

Reliability Estimation in Series Systems

Maximum Likelihood Techniques for Right-Censored and Masked
Failure Data

Alex Towell

Context & Motivation

Reliability in **series systems** is like a chain's strength – determined by its weakest link.

- Essential for system design and maintenance.

Main Goal: Estimate individual component reliability from *failure data*.

Challenges:

- *Masked* component-level failure data.
- *Right-censoring* in system-level failure data.

Our Response:

- Derive techniques to interpret such ambiguous data.
- Aim for precise and accurate reliability estimates for individual components using maximum likelihood estimation (MLE) and bootstrapped (BCa) confidence intervals.

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2023-10-10

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- **Chain Analogy:** Think of a series system as a chain. Its reliability, just like a chain's strength, is determined by its weakest link or component. When any component fails, the whole system does.
- **Reliability Importance:** Understanding the reliability of each component is essential for the design and maintenance of these systems.
- **Data Challenge:** The data we rely on can come with its own challenges. We sometimes encounter ambiguous data like right-censored information or masked component-level failures, where we don't know precisely which component failed.
- **Aim:** Our goal is to interpret such ambiguous data and provide accurate reliability estimates for each component, which includes providing correctly specified 95% CIs, meaning they cover the true parameter value around 95% of the time.
- To obtain good coverage, we bootstrap the confidence intervals

Likelihood Model for series systems.

- Accounts for *right-censoring* and *masked component failure*.

Specifications of Conditions:

- Assumptions about the masking of component failures.
- Simplifies and makes the model more tractable.

Simulation studies:

- Components with *Weibull* lifetimes.
- Evaluate MLE and confidence intervals under different scenarios.

R Library: Methods available on GitHub.

- github.com/queelius/wei.series.md.c1.c2.c3

└ Core Contributions

Our core contributions can be broken down into several parts:

- We've **derived a likelihood model** for series systems that accounts for **right-censoring** and **masked component failure**.
- We've **clarified** the conditions this model assumes about the masking of component failures. These conditions simplify the model and make it more tractable.
- We've **validated** our model with extensive simulations to gauge its performance under various simulation scenarios.
- The simulation is based on components with **Weibull** lifetimes.
- **R Library:** For those interested, we made our methods available in an R Library hosted on GitHub.

Section 1

Series System

Series System



Critical Components: Complex systems often comprise *critical* components. If any component fails, the entire system fails.

- We call such systems *series systems*.
- **Example:** A car's engine and brakes.

System Lifetime is dictated by its shortest-lived component:

$$T_i = \min(T_{i1}, \dots, T_{i5})$$

where:

- T_i is the lifetime of i^{th} system.
- T_{ij} is the j^{th} component of i^{th} system.

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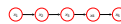
Reliability Estimation in Series Systems

└ Series System

└ Series System

- Many complex systems have **Critical Components** that are essential to their operation.
- If any of these components fail, the entire system fails. We call these **Series System**.
- Think of a **Car**- if the engine or brakes fail, it can't be operated.
- Its **Lifetime** is the lifetime of its **shortest-lived component**.
- For reference, we show the some notation we'll use throughout the talk. T_i is the system's lifetime and T_{ij} is the j^{th} component's lifetime.

Series System



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where:

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Reliability Function

Reliability Function represents the probability that a system or component functions beyond a specified time.

- Essential for understanding longevity and dependability.

Series System Reliability: Product of the reliability of its components:

$$R_{T_i}(t; \theta) = \prod_{j=1}^m R_j(t; \theta_j).$$

- Here, $R_{T_i}(t; \theta)$ and $R_j(t; \theta_j)$ are the reliability functions for the system i and component j , respectively.

Relevance:

- Forms the foundation for most reliability studies.
- Integral to our likelihood model, e.g., right-censoring events.

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Reliability Estimation in Series Systems

└ Series System

└ Reliability Function

- **Reliability Function** The reliability function tells us the chance a component or system functions past a specific time. It's our key metric for longevity.
- **Product of Component Reliability:** In a series system, the overall reliability is the product of its component reliabilities. So, if even one component has a low reliability, it can impact the whole system.
- **Relevance:** Why does this matter to us? This concept is foundational to our studies, especially when we're handling right-censored data.

Reliability Function

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- Essential for understanding longevity and dependability.

Series System Reliability: Product of the reliability of its components:

$$R_T(t; \theta) = \prod_{j=1}^m R_j(t; \theta_j).$$

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Hazard Function: Understanding Risks

Hazard Function: Measures the immediate risk of failure at a given time, assuming survival up to that moment.

- Reveals how the risk of failure evolves over time.
- Guides maintenance schedules and interventions.

Series System Hazard Function is the sum of the hazard functions of its components:

$$h_{T_i}(t; \theta) = \sum_{j=1}^m h_j(t; \theta_j).$$

- Components' risks are additive.

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Reliability Estimation in Series Systems

└ Series System

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Hazard Function: Let's shift focus to the hazard function. Essentially, it's a way to gauge the immediate risk of failure, especially if it's been functioning up until that point.

Series Hazard Function Lastly, the hazard function for a series system is just the sum of the hazard functions of its components.

Additive: We see that the component risks are additive.

Joint Distribution of Component Failure and System Lifetime

Our likelihood model depends on the **joint distribution** of the system lifetime and the component that caused the failure.

- **Formula:** Product of the failing component's hazard function and the system reliability function:

$$f_{K_i, T_i}(j, t; \theta) = h_j(t; \theta_j) R_{T_i}(t; \theta).$$

- **Single Point of Failure:** A series system fails due to one component's malfunction.
- **Representation:**
 - ▶ K_i : Component causing the i^{th} system's failure.
 - ▶ $h_j(t; \theta_j)$: Hazard function for the j^{th} component.

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Reliability Estimation in Series Systems

└ Series System

└ Joint Distribution of Component Failure and System Lifetime

- **Joint Distribution** In our likelihood model, understanding the joint distribution of a system's lifetime and the component that led to its failure is fundamental.
- **Formula:** It is the product of the failing component's hazard function and the system reliability function.
- **Unique Cause:** Which emphasizes that in a series system, failure can be attributed to a single component's malfunction.
- **Notation:** Here, K_i denotes the component responsible for the failure.

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Component Failure & Well-Designed Series Systems

The **marginal probability** of component failure helps predict the cause of failure.

- **Derivation:** Marginalize the joint distribution over the system lifetime:

$$\Pr\{K_i = j\} = E_{\theta} \left[\frac{h_j(T_i; \theta_j)}{h_{T_i}(T_i; \theta_I)} \right].$$

Well-Designed Series System: Components exhibit comparable chances of causing system failures.

- **Relevance:** Our simulation study employs a (reasonably) well-designed series system.

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Reliability Estimation in Series Systems

└ Series System

└ Component Failure & Well-Designed Series Systems

- **Marginal:** We can use this joint distribution to calculate the marginal probability of component failure.
- **Expected Value:** When we do so, we find that it is the expected value of the ratio of component and system hazard functions.
- **Well-Designed:** We say that a series system is *well-designed* if each components has a comparable chance of failing.
- **Relevance:** Our simulation study is based on a reasonably well-designed series system.

$$\Pr\{K_i = j\} = E_{\theta} \left[\frac{h_j(T_i; \theta_j)}{h_{T_i}(T_i; \theta_I)} \right].$$

Section 2

Likelihood Model

Likelihood Model

Likelihood measures how well our model parameters (θ) explain the data. Each system contributes to the **total likelihood** via its *likelihood contribution*:

$$L(\theta|\text{data}) = \prod_{i=1}^n L_i(\theta|\text{data}_i).$$

where **data_i** is the data for the i^{th} system and L_i is its contribution.

Our model handles the following data: **Right-Censored**: Experiment ends before failure (Event Indicator: $\delta_i = 0$). - Contribution is system reliability: $L_i(\theta) = R_{T_i}(\tau; \theta)$. **Masked Failure**: Failure observed, but the failed component is masked by a *candidate set*. More on its contribution later.

System	Right-Censored Lifetime	Event Indicator	Candidate Set
1	1.1	1	{1, 2}
2	5	0	\emptyset

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Reliability Estimation in Series Systems

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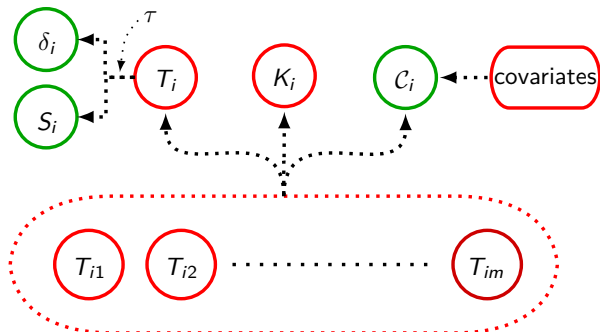
Let's talk about the likelihood model, which is a way of measuring how well our model explains the data.

- **Total likelihood** is the product of the likelihood contributions of each system.
- **Contributions**: Our likelihood model deals with right-censoring and masked cause of failure.
- **Right-Censoring** occurs when the experiment ends before the system fails. Its contribution is just the system reliability, since the event indicator tells us if the system was right-censored.
- **Masking** occurs when we observe a failure but we don't know the precise component cause. Instead, we observe a candidate set of components that could have failure. More on this later.
- Here's an example of observed data.
- **System 1**: We see that the system failed at 1.1. We don't know which component failed but we know it was either component 1 or

Data Generating Process

DGP is underlying process that generates the data:

- **Green** elements are observed, **Red** elements are latent.
- **Right-Censored** lifetime: $S_i = \min(T_i, \tau)$.
- **Event Indicator**: $\delta_i = 1_{\{T_i < \tau\}}$.
- **Candidate Set**: C_i related to components (T_{ij}) and other unknowns.

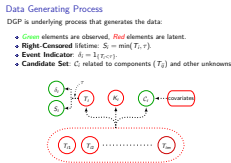


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Reliability Estimation in Series Systems

└ Likelihood Model

└ Data Generating Process



- **DGP**: Let's discuss the data generating process to motivate our model.
- **Graph**: Here's the graph: green is observed and red is latent.
- **Infer**: We don't get to see the red elements, but we can infer most of them from the green elements.
- **Green**: So, let's focus on the green elements.
- **Right-censoring** time is the minimum of the system lifetime and the right-censoring time.
- **Event** indicator is 1 if the system fails before the right-censoring time, 0 otherwise.
- The candidate sets are related to the component lifetimes and many other factors.
- **Difficult** to model. Seek a simple model that is valid under certain assumptions, which we discuss next.

Likelihood Contribution: Masked Failure Conditions

The candidate sets that **mask** the failed component are assumed to satisfy the following conditions:

Candidate Set Contains Failed Component: The candidate set includes the failed component.

Equal Probabilities Across Candidate Sets: The probability of the candidate set is constant across different components within it.

Masking Probabilities Independent of Parameters: The masking probabilities when conditioned on the system lifetime and the failed component aren't functions of the system parameter.

- Reasonable conditions in many industrial settings.

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Reliability Estimation in Series Systems

└ Likelihood Model

└ Likelihood Contribution: Masked Failure Conditions

- The right-censoring contribution is straightforward. But the masked failure contribution is a bit more complicated.
- **Masking** occurs when a system fails but the precise failed component is masked by a candidate set.
- **Tractable:** To make problem more tractable, we introduce certain conditions.
- **Condition 1:** The candidate set always includes the failed component.
- **Condition 2:** The probability of the candidate set is constant across different components within it.
- **Condition 3:** The masking probabilities when conditioned on the system lifetime and the failed component aren't functions of the system parameter.
- **Reasonable:** These conditions are often reasonable.

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• Reasonable conditions in many industrial settings.

Likelihood Contribution: Derivation for Masked Failures

Take the **joint distribution** of T_i , K_i , and C_i and marginalize over K_i :

$$f_{T_i, C_i}(t_i, c_i; \theta) = \sum_{j=1}^m f_{T_i, K_i}(t_i, j; \theta) \Pr_{\theta}\{C_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 1** to get a sum over candidate set:

$$f_{T_i, C_i}(t_i, c_i; \theta) = \sum_{j \in C_i} f_{T_i, K_i}(t_i, j; \theta) \Pr_{\theta}\{C_i = c_i | T_i = t_i, K_i = j\}.$$

Apply **Condition 2** to move probability outside the sum:

$$f_{T_i, C_i}(t_i, c_i; \theta) = \Pr_{\theta}\{C_i = c_i | T_i = t_i, K_i = j'\} \sum_{j \in C_i} f_{T_i, K_i}(t_i, j; \theta).$$

Apply **Condition 3** to remove the probability's dependence on θ :

$$f_{T_i, C_i}(t_i, c_i; \theta) = \beta_i \sum_{j \in C_i} f_{T_i, K_i}(t_i, j; \theta).$$

Result: $L_i(\theta) \propto \sum_{j \in C_i} f_{T_i, K_i}(t_i, j; \theta) = R_{T_i}(t_i; \theta) \sum_{j \in C_i} h_j(t_i; \theta_j).$

Reliability Estimation in Series Systems

└ Likelihood Model

└ Likelihood Contribution: Derivation for Masked Failures

- **Derive:** Here, we derive the likelihood contribution for masked failures.
- **Joint:** To start, we use the joint distribution of the system lifetime, the failed component, and the candidate set.
- **Marginalize:** We marginalize over the failed component, since we don't know which component failed.
- Apply Condition 1 to get a sum over the candidate set instead.
- Apply Condition 2 to move the probability outside the sum.
- Apply condition 3 to remove the probability's dependence on θ .
- The result: the likelihood contribution is proportional to the product of the system reliability and the sum of the component hazards in the masking set.

Likelihood Contribution: Derivation for Masked Failures
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Methodology: Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE): Maximize the likelihood function:

$$\hat{\theta} = \arg \max_{\theta} L(\theta).$$

Solution: Numerically solved system of equations for $\hat{\theta}$:

$$\nabla_{\theta} \log L(\hat{\theta}) = \mathbf{0}.$$

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Reliability Estimation in Series Systems

└ Likelihood Model

└ Methodology: Maximum Likelihood Estimation

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Solution: Numerically solved system of equations for $\hat{\theta}$:

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- **MLE:** We use the standard MLE approach.
- **ArgMax:** We find the parameter values that maximize the log-likelihood function.
- **Solution:** Since there is no closed-form solution, we numerically solve it.

Bootstrap Confidence Intervals (CIs)

Confidence Intervals (CI) quantify the uncertainty in our estimate.

Asymptotic Sampling Distribution of MLE is a popular choice for constructing CIs.

- **Challenge:** Asymptotic distribution may not be accurate for small sample sizes.
 - ▶ Particularly since we're dealing with right-censoring and masking.

Bootstrapped CIs: Resample data and obtain MLE for each.

- Use **percentiles** of bootstrapped MLEs for CIs.

Correctly Specified CIs:

- Desired: Coverage probability near 95%. ($> 90\%$ acceptable.)
- **Challenge:** Actual coverage may deviate.

BCa adjustments counteracts bias and skewness in estimates.

Reliability Estimation in Series Systems

└ Likelihood Model

└ Bootstrap Confidence Intervals (CIs)

- **Goal:** Need a way to measure the uncertainty in our estimate.
- **CIs** are a popular; they help us pin down the likely range of values for our parameters.
- **Bootstrap** the CIs, since there is a lot of bias and variability in our estimate due to the masking and censoring in our small data sets and the asymptotic distribution is not likely to be accurate.
- **Specified:** We want our CIs to be correctly specified, meaning they cover the true parameter value around 95
- **BCa:** But they may be too low or too high; we use the BCa method to adjust for bias and skewness in the estimate. A coverage probability above 90% is acceptable.

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Challenges with MLE on Masked Data

We discovered some challenges with the MLE.

Convergence Issues: Flat likelihood regions were observed due to the ambiguity in the masked data and small sample sizes.

Bootstrap Issues: Bootstrap relies on the Law of Large Numbers.

- It might not represent the true variability for small samples.
- Due to censoring and masking, the effective sample size is reduced.

Mitigation: In simulation study, we discard non-convergent samples for the MLE on original data, but keep all resamples for the bootstrap.

- Helps ensure robustness of the results, while acknowledging the inherent complexities introduced by masking and censoring.
- We report convergence rates in our simulation study.

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Reliability Estimation in Series Systems

└ Likelihood Model

└ Challenges with MLE on Masked Data

Like any model, ours has its challenges:

- **Masking:** Masking and censoring, combined with small sample sizes, can cause flat likelihood regions, which can lead to convergence issues.
- **Small:** For small samples, bootstrapping may not always capture the true variability in the data **Approach:** We take the following approach in our simulation study. **Discard:** We discard non-convergent samples for the MLE on original data but retain all MLEs for the resampled data. **Robustness:** This helps ensure the robustness of our results while acknowledging the inherent complexities introduced by masking and censoring. **Convergence Rate:** We report the convergence rate in our simulation study.

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Section 3

Simulation Study

$$T_{ij} \sim \text{Weibull}(k_j, \lambda_j)$$

- λ_j is the **scale** parameter
- k_j is the **shape** parameter:
 - $k_j < 1$: Indicates infant mortality.
 - $k_j = 1$: Indicates random failures.
 - $k_j > 1$: Indicates wear-out failures.

Recall that for a series system:

- **Series Reliability** is the product of the component reliabilities.
- **Hazard** is the sum of the component hazard functions.
- **Likelihood**: $L(\theta) \propto \prod_{i=1}^n R_{T_i}(t_i; \theta) \left[\sum_{j \in C_i} h_j(t_i; \theta_j) \right]^{\delta_i}$.

Reliability Estimation in Series Systems

└ Simulation Study

└ Series System: Weibull Components

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Series System: Weibull Components

The lifetime of the j^{th} component in the i^{th} system:

$$T_{ij} \sim \text{Weibull}(k_j, \lambda_j)$$

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- **Weibull**: We model a series system with Weibull components.
- **Component Functions**: Hazard and reliability functions are well-known for Weibull.
- **Shape** parameter tells us a lot about the failure characteristics.
- **Increasing**: When the function is increasing, think of it as wearing-out over time.
- **Decreasing**: If it's decreasing, it usually signals some early-life challenges.
- **Series System**: Recall that for a series system, the reliability is the product of the component reliabilities and the hazard function is the sum of the component hazard functions.
- **Likelihood**: The likelihood is the same as before, we've just reproduced it here.

Well-Designed Series System

Simulation study centered around series system with Weibull components:

Component	Shape	Scale	$\Pr\{K_i\}$
1	1.26	994.37	0.17
2	1.16	908.95	0.21
3	1.13	840.11	0.23
4	1.18	940.13	0.20
5	1.20	923.16	0.20

- Based on (Guo, Niu, and Szidarovszky 2013) which studies a 3-component series system.
 - ▶ We add components 4 and 5 to make the system more complex.
- **Probabilities** are comparable: it is *reasonably well-designed*.
 - ▶ Component 1 is most reliable, component 3 is least reliable.
- **Shape** parameters are greater than 1, indicating wear-out failures.

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Reliability Estimation in Series Systems

└ Simulation Study

└ Well-Designed Series System

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• We add components 4 and 5 to make the system more complex.

• **Probabilities** are comparable: it is *reasonably well-designed*.
• Component 1 is most reliable, component 3 is least reliable.

• **Shape** parameters are greater than 1, indicating wear-out failures.

- **Centered:** This study is centered around a series system with Weibull components.
- **Based:** It's based on a paper that studies a 3-component series system.
- **Added:** We added components 4 and 5 to make it more complex.
- **Probability:** We show the probability of each component being the cause of failure.
- **Well-Designed:** The probabilities are comparable, so no weak links. It's reasonably well-designed. Component 1 is most reliable, component 3 is least.
- **Parameters:** We show the shape and scale parameters for each component.
- **Wear-Out:** The shape parameters are greater than 1, indicating components are likely to fail due to wear-out.

Latent Component Lifetimes are generated for each system in the study.

Right-censoring: In our simulation study, we independently control the probability q (quantile) of right-censoring by finding the value τ that satisfies $\Pr\{T_i < \tau\} = q$.

- $S_i = \min(T_i, \tau)$ and $\delta_i = 1_{\{T_i < \tau\}}$.

Masking Component Failures: The *Bernoulli Masking Model* is used to mask component cause of failure, parameterized by masking probability p .

- p chosen independently: at the extremes, if $p = 0$ there is no masking, and if $p = 1$, there is total masking.
- We describe the process and how it satisfies the masking conditions next.

- **Data Generation:** We generate the latent component lifetimes for the series system we just discussed.
- **Observed Data:** Then, we generate the data we actually see, which is based on the component data.
- **Right-Censoring:** We control the probability of right-censoring by finding the value of τ that satisfies the quantile q . Then, we set the right-censoring time to be the minimum of the system lifetime and τ . The event indicator is 1 if the system fails before τ , 0 otherwise.
- **Masking:** We use a Bernoulli masking model to mask the component cause of failure. We parameterize the level of masking by the masking probability, p .
- We parameterize the level of masking by the masking probability, p , which specifies that each non-failed component has a p probability of masking the failed component by including it in the candidate set.

Data Generation: Satisfying Masking Conditions

We generate the candidate sets for each system in the study.

Satisfying Masking Conditions:

- **Condition 1:** The failed component deterministically placed in candidate set.
- **Condition 2:** By using a Bernoulli distribution with a constant probability p for all components, probability of a candidate set is constant as we vary which component failed within set.
- **Condition 3:** Masking only depends on the fixed parameter p and doesn't interact with the system parameter θ .

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Reliability Estimation in Series Systems

└ Simulation Study

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- **Masking:** We use a Bernoulli masking model for masking the failed component.
- This satisfies the masking failure conditions in the following ways:
- **Condition 1:** The failed component is deterministically placed in the candidate set.
- **Condition 2:** The probability of masking is the same for all components, so the probability of the candidate set is constant across components.
- **Condition 3:** The masking probability is independent of the parameters.

Objective: Evaluate the MLE and BCa confidence intervals' performance across various scenarios.

- **MLE Evaluation:**
 - ▶ **Accuracy:** Proximity of the MLE's expected value to the actual value.
 - ▶ **Precision:** Consistency of the MLE across samples.
- **BCa Confidence Intervals Evaluation:**
 - ▶ **Accuracy:** Confidence intervals (CIs) should cover true parameters around 95% of the time.
 - ★ Coverage probability (CP)
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Performance Metrics

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Scenario: Impact of Right-Censoring

Simulation Setup: - Vary the right-censoring quantile (q): 60% to 100%.
- Fixed the masking probability p to 21.5% - Fixed the sample size n to 90.

Background

- **Right-Censoring:** No failure observed.
- **Impact:** Reduces the effective sample size.
- **MLE:** Bias and precision affected by censoring.

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Reliability Estimation in Series Systems

- └ Simulation Study

- └ Scenario: Impact of Right-Censoring

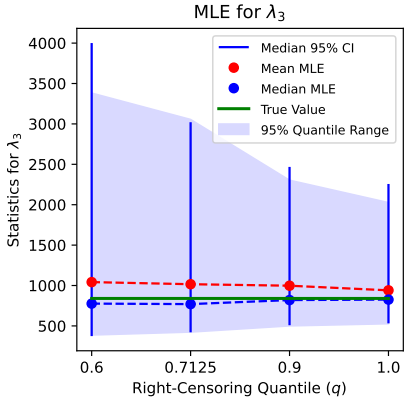
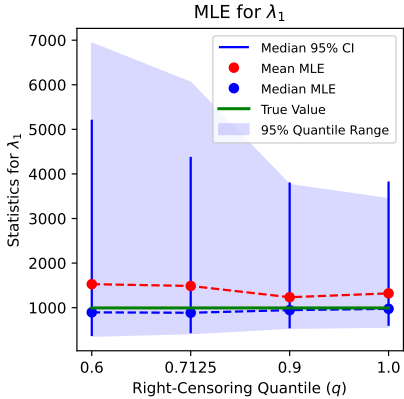
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Scale Parameters



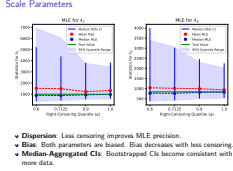
- **Dispersion:** Less censoring improves MLE precision.
- **Bias:** Both parameters are biased. Bias decreases with less censoring.
- **Median-Aggregated CIs:** Bootstrapped CIs become consistent with more data.

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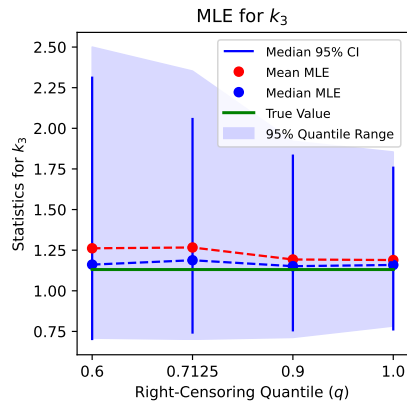
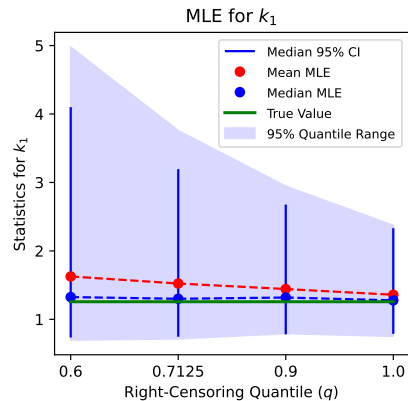
Reliability Estimation in Series Systems

└ Simulation Study

└ Scale Parameters



Shape Parameters

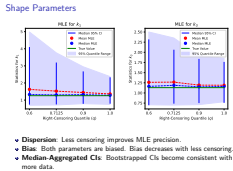


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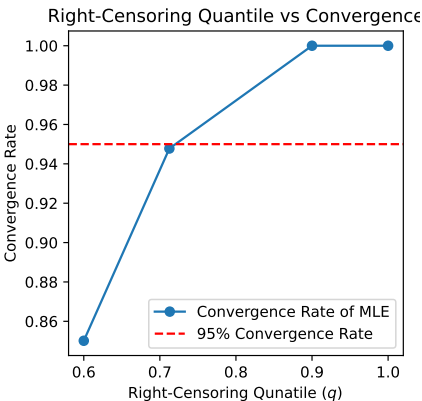
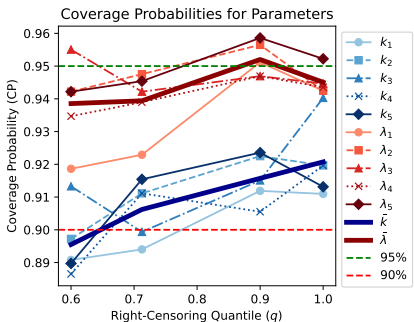
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Reliability Estimation in Series Systems

- └ Simulation Study
- └ Shape Parameters



Coverage Probability and Convergence Rate

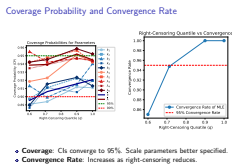


- **Coverage:** CIs converge to 95%. Scale parameters better specified.
- **Convergence Rate:** Increases as right-censoring reduces.

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Reliability Estimation in Series Systems
└ Simulation Study

└ Coverage Probability and Convergence Rate



Conclusion

- MLE precision improves, bias drops with decreased right-censoring.
- BCa CIs perform well, particularly for scale parameters.
- MLE of most reliable component more affected by right-censoring.

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Reliability Estimation in Series Systems

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Impact of Masking Probability

Simulation Setup: - Vary the masking probability p : 0.1 to 0.7. - Fixed the right-censoring quantile q to 0.825. - Fixed the sample size n to 90.

Background

- **Masking** adds ambiguity in identifying the failed component.
- Impacts on MLE:
 - ▶ **Ambiguity:** Higher p increases uncertainty in parameter adjustment.
 - ▶ **Bias:** Similar to right-censoring, but affected by both p and q .
 - ▶ **Precision:** Reduces as p increases.

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Reliability Estimation in Series Systems

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Impact of Masking Probability

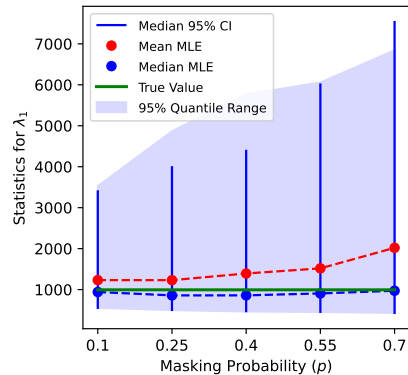
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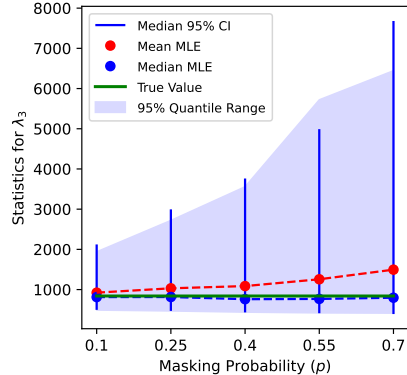
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Scale Parameters

MLE for λ_1



MLE for λ_3

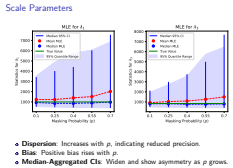


- **Dispersion:** Increases with p , indicating reduced precision.
- **Bias:** Positive bias rises with p .
- **Median-Aggregated CIs:** Widen and show asymmetry as p grows.

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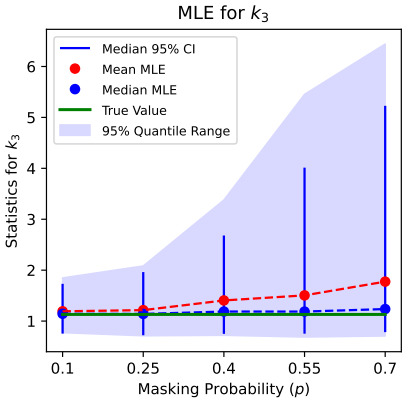
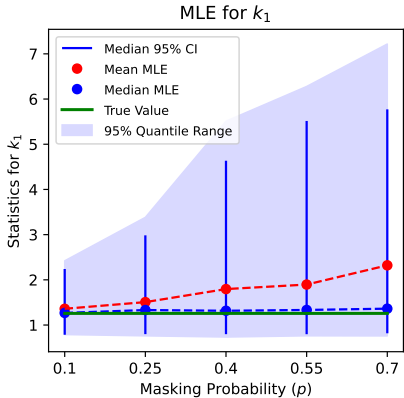
Reliability Estimation in Series Systems

- └ Simulation Study
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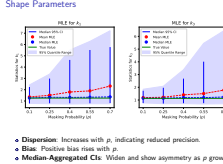


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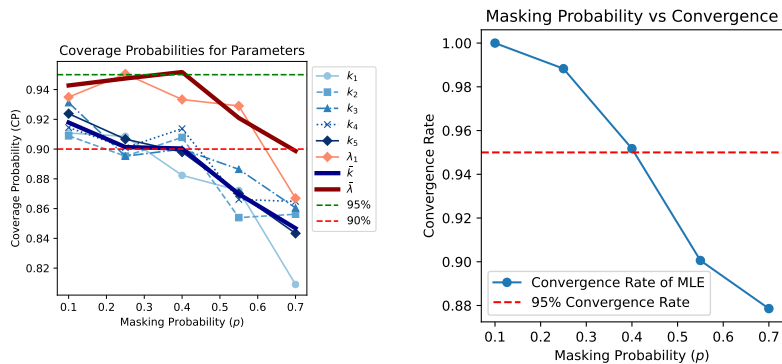
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Reliability Estimation in Series Systems

- Simulation Study
- Shape Parameters



Coverage Probability and Convergence Rate



Coverage: Caution advised for severe masking with small samples.

- Scale parameter CIs are correctly specified up to $p = 0.7$.
- Shape parameter CIs drop below 90% if $p > 0.4$.

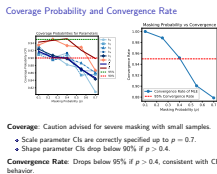
Convergence Rate: Drops below 95% if $p > 0.4$, consistent with CP behavior.

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Reliability Estimation in Series Systems

Simulation Study

Coverage Probability and Convergence Rate



Conclusion

- Masking influences MLE precision, coverage probability, and introduces bias.
- The bias is positive and increases with p , essentially adding a censoring-like effect to the components.
- Despite significant masking, scale parameters have commendable CI coverage.

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Reliability Estimation in Series Systems

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Impact of Sample Size

Assess the impact of sample size on MLEs and BCa CIs.

Simulation Setup: - Vary sample size n : 50 to 500. - Fixed the masking probability p to 0.215. - Fixed the right-censoring quantile q to 0.825.

Background

- **Sample Size:** Number of systems observed.
- **Impact:** More data reduces uncertainty in parameter estimation.
- **MLE:** Mitigates biasing effects of right-censoring and masking.

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Reliability Estimation in Series Systems

└ Simulation Study

└ Impact of Sample Size

Impact of Sample Size

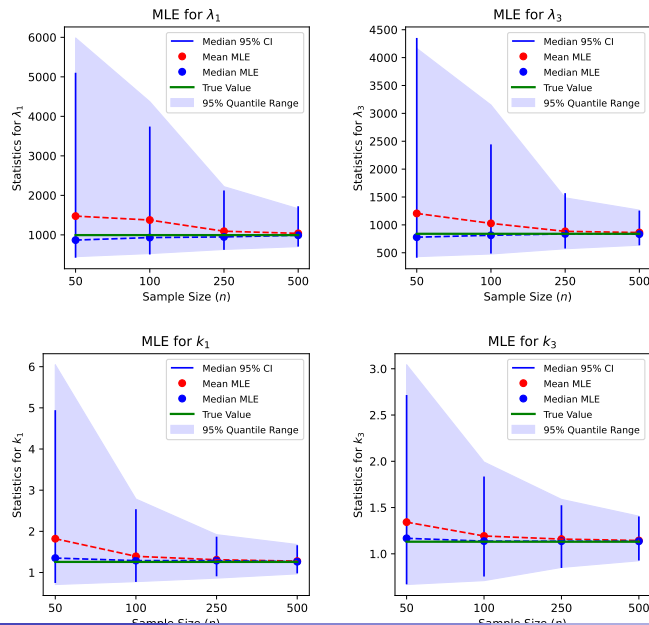
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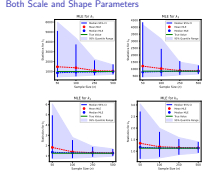
Both Scale and Shape Parameters



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Reliability Estimation in Series Systems

- Simulation Study
- Both Scale and Shape Parameters



- **Dispersion:**

- ▶ Dispersion reduces with n —indicating improved precision.
- ▶ Disparity observed between components k_1, λ_1 and k_3, λ_3 .

- **Bias:**

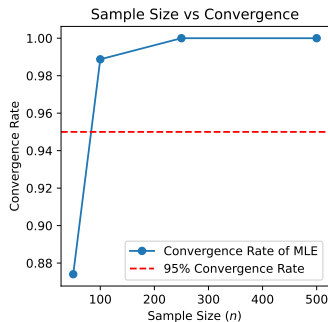
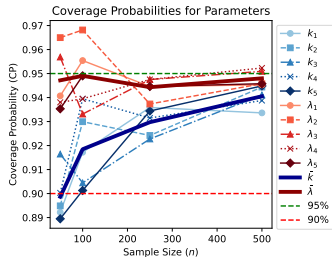
- ▶ High positive bias initially, but diminishes around $n = 250$.
- ▶ Enough sample data can counteract right-censoring and masking effects.

- **Median-Aggregated CIs:**

- ▶ CIs tighten as n grows—showing more consistency.
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Coverage Probability and Convergence Rate



• Calibration:

- ▶ CIs are mostly above 90% across sample sizes.
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- ▶ Scale parameters have better coverage than shape.

• Convergence Rate:

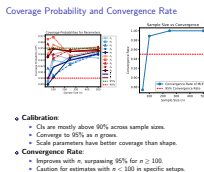
- ▶ Improves with n , surpassing 95% for $n \geq 100$.
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Reliability Estimation in Series Systems

Simulation Study

Coverage Probability and Convergence Rate



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- Sample size significantly mitigates challenges from right-censoring and masking.
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- BCa CIs become narrower and more reliable as sample size increases.

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Reliability Estimation in Series Systems

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Section 4

Conclusion

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Key Findings

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- Methods performed robustly despite masking and right-censoring challenges.

Simulation Insights

- Right-censoring and masking introduce positive bias; more reliable components are most affected.
- Shape parameters harder to estimate than scale.
- Large samples can counteract these challenges.

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Reliability Estimation in Series Systems

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Takeaways

- Framework offers a rigorous method for latent component property estimation from limited observational data.
- Techniques validated to provide practical insights in diverse scenarios.
- Enhanced capability for learning from obscured system failure data.

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Reliability Estimation in Series Systems

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Future Work and Discussion

Directions to enhance learning from masked data:

- **Relax Masking Conditions:** Assess sensitivity to violations and and explore alternative likelihood models.
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- **Homogenous Shape Parameter:** Analyze trade-offs with the full model.
- **Bootstrap Techniques:** Semi-parametric approaches and prediction intervals.
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Reliability Estimation in Series Systems

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└ Future Work and Discussion

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