Bernoulli masking probability from 10% to 70%.
- We fix the right-censoring probability at 17.5%.
- The **chances** of **censoring** are less than masking.
- We fix the **sample size** at 90, which was small enough to show the impact of masking on the MLE, but large enough so that the convergence rate was reasonable.

Shape Parameters

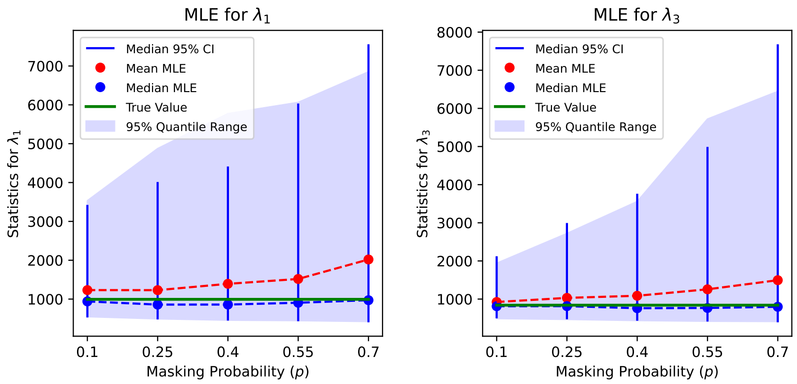


- **Dispersion**: Precision decreases with masking level (p).
- **Bias**: MLE *positively* biased and increases with masking level.
  - Applies a right-censoring like effect to the components.
- **Median CIs**: Tracks MLE dispersion.



- Here, we show the impact of **masking** on the MLE and confidence intervals, this time for the **shape parameters**.
- In **light solid blue**, we show the dispersion of the MLE. We see that as increases with masking level.
- Unlike for the scale parameter, the **more reliable** component on the left has only slightly more dispersion than the other component.
- In the **dashed red line**, we show the bias. The MLE is **positively** biased, and increases with masking level.
- In the **dark blue** vertical lines, we show the median confidence intervals.
- Again, we see they they **track** the MLE's dispersion.

Scale Parameters



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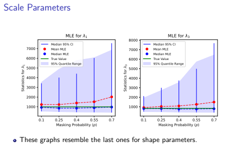
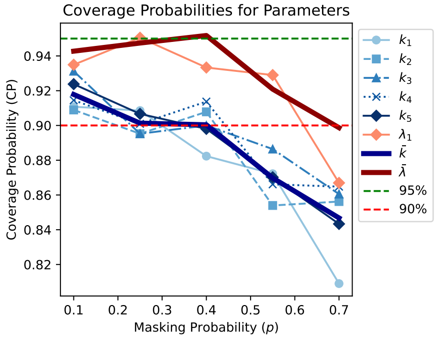
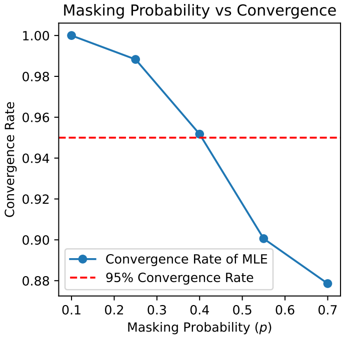


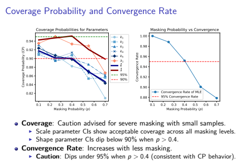
- We see similar results for the **scale parameters**.
- So, let's move on to evaluating the **accuracy** of the **confidence intervals**, where we do continue to see some differences.

Coverage Probability and Convergence Rate





- **Coverage**: Caution advised for severe masking with small samples.
  - Scale parameter CIs show acceptable coverage across all masking levels.
  - Shape parameter CIs dip below 90% when $p > 0.4$.
- **Convergence Rate**: Increases with less masking.
  - **Caution**: Dips under 95% when $p > 0.4$ (consistent with CP behavior).



Key Takeaways: Masking

The masking level of component failures profoundly affects the MLE:

- **MLE Precision**:
  - Decreases with more masking.
- **MLE Bias**:
  - Positive bias is amplified with increased masking.
  - Masking exhibits a right-censoring-like effect.
- **Convergence Rate**:
  - Commendable for Bernoulli masking levels $p \leq 0.4$.
    - *Extreme* masking: some masking occurs 90% of the time at $p = 0.4$.

The BCa confidence intervals show good coverage:

- **Scale** parameters maintain good coverage across all masking levels.
- **Shape** parameter coverage dip below 90% when $p > 0.4$.
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Scenario: Impact of Sample Size

Assess the mitigating affects of sample size.

- **Sample Size**: We vary the same size n from 50 to 500..
- **Right-Censoring**: q fixed at 82.5%
  - 17.5% chance of right-censoring.
- **Bernoulli Masking Probability**: p fixed at 21.5%
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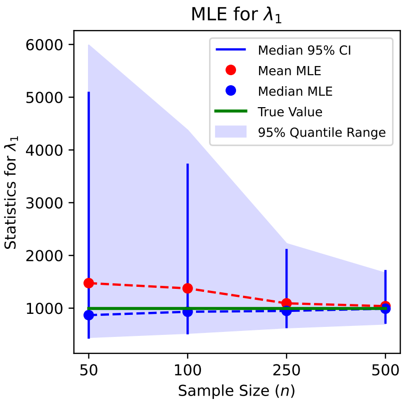
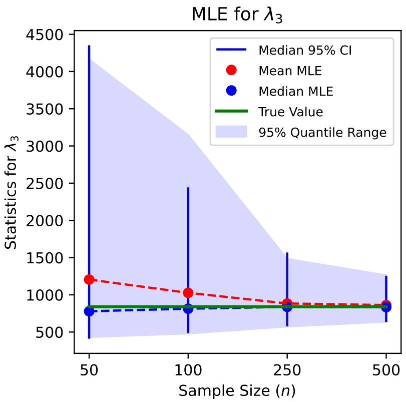
- We assess the impact of the **sample size** on the MLE and confidence intervals.
- We want to see how will it mitigate the challenges from right-censoring and masking.
- We **vary** the sample sie from size 50 to size 500.
- We fix the masking probability at 21.5% and the right-censoring probability at 17.5%.

Scale Parameters





- **Dispersion**: Increasing sample size improves MLE precision.
  - Extremely precise for $n \geq 250$.
- **Bias**: Large *positive* bias initially, but diminishes to zero.
  - Large samples counteract right-censoring and masking effects.
- **Median CIs**: Track MLE dispersion. Very tight for $n \geq 250$.

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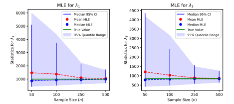

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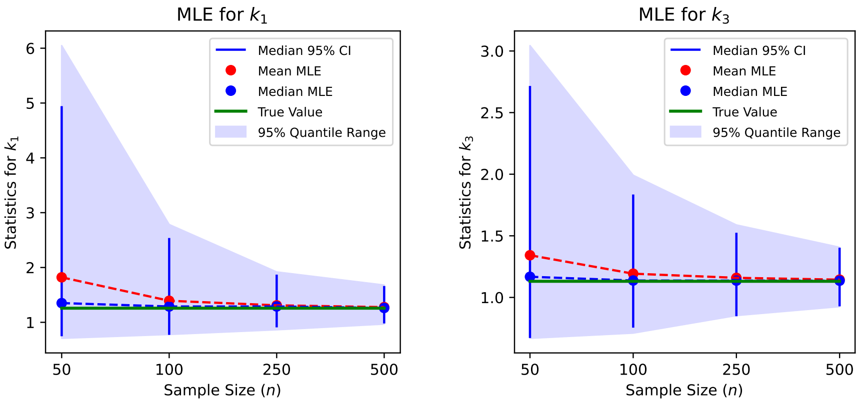

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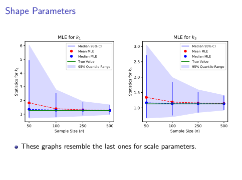

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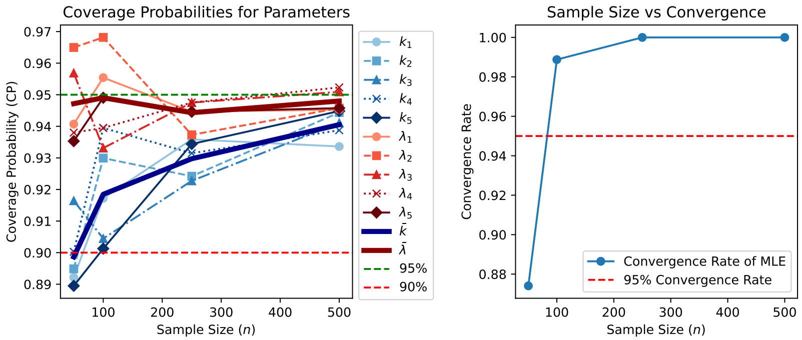
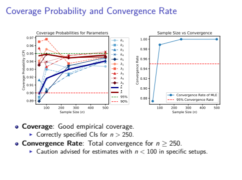
Coverage Probability and Convergence Rate



- **Coverage**: Good empirical coverage.
  - Correctly specified CIs for $n > 250$.
- **Convergence Rate**: Total convergence for $n \geq 250$.
  - Caution advised for estimates with $n < 100$ in specific setups.



Key Takeaways: Sample Size

Sample size has a notable impact on the MLE:

- **Precision**: Very precise for large samples ($n > 200$).
- **Bias**: Diminishes to near zero for large samples.
- **Coverage**: Correctly specified CIs for large samples.
- **Convergence**: Total convergence for large samples.

Summary

Larger samples lead to more accurate, unbiased, and reliable estimations.

- Mitigates the effects of right-censoring and masking.

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Section 4

Overall Conclusion

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- Methods were robust despite masking and right-censoring challenges.

MLE Performance:

- Right-censoring and masking introduce positive bias for our setup.
  - More reliable components are more affected.
- Shape parameters harder to estimate than scale parameters.
- Large samples can mitigate the affects of masking and right-censoring.

BCa Confidence Interval Performance:

- Width of CIs tracked MLE dispersion.
- Good empirical coverage in most scenarios.

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Future Work and Discussion

Directions to enhance learning from masked data:

- **Relax Masking Conditions**: Assess sensitivity to violations and and explore alternative likelihood models.
- **System Design Deviations**: Assess estimator sensitivity to deviations.
- **Homogenous Shape Parameter**: Analyze trade-offs with the full model.
- **Bootstrap Techniques**: Semi-parametric approaches and prediction intervals.
- **Regularization**: Data augmentation and penalized likelihood methods.
- **Additional Likelihood Contributions**: Predictors, etc.

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