CHAPTER 14

Model Building with Multiple Regression

This chapter introduces tools for building regression models and evaluating the effects on their fit of unusual observations or highly correlated predictors. It also shows ways of modeling variables that badly violate the assumptions of straight line relationships with a normal response variable.

Section 14.1 discusses criteria for selecting a regression model by deciding which of a possibly large collection of variables to include in the model. Section 14.2 introduces methods for checking regression assumptions and evaluating the influence of individual observations. Section 14.3 discusses effects of multicollinearity—such strong "overlap" among the explanatory variables that no one of them seems useful when the others are also in the model. Section 14.4 introduces a generalized model that can handle response variables having distributions other than the normal. Sections 14.5 and 14.6 introduce models for nonlinear relationships.

14.1 MODEL SELECTION PROCEDURES

Social research studies usually have several explanatory variables. For example, for modeling mental impairment, potential predictors include income, educational attainment, an index of life events, social and environmental stress, marital status, age, self-assessment of health, number of jobs held in previous five years, number of relatives who live nearby, number of close friends, membership in social organizations, frequency of church attendance, and so forth.

Usually, the regression model for the study includes some explanatory variables for theoretical reasons. Others may be included for exploratory purposes, to check whether they explain much variation in the response variable. The model might also include terms to allow for interactions. In such situations, it can be difficult to decide which variables to use and which to exclude.

Selecting Explanatory Variables for a Model

One possible strategy may be obvious: Include every potentially useful predictor and then delete those terms not making significant partial contributions at some preassigned α-level. Unfortunately, this usually is inadequate. Because of correlations among the explanatory variables, any one variable may have little unique predictive power, especially when the number of predictors is large. It is conceivable that few,

if any, explanatory variables would make significant *partial* contributions, given that all of the other explanatory variables are in the model.

Here are two general guidelines for selecting explanatory variables: First, include enough of them to make the model useful for theoretical purposes and to obtain good predictive power. Second, as a counterbalance to the first goal, keep the model simple. Having extra variables in the model that add little predictive power, perhaps because of overlapping a lot with the other variables, has disadvantages. The model may be more difficult to interpret because it has many more parameters to be estimated. This can result in inflated standard errors of the parameter estimates and may make it impossible to assess the partial contributions of variables that are important theoretically. To avoid multicollinearity (Section 14.3), it is helpful for the explanatory variables to be correlated with the response variable but not highly correlated among themselves.

Related to this second goal, it is best not to build complex models if the data set is small. If you have only 25 observations, you won't be able to untangle the complexity of relationships among 10 variables. Even with large data sets, it is difficult to build "believable" models containing more than about 10 explanatory variables, and with small to moderate sample sizes (say, 100 or less) it is safest to use relatively few predictors.

In attempting to obtain good predictive power, "Maximize R^2" is not a sensible criterion for selecting a model. Because R^2 cannot decrease as you add variables to a model, this approach would lead you to the most complex model in the set being considered.

Keeping these thoughts in mind, no unique or optimal approach exists for selecting predictors. For k potential predictors, since each can be either included or omitted (two possibilities for each variable), there are 2^k potential subsets. For $k = 2$, for example, there are $2^k = 2^2 = 4$ possible models: one with both x_1 and x_2, one with x_1 alone, one with x_2 alone, and one with neither variable. The set of potential models is too large to evaluate practically if k is even moderate; if $k = 7$ there are $2^7 = 128$ potential models.

Most software contains automated variable selection procedures that scan the explanatory variables to choose a subset for the model. These routines construct a model by sequentially entering or removing variables, one at a time according to some criterion. This takes much less time than fitting and comparing *all* 2^k possible regression models. For any particular sample and set of variables, however, different procedures may select different subsets of variables, and there is no guarantee of selecting a sensible model. Among the most popular automated variable selection methods are *backward elimination*, *forward selection*, and *stepwise regression*.

Backward Elimination

Backward elimination begins by placing all of the predictors under consideration in the model. It deletes one at a time until reaching a point where the remaining variables all make significant partial contributions to predicting y. For most software, the variable deleted at each stage is the one that is the least significant, having the largest P-value in the significance test for its effect.

Specifically, here's the sequence of steps for backward elimination: The initial model contains all potential explanatory variables. If all variables make significant partial contributions at some fixed α-level, according to the usual t test or F test, then that model is the final one. Otherwise, the explanatory variable having the largest P-value, controlling for the other variables in the model, is removed. Next, for the model with that variable removed, the partial contributions of the variables remaining

in the model are reassessed, controlling for the other variables still in the model. If they are all significant, that model is the final model. Otherwise, the variable having the largest P-value is removed. The process continues until each remaining predictor explains a significant partial amount of the variability in y.

EXAMPLE 14.1 Selecting Predictors of Home Selling Price

We refer to the 100 observations in the "house selling price" data file at the text Web site, which were introduced in Example 9.10 on page 278. We use y = selling price of home, with explanatory variables the size of the home (denoted SIZE), annual taxes (TAXES), number of bedrooms (BEDS), number of bathrooms (BATHS), and a dummy variable for whether the home is new (NEW). We use backward elimination with these variables as potential predictors, requiring a variable to reach significance at the $\alpha = 0.05$ level for inclusion in the model.

Table 14.1 shows the first stage of the process, fitting the model containing all the predictors. The variable making the least partial contribution to the model is BATHS. Its P-value ($P = 0.85$) is the largest, and R^2 (not shown in Table 14.1) decreases least by dropping it from the model (from 0.7934 to 0.7933). Although number of bathrooms is moderately correlated with the selling price ($r = 0.56$), the other predictors together explain most of the same variability in selling price. Once those variables are in the model, number of bathrooms is essentially redundant.

TABLE 14.1: Model Fit at Initial Stage of Backward Elimination for Predicting Home Selling Price

Variable	B	Std. Error	t	Sig
(Constant)	4525.75	24474.05		
SIZE	68.35	13.94	4.90	.000
NEW	41711.43	16887.20	2.47	.015
TAXES	38.13	6.81	5.60	.000
BATHS	-2114.37	11465.11	-.18	.854
BEDS	-11259.1	9115.00	-1.23	.220

When we refit the model after dropping BATHS, the only nonsignificant variable is BEDS, having a t statistic of -1.31 and P-value = 0.19. Table 14.2 shows the third stage, refitting the model after dropping BATHS and BEDS as predictors. Each variable now makes a significant contribution, controlling for the others in the model. Thus, this is the final model. Backward elimination provides the prediction equation

$$\hat{y} = -21{,}353.8 + 61.7(\text{SIZE}) + 46{,}373.7(\text{NEW}) + 37.23(\text{TAXES}).$$

Other things being equal, an extra thousand square feet of size increases the selling price by about 62 thousand dollars, and having a new home increases it by about

TABLE 14.2: Model Fit at Third Stage of Backward Elimination for Predicting Home Selling Price

Variable	B	Std. Error	Std. Coeff	t	Sig
(Constant)	-21353.8	13311.49			
SIZE	61.70	12.50	0.406	4.94	.000
NEW	46373.70	16459.02	0.144	2.82	.006
TAXES	37.23	6.74	0.466	5.53	.000

46 thousand. Using standardized variables, the equation is

$$\hat{z}_y = 0.406z_S + 0.144Z_N + 0.464z_T.$$

If we had included interactions in the original model, we would have ended up with a different final model. However, the model given here has the advantage of simplicity, and it has good predictive power ($R^2 = 0.790$, compared to 0.793 with all the predictors). In fact, the adjusted R^2 value (discussed later in this section) $R^2_{adj} = 0.783$ for this model is higher than $R^2_{adj} = 0.782$ for the original model. ■

Forward Selection and Stepwise Regression Procedures

Whereas backward elimination begins with *all* the potential explanatory variables in the model, ***forward selection*** begins with *none* of them. It adds one variable at a time to the model until reaching a point where no remaining variable not yet in the model makes a significant partial contribution to predicting y. At each step, the variable added is the one that is most significant, having the smallest P-value. For quantitative predictors, this is the variable having the largest t test statistic, or equivalently the one providing the greatest increase in R^2.

To illustrate, consider again the data on selling prices of homes. Table 14.3 depicts the process. The variable most highly correlated with selling price is TAXES, so it is added first. Once TAXES is in the model, SIZE provides the greatest boost to R^2, and it is significant ($P = 0.000$), so it is the second variable added. Once both TAXES and SIZE are in the model, NEW provides the greatest boost to R^2 and it is significant ($P = 0.006$), so it is added next. At this stage, BEDS gives the greatest boost to R^2 (from 0.790 to 0.793), but it does not make a significant contribution ($P = 0.194$), so the final model does not include it. In this case, forward selection reaches the same final model as backward elimination.

TABLE 14.3: Steps of Forward Selection for Predicting Home Selling Price

Step	Variables in Model	P-Value for New Term	R^2
0	None	—	0.000
1	TAXES	0.000	0.709
2	TAXES, SIZE	0.000	0.772
3	TAXES, SIZE, NEW	0.006	0.790
4	TAXES, SIZE, NEW, BEDS	0.194	0.793

Once forward selection provides a final model, not all the predictors appearing in it need necessarily be significantly related to y. The variability in y explained by a variable entered at an early stage may overlap with the variability explained by variables added later, so it may no longer be significant. Figure 14.1 illustrates. The figure portrays the portion of the total variability in y explained by each of three predictors. Variable x_1 explains a similar amount of variability, by itself, as x_2 or x_3. However, x_2 and x_3 between them explain much of the same variation that x_1 does. Once x_2 and x_3 are in the model, the unique variability explained by x_1 is minor.

Stepwise regression is a modification of forward selection that drops variables from the model if they lose their significance as other variables are added. The approach is the same as forward selection except that at each step, after entering the new variable, the procedure drops from the model any variables that no longer make

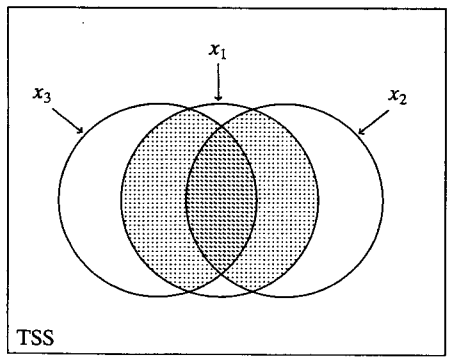


FIGURE 14.1: Variability in y Explained by x_1, x_2, and x_3. Shaded portion is amount explained by x_1 that is also explained by x_2 and x_3.

significant partial contributions. A variable entered into the model at some stage may eventually be eliminated because of its overlap with variables entered at later stages.

For the home sales data, stepwise regression behaves the same way as forward selection. At each stage, each variable in the model makes a significant contribution, so no variables are dropped. For these variables, backward elimination, forward selection, and backward elimination all agree. This need not happen.

Limitations and Abuses of Automatic Selection Procedures

It seems appealing to have a procedure that automatically selects variables according to established criteria. But any variable selection method should be used with caution and should not substitute for careful thought. There is no guarantee that the final model chosen will be sensible.

For instance, when it is not known whether explanatory variables interact in their effects on a response variable, one might specify all the pairwise interactions as well as the main effects as the potential explanatory variables. In this case, it is inappropriate to remove a main effect from a model that contains an interaction composed of that variable. Yet most software does not have this safeguard.

To illustrate, we used forward selection with the home sales data, including the five predictors from above as well as their 10 cross-product interaction terms. The final model had $R^2 = 0.866$, using four interaction terms (SIZE × TAXES, SIZE × NEW, TAXES × NEW, BATHS × NEW) and the TAXES main effect. It is inappropriate, however, to use these interactions as predictors without the SIZE, NEW, and BATHS main effects.

Also, a variable selection procedure may exclude an important predictor that really should be in the model according to other criteria. For instance, using backward elimination with the five predictors of home selling price and their interactions, TAXES was removed. In other words, at a certain stage, TAXES explained an insignificant part of the variation in selling price. Nevertheless, it is the best single predictor of selling price, having $r^2 = 0.709$ by itself. (Refer to step 1 of the forward selection process in Table 14.3.) Since TAXES is such an important determinant of selling price, it seems sensible that any final model should include it as a predictor.

Although P-values provide a guide for making decisions about adding or dropping variables in selection procedures, they are not the *true* P-values for the tests conducted. We add or drop a variable at each stage according to a minimum or maximum P-value, but the sampling distribution of the maximum or minimum of a set of t or F statistics differs from the sampling distribution for the statistic for an a priori chosen test.

For instance, suppose we add variables in forward selection according to whether the P-value is less than 0.05. Even if none of the potential predictors truly affect y, the probability is considerably larger than 0.05 that at least one of the separate test statistics provides a P-value below 0.05 (Exercise 14.48). At least one variable that is not really important may look impressive merely due to chance.

Similarly, once we choose a final model with a selection procedure, any inferences conducted with that model are highly approximate. In particular, P-values are likely to appear smaller than they should be and confidence intervals are likely to be too narrow, since the model was chosen that most closely reflects the data, in some sense. The inferences are more believeable if performed for that model with a new set of data.

Exploratory versus Explanatory (Theory Driven) Research

There is a basic difference between *explanatory* and *exploratory* modes of model selection. In ***explanatory research***, there is a theoretical model to test using multiple regression. We might test whether a hypothesized spurious association disappears when other variables are controlled, for example. In such research, automated selection procedures are usually not appropriate, because theory dictates which variables are in the model.

In ***exploratory research***, by contrast, the goal is not to examine theoretically specified relationships but merely to find a good set of predictors. This approach searches for predictors that give a large R^2, without concern about theoretical explanations. Thus, educational researchers might use a variable selection procedure to search for a set of test scores and other factors that predict well how students perform in college. They should be cautious about giving causal interpretations to the effects of the different variables. For example, possibly the "best" predictor of students' success in college is whether their parents use the Internet for voice communication (with a program such as SKYPE).

In summary, automated variable selection procedures are no substitute for careful thought in formulating models. For most scientific research, they are not appropriate.

Indices for Selecting a Model: Adjusted R^2, PRESS, and C_p

Instead of using an automated algorithm such as backward elimination to choose a method, we could ourselves try to identify a set of potentially adequate models and then use some established criterion to select among them. We have seen that maximizing R^2 is not a sensible criterion for selecting a model, because the most complicated model will have the largest R^2-value. This reflects that fact that R^2 has an upward bias as an estimator of the population value of R^2. This bias is small for large n but can be considerable with small n or with many predictors.

In comparing predictive power of different models, it is often more helpful to use *adjusted* R^2 instead of R^2. This equals

$$R^2_{\text{adj}} = \frac{s_y^2 - s^2}{s_y^2} = 1 - \frac{s^2}{s_y^2},$$

where $s^2 = \Sigma(y - \hat{y})^2/[n - (k + 1)]$ is the estimated conditional variance (i.e., the mean square error, MSE) and $s_y^2 = \Sigma(y - \bar{y})^2/(n - 1)$ is the sample variance of y. This is a less biased estimator of the population R^2. Unlike ordinary R^2, if a term is added to a model that is not especially useful, then R^2_{adj} may even *decrease*.

This happens when the new model has poorer predictive power, in the sense of a larger value of s^2, the MSE. For example, if the new model has the same value of SSE $= \sum(y - \hat{y})^2$, then s^2 will increase because the number of predictors k (which is in the denominator of s^2) has increased.

One possible criterion for selecting a model is to choose the one having the greatest value of R^2_{adj}. (This is, equivalently, the model with smallest MSE value.) Table 14.4 shows the R^2_{adj} values for five models for the house selling price data, in the order in which a model is built by forward selection (reverse order for backward elimination). According to this criterion, the selected model is the one with all the predictors except BATHS, which has $R^2_{adj} = 0.785$.

TABLE 14.4: Model Selection Criteria for Models for Home Selling Price

Variables in Model	R^2	R^2_{adj}	PRESS	C_p
TAXES	0.709	0.706	3.17	36.4
TAXES, SIZE	0.772	0.767	2.73	9.6
TAXES, SIZE, NEW	0.790	0.783	2.67	3.7
TAXES, SIZE, NEW, BEDS	0.793	0.785	2.85	4.0
TAXES, SIZE, NEW, BEDS, BATHS	0.793	0.782	2.91	6.0

Note: Actual PRESS equals value reported times 10^{11}.

Various other criteria have been proposed for selecting a model. Most of these criteria attempt to find the model for which the predicted values tend to be closest to the true expected values, in some average sense. One type of method for doing this uses ***cross-validation***. For a given model, you fit the model using some of the data and then analyze how well its prediction equation predicts the rest of the data. For example, the method described next analyzes how well each observation is predicted when all the observations except that one are used to fit the model.

Suppose we fit one of the models from Table 14.4 to the house selling price data using all the data except observation 1. Using the prediction equation we get, let $\hat{y}_{(1)}$ denote the predicted selling price for observation 1. That is, we find a prediction equation using the data for observations 2, 3, ..., 100, and then we substitute the values of the explanatory variables for observation 1 into that prediction equation to get $\hat{y}_{(1)}$. Likewise, let $\hat{y}_{(2)}$ denote the prediction for observation 2 when we fit the model to observations 1, 3, 4, ..., 100, leaving out observation 2. In general, for observation i, we leave it out in fitting the model and then use the resulting prediction equation to get $\hat{y}_{(i)}$. Then $(y_i - \hat{y}_{(i)})$ is a type of residual, measuring how far observation i falls from the value predicted for it using the prediction equation generated by the other 99 observations.

In summary, for a model for a data set with n observations, we fit the model n times, each time leaving out one observation and using the prediction equation to predict that observation. We then get n predicted values and corresponding prediction residuals. The ***predicted residual sum of squares***, denoted by PRESS, is

$$\text{PRESS} = \sum(y_i - \hat{y}_{(i)})^2.$$

The smaller the value of PRESS, the better the predictions tend to be, in a summary sense. According to this criterion, the best-fitting model is the one with the smallest value of PRESS.

Table 14.4 shows the PRESS values for five models for the house selling price data. According to this criterion, the selected model is the one with predictors TAXES, SIZE, and NEW, which has the minimum PRESS = 2.67. (The y-values were in dollars, so squared residuals tended to be huge numbers, and the actual PRESS values are the numbers reported multiplied by 10^{11}.) This was also the model selected by backward elimination and by forward selection.

A related approach reports a statistic that describes how well each model fits compared to the full model with all the predictors. Roughly speaking, it attempts to find the simplest model that has a relatively small expected value of $[\hat{y} - E(y)]^2$, which measures the distance between a predicted value and the true mean of y at the given values of the explanatory variables. When you have a full model that you believe has sufficient terms as to eliminate important bias, you can use this statistic to search for a simpler model that also has little bias. The statistic is denoted by C_p, where p denotes the number of parameters in the regression model (including the y-intercept). For a given number of parameters p, smaller values of C_p indicate a better fit. For the full model, necessarily $C_p = p$. A simpler model than the full one that has C_p close to p provides essentially as good a fit, apart from sampling error. Models having values of C_p considerably larger than p do not fit as well. In using C_p to help select a model, the goal is to have the smallest number of predictors necessary to give a value of C_p close to p. For that number of predictors, the selected model is the one with the minimum value of C_p.

Consider the models in Table 14.4. The full model shown on the last line of that table has five predictors and $p = 6$ parameters, so it has $C_p = 6.0$. The model removing BATHS has $p = 5$ parameters and has $C_p = 4.0$. The model removing BEDS has $p = 4$ parameters and has $C_p = 3.7$. Since C_p is then close to $p = 4$, this model seems to fit essentially as well as the full model, apart from sampling error. The simpler models listed in the table have C_p considerably larger than p ($C_p = 9.6$ with $p = 3$ and $C_p = 36.4$ with $p = 2$) and provide poorer fits.

Some software does not report PRESS or C_p but does present a measure that has a similar purpose. The ***AIC***, short for ***Akaike information criterion***, also attempts to find a model for which the $\{\hat{y}_i\}$ tend to be closest to $\{E(y_i)\}$ in an average sense. Its formula, not considered here, penalizes a model for having more parameters than are useful for getting good predictions. The AIC is also scaled in such a way that the lower the value, the better the model. The "best" model is the one with smallest AIC. An advantage of AIC it that it is also useful for models that assume nonnormal distributions for y, in which case a sum of squared errors may not be a useful summary.

14.2 REGRESSION DIAGNOSTICS

Once we have selected predictors for a model, how do we know that model fits the data adequately? This section introduces diagnostics that indicate (1) when model assumptions are grossly violated and (2) when certain observations are highly influential in affecting the model fit or inference about model parameters.

Recall that inference about parameters in a regression model makes these assumptions:

- The true regression function has the form used in the model (e.g., linear).
- The conditional distribution of y is normal.
- The conditional distribution of y has constant standard deviation throughout the range of values of the explanatory variables. This condition is called ***homoscedasticity***.
- The sample is randomly selected.

In practice, the assumptions are never perfectly fulfilled, but the regression model can still be useful. It is adequate to check that no assumption is grossly violated.

Examine the Residuals

Several checks of assumptions use the residuals, $y - \hat{y}$. They represent the deviations of the observations from the prediction equation values.

One type of check concerns the normality assumption. If the observations are normally distributed about the true regression equation with constant conditional standard deviation σ, then the residuals should be approximately normally distributed. To check this, plot the residuals about their mean value 0, using a histogram or stem-and-leaf plot. They should have approximately a bell shape about 0.

A standardized version of the residual equals the residual divided by a standard deviation that describes how much residuals vary because of ordinary sampling variability. In regression, this is called[1] a ***studentized residual***. Under the normality assumption, a histogram of these residuals should have the appearance of a standard normal distribution (bell shaped with mean of 0 and standard deviation of 1). If the model holds, studentized residuals between about 2 and 3 in absolute value may be worthy of notice, but about 5% are this large simply by chance.

If a studentized residual is larger than about 3 in absolute value, the observation is a potential outlier and should be checked. If an outlier represents a measurement error, it could cause a major bias in the prediction equation. Even if it is not an error, it should be investigated. It represents an observation that is not typical of the sample data, and it may have too much impact on the model fit. Consider whether there is some reason for the peculiarity. Sometimes the outliers differ from the other observations on some variable not included in the model, and once that variable is added, they cease to be outliers.

EXAMPLE 14.2 Residuals for Modeling Home Selling Price

For the data of Table 9.4 (page 278) with y = selling price, variable selection procedures in Example 14.1 (page 443) suggested the model having predictors SIZE of home, TAXES, and whether NEW. The prediction equation is

$$\hat{y} = -21{,}353.8 + 61.7(\text{SIZE}) + 46{,}373.7(\text{NEW}) + 37.2(\text{TAXES}).$$

Figure 14.2 is a histogram of the studentized residuals for this fit, as plotted by SPSS. No severe nonnormality seems to be indicated, since they are roughly bell shaped about 0. However, the plot indicates that two observations have relatively large residuals. On further inspection, we find that observation 6 had a selling price of \$499,900, which was \$168,747 higher than the predicted selling price for a new home of 3153 square feet with a tax bill of \$2997. The residual of \$168,747 has a studentized value of 3.88. Likewise observation 64 had a selling price of \$225,000, which was \$165,501 lower than the predicted selling price for a non-new home of 4050 square feet with a tax bill of \$4350. Its residual of $-$\$165,501 has a studentized value of -3.93.

A severe outlier on y can substantially affect the fit, especially when the values of the explanatory variables are not near their means. So we refitted the model without

[1]Some software, such as SPSS, reports also a *standardized residual* that is slightly different than this. It divides the ordinary residual by the square root of the mean square error, which is slightly larger than the standard error of the residual.

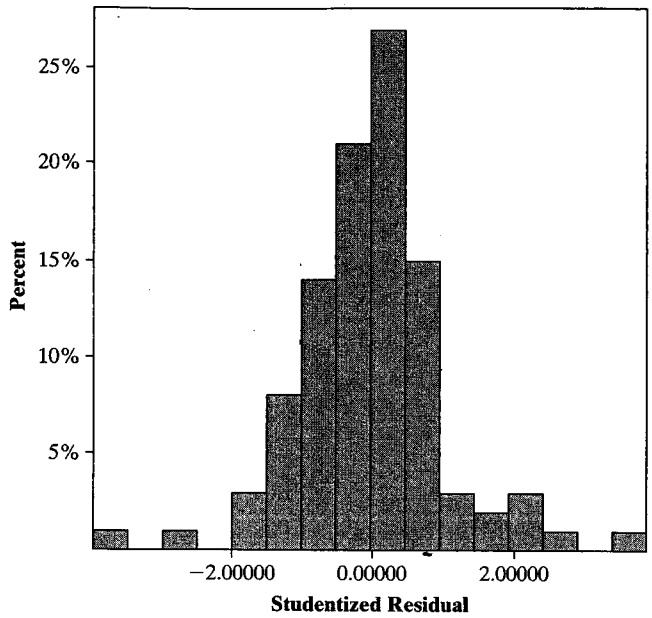


FIGURE 14.2: Histogram of Studentized Residuals for Multiple Regression Model Fitted to Housing Price Data, with Predictors Size, Taxes, and New

these two observations. The R^2-value changes from 0.79 to 0.83, and the prediction equation changes to

$$\hat{y} = -32226 + 68.9(\text{SIZE}) + 20436(\text{NEW}) + 38.3(\text{TAXES}).$$

The parameter estimates are similar for SIZE and TAXES, but the estimated effect of NEW drops from 46,374 to 20,436. Moreover, the effect of NEW is no longer significant, having a P-value of 0.17. Because the estimated effect of NEW is affected substantially by these two observations, we should be cautious in making conclusions about its effect. Of the 100 homes in the sample, only 11 were new. It is it difficult to make precise estimates about the NEW effect with so few new homes, and results are highly affected by a couple of unusual observations. ■

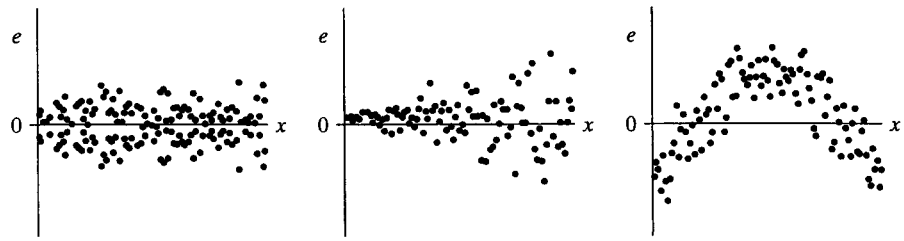
Plotting Residuals against Explanatory Variables

The normality assumption is not as important as the assumption that the model provides a good approximation for the true relationship between the predictors and the mean of y. If the model assumes a linear effect but the effect is actually strongly nonlinear, the conclusions will be faulty.

For bivariate models, the scatterplot provides a simple check on the form of the relationship. For multiple regression, it is also useful to construct a scatterplot of each explanatory variable against the response variable. This displays only the *bivariate* relationships, however, whereas the model refers to the *partial* effect of each predictor, with the others held constant. The *partial regression plot* introduced in Section 11.2 provides some information about this. It provides a summary picture of the partial relationship.

For multiple regression models, plots of the residuals (or studentized residuals) against the predicted values $\hat{y}$ or against each explanatory variable also help us check

for potential problems. If the residuals appear to fluctuate randomly about 0 with no obvious trend or change in variation as the values of a particular x_i increase, then no violation of assumptions is indicated. The pattern should be roughly like Figure 14.3a. In Figure 14.3c, y tends to be below $\hat{y}$ for very small and very large x_i-values (giving negative residuals) and above $\hat{y}$ for medium-sized x_i-values (giving positive residuals). Such a scattering of residuals suggests that y is actually nonlinearly related to x_i. Sections 14.5 and 14.6 show how to address nonlinearity.



(a) Assumptions satisfied (b) Nonconstant standard deviation (c) Nonlinear term needed

FIGURE 14.3: Possible Patterns for Residuals (e), Plotted against an Explanatory Variable x

In practice, most response variables can take only nonnegative values. For such responses, a fairly common occurrence is that the variability increases dramatically as the mean increases. For example, consider y = annual income (in dollars), using several predictors. For those subjects having $E(Y)$ = \$10,000, the standard deviation of income is probably much less than for those subjects having $E(Y)$ = \$200,000. Plausible standard deviations might be \$4000 and \$80,000. When this happens, the conditional standard deviation of y is not constant, whereas ordinary regression assumes that it is. An indication that this is happening is when the residuals are more spread out as the y_i-values increase. If we were to plot the residuals against a predictor that has a positive partial association with y, such as number of years of education, the residuals would then be more spread out for larger values of the predictor, as in Figure 14.3b.

Figure 14.4 is a residual plot for the model relating selling price of home to size, taxes, and whether new. It plots the residuals against size. There is some suggestion of more variability at the higher size values. It does seem sensible that selling prices would vary more for very large homes than for very small homes. A similar picture occurs when we plot the residuals against taxes.

If the change in variability is severe, then a method other than ordinary least squares provides better estimates with more valid standard errors. Section 14.4 presents a generalized regression model that allows the variability to be greater when the mean is greater.

In practice, residual patterns are rarely as neat as the ones in Figure 14.3. Don't let a few outliers or ordinary sampling variability influence too strongly your interpretation of a plot. Also, the plots described here just scratch the surface of the graphical tools now available for diagnosing potential problems. Fox (1991) described a variety of modern graphical displays.

Time Series Data

Some social research studies collect observations sequentially over time. For economic variables such as a stock index or the unemployment rate, for example, the observations often occur daily or monthly. The observations are then usually recorded

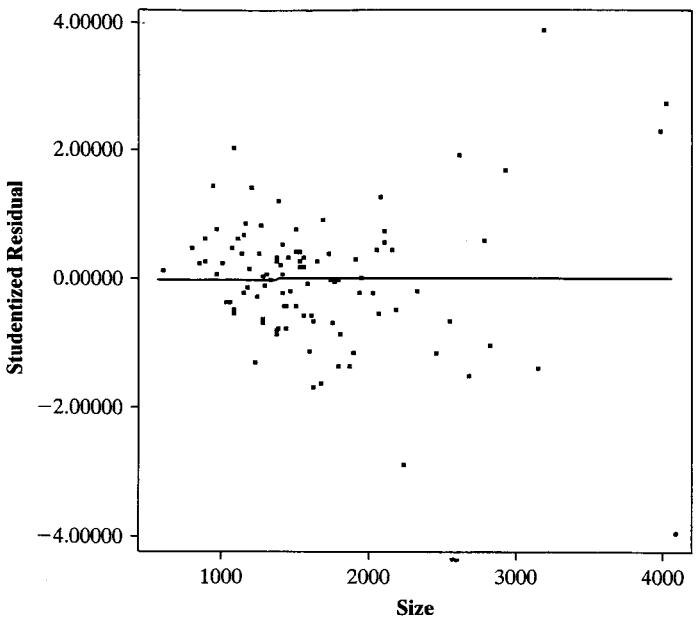


FIGURE 14.4: Scatterplot of Studentized Residuals of Home Selling Price Plotted against Size of Home, for Model with Predictors Size, Taxes, and Whether Home Is New

in sequence, rather than randomly sampled. Sampling subjects randomly from some population ensures that one observation is not statistically dependent on another, and this simplifies derivations of sampling distributions and their standard errors. However, neighboring observations from a time sequence are usually correlated rather than independent. For example, if the unemployment rate is relatively low in January 2008, it is probably also relatively low in February 2008.

A plot of the residuals against the time of making the observation checks for this type of dependence. Ideally, the residuals should fluctuate in a random pattern about 0 over time rather than showing a trend or periodic cycle. The methods presented in this text are based on independent observations and are inappropriate when time effects occur. For example, when observations next to each other tend to be positively correlated, the standard error of the sample mean is larger than the $\sigma/\sqrt{n}$ formula that applies for independent observations. Books specializing in time series or econometrics, such as Kennedy (2004), present methods for time series data.

Detecting Influential Observations: Leverage

Least squares estimates of parameters in regression models can be strongly influenced by an outlier, especially when the sample size is small. A variety of statistics summarize the influence each observation has. These statistics refer to how much the predicted values $\hat{y}$ or the model parameter estimates change when the observation is removed from the data set. An observation's influence depends on two factors: (1) how far the response on y falls from the overall trend in the sample, and (2) how far the values of the explanatory variables fall from their means.

The first factor on influence (how far an observation falls from the overall trend) is measured by the residual for the observation $y - \hat{y}$. The larger the residual, the farther the observation falls from the overall trend. We can search for observations

with large studentized residuals (say, larger than about 3 in absolute value) to find observations that may be influential.

The second factor on influence (how far the explanatory variables fall from their means) is summarized by the ***leverage*** of the observation. The leverage is a nonnegative statistic such that the larger its value, the greater weight that observation receives in determining the $\hat{y}$-values (hence, it also is sometimes called a *hat value*). The formula for the leverage in multiple regression is complex. For the bivariate model, the leverage for observation i simplifies to

$$h_i = \frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum(x - \bar{x})^2}.$$

So the leverage gets larger as the x-value gets farther from the mean, but it gets smaller as the sample size increases. When calculated for each observation in a sample, the average leverage equals p/n, where p is the number of parameters in the model.

Detecting Influential Observations: DFFIT and DFBETA

Most statistical software reports other diagnostics that depend on the studentized residuals and the leverages. Two popular ones are called ***DFFIT*** and ***DFBETA***.

For a given observation, DFBETA summarizes the *effect on the model parameter estimates* of removing the observation from the data set. For the effect β_j of x_j, DFBETA equals the change in the estimate $\hat{\beta}_j$ due to deleting the observation. The larger the absolute value of DFBETA, the greater the influence of the observation on that parameter estimate. Each observation has a DFBETA value for each parameter in the model.

DFFIT summarizes the *effect on the fit* of deleting the observation. It summarizes more broadly the influence of an observation, as each observation has a single DFFIT value whereas it has a separate DFBETA for each parameter. For observation i, DFFIT equals the change in the predicted value due to deleting that observation (i.e., $\hat{y}_i - \hat{y}_{(i)}$). The DFFIT value has the same sign as the residual. The larger its absolute value, the greater the influence that observation has on the fitted values. ***Cook's distance*** is an alternative measure with the same purpose, but it is based on the effect that observation i has on *all* the predicted values.

EXAMPLE 14.3 DFBETA and DFFIT for an Influential Observation

Example 14.2 (page 449) showed that observations 6 and 64 were influential on the equation for predicting home selling price using size of home, taxes, and whether the house is new. The prediction equation is

$$\hat{y} = -21{,}354 + 61.7(\text{SIZE}) + 46{,}373.7(\text{NEW}) + 37.2(\text{TAXES}).$$

For observation 6, the DFBETA values are 12.5 for size, 16,318.5 for new, and -5.7 for taxes. This means, for example, that if this observation is deleted from the data set, the effect of NEW changes from 46,373.7 to $46{,}373.7 - 16{,}318.5 = 30{,}055.2$. Observation 6 had a predicted selling price of $\hat{y} = 331{,}152.8$. Its DFFIT value is 29,417.0. This means that if observation 6 is deleted from the data set, then $\hat{y}$ at the predictor values for observation 6 changes to $331{,}152.8 - 29{,}417.0 = 301{,}735.8$. This analysis also shows that this observation is quite influential. ■

Some software reports standardized versions of the DFBETA and DFFIT measures. The standardized DFBETA divides the change in the estimate $\hat{\beta}_j$ due to

deleting the observation by the standard error of $\hat{\beta}_j$ for the adjusted data set. For observation i, the standardized DFFIT equals the change in the predicted value due to deleting that observation, divided by the standard error of $\hat{y}$ for the adjusted data set.

In practice, you scan or plot these diagnostic measures to see if some observations stand out from the rest, having relatively large values. Each measure has approximate cutoff points for noteworthy observations. A leverage larger than about $3p/n$ (three times the average leverage) indicates a potentially large influence. A standardized DFBETA larger than 1 suggests a substantial influence on that parameter estimate. However, the leverages, DFBETA, and DFFIT tend to decrease as n increases, so normally it is a good idea to examine observations having extreme values relative to the others. Individual data points tend to have less influence for larger sample sizes.

EXAMPLE 14.4 Influence Diagnostics for Crime Data

Table 9.1 in Chapter 9 (page 256) listed y = murder rate for the 50 states and the District of Columbia (D.C.). The data are shown again in Table 14.5. That table also showed data on x_1 = percentage of families below the poverty level and x_2 = percentage of single-parent families.

The least squares fit of the multiple regression model is

$$\hat{y} = -40.7 + 0.32x_1 + 3.96x_2.$$

Table 14.5 shows the influence diagnostics for the fit of this model, including the standardized versions of DFBETA and DFFIT. The studentized residuals all fall in a reasonable range except the one for the last observation (D.C.), which equals 14.2. The observed murder rate of 78.5 for D.C. falls far above the predicted value of 55.3, causing a large positive residual. This is an extreme outlier. In addition, the leverage for D.C. equals 0.54, more than three times as large as any other leverage and nine times the average leverage of $p/n = 3/51 = 0.06$. Since D.C. has both a large studentized residual and a large leverage, it has considerable influence on the model fit.

Not surprisingly, DFFIT for D.C. is much larger than for the other observations. This suggests that the predicted values change considerably if we refit the model after removing this observation. The DFBETA value for the single-family predictor x_2 is much larger for D.C. than for the other observations. This suggests that the effect of x_2 could change substantially with the removal of D.C. By contrast, DFBETA for poverty is not so large.

These diagnotics suggest that the D.C. observation has a large influence, particularly on the coefficient of x_2 and on the fitted values. The prediction equation for the model fitted without the D.C. observation is .

$$\hat{y} = -14.6 + 0.36x_1 + 1.52x_2.$$

Not surprisingly, the estimated effect of x_1 did not change much, but the coefficient of x_2 is now less than half as large. The standard error of the coefficient of x_2 also changes dramatically, decreasing from 0.44 to 0.26. ■

An observation with a large studentized residual does not have a major influence if its values on the explanatory variables do not fall far from their means. Recall that the leverage summarizes how far the explanatory variables fall from their means. For instance, New Mexico has a relatively large negative studentized residual (-2.25) but a relatively small leverage (0.046), so it does not have large values of DFFIT or DFBETA. Similarly, an observation far from the mean on the explanatory variables

TABLE 14.5: Influence Diagnostics for Model Using Poverty Rate and Single-Parent Percentage to Predict Murder Rate for 50 U.S. States and District of Columbia

Obs	Dep Var MURDER	Predict Value	Residual	Student Resid	Leverage h	Dffits	POVERTY Dfbeta	SINGLE Dfbeta
AK	9.0	18.88	-9.88	-2.04	0.162	-0.895	0.714	-0.761
AL	11.6	10.41	1.18	0.22	0.031	0.039	0.024	-0.011
AR	10.2	8.07	2.13	0.40	0.079	0.117	0.100	-0.069
AZ	8.6	12.16	-3.55	-0.65	0.022	-0.099	-0.005	-0.025
CA	13.1	14.63	-1.53	-0.28	0.034	-0.053	-0.027	-0.004
CO	5.8	10.41	-4.61	-0.87	0.060	-0.220	0.174	-0.134
CT	6.3	2.04	4.25	0.79	0.051	0.185	-0.130	0.015
DE	5.0	7.73	-2.73	-0.50	0.043	-0.107	0.079	-0.045
FL	8.9	6.97	1.92	0.35	0.048	0.080	0.059	-0.047
GA	11.4	15.12	-3.72	-0.69	0.042	-0.145	0.071	-0.105
HI	3.8	-2.07	5.87	1.11	0.059	0.279	-0.153	-0.058
IA	2.3	-1.74	4.04	0.75	0.045	0.164	-0.034	-0.081
ID	2.9	1.12	1.77	0.32	0.035	0.063	0.012	-0.040
IL	11.4	9.21	2.18	0.40	0.020	0.058	-0.013	0.011
IN	7.5	5.99	1.50	0.27	0.023	0.043	-0.014	-0.000
KS	6.4	2.71	3.68	0.68	0.029	0.117	0.013	-0.062
KY	6.6	7.79	-1.19	-0.22	0.088	-0.070	-0.061	0.043
LA	20.3	26.74	-6.44	-1.29	0.161	-0.568	-0.412	-0.055
MA	3.9	5.91	-2.01	-0.37	0.033	-0.068	0.042	-0.014
MD	12.7	9.95	2.74	0.51	0.060	0.130	-0.104	0.077
ME	1.6	4.72	-3.12	-0.57	0.031	-0.104	0.058	-0.008
MI	9.8	15.72	-5.92	-1.10	0.033	-0.204	0.035	-0.124
MN	3.4	2.23	1.16	0.21	0.029	0.037	-0.007	-0.013
MO	11.3	7.62	3.67	0.67	0.027	0.115	0.059	-0.049
MS	13.5	25.40	-11.90	-2.45	0.126	-0.933	-0.623	-0.151
MT	3.0	6.84	-3.84	-0.70	0.023	-0.108	-0.033	0.039
NC	11.3	7.87	3.42	0.62	0.020	0.090	0.009	-0.013
ND	1.7	-3.83	5.53	1.04	0.057	0.259	0.016	-0.184
NE	3.9	-0.15	4.05	0.75	0.039	0.153	-0.047	-0.056
NH	2.0	-1.07	3.07	0.57	0.044	0.123	-0.039	-0.047
NJ	5.3	0.82	4.47	0.83	0.035	0.158	-0.041	-0.058
NM	8.0	19.53	-11.53	-2.25	0.046	-0.499	-0.017	-0.308
NV	10.4	11.57	-1.17	-0.22	0.069	-0.060	0.048	-0.040
NY	13.3	14.85	-1.55	-0.28	0.028	-0.048	-0.005	-0.019
OH	6.0	8.62	-2.62	-0.48	0.022	-0.072	0.024	-0.015
OK	8.4	9.62	-1.22	-0.22	0.067	-0.061	-0.051	0.031
OR	4.6	7.84	-3.24	-0.59	0.027	-0.101	0.054	-0.029
PA	6.8	1.55	5.24	0.97	0.034	0.183	0.036	-0.115
RI	3.9	5.67	-1.77	-0.32	0.028	-0.056	0.029	-0.006
SC	10.3	13.99	-3.69	-0.68	0.038	-0.137	-0.084	0.008
SD	3.4	1.07	2.32	0.43	0.042	0.091	0.036	-0.067
TN	10.2	9.92	0.27	0.05	0.060	0.013	0.010	-0.006
TX	11.9	11.60	0.29	0.05	0.029	0.009	0.005	-0.001
UT	3.1	2.34	0.75	0.13	0.032	0.025	-0.010	-0.004
VA	8.3	3.21	5.08	0.94	0.039	0.192	-0.119	0.010
VT	3.6	6.08	-2.48	-0.46	0.040	-0.094	0.067	-0.028
WA	5.2	9.52	-4.32	-0.80	0.029	-0.139	0.078	-0.059
WI	4.4	4.53	-0.13	-0.02	0.023	-0.003	0.000	0.001
WV	6.9	3.60	3.29	0.66	0.178	0.307	0.274	-0.229
WY	3.4	6.34	-2.94	-0.54	0.021	-0.079	0.006	0.012
DC	78.5	55.28	23.22	14.20	0.536	15.271	-0.485	12.792

(i.e., with a large leverage) need not have a major influence, if it falls close to the prediction equation and has a small studentized residual. For instance, West Virginia has a relatively large poverty rate and its leverage of 0.18 is triple the average. However, its studentized residual is small (0.66), so it has little influence on the fit.

14.3 EFFECTS OF MULTICOLLINEARITY

In many social science studies using multiple regression, the explanatory variables "overlap" considerably. Each variable may be nearly redundant, in the sense that it can be predicted well using the others. If we regress an explanatory variable on the others and get a large R^2-value, this suggests that it may not be needed in the model once the others are there. This condition is called *multicollinearity*. This section describes the effects of multicollinearity and ways to diagnose it.

Multicollinearity Inflates Standard Errors

Multicollinearity causes inflated standard errors for estimates of regression parameters. To show why, we first consider the regression model $E(y) = \alpha + \beta_1 x_1 + \beta_2 x_2$. The estimate of β_1 has standard error

$$se = \frac{1}{\sqrt{1 - r^2_{x_1x_2}}}\left[\frac{s}{\sqrt{n-1}s_{x_1}}\right],$$

where s is the estimated conditional standard deviation of y and s_{x_1} denotes the sample standard deviation of x_1 values. The effect of the correlation $r_{x_1x_2}$ between the explanatory variables enters through the term $\sqrt{1 - r^2_{x_1x_2}}$ in the denominator. Other things being equal, the stronger that squared correlation, the larger the standard error of b_1. Similarly, the standard error of the estimator of β_2 also is larger with larger values of $r^2_{x_1x_2}$.

An analogous result applies for the model with multiple predictors. The standard error of the estimator of the coefficient β_j of x_j equals

$$se = \frac{1}{\sqrt{1 - R_j^2}}\left[\frac{s}{\sqrt{n-1}s_{x_j}}\right],$$

where s_{x_j} is the sample standard deviation of x_j and R_j denotes the multiple correlation from the regression of x_j on the other predictors. So when x_j overlaps a lot with the other predictors, in the sense that R_j^2 is large for predicting x_j using the other predictors, this se is relatively large. Then the confidence interval for β_j is wide, and the test of H_0: $\beta_j = 0$ has large P-value unless the sample size is very large.

The VIF and Other Indicators of Multicollinearity

The quantity $1/(1 - R_j^2)$ in the above se formula for the estimate of β_j in multiple regression is called a ***variance inflation factor*** (VIF). It represents the multiplicative increase in the variance (squared standard error) of the estimator due to x_j being correlated with the other predictors.

When any of the R_j^2-values from regressing each explanatory variable on the other explanatory variables in the model is close to 1, say above 0.90, multicollinearity exists. For example, if $R_j^2 > 0.90$, then VIF > 10 for the effect of that predictor. That

is, the variance of the estimate of β_j inflates by a factor of more than 10. The standard error inflates by a factor of more than $\sqrt{10} = 3.2$, compared to the standard error for uncorrelated predictors.

To describe the extent to which multicollinearity exists, most software can display the variance inflation factor

$$\text{VIF} = 1/(1 - R_j^2)$$

for each predictor. For example, for the model selected in Section 14.1 that predicts house selling price using taxes, size, and whether the house is new, SPSS reports as "collinearity statistics"

	VIF
TAXES	3.082
SIZE	3.092
NEW	1.192

The standard error for whether new is not affected much by correlation with the other predictors, but the other two standard errors multiply by a factor of roughly $\sqrt{3.1} = 1.76$.

Suppose a predictor is in the model primarily as a control. That is, we want to control its effects in studying effects of the variables of primary interest, but we do not need precise estimates of its effect on the response variable. Then, it is not crucial to worry about the VIF value for this predictor variable.

Even without checking VIFs, various types of behavior in a regression analysis can indicate potential problems due to multicollinearity. A warning sign occurs when the estimated coefficient for a predictor already in the model changes substantially when another variable is introduced. For example, perhaps the estimated coefficient of x_1 is 2.4 for the bivariate model, but when x_2 is added to the model, the coefficient of x_1 changes to 25.9.

Another indicator of multicollinearity is when a highly significant R^2 exists between y and the explanatory variables, but individually each partial regression coefficient is not significant. In other words, H_0: $\beta_1 = \cdots = \beta_k = 0$ has a small P-value in the overall F test, but H_0: $\beta_1 = 0$, H_0: $\beta_2 = 0$, and so forth do not have small P-values in the separate t tests. Thus, it is difficult to assess individual partial effects when severe multicollinearity exists. Other indicators of multicollinearity are surprisingly large standard errors or standardized regression coefficients that are larger than 1 in absolute value (which is impossible for partial correlations).

Since a regression coefficient in a multiple regression model represents the effect of an explanatory variable when other variables are held constant, it has less meaning when multicollinearity exists. If $|r_{x_1x_2}|$ is high, then as x_1 changes, x_2 also tends to change in a linear manner, and it is somewhat artificial to envision x_1 or x_2 as being held constant. Thus, the coefficients have dubious interpretations when multicollinearity exists.

Remedial Actions when Multicollinearity Exists

Here are some remedial measures to reduce the effects of multicollinearity. First, since it may not make sense to study partial effects when the explanatory variables are highly correlated, you could use simple bivariate regression models to analyze the relationship between y and each x_i separately.

A better solution is to choose a subset of the explanatory variables, removing those variables that explain a small portion of the remaining unexplained variation in y. If x_4 and x_5 have a correlation of 0.96, it is only necessary to include one of them in the

model. You could use an automated variable selection procedure to select a subset of variables, but this is primarily helpful for purely exploratory research.

Alternatively, when several predictors are highly correlated and are indicators of a common feature, you could construct a summary index by combining responses on those variables. For example, suppose that a model for predicting y = opinion about president's performance in office uses 12 predictors, of which three refer to the subject's opinion about whether a woman should be able to obtain an abortion (1) when she cannot financially afford another child, (2) when she is unmarried, and (3) anytime in the first three months. Each of these items is scaled from 1 to 5, with a 5 being the most conservative response. They are likely to be highly positively correlated, contributing to multicollinearity. A possible summary measure for opinion about abortion averages (or sums) the responses to these items. Higher values on that summary index represent more conservative responses. If the items were measured on different scales, we could first standardize the scores before averaging them. Socioeconomic status is a variable of this type, summarizing the joint effects of education, income, and occupational prestige.

Often multicollinearity occurs when the predictors include interaction terms. Since cross-product terms are composed of other predictors in the model, it is not surprising that they tend to be highly correlated with the other terms. Section 11.5 noted that the effects of this are diminished if we center the predictors by subtracting their sample means before entering them in the interaction model.

Other procedures, beyond the scope of this chapter, can handle multicollinearity. For example, *factor analysis* (introduced in Chapter 16) is a method for creating artificial variables from the original ones in such a way that the new variables can be uncorrelated. In most applications, though, it is more advisable to use a subset of the variables or create some new variables directly, as just discussed.

Multicollinearity does not adversely affect all aspects of regression. Although multicollinearity makes it difficult to assess *partial* effects of explanatory variables, it does not hinder the assessment of their *joint* effects. If newly added explanatory variables overlap substantially with ones already in the model, then R and R^2 will not increase much, but the fit will not be poorer. So the presence of multicollinearity does not diminish the predictive power of the equation. For further discussion of the effects of multicollinearity and methods for dealing with it, see DeMaris (2004), Fox (1991), and Kutner et al. (2004).

14.4 GENERALIZED LINEAR MODELS

The models presented in this book are special cases of ***generalized linear models***. This is a broad class of models that includes ordinary regression models for response variables assumed to have a normal distribution, alternative models for continuous variables that do not assume normality, and models for discrete response variables including categorical variables. This section introduces generalized linear models. We use the acronym *GLM*.

Nonnormal Distributions for a Response

As in other regression models, a GLM identifies a response variable y and a set of explanatory variables. The regression models discussed in the past six chapters are GLMs that assume that y has a normal distribution.

In many applications, the potential outcomes for y are binary rather than continuous. Each observation might be labeled as a *success* or *failure*, as in the methods for proportions presented in Sections 5.2, 6.3, and 7.2. For instance, consider a study of

factors that influence votes in presidential elections. For each subject, the response variable indicates the preferred candidate in the previous presidential election—the Democratic or the Republican candidate. The study uses predictors in a model for subjects' decisions about the preferred candidate. In this case, models usually assume a *binomial* distribution for y. The next chapter presents a GLM for binary data, called *logistic regression*.

In some applications, each observation is a count. For example, consider a study of factors associated with family size. The response for a given married couple is their number of children. The study constructs a model that uses several explanatory variables to predict the number of children. Two distributions not discussed in this text, called the *Poisson* and the *negative binomial*, are often assumed for y in GLMs for count data.

Binary outcomes and counts are examples of discrete variables. Regression models that assume normal distributions are not optimal for models with discrete responses. Even when the response variable is continuous, the normal distribution is not necessarily optimal. When each observation must take a positive value, for instance, the distribution is often skewed to the right with greater variability when the mean is greater. In that case, a GLM can assume a *gamma* distribution for y, as discussed later in this section.

The Link Function for a GLM

Denote the expected value of y, which is the mean of its probability distribution, by $\mu = E(y)$. As in ordinary regression models, in a GLM μ varies according to values of explanatory variables, which enter linearly as predictors on the right-hand side of the model equation. However, a GLM allows a function $g(\mu)$ of the mean rather than just the mean μ itself on the left-hand side. The GLM formula states that

$$g(\mu) = \alpha + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k.$$

The function $g(\mu)$ is called the ***link function***, because it links the mean of the response variable to the explanatory variables.

The simplest possible link function is $g(\mu) = \mu$. This models the mean directly and is called the ***identity link***. It specifies a linear model for the mean response,

$$\mu = \alpha + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k.$$

This is the form of ordinary regression models.

Other link functions permit the mean to relate nonlinearly to the predictors. For instance, the link function $g(\mu) = \log(\mu)$ models the log of the mean. The log function applies to positive numbers, so this ***log link*** is appropriate when μ cannot be negative, such as with count data. A GLM that uses the log link is often called a ***loglinear model***. It has form

$$\log(\mu) = \alpha + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k.$$

The final section of this chapter shows an example of this model.

For binary data, the most common link function is $g(\mu) = \log[\mu/(1 - \mu)]$. This is called the ***logit link***. It is appropriate when μ falls between 0 and 1, such as a probability, in which case $\mu/(1 - \mu)$ is the *odds*. When y is binary, this link is used in models for the probability of a particular outcome, for instance, to model the probability that a subject votes for the Democratic candidate. A GLM using the logit link is called a ***logistic regression model***.

GLMs for a Response Assuming a Normal Distribution

Ordinary regression models are special cases of GLMs. They assume a normal distribution for y and model the mean directly, using the identity link, $g(\mu) = \mu$. A GLM generalizes ordinary regression in two ways: First, y can have a distribution other than the normal. Second, it can model a function of the mean. Both generalizations are important, especially for discrete responses.

Before GLMs were developed in the 1970s, the traditional way of analyzing "nonnormal" data involved transforming the y-values. The goal is to find transformed values that have an approximately normal distribution, with constant standard deviation at all levels of the predictors. Square root or log transforms are often applied to do this. If the goals of normality and constant variation are achieved, then ordinary regression methods using least squares are applicable with the transformed data. In practice, this usually does not work well. A transform that produces constant variation may not produce normality, or else simple linear models for the explanatory variables may fit poorly on that scale. Moreover, conclusions that refer to the mean response on the scale of the transformed variable are usually less relevant, and there can be technical problems such as taking logarithms of 0.

With the GLM approach, it is not necessary to transform data and use normal methods. This is because the GLM fitting process utilizes a powerful estimation method (***maximum likelihood***, see Section 5.1) for which the choice of distribution for y is not restricted to normality. In addition, in GLMs the choice of link function is separate from the choice of distribution for y. If a certain link function makes sense for a particular type of data, it is not necessary that it also stabilize variation or produce normality.

We introduce the concept of GLMs to unify a wide variety of statistical methods. Ordinary regression models as well as models for discrete data (Chapter 15) are special cases of one highly general model. In fact, the same fitting method yields parameter estimates for all GLMs. Using GLM software, there is tremendous flexibility and power in the model-building process. You pick a probability distribution that is most appropriate for y. For instance, you might select the normal option for a continuous response or the binomial option for a binary response. You specify the variables that are the predictors. Finally, you pick the link function, determining which function of the mean to model. The appendix provides examples.

The next chapter introduces the most important GLM for binary response variables—the *logistic regression* model. The next subsection shows the use of GLMs for data with nonconstant variation, and the final section of the chapter shows a GLM for modeling the log link of the mean as a way of handling nonlinearity.

GLMs for a Response Assuming a Gamma Distribution

For Example 14.2, on selling prices of homes, Figure 14.4 (page 452) showed a tendency for greater variability at higher home size values. Small homes show little variability in selling price, whereas large homes show high variability. Large homes are the ones that tend to have higher selling prices, so variability in y increases as its mean increases.

This phenomenon often happens for positive-valued response variables. When the mean response is near 0, less variation occurs than when the mean response is high. For such data, least squares is not optimal. It is identical to maximum likelihood for a GLM in which y is assumed to have a normal distribution with the same standard deviation σ at all values of predictors.

An alternative approach for data of this form assumes a distribution for y for which the standard deviation increases as the mean increases (i.e., that permits *heteroscedasticity*). The family of ***gamma distributions*** has this property. Its standard deviation increases proportionally to the mean: When the mean doubles, the standard deviation doubles. The gamma distribution is concentrated on the positive part of the line. It exhibits skewness to the right, like the chi-squared distribution, which is a special case of the gamma. (Technically, GLMs use the gamma family that has a constant *shape* parameter determining the shape of the distribution. Software estimates this parameter. SPSS reports the inverse of the estimated shape parameter. It refers to it as a *scale* parameter, terminology that conflicts with what's called the scale parameter in other sources.)

With GLMs, you can fit a regression model assuming a gamma distribution for y instead of a normal distribution. Even if the data are close to normal, this alternative fit is more appropriate than the least squares fit when the standard deviation increases proportionally to the mean.

EXAMPLE 14.5 Gamma GLM for Home Selling Price

The least squares fit of the model to the data on y = selling price using predictors size of home, taxes, and whether new, discussed in Example 14.1 (page 443), is

$$\hat{y} = -21{,}353.8 + 61.7(\text{SIZE}) + 46{,}373.7(\text{NEW}) + 37.2(\text{TAXES}).$$

However, Example 14.2 (page 449) showed that two outlying observations had a substantial effect on the NEW estimate. Figure 14.4 showed that the variability in selling prices seems to increase as its mean does. This suggests that a model assuming a gamma distribution may be more appropriate. For the gamma distribution, the standard deviation increases as the mean does.

We can use software[2] to fit the GLM assuming a gamma distribution for y. For the GLM fit of the same model form, but assuming a gamma distribution, we get

$$\hat{y} = -940.0 + 48.7(\text{SIZE}) + 32{,}868.0(\text{NEW}) + 37.9(\text{TAXES}).$$

The estimated effect of TAXES is similar, but the estimated effect of SIZE is weaker and the estimated effect of NEW is much weaker. Moreover, the effect of NEW is no longer significant, as the ratio of the estimate to the standard error is 1.53. This result is similar to what Example 14.3 (page 453) obtained after deleting observation 6, an outlier corresponding to a large, new house with an unusually high selling price. The outliers are not as influential for the gamma fit, because that model expects more variability in the data when the mean is larger.

SPSS reports a scale parameter estimate of 0.0707. The larger this value, the greater the degree of skew in the gamma distributions estimated for the model. The estimated standard deviation $\hat{\sigma}$ of the conditional distribution of Y relates to the estimated conditional mean $\hat{\mu}$ by

$$\hat{\sigma} = \sqrt{\text{scale}}\hat{\mu} = \sqrt{0.0707}\hat{\mu} = 0.266\hat{\mu}.$$

For example, at predictor values such that the estimated mean selling price is $\hat{\mu}$ = \$100,000, the estimated standard deviation of selling prices is $\hat{\sigma} = 0.266(\$100{,}000) = \$26{,}600$. By contrast, at predictor values such that $\hat{\mu}$ = \$400,000, $\hat{\sigma} = 0.266(\$400{,}000) = \$106{,}400$, four times as large. (SAS identifies the scale parameter as the

[2]Such as the Generalized Linear Models option in the Analyze menu of SPSS, or PROC GENMOD in SAS.

inverse of what SPSS does, so it reports an estimated scale parameter of $1/0.0707 = 14.14$. The parameter that SAS estimates is actually the ***shape parameter*** for the gamma distribution. So, with SAS, you use $\hat{\sigma} = \hat{\mu}/\sqrt{\text{scale}} = \hat{\mu}/\sqrt{14.14}$.) ■

The traditional method of dealing with variability that increases with the mean is to transform the data, applying the log or square root to the y-values. Then the variability is more nearly constant, and least squares works well. There is a fundamental flaw with this approach. If the original relationship is linear, it is no longer linear after applying the transformation. If we fit a straight line and then transform back to the original scale, the fit is no longer linear. Although this approach is still used in many statistical methods textbooks, the gamma GLM approach is more elegant and preferable because of maintaining the linear relationship. Another approach that is preferable to transforming the data is the use of ***weighted least squares***, which gives more weight to observations over regions that show less variability. For further details about generalized linear models, see Gill (2000) and King (1989).

14.5 NONLINEAR RELATIONSHIPS: POLYNOMIAL REGRESSION

The ordinary regression model assumes that relationships are linear. The multiple regression model assumes that the partial relationship between the mean of y and each quantitative explanatory variable is linear, controlling for other explanatory variables. Although social science relationships are not *exactly* linear, the degree of nonlinearity is often so minor that they can be reasonably well approximated with linear equations.

Occasionally, though, such a model is inadequate, even for approximation. A scatterplot may reveal a highly nonlinear relationship. Alternatively, the theoretical formulation of an expected relationship might predict a nonlinear relationship. For example, you might expect y = annual medical expenses to have a curvilinear relationship with x = age, being relatively high for the very young and the very old but lower for older children and young adults (Figure 14.5a). The relationship between x = per capita income and y = life expectancy for a sample of countries might be approximately a linearly increasing one, up to a certain point. However, beyond a certain level, additional income would probably result in little, if any, improvement in life expectancy (Figure 14.5b).

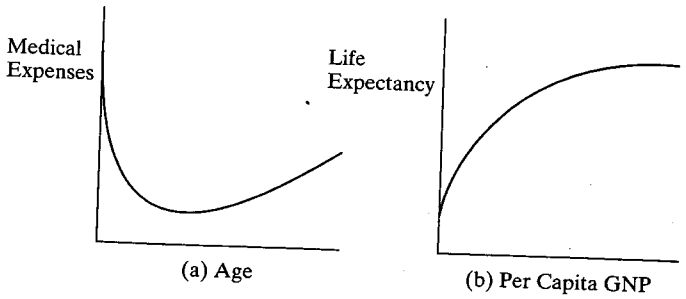


FIGURE 14.5: Two Nonlinear Relationships

Undesirable consequences may result from using straight line regression to describe relationships that are curvilinear. Measures of association designed for linearity, such as the correlation, may underestimate the true association. Estimates of the mean of y at various x-values may be badly biased, since the prediction line may poorly

approximate the true regression curve. This section and the following one present ways of modeling nonlinear relationships.

Two approaches are commonly used. The first of these uses a *polynomial* regression function. The class of polynomial functions includes a diverse set of functional patterns, including straight lines. The second approach uses a generalized linear model with a link function such as the logarithm. For example, for certain curvilinear relationships, the logarithm of the mean of the response variable is linearly related to the explanatory variables. The final section of the chapter discusses this second approach.

Quadratic Regression Models

A ***polynomial regression function*** for a response variable y and single explanatory variable x has form

$$E(y) = \alpha + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k.$$

In this model, x occurs in powers from the first ($x = x^1$) to some integer k. For $k = 1$, this is the straight line $E(y) = \alpha + \beta_1 x$. The index k, the highest power in the polynomial equation, is called the ***degree*** of the polynomial function.

The polynomial function most commonly used for nonlinear relationships is the *second-degree polynomial*

$$E(y) = \alpha + \beta_1 x + \beta_2 x^2.$$

This is called a ***quadratic regression model***. The graph of this function is parabolic, as Figure 14.6 portrays. It has a single bend, either increasing and then decreasing or else decreasing and then increasing. The shape of the parabolic curve is symmetric about a vertical axis, with its appearance when increasing a mirror image of its appearance when decreasing.

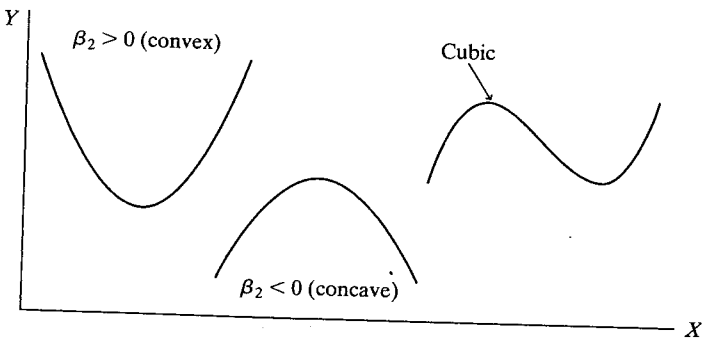


FIGURE 14.6: Graphs of Two Second-Degree Polynomials (Quadratic Functions) and a Third-Degree Polynomial (Cubic Function)

If a scatterplot reveals a pattern of points with one bend, then a second-degree polynomial usually improves upon the straight line fit. A third-degree polynomial $E(y) = \alpha + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$, called a ***cubic function***, is a curvilinear function having *two* bends. See Figure 14.6. In general, a kth-degree polynomial has $(k - 1)$ bends. Of the polynomial models, the linear and quadratic equations are most useful. Rarely is it necessary to use higher than a second-degree polynomial to describe the trend.

EXAMPLE 14.6 Fertility Predicted Using Gross Domestic Product (GDP)

Table 14.6 shows values reported by the United Nations in 2005 for several nations on y = fertility rate (the mean number of children per adult woman) and x = per capita gross domestic product (GDP, in dollars). Fertility tends to decrease as GDP increases. However, a straight line model may be inadequate, since it might predict negative fertility for sufficiently high GDP. In addition, some demographers predict that after GDP passes a certain level, fertility rate may increase, since the nation's wealth makes it easier for a parent to stay home and take care of children rather than work. In the following model fitting, we will measure GDP in tens of thousands of dollars (e.g., for the U.S., 3.7648 rather than 37,648) to make the coefficients more interpretable.

TABLE 14.6: Data on Fertility Rate and Per Capita Gross Domestic Product GDP (in Dollars)

Nation	GDP	Fertility Rate	Nation	GDP	Fertility Rate	Nation	GDP	Fertility Rate
Algeria	2090	2.5	Germany	29115	1.3	Pakistan	555	4.3
Argentina	3524	2.4	Greece	15608	1.3	Philippines	989	3.2
Australia	26275	1.7	India	564	3.1	Russia	3018	1.3
Austria	31289	1.4	Iran	2066	2.1	S Africa	3489	2.8
Belgium	29096	1.7	Ireland	38487	1.9	Saudi Ar.	9532	4.1
Brazil	2788	2.3	Israel	16481	2.9	Spain	20404	1.3
Canada	27079	1.5	Japan	33713	1.3	Sweden	33676	1.6
Chile	4591	2.0	Malaysia	4187	2.9	Switzerland	43553	1.4
China	1100	1.7	Mexico	6121	2.4	Turkey	3399	2.5
Denmark	39332	1.8	Netherlands	31532	1.7	UK	30253	1.7
Egypt	1220	3.3	New Zealand	19847	2.0	US	37648	2.0
Finland	31058	1.7	Nigeria	428	5.8	Viet Nam	482	2.3
France	29410	1.9	Norway	48412	1.8	Yemen	565	6.2

Source: *Human Development Report, 2005* available at hdr.undp.org/statistics/data

Figure 14.7, a scatterplot for the 39 observations, shows a clear decreasing trend. The linear prediction equation is $\hat{y} = 3.04 - 0.415x$, and the correlation equals -0.56. This prediction equation gives absurd predictions for very large x-values; $\hat{y}$ is negative for $x > 7.3$ (i.e., \$73,000). However, the predicted values are positive over the range of x-values for this sample.

To allow for potential nonlinearity and for the possibility that fertility rate may increase for sufficiently large GDP, we could fit a quadratic regression model to these data. We would use the second-degree polynomial, rather than higher, because we expect at most one bend in the relationship, that is, a decrease followed potentially by an increase. ■

Interpreting and Fitting Quadratic Regression Models

The quadratic regression model

$$E(y) = \alpha + \beta_1 x + \beta_2 x^2$$

plotted for the possible values of α, β_1, and β_2 describes the possible parabolic shapes. Unlike straight lines, for which the slope remains constant over all x-values, the mean change in y for a one-unit increase in x *depends on the value of* x. For example, a straight line drawn tangent to the parabola in Figure 14.8 has positive slope for small values of x, zero slope where the parabola achieves its maximum value, and negative

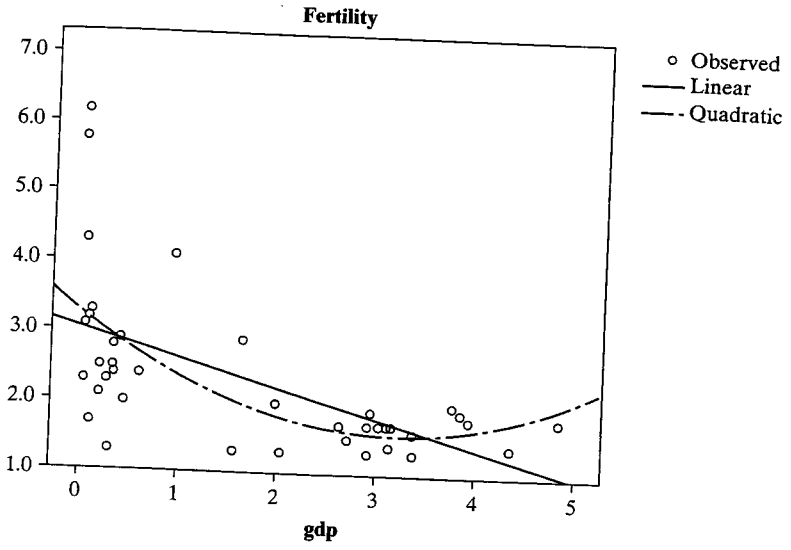


FIGURE 14.7: Scatterplot and Best-Fitting Straight Line and Second-Degree Polynomial for Data on Fertility Rate and Per Capita GDP

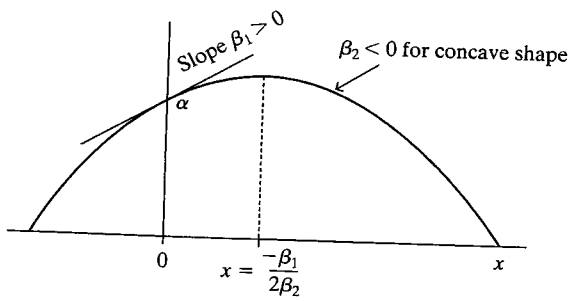


FIGURE 14.8: Interpretation of Parameters of Second-Degree Polynomial $E(y) = \alpha + \beta_1 x + \beta_2 x^2$

slope for large values of x. The rate of change of the line varies to produce a curve having a smooth bend.

The sign of the coefficient β_2 of the x^2-term determines whether the function is bowl shaped (opens up) relative to the x-axis or mound shaped (opens down). Bowl-shaped functions (also called *convex* functions) have $\beta_2 > 0$. Mound-shaped functions (also called *concave* functions) have $\beta_2 < 0$. See Figure 14.8.

As usual, the coefficient α is the y-intercept. The coefficient β_1 of x is the slope of the line that is tangent to the parabola as it crosses the y axis. If $\beta_1 > 0$, for example, then the parabola is sloping upward at $x = 0$ (as Figure 14.8 shows). At the point at which the slope is zero, the relationship changes direction from positive to negative or from negative to positive. This happens at $x = -\beta_1/(2\beta_2)$. This is the x-value at which the mean of y takes its maximum if the parabola is mound-shaped and its minimum if it is bowl shaped.

To fit quadratic regression models, we treat them as a special case of the multiple regression model

$$E(y) = \alpha + \beta_1 x_1 + \beta_2 x_2 = \alpha + \beta_1 x + \beta_2 x^2$$

with two explanatory variables. We identify x_1 with the explanatory variable x and x_2 with its square, x^2. The data for the model fit consist of the y-values for the subjects in the sample, the x-values (called x_1), and an artificial variable (x_2) consisting of the squares of the x-values. Software can create these squared values for us. It then uses least squares to find the best fitting function out of the class of all second-degree polynomials.

EXAMPLE 14.7 Quadratic Regression for Fertility and GDP

Table 14.7 shows part of a printout for the quadratic regression of y = fertility rate on x = GDP. Here, GDP2 denotes an artificial variable constructed as the square of GDP. The prediction equation is

$$\hat{y} = 3.28 - 1.054x + 0.163x^2.$$

Figure 14.7 plots the linear and quadratic prediction equations in the scatter diagram. Since the coefficient 0.163 of x^2 is positive, the graph is bowl shaped (convex). Also, since the coefficient -1.054 of x is negative, the curve is decreasing as it crosses the y-axis.

TABLE 14.7: Part of Printout for Second-Degree Polynomial Model for y = Fertility Rate and x = GDP

Variable	B	Std. Error	t	Sig
INTERCEP	3.278	.257	12.750	.000
GDP	-1.054	0.366	-2.880	.007
GDP2	.163	0.090	1.810	.079
R-square	0.375			

A bowl-shaped quadratic equation takes its minimum at $x = -\beta_1/(2\beta_2)$. For these data, we estimate this point to be $x = 1.054/[2(0.163)] = 3.23$. The predicted fertility rate increases as GDP increases above this point (i.e., \$32,300). ■

Description and Inference about the Nonlinear Effect

For a polynomial model, R^2 for multiple regression describes the strength of the association. In this context, it describes the proportional reduction in error obtained from using the polynomial model, instead of $\bar{y}$, to predict y. Comparing this measure to r^2 for the straight line model indicates how much better a fit the curvilinear model provides. Since the polynomial model has additional terms besides x, R^2 always is at least as large as r^2. The difference $R^2 - r^2$ measures the additional reduction in prediction error obtained by using the polynomial instead of the straight line.

For Table 14.6, the best-fitting straight line prediction equation has $r^2 = 0.318$. That line is also plotted in Figure 14.7. From Table 14.7 for the quadratic model, $R^2 = 0.375$. The best quadratic equation explains about 6% more variability in y than does the best-fitting straight line equation.

If $\beta_2 = 0$, the quadratic regression equation $E(y) = \alpha + \beta_1 x + \beta_2 x^2$ reduces to the linear regression equation $E(y) = \alpha + \beta_1 x$. Therefore, to test the null hypothesis that the relationship is linear against the alternative that it is quadratic, we test H_0: $\beta_2 = 0$. The usual t test for a regression coefficient does this, dividing the estimate of β_2 by its standard error. The assumptions for applying inference are the same as for ordinary regression: randomization for gathering the data, a conditional distribution of y-values that is normal about the mean, with constant standard deviation σ at all x-values.

The set of nations in Table 14.6 is not a random sample of nations, so inference is not relevant. If it had been, the printout in Table 14.7 shows that $t = 0.163/0.090 = 1.81$, with $df = 37$. The P-value for testing H_0: $\beta_2 = 0$ against H_a: $\beta_2 \neq 0$ is $P = 0.08$. In this sense, the quadratic prediction equation apparently provides weak evidence of a better fit than the straight line equation.

Cautions in Using Polynomial Models

Some cautions are in order before you take the conclusions in this example too seriously. The scatterplot (Figure 14.7) suggests that the variability in fertility rates is considerably higher for nations with low GDPs than it is for nations with high GDPs. The fertility rates show much greater variability when their mean is higher. A GLM that permits nonconstant standard deviation by assuming a gamma distribution for y, discussed in Section 14.4, provides somewhat different results, including stronger evidence of nonlinearity (Exercise 14.14).

In fact, before we conclude that fertility rate increases above a certain value, we should realize that other models for which this does not happen are also consistent with these data. For instance, Figure 14.7 suggests that a "piecewise linear" model that has a linear decrease until GDP is about \$25,000 and then a separate, nearly horizontal, line beyond that point fits quite well. A more satisfactory model for these data is one discussed in the next section of this chapter for *exponential regression*. Unless a data set is very large, several models may be consistent with the data.

In examining scatterplots, be cautious not to read too much into the data. Don't let one or two outliers suggest a curve in the trend. Good model building follows the principle of ***parsimony***: Models should have no more parameters than necessary to represent the relationship adequately. One reason is that simple models are easier to understand and interpret than complex ones. Another reason is that when a model contains unnecessary variables, the standard errors of the estimates of the regression coefficients tend to inflate, hindering efforts at making precise inferences. Estimates of the conditional mean of y also tend to be poorer than those obtained with well-fitting simple models.

When a polynomial regression model is valid, the regression coefficients do not have the partial slope interpretation usual for coefficients of multiple regression models. It does not make sense to refer to the change in the mean of y when x^2 is increased one unit and x is held constant. Similarly, it does not make sense to interpret the partial correlations $r_{yx^2 \cdot x}$ or $r_{yx \cdot x^2}$ as measures of association, controlling for x or x^2. However, the coefficient $r^2_{yx^2 \cdot x}$ does measure the proportion of the variation in y unaccounted for by the straight line model that is explained by the quadratic model. In Example 14.6 (page 464), applying the formula for $r^2_{yx_2 \cdot x_1}$ from Section 11.7 yields

$$r^2_{yx^2 \cdot x} = \frac{R^2 - r^2_{yx}}{1 - r^2_{yx}} = \frac{0.375 - 0.318}{1 - 0.318} = 0.08.$$

Of the variation in y unexplained by the linear model, about 8% is explained by the introduction of the quadratic term.

Nonlinear relationships are also possible when there are several explanatory variables. For example, the model

$$E(y) = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_2^2$$

allows nonlinearity in x_2. For fixed x_1, the mean of y is a quadratic function of x_2. For fixed x_2, the mean of y is a linear function of x_1 with slope β_1. This model is a special

case of multiple regression with three explanatory variables, in which x_3 is the square of x_2. Models allowing both nonlinearity and interaction are also possible.

Nonparametric Regression*

Recent advances make it possible to fit models to data without assuming particular functional forms, such as straight lines or parabolas, for the relationship. These approaches are *nonparametric*, in terms of having fewer (if any) assumptions about the functional form and the distribution of y. It is helpful to look at a plot of a fitted nonparametric regression model to learn about trends in the data.

One nonparametric regression method, called ***generalized additive modeling***, is a further generalization of the generalized linear model. It has the form

$$g(\mu) = f_1(x_1) + f_2(x_2) + \cdots + f_k(x_k),$$

where $f_1, \ldots, f_k$ are unspecified and potentially highly complex functions. The GLM is the special case in which each of these functions is linear. The estimated functional form of the relationship for each predictor is determined by a computer algorithm, using the sample data. As in GLMs, with this model you can select a particular link function g and also a distribution for y. This model is useful for smoothing data to reveal overall trends.

Nonparametric regression is beyond the scope of this text. At this time, some statistical software do not yet have routines for generalized additive models. Many have related nonparametric smoothing methods that usually provide similar results. Popular smoothers are ***LOESS*** (sometimes also denoted by LOWESS and called "locally weighted scatterplot smoothing") and ***kernel*** methods that get the prediction at a particular point by smoothly averaging nearby values. The smoothed value is found by fitting a low-degree polynomial while giving more weight to observations near the point and less weight to observations further away. You can achieve greater smoothing by choosing a larger *bandwidth*, essentially by letting the weights die out more gradually as you move away from each given point.

Figure 14.9 shows two plots of nonparametric regression fits for the fertility rate data of Table 14.6. The first plot employs greater smoothing and has a curved,

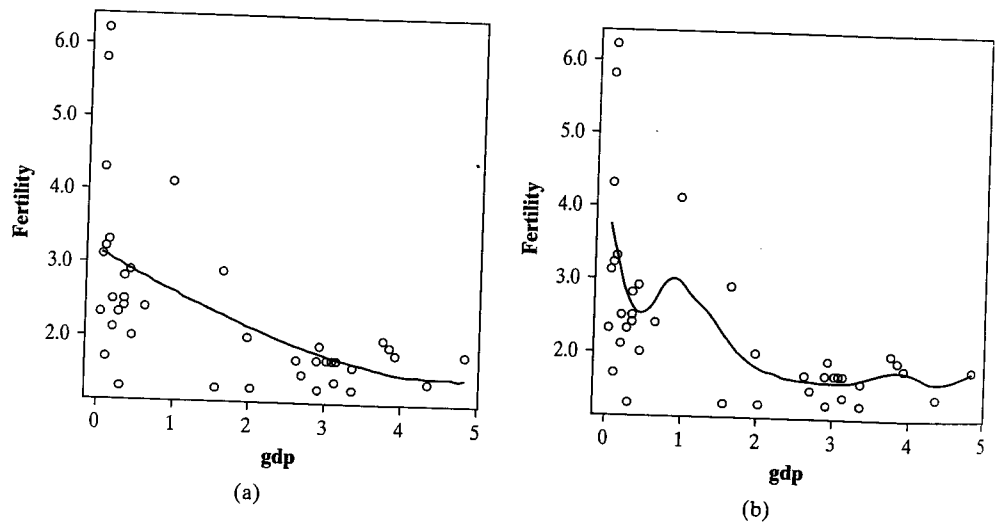


FIGURE 14.9: Fits of Nonparametric Regression Model to Smooth the Fertility Rate Data of Table 14.6. Fit (a) employs greater smoothing (bandwidth = 5 in SPSS) than fit (b) (bandwidth = 1 in SPSS).

decreasing trend. It is evident that the response may not eventually increase, as a quadratic model predicts. The next section discusses a model that provides a more satisfactory fit for these data.

14.6 EXPONENTIAL REGRESSION AND LOG TRANSFORMS*

Although polynomials provide a diverse collection of functions for modeling nonlinearity, other mathematical functions are often more appropriate. The most important case is when the mean of the response variable is an *exponential* function of the explanatory variable.

Exponential Regression Function

An ***exponential regression*** function has the form $E(y) = \alpha\beta^x$.

In this equation, the explanatory variable appears as the exponent of a parameter. Unlike a quadratic function, an exponential function can take only positive values, and it continually increases (if $\beta > 1$) or continually decreases (if $\beta < 1$). In either case, it has a convex shape, as Figure 14.10 shows. We provide interpretations for the model parameters later in this section.

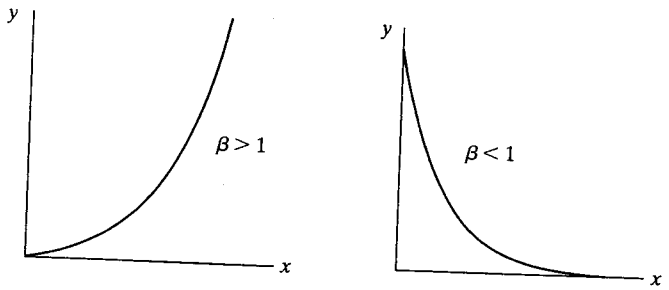


FIGURE 14.10: The Exponential Regression Function $E(y) = \alpha\beta^x$

For the exponential regression function, the *logarithm* of the mean is linearly related to the explanatory variable. That is, if $\mu = E(y) = \alpha\beta^x$, then

$$\log(\mu) = \log\alpha + (\log\beta)x.$$

The right-hand side of this equation has the straight line form $\alpha' + \beta' x$ with intercept $\alpha' = \log(\alpha)$ the log of the α parameter and slope $\beta' = \log(\beta)$ the log of the β parameter for the exponential regression function. This model form is the special case of a generalized linear model (GLM) using the log link function. If the model holds, a plot of the log of the y-values should show approximately a linear relation with x. (Don't worry if you have forgotten your high school math about logarithms. You will not need to know this in order to understand how to fit or interpret the exponential regression model.)

It is simple to use GLM software to estimate the parameters in the model $\log[E(y)] = \alpha' + \beta' x$. The antilogs of these estimates are the estimates for the parameters in the exponential regression model $E(y) = \alpha\beta^x$, as shown Example 14.8.

EXAMPLE 14.8 Exponential Population Growth

Exponential regression is often used to model the growth of a population over time. If the rate of growth remains constant, in percentage terms, then the size of that

population grows exponentially fast. Suppose that the population size at some fixed time is α and the growth rate is 2% per year. After 1 year, the population is 2% larger than at the beginning of the year. This means that the population size grows by a multiplicative factor of 1.02 each year. The population size after 1 year is $\alpha(1.02)$. Similarly, the population size after 2 years is

$$(\text{Population size at end of 1 year})(1.02) = [\alpha(1.02)]1.02 = \alpha(1.02)^2.$$

After 3 years, the population size is $\alpha(1.02)^3$. After x years, the population size is $\alpha(1.02)^x$. The population size after x years follows an exponential function $\alpha\beta^x$ with parameters given by the initial population size α and the rate of growth factor, $\beta = 1.02$, corresponding to 2% growth.

Table 14.8 shows the U.S. population size (in millions) at 10-year intervals beginning in 1890. Figure 14.11 plots these values over time. Table 14.8 also shows the natural logarithm of the population sizes. (This uses the base e, where $e = 2.718\ldots$ is an irrational number that appears often in mathematics. The model makes sense with logs to any base, but software fits the GLM using natural logs, denoted by $\log_e$ or by *LN*.) Figure 14.12 plots these log of population size values over time. The log population sizes appear to grow approximately linearly. This suggests that population growth over this time period was approximately exponential, with a constant rate of growth. We now estimate the regression curve, treating time as the explanatory variable x.

TABLE 14.8: Population Sizes and Log Population Sizes by Decade from 1890 to 2000, with Predicted Values for Exponential Regression Model

Year	No. Decades since 1890 x	Population Size y	$\log_e(y)$	$\hat{y}$
1890	0	62.95	4.14	71.5
1900	1	75.99	4.33	81.1
1910	2	91.97	4.52	92.0
1920	3	105.71	4.66	104.4
1930	4	122.78	4.81	118.5
1940	5	131.67	4.88	134.4
1950	6	151.33	5.02	152.5
1960	7	179.32	5.19	173.0
1970	8	203.30	5.31	196.3
1980	9	226.54	5.42	222.6
1990	10	248.71	5.52	252.6
2000	11	281.42	5.64	286.6

Source: U.S. Census Bureau.

For convenience, we identify the time points 1890, 1900, ..., 2000 as times 0, 1, ..., 11; that is, x represents the number of decades since 1890. We use software to estimate the generalized linear model $\log(\mu) = \alpha' + \beta' x$, assuming a normal distribution for y. The prediction equation, for natural logs, is

$$\log_e(\hat{\mu}) = 4.2698 + 0.1262x.$$

Antilogs of these estimates are the parameter estimates for the exponential regression model. For natural logs, the antilog function is the exponential function e^x. That is,

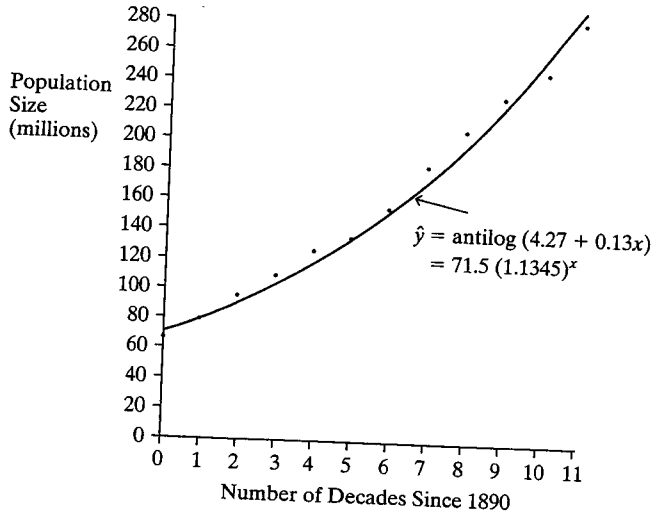


FIGURE 14.11: U.S. Population Size since 1890

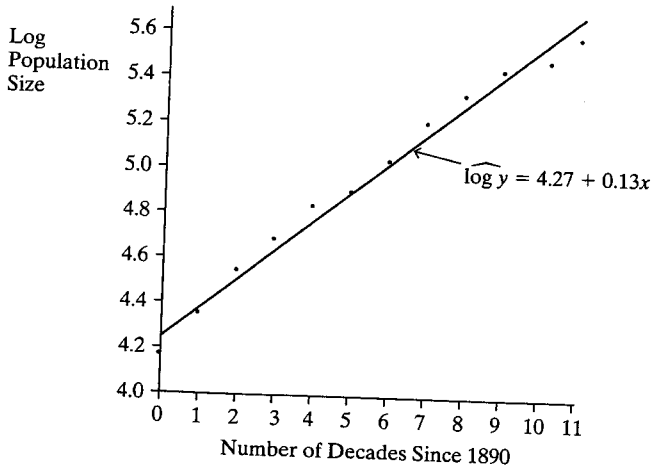


FIGURE 14.12: Log Population Sizes since 1890

antilog(4.2698) $= e^{4.2698} = 71.507$, and antilog(0.1262) $= e^{0.1262} = 1.1345$. (Most calculators have an e^x key that provides these antilogs.) Thus, for the exponential regression model $E(y) = \alpha\beta^x$, the estimates are $\hat{\alpha} = 71.507$ and $\hat{\beta} = 1.1345$. The prediction equation is

$$\hat{y} = \hat{\alpha}\hat{\beta}^x = 71.507(1.1345)^x.$$

The predicted initial population size (in 1890) is $\hat{\alpha} = 71.5$ million. The predicted population size x decades after 1890 equals $\hat{y} = 71.507(1.1345)^x$. For instance, $x = 11$, and the predicted population size is $\hat{y} = 71.507(1.1345)^{11} = 286.6$ million. Table 14.8 shows the predicted values for each decade. Figure 14.11 plots the exponential prediction equation.

The predictions are quite good. The total sum of squares of population size values about their mean equals TSS = 55,481, whereas the sum of squared errors

about the prediction equation is $SSE = 275$. The proportional reduction in error is $(55{,}481 - 275)/55{,}481 = 0.995$. ■

A caution: The fit of the model $\log[E(y)] = \alpha' + \beta' x$ that you get with GLM software will *not* be the same as you get by taking logarithms of all the y-values and then fitting a straight line model using least squares. The latter approach[3] gives the fit for the model $E[\log(y)] = \alpha' + \beta' x$. For that model, taking antilogs does not take you back to $E(y)$, because $E[\log(y)]$ is not equivalent to $\log[E(y)]$. So in software it is preferable to use a generalized linear modeling option rather than an ordinary regression option.

Interpreting Exponential Regression Models

Now let's take a closer look at how to interpret parameters in the exponential regression model, $E(y) = \alpha\beta^x$. The parameter α represents the mean of y when $x = 0$. The parameter β represents the ***multiplicative*** change in the mean of y for a one-unit increase in x. The mean of y at $x = 10$ equals β *multiplied* by the mean of y at $x = 9$. For instance, for the equation $\hat{y} = 71.507(1.1345)^x$, the predicted population size at a particular date equals 1.1345 times the predicted population size a decade earlier.

By contrast, the parameter β in the *linear* model $E(y) = \alpha + \beta x$ represents the ***additive*** change in the mean of y for a one-unit increase in x. In the linear model, the mean of y at $x = 10$ equals β *plus* the mean of y at $x = 9$. The prediction equation for the linear model (i.e., identity link) fitted to Table 14.8 equals $\hat{y} = 49.56 + 19.50x$. This model predicts that the population size increases by 19.50 million people every decade.

In summary, for the linear model, $E(y)$ changes by the same *quantity* for each one-unit increase in x, whereas for the exponential model, $E(y)$ changes by the same *percentage* for each one-unit increase. For the exponential regression model with Table 14.8, the predicted population size is multiplied by 1.1345 each decade. This equation corresponds to a predicted 13.45% growth per decade.

Suppose the growth rate is 15% per decade, to choose a rounder number. This corresponds to a multiplicative factor of 1.15. After five decades, the population grows by a factor of $(1.15)^5 = 2.0$. That is, after five decades, the population size doubles. If the rate of growth remained constant at 15% per decade, the population would double every 50 years. After 100 years, the population size would be quadruple the original size; after 150 years it would be 8 times as large; after 200 years it would be 16 times its original size; and so forth.

The exponential function with $\beta > 1$ has the property that its doubling time is a constant. As can be seen from the sequence of population sizes at 50-year intervals, this is an extremely fast increase even though the annual rate of growth (1.4% annually for a decade increase of 15%) seems small. In fact, the world population has been following an exponential growth pattern, with recent rate of growth over 15% per decade.

EXAMPLE 14.9 Exponential Regression for Fertility Rate Data

When $\beta < 1$ in the exponential regression model, $\beta' = \log(\beta) < 0$ in the log transformed GLM. In this case, the mean of y *decreases* exponentially fast as x increases. The curve then looks like the second curve in Figure 14.10.

In Example 14.6 with Table 14.6 (page 464), we modeled y = fertility rate for several countries, with x = per capita GDP. The nonparametric regression curve

[3]For example, as SPSS would give by selecting *Regression* in the *Analyze* menu, followed by the choice of *Curve Estimation* with the *Exponential* option.

(Figure 14.9) had appearance much like an exponentially decreasing curve. In fact, the exponential regression model provides a good fit for those data. Using the GLM with log link for y = fertility rate and x = per capita GDP and assuming a normal distribution for y, we get the prediction equation

$$\log_e(\hat{\mu}) = 1.148 - 0.206x.$$

Taking antilogs yields the exponential prediction equation

$$\hat{y} = \hat{\alpha}\hat{\beta}^x = e^{1.148}(e^{-0.206})^x = 3.15(0.81)^x.$$

The predicted fertility rate at GDP value $x + 1$ equals 81% of the predicted fertility rate at GDP value x; that is, it decreases by 19% for a $10,000 increase in per capita GDP.

With this fit, the correlation between the observed and predicted fertility rates equals 0.59, nearly as high as the value of 0.61 achieved with the quadratic model, which has an extra parameter. If we expect fertility rate to decrease continuously as GDP increases, the exponential regression model is a more realistic model than the quadratic regression model of Section 14.5, which predicted increasing fertility above a certain GDP level. Also, unlike the straight line model, the exponential regression model cannot yield negative predicted fertility rates.

Since the scatterplot in Figure 14.7 suggests greater variability when the mean fertility rate is higher, it may be even better to assume a gamma distribution for y with this exponential regression model. The prediction equation is then

$$\log_e(\hat{\mu}) = 1.112 - 0.177x, \quad \text{for which} \quad \hat{y} = e^{1.112}(e^{-0.177})^x = 3.04(0.84)^x.$$

This gives a slightly shallower rate of decrease than the fit $3.15(0.81)^x$ obtained assuming a normal response. ■

Transforming the Predictor to Achieve Linearity

Other transformations of the response mean or of explanatory variables are useful in some situations. For example, suppose y tends to increase or decrease over a certain range of x-values, but once a certain x-value has been reached, further increases in x have less effect on y, as in Figure 14.5b. For this concave increasing type of trend, x behaves like an exponential function of y. Taking the logarithms of the x-values often linearizes the relationship. Another possible transform for this case is to invert the x-values (i.e., use $1/x$ as the explanatory variable).

14.7 CHAPTER SUMMARY

This chapter discussed issues about building regression models. We have seen how to check assumptions of the basic regression model and how to ease some restrictions of this model.

- When a large number of terms might serve as explanatory variables, the ***backward elimination*** and ***forward selection*** procedures use a sequential algorithm to select variables for the model. These are exploratory in purpose and should be used with caution.
- Plots of the ***residuals*** check whether the model is adequate and whether the assumptions for inferences are reasonable. Observations having a large leverage and large studentized residual have a strong influence on the model fit. The

BATHS	19.203	5.650	3.40	0.0010
NEW	18.984	3.873	4.90	0.0001

DFBETA and DFFIT diagnostics describe which observations have a strong influence on the parameter estimates and the model fit.

- ***Multicollinearity***, the condition by which the set of explanatory variables contains some redundancies, causes inflation of standard errors of estimated regression coefficients and makes it difficult to evaluate partial effects.
- ***Generalized linear models*** allow the response variable to have a distribution other than the normal, such as the binomial for binary data and the gamma for positive responses having greater variation at greater mean values. Such models also permit modeling a function of the mean, called the ***link function***.
- ***Nonlinear*** relationships are modeled through the use of ***polynomial*** (particularly ***quadratic***) functions and ***exponential*** functions. Quadratic functions have a parabolic appearance, whereas exponential functions have a convex increasing or convex decreasing appearance. The ***exponential regression model*** is a generalized linear model for the log of the mean.

PROBLEMS

Practicing the Basics

14.1. For Example 11.2 (page 326) on y = mental impairment, x_1 = life events, and x_2 = SES, Table 11.5 showed the output

	B	Std. Error	t	Sig.
(Constant)	28.230	2.174	12.984	.000
LIFE	.103	.032	3.177	.003
SES	-.097	.029	-3.351	.002

and Table 11.8 showed the output for the interaction model,

	B	Std. Error	t	Sig
(Constant)	26.036649	3.948826	6.594	0.0001
LIFE	0.155865	0.085338	1.826	0.0761
SES	-0.060493	0.062675	-0.965	0.3409
LIFE*SES	-0.000866	0.001297	-0.668	0.5087

Table 11.4 showed that SES had P-value 0.011 in the bivariate model containing only that predictor, and Table 11.3 showed that LIFE had P-value of 0.018 in the bivariate model containing only that predictor. Select explanatory variables from the set $x_1, x_2, x_3 = x_1x_2$, with $\alpha = 0.05$

(a) Using backward elimination

(b) Using forward selection

14.2. Table 11.21 on page 363 showed results of a multiple regression using nine predictors of the quality of life in a country.

(a) In backward elimination with these nine predictors, can you predict which variable would be deleted (i) first? (ii) second? Explain.

(b) In forward selection with these nine predictors, can you predict which variable would be added first? Explain.

14.3. For the "house selling price 2" data file at the text Web site, Table 14.9 shows a correlation matrix and a model fit using four predictors of selling price. With these four predictors.

(a) For backward elimination, which variable would be deleted first? Why?

TABLE 14.9

Correlation coefficients

	price	size	beds	baths	new
price	1.00000	0.89881	0.59027	0.71370	0.35655
size	0.89881	1.00000	0.66911	0.66248	0.17629
beds	0.59027	0.66911	1.00000	0.33380	0.26721
baths	0.71370	0.66248	0.33380	1.00000	0.18207
new	0.35655	0.17629	0.26721	0.18207	1.00000

Variable	Estimate	Std. Error	t	Sig
INTERCEP	-41.795	12.104	3.45	0.0009
SIZE	64.761	5.630	11.50	0.0001
BEDS	-2.766	3.960	0.70	0.4868
BATHS	19.203	5.650	3.40	0.0010
NEW	18.984	3.873	4.90	0.0001