Vasicek Interest Rate Model for Australia

This assignment aims to clarify the understanding of one of the original interest rate models widely used in practice in applications like interest rate derivatives etc. In this short exercise the focus is on understanding the implementation issues of such a model in the context of Australian data. You need to read this document carefully before proceeding with any implementation attempt.

Vasicek's pioneering work (1977) is the first account of a bond pricing model that incorporates stochastic interest rate. The model essentially assumes a particular form of the short-term interest rate dynamic as,

$$dr = a(b - r)dt + \sigma_r dz, \quad dz \sim N\left(0, \sqrt{dt}\right) \tag{1}$$

where r is understood to be a function of the time t. From this, zero coupon bond prices of different maturities are developed. Details of this are beyond the scope of this course. Mamon (2004) is a good account of those relevant details. Here we are simply interested in implementing in Excel environment.

In terms of the notations in the literature the zero coupon bond price $P(t,s)$ at time t maturing at time s (s>t) is:

$$P(t,s) = A(t,s) \times \exp\left[-B(t,s) \times r(t)\right] \tag{2}$$

where A(t,s) and B(t,s) are defined by the following equations and these contain parameters of the model in equation (1) i.e. $\begin{pmatrix} a & b & \sigma_r \end{pmatrix}$. These are:

$$B(t,s) = \frac{1}{a}\left[1 - e^{-a(s-t)}\right] \tag{3}$$

$$\ln A(t,s) = \left[B(t,s) - (s-t)\right]\left(b - \frac{\sigma_r^2}{2a^2}\right) - \frac{\sigma_r^2 B^2(t,s)}{4a} \tag{4}$$

Also, from equation (2) the yield to maturity for various maturities may be obtained by,

$$y(t,s) = -\frac{1}{(s-t)}\ln\left[P(t,s)\right]. \tag{5}$$

It is obvious that to implement Vasicek model we need three parameters $\begin{pmatrix} a & b & \sigma_r \end{pmatrix}$. Thus, the zero coupon bond prices and the yield to maturity also require these parameter estimates. In the literature various authors have suggested suitable ways to achieve this. Here we will adopt the maximum likelihood approach (steps outlined later in this document). Further details may be found in Ferguson and Platen (2015).

In the accompanying spreadsheet "S1_2018_ACTL2111_Vasicek_Aus_Data" you are given interest rate data from the Australian market. To estimate the Vasicek model parameters, you need to set up the maximum likelihood procedure using this data and following the equation (1) above. Once the Vasicek model parameters are estimated you can then compute the zero coupon bond prices (equation 2) as well as the yield to maturity (equation 5).

Your task is to generate the term structure of interest rates for terms ranging from 1-10 years, 20 and 30 years. In other words the maturity periods are similar to that of US interest rate market.

We are aiming here for you develop some understanding of the interest rate modeling via this simple exercise and at the same time develop good Excel skill. This is the stepping stone into the complex, but exciting area of interest rate derivatives.

References:

Ferguson, K. and Platen, E. (2015), Application of Maximum Likelihood Estimation to Stochastic Short Rate Models, Quantitative Finance Research Centre, UTS, Research Paper 361.

Mamon, R. S. (2004), Three Ways to Solve for Bond Prices in the Vasicek Model, Journal Of Applied Mathematics and Decision Sciences, 8(1), 1–14.

Vasicek, O. (1977), An Equilibrium characterization of the Term Structure, Journal of Financial Economics, 5: 177-188.

Maximum Likelihood Estimates of the Vasicek Parameters

The discrete version of equation (1) is:

$$\Delta r_t = r_t - r_{t-1} = a\left(b - r_{t-1}\right)\Delta t + \sigma_r \Delta z, \quad \Delta z \sim N\left(0, \sqrt{\Delta t}\right).$$

This is clearly states that the change in interest rate Δr is normally distributed. For the observation at the time period, t, the mean and the variance may be written as:

$$\text{Mean} = \mu = a\left(b - r_{t-1}\right)\Delta t, \ \text{Variance} = \sigma^2 = \sigma_r^2 \Delta t. \tag{6}$$

For a normally distributed random variable x, with mean μ and variance σ^2, the probability density function is:

$$f\left(x\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x-\mu\right)^2}{2\sigma^2}}. \tag{7}$$

In term of equation (6) the probability density function becomes:

$$f\left(\Delta r_t\right) = \frac{1}{\sqrt{2\pi\sigma_r^2 \Delta t}} e^{-\frac{\left(\Delta r_t - a\left(b - r_{t-1}\right)\Delta t\right)^2}{2\sigma_r^2 \Delta t}} \tag{8}$$

You can thus write the density function for each observation and create log density and finally log likelihood function. You maximize that log likelihood function with respect to the three parameters $\left(a \quad b \quad \sigma_r\right)$ to obtain their estimates. These estimated parameters may then be used to compute bond prices using Vasicek formula given earlier.

Main Deliverables:

You will need to submit a spreadsheet that clearly shows:

- The parameters estimated by the maximum likelihood method, as explained earlier in this document, using the data provided for this purpose,
- A suitable table that shows the yield to maturity by Vasicek model for the periods 1 to 10 years (increment of 1 year), 20 years and 30 years, and a plot of yield versus maturity period with meaningful labels for the axes.

Clarification of equation (2) in Vasicek model:

In terms of the notations in the literature the zero coupon bond price $P(t,s)$ at time t maturing at time s (s>t) is:

$$P(t,s) = A(t,s) \times \exp\left[-B(t,s) \times r(t)\right] \tag{1}$$

In this exercise we are pricing the bonds at time (now) i.e. t = 0. So, the very first entry in the interest rate data is treated as time zero value of r(t).

The zero coupon bond price for another maturity, say 4 years, s = 4. This maturity dependency is captured by the functions A and B. In this instance you calculate A and B as A(0,4) and B(0,4) using the formulas given.

Basic Form of Likelihood Expressions for Time Series Data:

Univariate time series with normally distributed error:

Time series: $x_t = \mu + \varepsilon_t,\ \varepsilon_t \sim N\left(0, \sigma^2\right)$

The time series is characterised by mean μ and variance σ^2:

Under the above assumption of the random error term, the density function for any observation:

$$f\left(x_t\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x_t - \mu\right)^2}{2\sigma^2}}$$

In most cases, we use log (ln) of the density function.

$$\ln f\left(x_t\right) = \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(x_t-\mu\right)^2}{2\sigma^2}}\right) = \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \ln\left(e^{-\frac{\left(x_t-\mu\right)^2}{2\sigma^2}}\right)$$

$$= \ln\left(2\pi\sigma^2\right)^{-\frac{1}{2}} - \frac{\left(x_t - \mu\right)^2}{2\sigma^2} = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln\sigma^2 - \frac{1}{2}\frac{\left(x_t - \mu\right)^2}{\sigma^2}$$

This is log density and when it is aggregated over all observations in the time series of data it is called log likelihood function.

Assume that we have 100 observations of x_t , t=1,2,3..100.

$$\text{So, } f\left(x_t\right)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\left(x_t-\mu\right)^2}{2\sigma^2}}\text{, for each } x_t$$

If we assume that all observations are independent, then likelihood function is simply the product of all these density functions.

$$\text{likh}=\prod_{t=1}^{100} f\left(x_t\right)$$

When we use log likelihood function then this product becomes summation, i.e.

$$\ln\ \text{likh}=\sum_{t=1}^{100}\ln f\left(x_t\right)$$

Thus,

$$\ln \text{likh} = -\frac{1}{2}\sum_{t=1}^{100} \ln 2\pi - \frac{1}{2}\sum_{t=1}^{100} \ln \sigma^2 - \frac{1}{2}\sum_{t=1}^{100} \frac{(x_t - \mu)^2}{\sigma^2}$$

This log likelihood function is the template for most empirical models you are likely to come across. Depending on the problem being examined, definitions and forms of x, μ, and σ will change.

For the assignment problem you need to carefully specify what μ should be given the description of Vasicek model.