Positive and monotone fragments of FO and LTL

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Context

A wonderful world of words

First-Order Logic over finite words (FO):

- Signature: $(<, \Sigma)$ where $\Sigma = \{a, b, c, \dots\}$ is a set of **unary predicates**.
- Syntax of FO:

$$\phi, \psi ::= a(x) \mid x = y \mid x < y \mid \bot \mid \top \mid \\ \phi \vee \psi \mid \phi \wedge \psi \mid \exists x, \phi(x) \mid \forall x, \phi(x) \mid \neg \phi.$$

$$\circ \qquad \circ \qquad \circ \qquad \circ \qquad \circ \qquad \circ \qquad \circ$$

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$$\begin{array}{ccccccc} & & & & & c & \\ & b & & & & b & \\ a & a & & b & a & a & a \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ \end{array}$$

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position x is to the left of position y

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 & & & & & c & \\
 & b & & & & b & \\
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- Example:

$$\exists x, (a(x) \wedge \forall y > x, b(y) \vee c(y))$$

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$$bcc\binom{b}{d}\binom{b}{c}\binom{a}{b}cc\binom{b}{d}bb \models \exists x, (a(x) \wedge \forall y > x, b(y) \vee c(y))$$

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- $\subset$ extended to A^*: $abc \leq_{A^*} a\binom{b}{c}\binom{a}{c}$.

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- FO^+: **positive** formulas i.e. without $\neg$.
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- In FO^+: $\exists x, a(x)$
  Not in FO^+: $\forall x, a(x) \implies b(x)$
- $\phi \in \mathrm{FO}^+ \implies [\![\phi]\!]$ monotone.

Lyndon on finite structures

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on finite words

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- Our goals:
    1. Positivity VS monotonicity on fragments.
    2. Preservation of robustness properties.

Equivalences between positive fragments of FO and LTL

- FO^3: only 3 variables.
- Example:

  $\exists x < y < z, b(x) \wedge c(y) \wedge b(z) \wedge (\forall y, x < y < z \implies a(y))$

					c				
		b	a	a	a	a	a	b	
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
		x			y			z	

- FO^3: only 3 variables.
- Example:
  $\exists x < y < z, b(x) \land c(y) \land b(z) \land (\forall y, x < y < z \implies a(y))$

$$\begin{array}{cccccccccc}
 & & & & & c & & & & \\
 & & b & a & a & a & a & a & b & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
 & & x & & & y & & & z &
\end{array}$$

- FO^2: only 2 variables.
- Example: $\exists x, y, x < y \land a(x) \land b(y) \land (\exists x, x < y \land c(y))$

$$\begin{array}{cccccccccc}
 & & a & & & b & & & c & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
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FO^3 and FO^2

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$$
\begin{array}{cccccccccc}
 & & & & & c & & & & \\
 & & b & a & a & a & a & a & b & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
 & & x & & & y & & & z &
\end{array}
$$

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$$
\begin{array}{cccccccccc}
 & & a & & & b & & & c & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
 & & x & & & y & & & x &
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$$

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- Note: The signature of FO^2 can be enriched with $y = x + 1$.
- $\mathrm{FO}^3 \equiv \mathrm{FO}$ (Kamp 1968).
- $\mathrm{FO}^2 \subsetneq \mathrm{FO}^3$.

$\mathrm{FO}^{+} \equiv \mathrm{FO}^{3+} \equiv \mathrm{LTL}^{+}$

Linear Temporal Logic over finite words (LTL):

- Syntax of LTL:

$$\phi, \psi \ ::= \ a \mid \bot \mid \top \mid \phi \vee \psi \mid \phi \wedge \psi \mid \mathrm{X}\phi \mid \phi \mathrm{U} \psi \mid \neg\phi$$

$$\begin{array}{cccccccccc} \phi & \phi & \phi & \phi & \psi & & & & \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \models \phi \mathrm{U} \psi \\ t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & \end{array}$$

- Example: $\binom{a}{c} b \binom{b}{c} b b \binom{a}{b} c c \models \mathrm{X}(b \mathrm{U} a)$
- $\mathrm{FO} \equiv \mathrm{FO}^{3} \equiv \mathrm{LTL}$ (Kamp 1968).

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$\mathrm{FO}^{2+} \equiv \mathrm{UTL}^{+}$

Unary Temporal Logic over finite words (UTL):

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$$\begin{array}{cccccccccc} & & & \phi & \mathrm{Y}\phi & & & & & \\ \circ & \circ & \circ & \circ & \bullet & \circ & \circ & \circ & \circ & \circ \\ t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 & t_9 \end{array}$$

Unary Temporal Logic over finite words (UTL):

				$\mathrm{F}\phi$			ϕ		
$\circ$	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$
t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9

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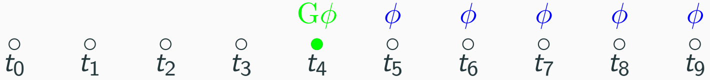
Unary Temporal Logic over finite words (UTL):

$$\begin{array}{cccccccccc} & \phi & & & \mathrm{P}\phi & & & & & \\ \circ & \circ & \circ & \circ & \bullet & \circ & \circ & \circ & \circ & \circ \\ t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 & t_9 \end{array}$$

Unary Temporal Logic over finite words (UTL):



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Unary Temporal Logic over finite words (UTL):

ϕ	ϕ	ϕ	ϕ	$\mathrm{H}\phi$					
t_0	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9

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Unary Temporal Logic over finite words (UTL):

- $\mathrm{FO}^2[<, +1] \equiv \mathrm{UTL}$ and $\mathrm{FO}^2[<] \equiv \mathrm{UTL}[\mathrm{P}, \mathrm{F}, \mathrm{H}, \mathrm{G}]$ (Etessami, Vardi, and Wilke 1997).

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- $\mathrm{FO}^{2+}[<, +1] \equiv \mathrm{UTL}^{+}$ and $\mathrm{FO}^{2+}[<] \equiv \mathrm{UTL}^{+}[\mathrm{P}, \mathrm{F}, \mathrm{H}, \mathrm{G}]$ (this work).

Open problem

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Can any FO^2-definable monotone language be defined by an FO^{2+} formula? Is definability by FO^{2+} decidable?

Elements of answer

- FO^{2+}[between] $\neq$ FO^2[between]-monotone (between: Krebs et al. 2016).

$$\circ \quad \underset{x}{\circ} \quad \circ \quad \overset{a}{\circ} \quad \circ \quad \circ \quad \underset{y}{\circ} \quad \circ \quad \models a(x, y)$$

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- FO^{2+}(no alternation) $\neq \mathrm{FO}^2$(no alternation)-monotone.
- $\mathrm{FO}^{2+} \overset{?}{\subseteq} \mathrm{FO}^2$-monotone $= \Sigma_2^+ \cap \Pi_2^+ \subset \mathrm{FO}^+$.

Thank you for your attention!

- Lyndon (1959). **"Properties preserved under homomorphism."**. In: *Pacific J. Math.*
- Ajtai and Gurevich (1987). **"Monotone versus Positive"**. In: *J. ACM*.
- Stolboushkin (1995). **"Finitely Monotone Properties"**. In: *LICS*. IEEE Computer Society.
- Kuperberg (2023). **"Positive First-order Logic on Words and Graphs"**. In: *Logical Methods in Computer Science*.
- Kamp (1968). **"Tense Logic and the Theory of Linear Order"**. PhD thesis. University of California Los Angeles.
- Etessami, Vardi, and Wilke (1997). **"First-Order Logic with Two Variables and Unary Temporal Logic"**. In: *BRICS Report Series*.
- Krebs et al. (2016). **"Two-variable Logic with a Between Relation"**. In: *LICS*. ACM.