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Technical Note

# Reliability estimations of components from masked system life data

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**Abstract**

This paper introduces estimations of reliability values for the individual components in a series system using masked system life data. In particular, we compute the maximum likelihood and Bayes estimates of component reliabilities when the system components have constant failure rates. In obtaining Bayes estimates, it is assumed that the component reliabilities are independent random variables having piecewise linear prior distributions. The model is illustrated for a two-component series. A numerical simulation study is presented to show how one can utilize the present approach to compute estimations of component reliabilities for a practical problem. Further, we investigate the comparison between the maximum likelihood and Bayes estimates, based on the respective percentage errors. 

*Keywords*: Masked data; Maximum likelihood estimation; Bayes estimation; Piecewise linear distribution family

## 1. Introduction

One of interesting problems in the reliability analysis, is how to estimate the reliability of individual components in a multi-component system. Such estimates can be extremely useful since they reflect the component reliability after assembly into an operational system, see Ref. [1]. Under appropriate conditions, these estimates can be used to predict the reliability of a new configuration of components.

Consider a system of $J$ components that fails whenever at least one of the components fails. Due to some certain environmental conditions, the exact cause of system failure might be unknown. Instead, it may only be ascertained that the cause of system failure is due to that component which belongs to some subset of the system components. Such type of observation is referred as being masked, see Ref. [1].

In practice, masking occurs frequently in multi-component systems when the exact cause of system failure is unknown. For example, consider a large computer system that has failed. It is often more cost effective to isolate the cause of system failure to a single circuit card that can be quickly replaced with a new one. Indeed such circuit card contains many components. The observable quantities in this case are: (i) the system lifetime, and (ii) a subset of system components that may cause system failure.

Various studies used masked data to estimate the unknown parameters that were indexed to the lifetime distributions of the system components in the case of series systems. In particular, [2] considered the two-component series system when the underlying components have constant failure rates. He derived closed form expressions for the MLE of the components failure rates. Usher and Hodgson [1] and Lin et al. [3] extended Miyakawa's results to a three-component series system. Usher [4] used the masked data to obtain the MLE of those parameters that indexed to the component lifetime distributions in the case of a two-component series system when the lifetime of each component has Weibull distribution. Lin et al. [5] used the Bayes procedure to estimate the parameters that were indexed to the component lifetime distributions. They illustrated their approach on a two-component series system when each component has exponentially distributed lifetime. They have not introduced closed forms for the required Bayes estimates. Instead they presented numerical example to illustrate their approach.

In this paper we use both maximum likelihood and Bayes procedures to estimate component reliabilities in a series system using the masked data when the underlying components have exponentially distributed lifetimes. It is assumed in the Bayes procedure that the component reliabilities are independent random variables having prior distributions that belong to a piecewise linear family of distributions.

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**Nomenclature**

Acronyms (The singular and plural of an acronym are spelled the same.)

| | |
|---|---|
| MLE | maximum likelihood function |
| pdf | probability density function |
| *Rf* | reliability function |

Notation

| | |
|---|---|
| $n$ | number of observed systems |
| $J$ | number of system components |
| $T_i$ | the life time of system $i$ |
| $T_{ij}$ | the life time of component $j$ in system $I$ |
| $f_j(t)$, $R_j(t)$ | the *pdf* and *Rf* of component $j, j = 1, 2, \ldots, J$ |
| $r_j$ | the value of reliability function of component $j$ at time $t_0$. That is, $r_j = R_j(t_0)$ |
| r | set of $r_1$, $r_2$,…,$r_J$ |
| $\lambda_j$ | the failure rate of component $j$ |
| $\lambda$ | set of the unknown parameters $\lambda_1$, $\lambda_2$,…,$\lambda_J$ |
| $S_i$ | set of components that may cause the system failure at time $T_i$ |
| $\zeta(\cdot)$ | indicator function: $\zeta(\text{True}) = 1$, $\zeta(\text{False}) = 0$ |
| Data | the set of available observations $\{(T_i,S_i), i = 1,2,\ldots,n\}$ |
| $\delta_{ij}$ | Kroneker delta: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$ |

The model is illustrated on a two-component series system.

The generalized likelihood function for a series system of $J$ components each having exponentially distributed lifetime is presented in Section 2. We also illustrate its use to find the MLE of component reliabilities. The special case when $J = 2$ is discussed. Section 3 presents the Bayes estimates of component reliabilities. A numerical simulation study is presented in Section 4. Maximum likelihood and Bayes estimates of the reliability of each component are calculated based on the generated data. The two-sided Bayes probability interval estimates of the reliability of components are calculated. Also we investigate the influence of the level of masking of failure causes on the accuracy of the point and interval estimations.

Throughout the paper we consider the following assumptions.

*Assumptions* I.

I.1 The system consists of $J$ independent components connected in series.

I.2 $n$ identical systems are put on the life test. Test is terminated if the system has failed.

I.3 The random variables $T_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, J$, are independent with $T_{1j}$, $T_{2j}$,…,$T_{nj}$ being identically distributed and having a constant failure rate $\lambda_j$.

I.4 The observable quantities that we call data are $\{t_i, S_i\}$.

I.5 Masking is *s*-independent of the failure cause.

## 2. Maximum likelihood estimations

Under the group of assumptions I, the likelihood function becomes, see Ref. [6]

$$L(\text{data}; \boldsymbol{\theta}) = \prod_{i=1}^{n}\left[\sum_{j\in S_i} f_j(t_i) \prod_{l\in J_j} R_l(t_i)\right], \tag{1}$$

$$J_j = 1, 2, \ldots, j-1, j+1, \ldots, J,$$

where $\boldsymbol{\theta}$ denotes a vector of unknown parameters.

For the exponential case, $f_j(t)$ and $R_j(t)$ are given by

$$f_j(t) = \lambda_j \exp(-\lambda_j t) \quad \text{and} \quad R_j(t) = \exp(-\lambda_j t), \qquad \lambda_j > 0; \quad t \geq 0. \tag{2}$$

Substituting from Eq. (2) into Eq. (1), we get

$$L(\text{data}; \lambda) = \prod_{i=1}^{n}\left[\sum_{j\in S_i} \lambda_j \prod_{j=1}^{J} \exp(-\lambda_j t_i)\right] = \exp\left(-T\sum_{j=1}^{J}\lambda_j\right)\prod_{i=1}^{n}\left(\sum_{j\in S_i}\lambda_j\right), \tag{3}$$

where $T = \sum_{i=1}^{n} t_i$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_J)$.

At a fixed time, say $t_0$, the reliability of the component $j$, $(j = 1, 2, \ldots, J)$, becomes

$$r_j = R_j(t) = \exp(-\lambda_j t_0). \tag{4}$$

Using Eq. (4) together with Eq. (3), one can derive the likelihood function as a function of $r$ in the following form:

$$L(\text{data}; r) = \frac{1}{t_0^n}\left[\prod_{j=1}^{J} r_j^{T/t_0}\right]\prod_{i=1}^{n}\left[\sum_{j\in S_i}(-\ln r_j)\right]. \tag{5}$$

Therefore, the log-likelihood function becomes

$$\ln L = -n \ln t_0 + \frac{T}{t_0}\sum_{j=1}^{J}\ln r_j + \sum_{i=1}^{n}\ln\left(\sum_{j\in S_i}(-\ln r_j)\right). \tag{6}$$

The MLE of $r_1$, $r_2$,…,$r_J$ can be obtained by solving the set of likelihood equations

$$\frac{\partial \ln L}{\partial r_j} = 0, \qquad j = 1, 2, \ldots, J. \tag{7}$$

Using Eqs. (6) and (7), one can deduce the set of likelihood equations in the following form:

$$\frac{T}{t_0}\sum_{m=1}^{J}\frac{\delta_{mj}}{r_m} + \sum_{i=1}^{n}\frac{\sum_{m\in S_i}\frac{\delta_{mj}}{r_m}}{\sum_{m\in S_i}\ln r_m} = 0, \qquad j = 1, 2, \ldots, J. \tag{8}$$

The MLE of component reliabilities $\hat{r}_1$, $\hat{r}_2$,…,$\hat{r}_J$ can be obtained by solving the set of equations given by Eq. (8) with respect to $r_1, r_2, \ldots, r_J$. As seen such set of equations has no closed form solution in the general case. For this reason let us consider the simple case of $J = 2$. In this case, we need the following supplementary assumptions. Suppose $n$ of such system were put on the life test. Let $n_1$ and $n_2$ denote the number of system failures for which the components 1 and 2 cause the failure, respectively. It means that $n_1$ and $n_2$ are the number of observations where $S_i = \{1\}$ and $S_i = \{2\}$, respectively. Let $n_{12}$ denote the number of masked observations, that is, $n_{12}$ represents the number of observations where $S_i = \{1,2\}$.

Based on the above notations, the likelihood and log-likelihood functions, respectively, become

$$L(\text{data}; r) = \frac{1}{t_0^n} r_1^{T/t_0} r_2^{T/t_0} (-\ln r_1)^{n_1} (-\ln r_2)^{n_2} (-\ln r_1 - \ln r_2)^{n_{12}}, \quad (9)$$

and

$$\ln L = -n \ln t_0 + \frac{T}{t_0}[\ln r_1 + \ln r_2] + n_1 \ln(-\ln r_1) + n_2 \ln(-\ln r_2) + n_{12} \ln(-\ln r_1 - \ln r_2).$$

Then the set of likelihood equations reduces to

$$\frac{T}{t_0} + \frac{n_j}{\ln r_j} + \frac{n_{12}}{\ln r_1 + \ln r_2} = 0, \qquad j = 1,2. \quad (10)$$

By solving the above set of equations with respect to $r_1$ and $r_2$, we can obtain their MLE in the following closed form:

$$\hat{r}_j = \exp\left\{-\frac{n n_j t_0}{(n_1 + n_2)T}\right\}, \quad (11)$$

$j = 1,2$ and $n = n_1 + n_2 + n_{12}$.

These estimators can be obtained by using the MLE of $\lambda_j$, introduced by Usher and Hodgson [1], and the relations between $r_j$ and $\lambda_j$ given by Eq. (4).

The set of solutions given by Eq. (11) is not defined for that case in which the data is completely masked ($n_1 = 0$, $n_2 = 0$ and $n_{12} = n$).

## 3. Bayes analysis

As we have noted in Section 2, the MLE of the component reliabilities are undefined in the case in which the available data is completely masked. For this reason, we look for another procedure that permits us to get estimations of component reliabilities for all available data types. As we shall see, the Bayes approach introduces not only estimations of component reliabilities using any type of masked data but also gives such estimations in closed forms.

To present the Bayes analysis we need the following additional assumptions.

*Assumptions* II

II.1 $r_1, r_2, \ldots, r_J$ are independent random variables.

II.2 The prior pdf of $r_j$, $j = 1,2,\ldots,J$, belongs to a family of piecewise linear distributions.

The main motivation for using such family of piecewise linear distributions is that, while permitting a variety of prior 'shapes' depending upon the number and positioning of the line segments, the resulting Bayes point and interval estimators for the reliability can be expressed in terms of standard incomplete gamma functions.

Let $g_j$ be the pdf of $r_j$, $j = 1,2,\ldots,J$.The assumption II.2 means that $g_j$ has the following form, see Ref. [7]:

$$g_j(u) = \begin{cases} m_{j,i}u + \tau_{j,i}, & \text{for } u \in (a_{j,i}, a_{j,i+1}],\ i = 1,2,\ldots,k-1; \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

where $0 \le a_{j,1} \le a_{j,2} \le \cdots \le a_{j,k} \le 1$, $m_{j,i} = (v_{j,i+1} - v_{j,i})/(a_{j,i+1} - a_{j,i})$, $i = 1,2,\ldots,k-1$, $\tau_{j,i} = v_{j,i} - m_{j,i}a_{j,i}$; $v_{j,i} = b_{j,i}/A_j$, $b_{j,i} \ge 0$, $i = 1,2,\ldots,k$

$$A_j = \frac{1}{2}\sum_{i=1}^{k-1} (b_{j,i+1} + b_{j,i})(a_{j,i+1} - a_{j,i}). \quad (13)$$

For a fixed $j$, ($j = 1,2,\ldots,J$), and given the coordinates of the line segments $(a_{j,i}, b_{j,i})$, $i = 1,2,\ldots,k$, one can use Eqs. (12) and (13) to write $g_j$ in the following form:

$$g_j(u) = \sum_{i=1}^{k-1} (m_{j,i}u + \tau_{j,i})\zeta(u \in (a_{j,i}, a_{j,i+1}]). \quad (14)$$

Based on assumption II.2, the joint prior pdf of ($r_1$, $r_2,\ldots,r_J$) becomes

$$g(r_1, r_2, \ldots, r_J) = \prod_{J=1}^{J}\sum_{i=1}^{k-1} (m_{j,i}r_j + \tau_{j,i})\zeta(r_j \in (a_{j,i}, a_{j,i+1}]). \quad (15)$$

In what follows, we shall illustrate the methodology of obtaining the Bayes estimates of the component reliabilities on two component series system.

### 3.1. Two-component system

Theorem 1 gives the form of the joint posterior pdf of $r = (r_1, r_2)$ given the available observation. The proof is presented in the Appendix A.

**Theorem 1.** *Under assumptions I and II, the joint posterior pdf of r becomes*

$$g(r_1, r_2|\text{data}) = \frac{f_N(\text{data}; r_1, r_2)}{f_d(\text{data})}, \quad (16)$$

*where*

$$f_N(\text{data}; r_1, r_2) = (r_1 r_2)^{T/t_0}(-\ln r_1)^{n_1}(-\ln r_2)^{n_2}(-\ln r_1 - \ln r_2)^{n_{12}} \times \prod_{j=1}^{2}\sum_{i=1}^{k-1} (m_{j,i}r_j + \tau_{j,i})\zeta(r_j \in (a_{j,i}, a_{j,i+1}]), \quad (17)$$

$$f_d(\text{data}) = \sum_{l=0}^{n_{12}} \binom{n_{12}}{l} I_1(l)I_2(l). \quad (18)$$

*$I_1$ and $I_2$ are given in Appendix A.*

Theorem 2 establishes the Bayes estimates of $r_1$ and $r_2$. The proof for this theorem is given in Appendix A.

**Theorem 2.** *Under the squared error loss, the Bayes estimates of* $r_1$ and $r_2$ *are*

$$\hat{r}_1 = \frac{1}{f_d(\text{data})}\sum_{l=0}^{n_{12}}\binom{n_{12}}{l} I_3(l)I_2(l) \quad \text{and}$$
$$\hat{r}_2 = \frac{1}{f_d(\text{data})}\sum_{l=0}^{n_{12}}\binom{n_{12}}{l} I_1(l)I_4(l), \tag{19}$$

*where* $I_i$, $i = 1,2,3,4$ *are give in Appendix A.*

Corollary 1 gives the marginal posterior probability density function of $r_j$, $(j = 1,2)$. The proof of this corollary could be simply derived from Theorem 1.

**Corollary 1.** *The marginal posterior pdf of* $r_j$, $(j = 1,2)$, *has the form*

$$g_j(r_j|\text{data}) = \frac{r_j^{T/t_0} g_j(r_j)}{f_d(\text{data})}\sum_{l=0}^{n_{12}}\binom{n_{12}}{l} I_{1+\delta_{1j}}(l)h_j^{(l)}(r_j), \tag{20}$$

$r_j \in (0,1)$,

*where*

$h_j^{(l)}(r_j) = (\ln r_j)^{n_j + l\delta_{ij} + (n_{12}-l)\delta_{2j}}$.

*Bayesian interval estimation.* The Bayesian approach to interval estimation is more direct than the classical approach based on confidence intervals [8]. Once the marginal posterior pdf of $r_j$, $(j = 1,2)$, given the available data has been obtained, a symmetric $100(1-\alpha)\%$ two-sided Bayes probability interval [$100(1-\alpha)\%$ TBPI] estimate of $r_j$ is easily obtained by solving the two equations

$$\int_0^{u_j} g_j(r_j|\text{data})\,\mathrm{d}r_j = \alpha/2 \quad \text{and} \quad \int_{v_j}^1 g_j(r_j|\text{data})\,\mathrm{d}r_j = \alpha/2 \tag{21}$$

for $u_j$ and $v_j$, so that $P(u_j < r_j < v_j) = 1-\alpha$.

Based on the above introductory and using the marginal pdf of $r_j$ we can prove Corollary 2.

**Corollary 2.** *The* $100(1-\alpha)\%$ *TBPI estimate of* $r_j$, $(j = 1,2)$, *is the solution of the following system of equations with respect to* $u_j$ *and* $v_j$

$$\frac{1}{f_d(\text{data})}\sum_{l=0}^{n_{12}}\binom{n_{12}}{l} I_{1+\delta_{1j}}(l) \times \int_0^{u_j} r_j^{T/t_0}(\ln r_j)^{n_j + l\delta_{1j} + (n_{12}-l)\delta_{2j}} g_j(r_j)\,\mathrm{d}r_j = \alpha/2,$$
$$\frac{1}{f_d(\text{data})}\sum_{l=0}^{n_{12}}\binom{n_{12}}{l} I_{1+\delta_{1j}}(l) \times \int_0^{v_j} r_j^{T/t_0}(\ln r_j)^{n_j + l\delta_{1j} + (n_{12}-l)\delta_{2j}} g_j(r_j)\,\mathrm{d}r_j = 1-\alpha/2. \tag{22}$$

Unfortunately, the system of equations given by Eq. (22) has no closed form solution in $u_j$ and $v_j$. Therefore a numerical technique method is required to solve such system. In Section 3, we calculate 90% TBPI estimates of $r_j$, $(j = 1,2)$, for a numerical simulation example.

Similarly we can extend the obtained results to a series system consists of more than two components.

## 4. Simulation study and conclusion

To illustrate how one can utilize the obtained theoretical results, we present a numerical example. In this example we consider a series system of two independent components each has exponentially distributed lifetimes. To simulate data it is assumed that 30 ($n = 30$) independent and identical such systems were put on the life test. The lifetime of each system and the subset of components that may cause system failure were observed. The true cause of system failure was found by observing the minimum lifetime of the two components. The simulated data are presented in Table 1. These data were generated when the lifetime of the system components are exponentially distributed with $\lambda_1 = 0.18$ and $\lambda_2 = 0.09$. The total time on test was $T = \sum_{i=1}^{30} t_i = 108.052$.

The exact value of $r_1$ and $r_2$ at time $t_0 = 1.5$ are 0.763 and 0.874, respectively. According to Eq. (11), the MLE of $r_1$ and $r_2$ at time $t_0 = 1.5$ are calculated.

To get the Bayes estimates of $r_1$ and $r_2$ at time $t_0 = 1.5$, we assume that the prior pdf of $r_1$ and $r_2$ are

$$g_1(r_1) = \begin{cases} 0.507r_1 + 0.225 & \text{for } 0.123 < r_1 \le 0.280, \\ 2.603r_1 - 0.360 & \text{for } 0.280 < r_1 \le 0.426, \\ 0.807r_1 + 0.405 & \text{for } 0.426 < r_1 \le 0.517, \\ 17.48r_1 - 8.216 & \text{for } 0.517 < r_1 \le 0.550, \\ 1.359r_1 + 0.644 & \text{for } 0.550 < r_1 \le 0.682, \\ 3.785r_1 - 1.011 & \text{for } 0.682 < r_1 \le 0.722, \\ 45.87r_1 - 31.395 & \text{for } 0.722 < r_1 \le 0.732, \\ 0.192r_1 + 2.044 & \text{for } 0.732 < r_1 \le 0.835, \\ 0.021r_1 + 2.187 & \text{for } 0.835 < r_1 \le 0.949, \\ 0 & \text{for } r_1 < 0 \text{ or } r_1 > 1, \end{cases}$$

and

$$g_2(r_2) = \begin{cases} 1.818r_2 - 0.046 & \text{for } 0.027 < r_2 \le 0.196, \\ 1.017r_2 + 0.112 & \text{for } 0.196 < r_2 \le 0.456, \\ 7.366r_2 - 2.784 & \text{for } 0.456 < r_2 \le 0.472, \\ 17.22r_2 - 7.431 & \text{for } 0.472 < r_2 \le 0.501, \\ 4.068r_2 - 0.845 & \text{for } 0.501 < r_2 \le 0.573, \\ 0.160r_2 + 1.393 & \text{for } 0.573 < r_2 \le 0.739, \\ 3.897r_2 - 1.369 & \text{for } 0.739 < r_2 \le 0.839, \\ 20.30r_2 - 15.14 & \text{for } 0.839 < r_2 \le 0.847, \\ 1.085r_2 + 1.136 & \text{for } 0.847 < r_2 \le 0.983, \\ 0 & \text{for } r_2 < 0 \text{ or } r_2 > 1. \end{cases}$$

Table 1
Simulated system-lifetime data for the numerical example

| System I | Lifetime $T_i$ | Masking-level of failure causes (%)$S_i$ | | | | |
|---|---|---|---|---|---|---|
| | | 0 | 10 | 30 | 50 | 70 |
| 1 | 10.87 | {2} | {2} | {1,2} | {1,2} | {1,2} |
| 2 | 4.331 | {2} | {2} | {2} | {2} | {2} |
| 3 | 2.979 | {1} | {1} | {1,2} | {1,2} | {1,2} |
| 4 | 5.827 | {1} | {1} | {1} | {1} | {1} |
| 5 | 1.083 | {1} | {1} | {1} | {1,2} | {1,2} |
| 6 | 1.969 | {2} | {2} | {2} | {2} | {2} |
| 7 | 1.899 | {1} | {1,2} | {1,2} | {1,2} | {1,2} |
| 8 | 3.290 | {2} | {2} | {1,2} | {1,2} | {1,2} |
| 9 | 8.677 | {2} | {2} | {2} | {1,2} | {1,2} |
| 10 | 3.279 | {2} | {2} | {2} | {2} | {2} |
| 11 | 0.064 | {1} | {1} | {1} | {1} | {1} |
| 12 | 3.421 | {1} | {1} | {1,2} | {1,2} | {1,2} |
| 13 | 6.199 | {2} | {2} | {2} | {2} | {1,2} |
| 14 | 3.510 | {1} | {1} | {1} | {1,2} | {1,2} |
| 15 | 2.822 | {1} | {1} | {1,2} | {1,2} | {1,2} |
| 16 | 7.328 | {1} | {1} | {1} | {1} | {1,2} |
| 17 | 3.174 | {2} | {2} | {2} | {2} | {2} |
| 18 | 15.91 | {1} | {1} | {1} | {1,2} | {1,2} |
| 19 | 1.357 | {1} | {1} | {1} | {1} | {1} |
| 20 | 3.634 | {1} | {1,2} | {1,2} | {1,2} | {1,2} |
| 21 | 0.736 | {1} | {1} | {1} | {1} | {1,2} |
| 22 | 0.251 | {1} | {1} | {1} | {1} | {1} |
| 23 | 3.430 | {1} | {1} | {1} | {1} | {1} |
| 24 | 4.289 | {1} | {1} | {1} | {1,2} | {1,2} |
| 25 | 0.377 | {2} | {1,2} | {1,2} | {1,2} | {1,2} |
| 26 | 1.383 | {1} | {1} | {1} | {1} | {1} |
| 27 | 0.018 | {1} | {1} | {1,2} | {1,2} | {1,2} |
| 28 | 2.733 | {1} | {1} | {1} | {1} | {1,2} |
| 29 | 2.243 | {2} | {2} | {2} | {2} | {1,2} |
| 30 | 0.968 | {1} | {1} | {1,2} | {1,2} | {1,2} |

Then the Bayes estimates of $r_1$, $r_2$ can be calculated by using Eq. (19).

As a measure of accuracy of the obtained estimates, we calculate the respective percentage errors that are given by

$$\mathrm{PE} = \frac{|\text{estimated value} - \text{exact value}|}{\text{exact value}} 100\%.$$

The Bayes and MLE of $r_1$, $r_2$ at different levels of masking 0% (no masking), 10, 30, 50, 70% of failure causes and their respective percentage errors are presented in Table 2. The author would like to recall that the data of two-component system is masked if $S_i = \{1, 2\}$, it means that either component 1 or 2 may cause the system failure. Then for example, 10% masking data can be obtained by randomly masking three observations among the 30 ($n = 30$) observations, as shown in column 4 in Table 1.

The joint posterior pdf of $r_1$, $r_2$ is obtained according to Eq. (16) as an example when $(n_1, n_2, n_{12}) = (12, 8, 10)$. Then the marginal posterior pdf of $r_1$ and $r_2$ could be derived according to Eq. (20). Figs. 1 and 2 show the prior and marginal posterior pdf of $r_1$ and $r_2$, respectively.

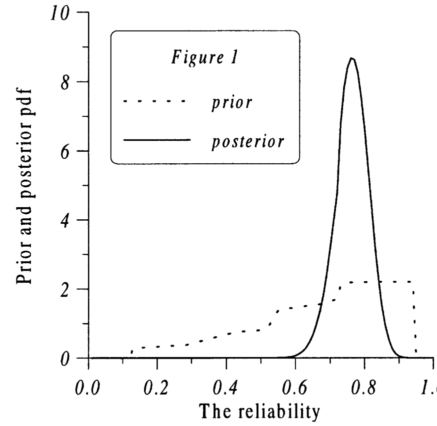


Fig. 1.

The 90% TBPI estimates of $r_j$, ($j = 1, 2$), are obtained by solving the system (22) with respect to the lower limit $u_j$ and upper limit $v_j$. The 90% TBPI estimates of $r_j$ and the respective length of the interval, defined by $\mathrm{IL}_j = v_j - u_j$, at different levels of masking of failure causes are presented in Table 3.

The results presented in Table 2 show that the estimated reliability of $r_1$ increases as the masking level is increased while that of $r_2$ decreases as the level of masking is increased. To explain such variations recall that the exact value of $r_1$ is 0.763. When there is no masking the Bayes estimate and MLE of $r_1$ are 0.766 and 0.768, respectively, which are both greater than the exact value. When the masking level is increased these estimates get even larger as one would expect since increasing masking level yields less accurate estimates. Similarly, the exact value of $r_2$ is 0.874 and the Bayes estimate and MLE are 0.853 and 0.858, respectively, which are both smaller than the exact value. When the masking level is increased these estimates become even smaller as one would expect since increasing masking level yields less accurate estimates.

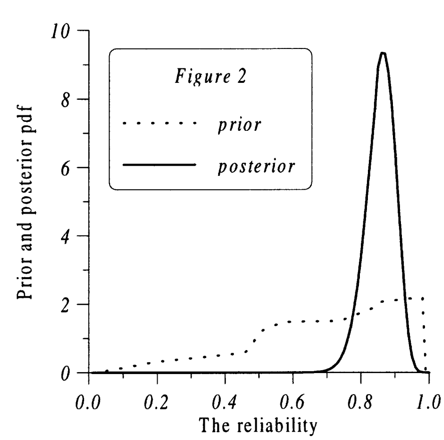


Fig. 2.

Table 2
Estimations of $r_1$, $r_2$ and their respective percentage errors

| Masking level (%) | $n_{12}$ | $n_1$ | $n_2$ | Bayes | | | | ML | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Estimation | | PE | | Estimation | | *PE* | |
| | | | | $r_1$ | $r_2$ | $r_1$ | $r_2$ | $r_1$ | $r_2$ | $r_1$ | $r_2$ |
| 0 | 0 | 19 | 11 | 0.766 | 0.853 | 0.393 | 2.403 | 0.768 | 0.858 | 0.655 | 1.831 |
| 10 | 3 | 17 | 10 | 0.767 | 0.852 | 0.524 | 2.517 | 0.769 | 0.857 | 0.786 | 1.945 |
| 30 | 10 | 12 | 8 | 0.777 | 0.841 | 1.835 | 3.776 | 0.779 | 0.847 | 2.097 | 3.089 |
| 50 | 15 | 8 | 7 | 0.796 | 0.820 | 4.325 | 6.178 | 0.801 | 0.823 | 4.980 | 5.835 |
| 70 | 20 | 4 | 6 | 0.834 | 0.782 | 9.305 | 10.53 | 0.847 | 0.779 | 10.01 | 10.89 |

Table 3
$100(1-\alpha)\%$ TBPI estimates of $r_1$ and $r_2$ and respective lengths

| Masking level (%) | 90% TBPI estimate of | | $\mathrm{IL}_j = v_j - u_j$ | |
|---|---|---|---|---|
| | $r_1$ | $r_2$ | $r_1$ | $r_2$ |
| 0 | (0.690, 0.834) | (0.784, 0.912) | 0.144 | 0.128 |
| 10 | (0.690, 0.838) | (0.780, 0.911) | 0.148 | 0.131 |
| 30 | (0.696, 0.851) | (0.762, 0.908) | 0.155 | 0.146 |
| 50 | (0.712, 0.876) | (0.731, 0.897) | 0.164 | 0.166 |
| 70 | (0.739, 0.917) | (0.683, 0.874) | 0.178 | 0.191 |

It seems from the above results that:

1. The percentage error associated with point estimation of either $r_1$ or $r_2$ is increasing (as it would expected) with increasing the level of masking of failure causes, see Table 2.
2. The length of 90% TBPI estimate of either $r_1$ or $r_2$ is increasing with increasing the level of masking of failure causes, see Table 3.

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## Appendix A

To prove Theorems 1 and 2 let us introduce the following notation:

$$\Gamma(m,n,a,b) = \int_a^b u^m (\ln u)^n \, \mathrm{d}u, \qquad a,b,m,n > 0. \tag{A1}$$

Using integration by parts we can write

$$\Gamma(m,n,a,b) = (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}} (b^{m+1} - a^{m+1}) + \frac{1}{m+1} \sum_{i=0}^{n-1} (-1)^i [b^{m+1} (\ln b)^{n-i} - a^{m+1} (\ln a)^{n-i}]. \tag{A2}$$

**Proof of Theorem 1.** One can obtain the joint posterior pdf of $r = (r_1, r_2)$, given the observed *data* and assuming the joint prior pdf of $r$, using Bayes theorem, see Ref. [8]

$$g(r_1, r_2|\text{data}) = \frac{L(\text{data}; r_1, r_2) g(r_1, r_2)}{\int_0^1 \int_0^1 L(\text{data}; r_1, r_2) g(r_1, r_2) \, \mathrm{d}r_1 \, \mathrm{d}r_2}, \tag{A3}$$

$0 < r_1, r_2 < 1.$

Substituting from Eqs. (9) and (14) into Eq. (A3) we get

$$g(r_1, r_2|\text{data}) = \frac{f_N(\text{data}; r_1, r_2)}{f_d(\text{data})},$$

where $f_N$ (data;$r_1$,$r_2$) is given by Eq. (17) and

$$f_d(\text{data}) = \int_0^1 \int_0^1 (r_1 r_2)^{T/t_0} (-\ln r_1)^{n_1} (-\ln r_2)^{n_2} (-\ln r_1 - \ln r_2)^{n_{12}} \prod_{j=1}^{2} \sum_{i=1}^{k-1} (m_{j,i} r_j + \tau_{j,i}) \zeta(r_j \in (a_{j,i}, a_{j,i+1}]) \, \mathrm{d}r_1 \, \mathrm{d}r_2.$$

Using the binomial expansion for $(\ln r_1 + \ln r_2)^{n_{12}}$, then

$$f_d(\text{data}) = \sum_{l=0}^{n_{12}} \binom{n_{12}}{l} \int_0^1 \int_0^1 (r_1 r_2)^{T/t_0} (-\ln r_1)^{n_1 + l} \times (-\ln r_2)^{n_2 + n_{12} - l} \prod_{j=1}^{2} \sum_{i=1}^{k-1} (m_{j,i} r_j + \tau_{j,i}) \zeta(r_j \in (a_{j,i}, a_{j,i+1}]) \, \mathrm{d}r_1 \, \mathrm{d}r_2.$$

We can write $f_d$ (data) as follows:

$$f_d(\text{data}) = \sum_{l=0}^{n_{12}} \binom{n_{12}}{l} I_1(l) I_2(l),$$

where $I_j(l)$, $j = 1, 2$, are

$$I_j(l) = \int_0^1 r_j^{T/t_0} (\ln r_j)^{n_j + l\delta_{1j} + (n_{12} - l)\delta_{2j}} \times \left\{ \sum_{i=1}^{k-1} (m_{j,i} r_j + \tau_{j,i}) \zeta(r_j \in (a_{j,i}, a_{j,i+1}]) \right\} \mathrm{d}r_j$$

$$= \sum_{i=1}^{k-1} \int_{a_{j,i}}^{a_{j,i+1}} (m_{j,i} r_j + \tau_{j,i}) r_j^{T/t_0} (\ln r_j)^{n_j + l\delta_{1j} + (n_{12} - l)\delta_{2j}} \, \mathrm{d}r_j.$$

By using Eq. (A1), $I_j(l)$, $j = 1, 2$, becomes

$$I_j(l) = \sum_{i=1}^{k-1} \{ m_{j,i} \Gamma(T/t_0 + 1, n_j + l\delta_{1j} + (n_{12} - l)\delta_{2j}, a_{j,i}, a_{j,i+1}) + \tau_{j,i} \Gamma(T/t_0, n_j + l\delta_{1j} + (n_{12} - l)\delta_{2j}, a_{j,i}, a_{j,i+1}) \}. \quad \text{(A4)}$$

**Proof of Theorem 2.** Under the squared error, the Bayes estimate of $r_j$ is defined as the posterior mean [8]. That is

$$\hat{r}_j = \int_0^1 \int_0^1 r_j g(r_1, r_2 | \text{data}) \, \mathrm{d}r_1 \, \mathrm{d}r_2, \qquad j = 1, 2.$$

Similar to the Proof of Theorem 1, we can derive

$$\hat{r}_1 = \frac{1}{f_d(\text{data})} \sum_{l=0}^{n_{12}} \binom{n_{12}}{l} I_3(l) I_2(l) \qquad \text{and}$$

$$\hat{r}_2 = \frac{1}{f_d(\text{data})} \sum_{l=0}^{n_{12}} \binom{n_{12}}{l} I_1(l) I_4(l),$$

where $I_j(l)$, $j = 3, 4$, are given by

$$I_j(l) = \sum_{i=1}^{k-1} \{ m_{j,i} \Gamma(T/t_0 + 2, n_j + l\delta_{1j} + (n_{12} - l)\delta_{2j}, a_{j,i}, a_{j,i+1}) + \tau_{j,i} \Gamma(T/t_0 + 1, n_j + l\delta_{1j} + (n_{12} - l)\delta_{2j}, a_{j,i}, a_{j,i+1}) \}. \quad \text{(A5)}$$

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