In [181…

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from scipy.interpolate import interp1d
from scipy.integrate import cumulative_trapezoid
from matplotlib.patches import Ellipse
from numpy.polynomial import Polynomial
from pandas.plotting import autocorrelation_plot
import corner
from IPython.display import display, HTML
# from functools import lru_cache

# Base Constants
H_0 = 70  # km/s/Mpc
c = 2.99792458e5  # km/s
M = -19.3146267582  # SN Absolute Magnitude
M_n = 25 - 5*np.log10(H_0) #Script_M
Omega_R = 1e-8

# Test Guess Constants
Omega_K = 0.0001
Omega_M = 0.27 - Omega_R - Omega_K
Omega_DE = 1 - Omega_M
w = -1

# Supernova data
data = pd.read_csv('SCPUnion2.1_mu_vs_z.csv', sep='\t', comment='#', usecols=[0, 1, 2, 3], names=['SNe', 'z', '

z = data['z'].values
m = data['m'].values
m_err = data['m_err'].values

# Covariance matrix
data_cov = (pd.read_csv('SCPUnion2.1_covmat_nosys.csv', sep=r'\s+', header=None)).values

# Calculation of the cumulative integral r(z)
def r_scalar_incremental(z, Omega_M, Omega_DE, w, Omega_K):

    # Creating a finer array from the sorted and unique z values
    z_min, z_max = 0, np.max(z)
    z_fine = np.linspace(z_min, z_max, 580)

    # Interpolate original integrand values to the finer z array
    original_integrand = 1 / (H_0 * np.sqrt(np.abs(Omega_M * (1 + z_fine)**3 + Omega_DE * (1 + z_fine)**(3 * (1

    # Adjust integrand based on Omega_K
    if Omega_K == 0:
        integrand_values_fine = c * original_integrand
    elif Omega_K < 0:
        integrand_values_fine = c * (1 / (H_0 * np.sqrt(np.abs(Omega_K)))) * np.sinh(original_integrand / (1 /
    else:
        integrand_values_fine = c * (1 / (H_0 * np.sqrt(Omega_K))) * np.sin(original_integrand / (1 / (H_0 * np

    # Compute the cumulative integral using the finer differential elements and integrand
    cumulative_integral_fine = cumulative_trapezoid(integrand_values_fine, z_fine, initial=0)

    interpolator = interp1d(z_fine, cumulative_integral_fine, kind='cubic')
    r_values = interpolator(z)

    return r_values

# Calculation of d_l for each z_i
def d_L(z, Omega_M, Omega_DE, w, Omega_K):
    return (1 + z) * r_scalar_incremental(z, Omega_M, Omega_DE, w, Omega_K)

# Calculation of m(z, {pj})
def m_th(z_i, Omega_M, Omega_DE, w, M_n, Omega_K):
    return 5 * np.log10(H_0*d_L(z_i, Omega_M, Omega_DE, w, Omega_K)) + M_n

# Calculation of (m - m^th)
def m_diff_vector(z_i, m, Omega_M, Omega_DE, w, M_n, Omega_K):
    return m - m_th(z_i, Omega_M, Omega_DE, w, M_n, Omega_K)

m_diff_vector(z, m, Omega_M, Omega_DE, w, M_n, Omega_K)
```

Out[181…

```
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```

In [143…

```python
# FIM Case 1

def fisher_matrix(z, m_err, Omega_M, w, M_n):
    # Define parameters
    params = {'Omega_M': Omega_M, 'Omega_DE': 1-Omega_M-Omega_R-Omega_K, 'w': w, 'M_n': M_n}
    vary_params_case_one = ['Omega_M', 'w', 'M_n']
    param_keys = list(vary_params_case_one)
    n_params = len(param_keys)
    fisher = np.zeros((n_params, n_params))
    h = 1e-3

    for i in range(n_params):
        for j in range(n_params):
            base_params = params.copy() #remove

            # Modify parameters i and j
            base_params[param_keys[i]] += h
            m_th1 = m_th(z, **base_params, Omega_K=0)

            base_params[param_keys[i]] -= 2 * h
            m_th2 = m_th(z, **base_params, Omega_K=0)

            # Reset to original
            base_params[param_keys[i]] += h

            # Central difference for partial derivatives
            dm_dpi = (m_th1 - m_th2) / (2 * h)
```

```
            if i == j:  # Diagonal elements, derivative wrt same parameter
                fisher[i, j] = np.sum((dm_dpi**2) / (m_err**2))
            else:  # Off-diagonal elements, mixed partial derivatives
                base_params[param_keys[j]] += h
                m_th1 = m_th(z, **base_params, Omega_K=0)

                base_params[param_keys[j]] -= 2 * h
                m_th2 = m_th(z, **base_params, Omega_K=0)

                dm_dpj = (m_th1 - m_th2) / (2 * h)
                fisher[i, j] = np.sum((dm_dpi * dm_dpj) / (m_err**2))

    return fisher

F_flat = fisher_matrix(z, m_err, Omega_M, w, M_n)

print("Fisher Matrix (Flat Universe):")
print(F_flat)
```

```
Fisher Matrix (Flat Universe):
[[ 38488.08984695   6056.51401578 -26831.98300159]
 [  6056.51401578   1156.18464212  -3863.50573659]
 [-26831.98300159  -3863.50573659  19434.07720999]]
```

In [144…

```
# Calculation of Chi^2 (With FIM variance)

inv_cov_matrix = np.linalg.inv(data_cov)

def chi_squared(inv_cov, z, m, Omega_M, Omega_DE, w, M_n, Omega_K):

    m_diff = m_diff_vector(z, m, Omega_M, Omega_DE, w, M_n, Omega_K)
    # Calculate the chi-squared value
    chi_squared1 = np.dot(m_diff, np.dot(inv_cov, m_diff))

    return chi_squared1

# Redefine m_diff_vector with outside array for z
# Add data as a func arg
chi_squared(inv_cov_matrix, z, m, Omega_M, Omega_DE, w, M_n, Omega_K)
```

Out[144… 562.5625084965494

In [145…

```
# Calculation of the likelihood for Omega_M, w, and M_n

cov = data_cov
sign, log_det = np.linalg.slogdet(cov)
n = cov.shape[0]

def log_likelihood_w_m_mn(inv_cov, log_det, n, z, m, Omega_M, Omega_DE, w, M_n, Omega_K):
    # Calculate the log of the determinant and the sign of the determinant for the regularized matrix

    log_likelihood4 = -0.5 * n * np.log(2 * np.pi) - 0.5 * log_det - 0.5 * chi_squared(inv_cov, z, m, Omega_M, (

    return log_likelihood4

print(log_likelihood_w_m_mn(inv_cov_matrix, log_det, n, z, m, Omega_M, Omega_DE, w, M_n, Omega_K))
```

```
118.57234070013243
```

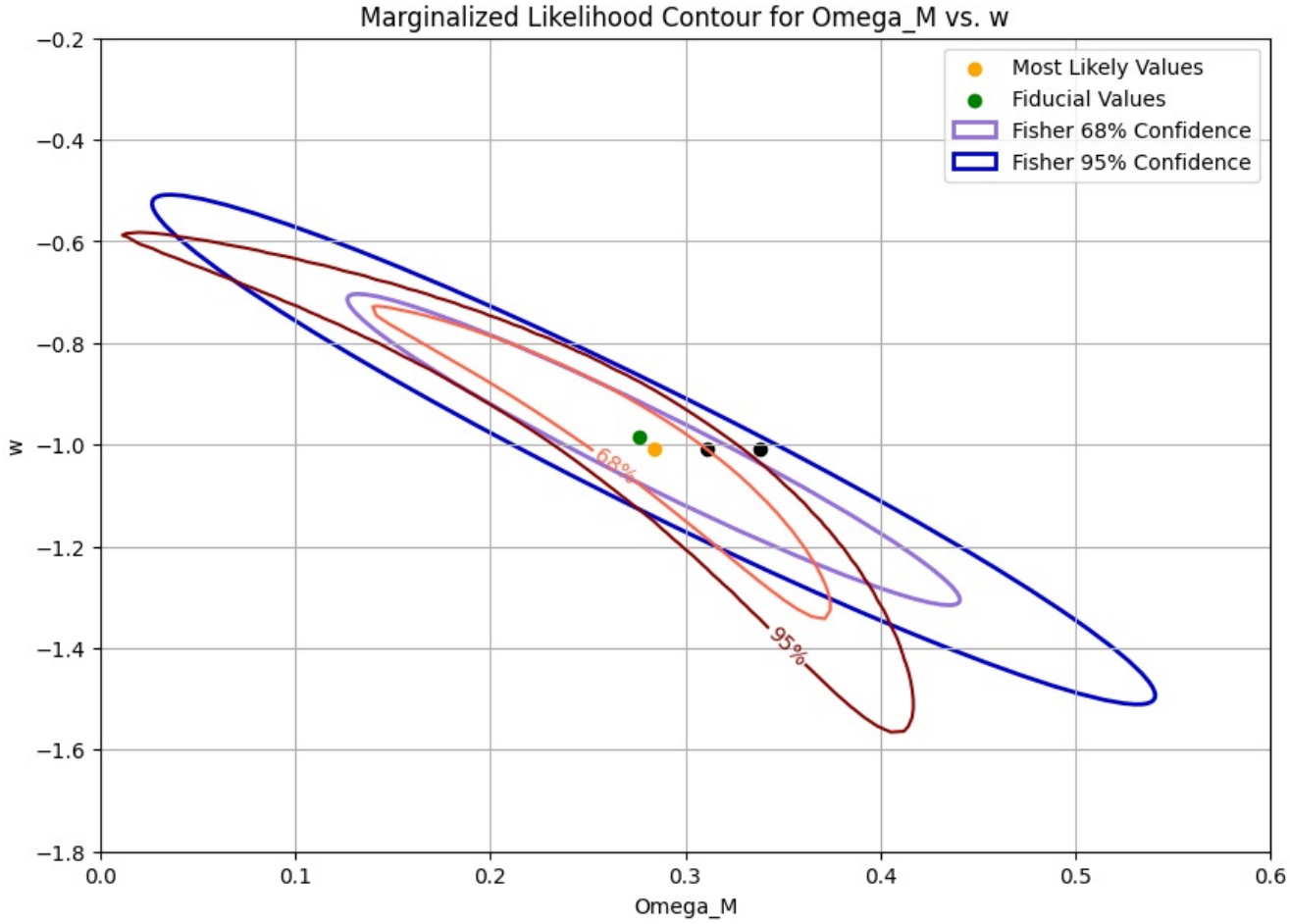
In [158…

```
# Case 1: Varying w, Omega_M, M_n

# Fixed values
fixed_omega_m = 0.276 # From Union 2.1 paper
fixed_w = -0.985 # From Union 2.1 paper
Omega_K = 0.0001
k = 0

# Ranges for Omega_M, w, and H_0
omega_m_min1, omega_m_max1 = 0, 0.5
w_min1, w_max1 = -1.8, -0.5
mn_min1, mn_max1 = 15.5, 16

# Values for Omega_M, w, and H_0
omega_m_range1 = np.linspace(omega_m_min1, omega_m_max1, 75)
w_range1 = np.linspace(w_min1, w_max1, 75)
mn_range1 = np.linspace(mn_min1, mn_max1, 50)

# Meshgrids
omega_m_grid1, w_grid1, mn_grid1 = np.meshgrid(omega_m_range1, w_range1, mn_range1, indexing='ij')

raveled_m_grid1 = np.ravel(omega_m_grid1)
raveled_w_grid1 = np.ravel(w_grid1)
raveled_mn_grid1 = np.ravel(mn_grid1)
```

```
# Calculate likelihoods, normalize
likelihoods_w_m_mn = np.array([log_likelihood_w_m_mn(inv_cov_matrix, log_det, n, z, m, Omega_M, 1 - Omega_M - Or
                        for Omega_M, w, M_n in zip(raveled_m_grid1, raveled_w_grid1, raveled_mn_grid1)])

likelihoods_w_m_mn = likelihoods_w_m_mn.reshape(omega_m_grid1.shape)
for row in likelihoods_w_m_mn: # Resolve bug in abnormal increase in likelihood
    row[0] = row[1]

likelihoods_w_m_mn_norm = np.exp(likelihoods_w_m_mn - np.max(likelihoods_w_m_mn))

# Marginalize over M_n
marg_likelihoods_w_m_mn = np.sum(likelihoods_w_m_mn_norm, axis=2)

# Most likely values
index_max_w_m_mn = np.argmax(marg_likelihoods_w_m_mn)
max_index_w_m_mn = np.unravel_index(index_max_w_m_mn, marg_likelihoods_w_m_mn.shape)

most_likely_omega_m1 = omega_m_grid1[max_index_w_m_mn][0]
most_likely_w1 = w_grid1[max_index_w_m_mn][0]

print(f"Most likely Omega_M: {most_likely_omega_m1}")
print(f"Most likely w: {most_likely_w1}")

# FIM Ellipses
fisher_at_most_likely = fisher_matrix(z, m_err, most_likely_omega_m1, most_likely_w1, M_n)

def plot_fisher_confidence_ellipse(ax, fisher_matrix, mean, confidence_level):
    cov_mat = np.linalg.inv(fisher_matrix)
    marg_cov_mat_mn = np.delete((np.delete(cov_mat, 2, axis=1)), 2, axis=0)

    fixed_fisher_mat_w = np.delete((np.delete(fisher_matrix, 1, axis=1)), 1, axis=0)
    fixed_cov_mat_w = np.linalg.inv(fixed_fisher_mat_w)

    sig_x_2 = marg_cov_mat_mn[0,0]
    sig_xy = marg_cov_mat_mn[1,0]
    sig_y_2 = marg_cov_mat_mn[1,1]

    term_1 = (sig_x_2 + sig_y_2) / 2
    term_2 = np.sqrt(((sig_x_2 - sig_y_2)**2 / 4) + sig_xy**2)

    a = np.sqrt(term_1 + term_2)
    b = np.sqrt(term_1 - term_2)
    theta = np.degrees(np.arctan2((2 * sig_xy), (sig_x_2 - sig_y_2)) / 2)

    if confidence_level == 2.30:
        lbl = 'Fisher 68% Confidence'
        color = 'mediumpurple'
    else:
        lbl = 'Fisher 95% Confidence'
        color = 'mediumblue'

    ellipse = Ellipse(xy=mean, width=2*np.sqrt(confidence_level)*a, height=2*np.sqrt(confidence_level)*b, angle=
    ax.add_patch(ellipse)

    std = np.sqrt(fixed_cov_mat_w[0,0])
    boundary_right_68 = most_likely_omega_m1 + std
    boundary_right_95 = most_likely_omega_m1 + 2*std

    return boundary_right_68, boundary_right_95

# Confidence levels
sorted_likelihoods_w_m_mn = np.sort(marg_likelihoods_w_m_mn.ravel())
cumulative_sum_w_m_mn = np.cumsum(sorted_likelihoods_w_m_mn)
norm_cumulative_sum_w_m_mn = cumulative_sum_w_m_mn / cumulative_sum_w_m_mn[-1]

level_68_w_m_mn = sorted_likelihoods_w_m_mn[np.searchsorted(norm_cumulative_sum_w_m_mn, 1 - 0.68)]
level_95_w_m_mn = sorted_likelihoods_w_m_mn[np.searchsorted(norm_cumulative_sum_w_m_mn, 1 - 0.95)]

# Contour plot
omega_m_grid1_slice = omega_m_grid1[:, :, 0]
w_grid1_slice = w_grid1[:, :, 0]
plt.figure(figsize=(10, 7))
ax = plt.gca()
cp = plt.contour(omega_m_grid1_slice, w_grid1_slice, marg_likelihoods_w_m_mn, levels=[level_95_w_m_mn, level_68_
plt.scatter([most_likely_omega_m1], [most_likely_w1], color='orange', label='Most Likely Values', zorder=5)
plt.clabel(cp, inline=True, fontsize=10, fmt={level_68_w_m_mn: '68%', level_95_w_m_mn: '95%'})
plt.scatter([fixed_omega_m], [fixed_w], color='green', label='Fiducial Values', zorder=5)

# Plot Fisher confidence ellipses
boundary_right_sig1, boundary_right_sig2 = plot_fisher_confidence_ellipse(ax, fisher_at_most_likely, (most_like
plot_fisher_confidence_ellipse(ax, fisher_at_most_likely, (most_likely_omega_m1, most_likely_w1), 6.17)
```

```
plt.scatter(boundary_right_sig1, most_likely_w1, color='black')
plt.scatter(boundary_right_sig2, most_likely_w1, color='black')
plt.title('Marginalized Likelihood Contour for Omega_M vs. w')
plt.xlabel('Omega_M')
plt.ylabel('w')
plt.xlim(0, 0.6)
plt.ylim(-1.8, -0.2)
plt.legend()
plt.grid(True)
plt.show()

# Differences in computed values to Union 2.1 values:
omega_m_diff = np.abs(fixed_omega_m - most_likely_omega_m1)
w_diff = np.abs(fixed_w - most_likely_w1)
print(f"Diff in Omega_M: {omega_m_diff}")
print(f"Diff in w: {w_diff}")

# Percentage error
omega_m_pe = omega_m_diff/fixed_omega_m * 100
w_pe = w_diff/fixed_w * 100
print(f"% err in Omega_M: {omega_m_pe}%")
print(f"% err in w: {w_pe}%")
```

```
Most likely Omega_M: 0.28378378378378377
Most likely w: -1.0094594594594595
```




```
Diff in Omega_M: 0.007783783783783749
Diff in w: 0.024459459459459487
% err in Omega_M: 2.8202115158636767%
% err in w: -2.4831938537522324%
```

In [6]:
```
# Case 1 MCMC

Omega_K = 0.0001

# Pre-generate necessary arrays
def pre_arrays(samples, fish_mat):
    inv_fish = np.linalg.inv(fish_mat)
    array0 = np.random.multivariate_normal(mean=[0,0,0], cov=inv_fish, size=samples)
    array1 = array0[:, 0]
    array2 = array0[:, 1]
    array3 = array0[:, 2]
    array4 = np.ones(samples)
    array5 = np.ones(samples)
    array6 = np.ones(samples)
    array7 = np.ones(samples)
    return array1, array2, array3, array4, array5, array6, array7
```

In [7]:
```python
# Gelman-Rubin R function

def gelman_rubin(chains):
    _, num_samples = chains.shape

    chain_means = np.mean(chains, axis=1)

    # Between-chain variance
    B = num_samples * np.var(chain_means, ddof=1)

    # Within-chain variance
    W = np.mean([np.var(chain, ddof=1) for chain in chains])

    # Estimate of target variance
    V = W * (num_samples - 1) / num_samples + B / num_samples

    # Gelman-Rubin R stat
    R = np.sqrt(V / W)

    return R
```

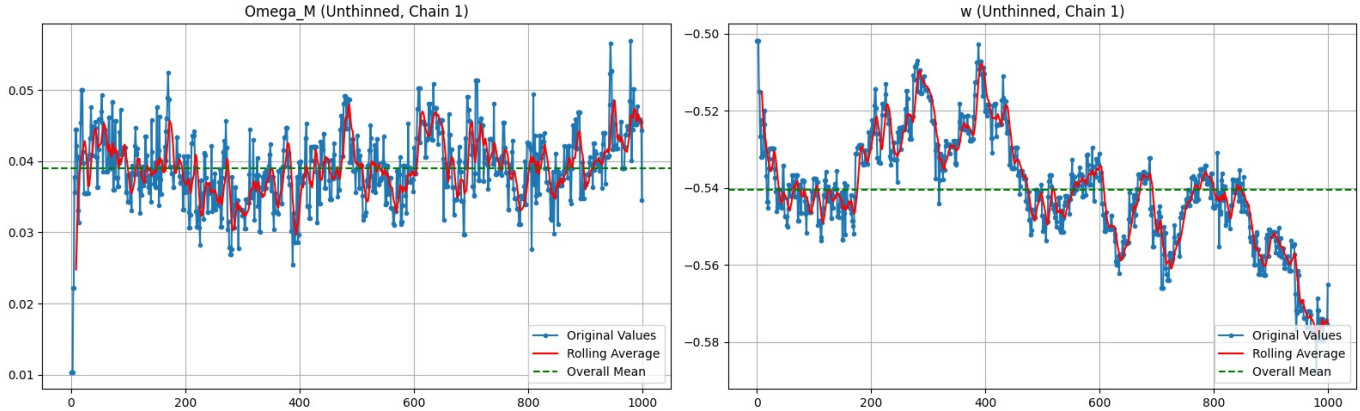
In [8]:
```python
# Step accept/reject

def accept_or_reject(samples, burn_in, fisher_mat, Omega_M, w, M_n):

    steps_Omega_M, steps_w, steps_M_n, Omega_M_values, w_values, M_n_values, log_likelihoods = pre_arrays(sample

    total_accepted_count = 0
    accepted_count = 0

    for i in range(samples):
        # Change to array for 3 parameters
        new_Omega_M = Omega_M + steps_Omega_M[i]
        new_w = w + ( steps_w[i] / 3 )
        new_M_n = M_n + ( steps_M_n[i] / 2 )
        likelihood = 0

        norm_num = np.exp(log_likelihood_w_m_mn(inv_cov_matrix, log_det, n, z, m, new_Omega_M, 1 - new_Omega_M,
        norm_denom = np.exp(log_likelihood_w_m_mn(inv_cov_matrix, log_det, n, z, m, Omega_M, 1 - Omega_M, w, M_

        if new_Omega_M >= 1 or new_Omega_M <= 0 or new_w > 0:
            r = 0
        else:
            r = norm_num / norm_denom

        # Accept or reject the new value
        if np.random.rand() < r:
            Omega_M, w, M_n, likelihood = new_Omega_M, new_w, new_M_n, norm_num
            total_accepted_count += 1

            if i >= burn_in:
                accepted_count += 1
        else:
            likelihood = norm_denom

        # Store values for all steps
        Omega_M_values[i] = Omega_M
        w_values[i] = w
        M_n_values[i] = M_n
        log_likelihoods[i] = np.log(likelihood)

        # Tracking
        if (i + 1) % 50000 == 0 and i >= burn_in:
            interim_percent_accepted = (accepted_count / ((i + 1) - burn_in)) * 100
            print(f"% of accepted values: {np.round(interim_percent_accepted, 3)} % @ {i+1}th iteration")

    # Burn-in
    Omega_M_values = Omega_M_values[burn_in:]
    w_values = w_values[burn_in:]
    M_n_values = M_n_values[burn_in:]
    log_likelihoods = log_likelihoods[burn_in:]

    percent_accepted = (accepted_count / (samples - burn_in)) * 100

    return Omega_M_values, w_values, M_n_values, log_likelihoods, percent_accepted


num_samples = 1000
burn_in_period = 15
selected_thinning_value = 2
fisher_matrix = fisher_at_most_likely
start_Omega_M = 0.28
start_w = -1
```

```
start_M_n = M_n

chain_1_Omega_M_values, chain_1_w_values, chain_1_M_n_values, chain_1_log_likelihoods, chain_1_percent_accepted

# Will deviating the starting value from its true value increase/decrease the % of accepted values?
```

```
% of accepted values: 20.91 % @ 200000th iteration
% of accepted values: 20.925 % @ 250000th iteration
% of accepted values: 21.005 % @ 300000th iteration
% of accepted values: 20.857 % @ 350000th iteration
% of accepted values: 20.89 % @ 400000th iteration
% of accepted values: 20.863 % @ 450000th iteration
% of accepted values: 20.845 % @ 500000th iteration
% of accepted values: 20.888 % @ 550000th iteration
% of accepted values: 20.923 % @ 600000th iteration
% of accepted values: 20.901 % @ 650000th iteration
% of accepted values: 20.858 % @ 700000th iteration
% of accepted values: 20.866 % @ 750000th iteration
% of accepted values: 20.853 % @ 800000th iteration
% of accepted values: 20.842 % @ 850000th iteration
% of accepted values: 20.859 % @ 900000th iteration
% of accepted values: 20.858 % @ 950000th iteration
% of accepted values: 20.883 % @ 1000000th iteration
```

In [9]:
```
# Chain 1 Thinning

thinned_chain_1_Omega_M_values = chain_1_Omega_M_values[::selected_thinning_value]
thinned_chain_1_w_values = chain_1_w_values[::selected_thinning_value]
thinned_chain_1_M_n_values = chain_1_M_n_values[::selected_thinning_value]
thinned_chain_1_log_likelihoods = chain_1_log_likelihoods[::selected_thinning_value]
```

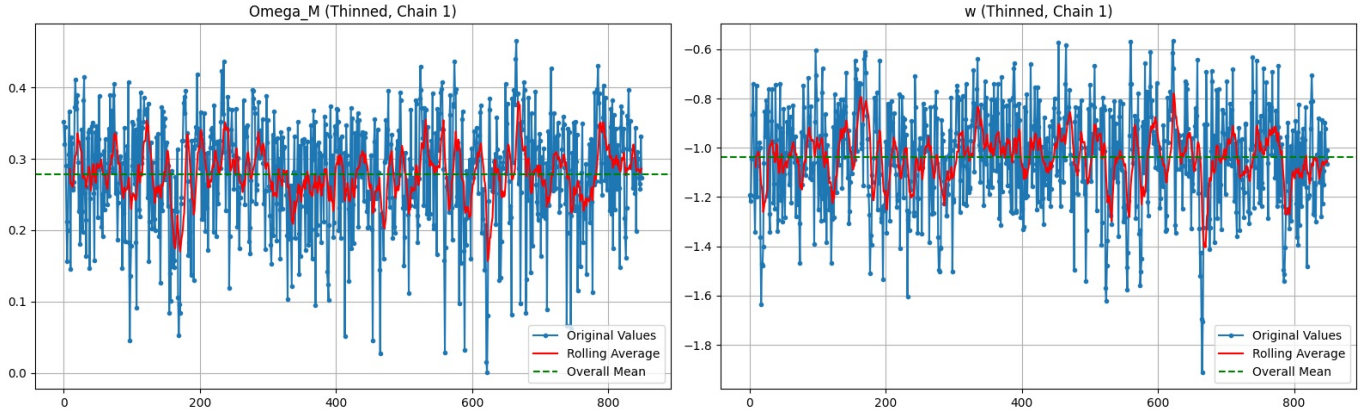
In [ ]:
```
# Chain 1 Unthinned Averages

fig, axs = plt.subplots(1, 2, figsize=(16, 5))  # Adjust the subplots for two plots side by side.

# Plot for Omega_M values
axs[0].plot(chain_1_Omega_M_values, marker='.', label='Original Values')
rolling_avg_Omega_M = pd.Series(chain_1_Omega_M_values).rolling(window=10).mean()
axs[0].plot(rolling_avg_Omega_M, color='red', label='Rolling Average')
mean_value_Omega_M = pd.Series(chain_1_Omega_M_values).mean()
axs[0].axhline(y=mean_value_Omega_M, color='green', linestyle='--', label='Overall Mean')
axs[0].grid()
axs[0].legend(loc='lower right')
axs[0].set_title('Omega_M (Unthinned, Chain 1)')

# Plot for w values
axs[1].plot(chain_1_w_values, marker='.', label='Original Values')
rolling_avg_w = pd.Series(chain_1_w_values).rolling(window=10).mean()
axs[1].plot(rolling_avg_w, color='red', label='Rolling Average')
mean_value_w = pd.Series(chain_1_w_values).mean()
axs[1].axhline(y=mean_value_w, color='green', linestyle='--', label='Overall Mean')
axs[1].grid()
axs[1].legend(loc='lower right')
axs[1].set_title('w (Unthinned, Chain 1)')

plt.tight_layout()
plt.show()

print(f'Unthinned Mean for Omega_M is: {np.round(mean_value_Omega_M, 3)}')
print(f'Unthinned Mean for w is: {np.round(mean_value_w, 3)}')

# Check chart to determine burn in and thinning # Don't use this one though ofc
```




```
Unthinned Mean for Omega_M is: 0.039
Unthinned Mean for w is: -0.54
```

In [11]:
```
# Chain 1 Thinned Averages
```

```
fig, axs = plt.subplots(1, 2, figsize=(16, 5))  # Adjust the subplots for two plots side by side.

# Plot for Omega_M values
axs[0].plot(thinned_chain_1_Omega_M_values, marker='.', label='Original Values')
rolling_avg_Omega_M = pd.Series(thinned_chain_1_Omega_M_values).rolling(window=10).mean()
axs[0].plot(rolling_avg_Omega_M, color='red', label='Rolling Average')
mean_value_Omega_M = pd.Series(thinned_chain_1_Omega_M_values).mean()
axs[0].axhline(y=mean_value_Omega_M, color='green', linestyle='--', label='Overall Mean')
axs[0].grid()
axs[0].legend(loc='lower right')
axs[0].set_title('Omega_M (Thinned, Chain 1)')

# Plot for w values
axs[1].plot(thinned_chain_1_w_values, marker='.', label='Original Values')
rolling_avg_w = pd.Series(thinned_chain_1_w_values).rolling(window=10).mean()
axs[1].plot(rolling_avg_w, color='red', label='Rolling Average')
mean_value_w = pd.Series(thinned_chain_1_w_values).mean()
axs[1].axhline(y=mean_value_w, color='green', linestyle='--', label='Overall Mean')
axs[1].grid()
axs[1].legend(loc='lower right')
axs[1].set_title('w (Thinned, Chain 1)')

plt.tight_layout()
plt.show()

print(f'Thinned Mean for Omega_M is: {np.round(mean_value_Omega_M, 3)}')
print(f'Thinned Mean for w is: {np.round(mean_value_w, 3)}')
```

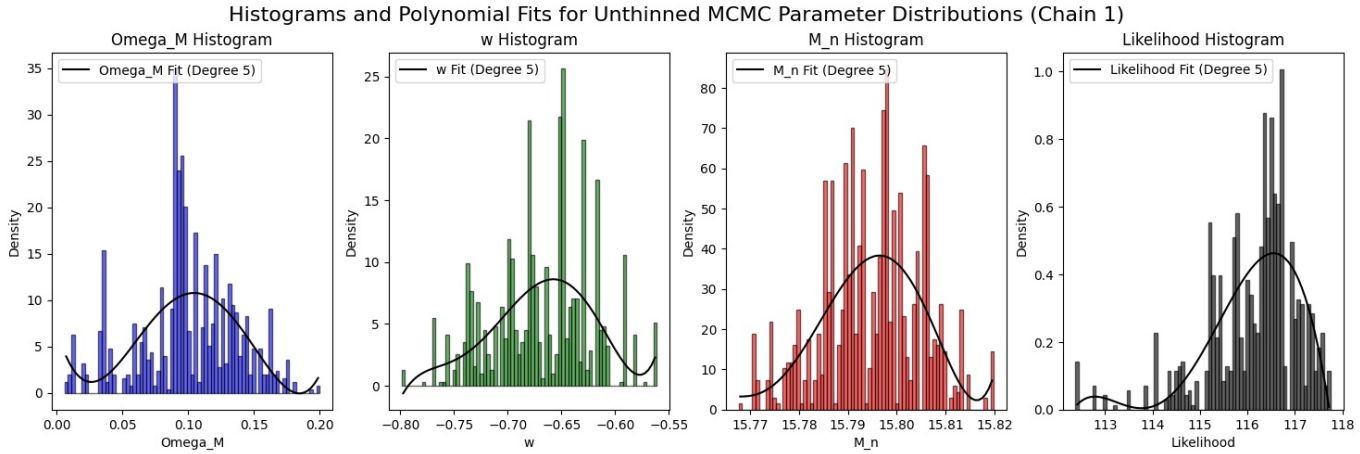



```
Thinned Mean for Omega_M is: 0.278
Thinned Mean for w is: -1.039
```

In [262…

```
# Chain 1 Unthinned Histograms

def plot_histograms_with_fit(degree, degree_max, params, chain_1_all_values):

    # Begin plot
    fig, axes = plt.subplots(1, len(params), figsize=(18, 5))
    num_bins = 75
    colors = ['blue', 'green', 'red', 'black']

    def fit_polynomial_and_find_peak(values, ax, poly_degree, degree_max, param_name, color):
        # Histogram data
        counts, bin_edges = np.histogram(values, bins=num_bins, density=True)
        bin_centers = (bin_edges[:-1] + bin_edges[1:]) / 2

        # Prepare array for peak values and fit polynomials
        peak_value_array = np.zeros(degree_max + 1)

        # Fit polynomial and find peaks for each degree
        for i in range(1, degree_max + 1):
            p = Polynomial.fit(bin_centers, counts, deg=i)
            fitted_values = p(bin_centers)
            peak_index = np.argmax(fitted_values)
            peak_value = bin_centers[peak_index]
            peak_value_array[i] = peak_value

        # Peak value Estimations
        avg_peak_value = np.mean(peak_value_array[2:])
        std_peak_value = np.std(peak_value_array[2:])

        # Fit and plot the polynomial of the specified degree (poly_degree) for visualization
        selected_poly = Polynomial.fit(bin_centers, counts, deg=poly_degree)
        selected_fitted_values = selected_poly(bin_centers)

        # Plot histogram and the selected polynomial fit
        ax.hist(values, bins=num_bins, color=color, alpha=0.6, edgecolor='black', density=True)
        ax.plot(bin_centers, selected_fitted_values, color='black', label=f'{param_name} Fit (Degree {poly_degr
        ax.set_title(f'{param_name} Histogram')
```

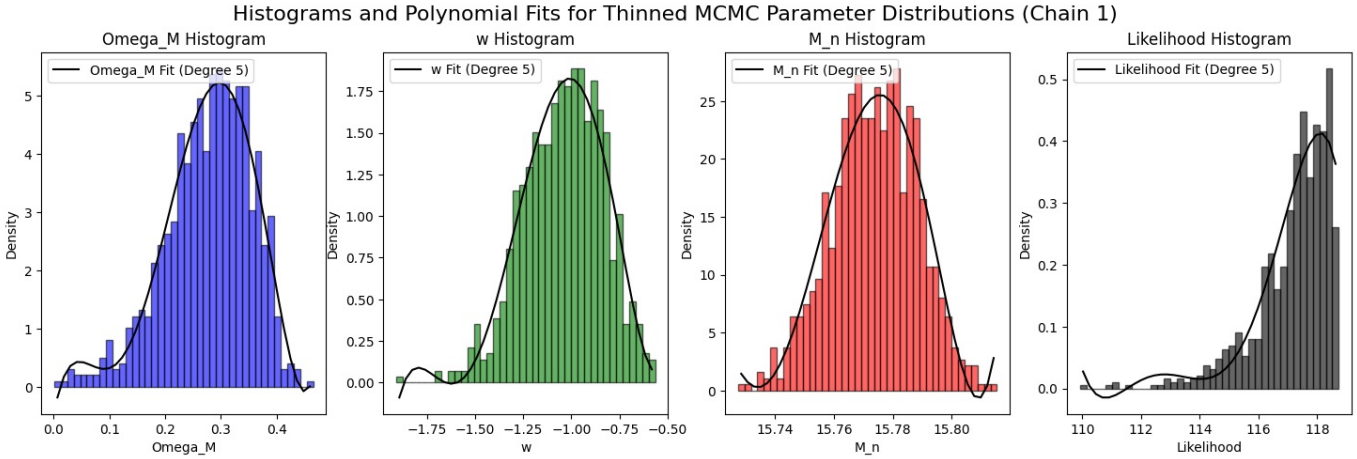
```
        ax.set_xlabel(param_name)
        ax.set_ylabel('Density')
        ax.legend(loc='upper left')

        return avg_peak_value, std_peak_value

    # Plot calls
    converged_values = np.zeros(len(params))
    converged_values_errors = np.zeros(len(params))

    for i, param in enumerate(params):
        converged_values[i], converged_values_errors[i] = fit_polynomial_and_find_peak(chain_1_all_values[i], a

    fig.suptitle('Histograms and Polynomial Fits for Unthinned MCMC Parameter Distributions (Chain 1)', fontsiz

    plt.show()

    return converged_values, converged_values_errors

# Plot histograms and fit polynomial curves for the 3 parameters
poly_degree = 5
max_degree = 12
params = ['Omega_M', 'w', 'M_n', 'Likelihood']
chain_1_all_values = np.array([chain_1_Omega_M_values, chain_1_w_values, chain_1_M_n_values, chain_1_log_likeli

conv_vals, conv_vals_errs = plot_histograms_with_fit(poly_degree, max_degree, params, chain_1_all_values)
```



In [259…

```
# Chain 1 Thinned Histograms

def plot_histograms_with_fit(degree, degree_max, params, chain_1_all_values):

    # Begin plot
    fig, axes = plt.subplots(1, len(params), figsize=(18, 5))
    num_bins = 40
    colors = ['blue', 'green', 'red', 'black']

    def fit_polynomial_and_find_peak(values, ax, poly_degree, degree_max, param_name, color):
        # Histogram data
        counts, bin_edges = np.histogram(values, bins=num_bins, density=True)
        bin_centers = (bin_edges[:-1] + bin_edges[1:]) / 2

        # Prepare array for peak values and fit polynomials
        peak_value_array = np.zeros(degree_max + 1)

        # Fit polynomial and find peaks for each degree
        for i in range(1, degree_max + 1):
            p = Polynomial.fit(bin_centers, counts, deg=i)
            fitted_values = p(bin_centers)
            peak_index = np.argmax(fitted_values)
            peak_value = bin_centers[peak_index]
            peak_value_array[i] = peak_value

        # Peak value Estimations
        avg_peak_value = np.mean(peak_value_array[2:])
        std_peak_value = np.std(peak_value_array[2:])

        # Fit and plot the polynomial of the specified degree (poly_degree) for visualization
        selected_poly = Polynomial.fit(bin_centers, counts, deg=poly_degree)
        selected_fitted_values = selected_poly(bin_centers)

        # Plot histogram and the selected polynomial fit
        ax.hist(values, bins=num_bins, color=color, alpha=0.6, edgecolor='black', density=True)
        ax.plot(bin_centers, selected_fitted_values, color='black', label=f'{param_name} Fit (Degree {poly_degr
        ax.set_title(f'{param_name} Histogram')
        ax.set_xlabel(param_name)
```

```
        ax.set_ylabel('Density')
        ax.legend(loc='upper left')

        return avg_peak_value, std_peak_value

    # Plot calls
    converged_values = np.zeros(len(params))
    converged_values_errors = np.zeros(len(params))

    for i, param in enumerate(params):
        converged_values[i], converged_values_errors[i] = fit_polynomial_and_find_peak(chain_1_all_values[i], a

    fig.suptitle('Histograms and Polynomial Fits for Thinned MCMC Parameter Distributions (Chain 1)', fontsize=

    plt.show()

    return converged_values, converged_values_errors

# Plot histograms and fit polynomial curves for the 3 parameters
poly_degree = 5
max_degree = 12
params = ['Omega_M', 'w', 'M_n', 'Likelihood']
chain_1_all_thinned_values = np.array([thinned_chain_1_Omega_M_values, thinned_chain_1_w_values, thinned_chain_

thinned_conv_vals, thinned_conv_vals_errs = plot_histograms_with_fit(poly_degree, max_degree, params, chain_1_a
```



In [14]:
```
# Chain 1 Unthinned Results, Using Polynomial Averaging

for i in range(len(conv_vals)):
    print(f"Estimated Unthinned {params[i]} by Polynomial Averaging: {np.round(conv_vals[i], 3)} ± {np.round(co
```

```
Estimated Unthinned Omega_M by Polynomial Averaging: 0.297 ± 0.013
Estimated Unthinned w by Polynomial Averaging: -1.0 ± 0.027
Estimated Unthinned M_n by Polynomial Averaging: 15.775 ± 0.003
Estimated Unthinned Likelihood by Polynomial Averaging: 118.315 ± 0.258
```

In [15]:
```
# Chain 1 Thinned Results, Using Polynomial Averaging

for i in range(len(thinned_conv_vals)):
    print(f"Estimated Thinned {params[i]} by Polynomial Averaging: {np.round(thinned_conv_vals[i], 3)} ± {np.ro
```

```
Estimated Thinned Omega_M by Polynomial Averaging: 0.299 ± 0.013
Estimated Thinned w by Polynomial Averaging: -0.985 ± 0.029
Estimated Thinned M_n by Polynomial Averaging: 15.774 ± 0.001
Estimated Thinned Likelihood by Polynomial Averaging: 118.286 ± 0.237
```

In [16]:
```
# Chain 1 Thinned/Unthinned Percent Errors, Using Polynomial Averaging

likely_error_thinned_unthinned_diff_chain_1 = np.zeros(len(params))

for i in range(len(params)):
    likely_error_thinned_unthinned_diff_chain_1[i] = ( np.abs(conv_vals[i] - thinned_conv_vals[i]) / thinned_co

    print(f'Percent error (Polynomial Averaging) between thinned and unthinned estimates for {params[i]}: {np.r
```

```
Percent error (Polynomial Averaging) between thinned and unthinned estimates for Omega_M: 0.884 %
Percent error (Polynomial Averaging) between thinned and unthinned estimates for w: -1.539 %
Percent error (Polynomial Averaging) between thinned and unthinned estimates for M_n: 0.003 %
Percent error (Polynomial Averaging) between thinned and unthinned estimates for Likelihood: 0.024 %
```

In [17]:
```
# Chain 1 Unthinned Results, Using Indexing

likely_log_likelihood_chain_1_index = np.argmax(chain_1_log_likelihoods)

tolerance = 0.002
```

```
for i, log_likelihood in enumerate(chain_1_log_likelihoods):
    if abs(log_likelihood - chain_1_log_likelihoods[likely_log_likelihood_chain_1_index]) < tolerance:
        likely_omega_m_chain_1 = chain_1_Omega_M_values[i]
        likely_w_chain_1 = chain_1_w_values[i]
        likely_M_n_chain_1 = chain_1_M_n_values[i]
        likely_log_likelihood_chain_1 = chain_1_log_likelihoods[i]

print(f'The most likely unthinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_omeg
print(f'The most likely unthinned w value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_w_chain_1
print(f'The most likely unthinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_M_n_cha
print(f'The most likely unthinned log likelihood (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_lo
```

```
The most likely unthinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: 0.284
The most likely unthinned w value (Likelihood Indexing) for MCMC Chain 1 is: -1.018
The most likely unthinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: 15.774
The most likely unthinned log likelihood (Likelihood Indexing) for MCMC Chain 1 is: 118.74

The most likely unthinned w value (Likelihood Indexing) for MCMC Chain 1 is: -1.018
The most likely unthinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: 15.774
The most likely unthinned log likelihood (Likelihood Indexing) for MCMC Chain 1 is: 118.74
```

In [18]:
```
# Chain 1 Thinned Results, Using Indexing

thinned_likely_log_likelihood_chain_1_index = np.argmax(thinned_chain_1_log_likelihoods)

tolerance = 0.002

for i, thinned_log_likelihood in enumerate(thinned_chain_1_log_likelihoods):
    if abs(thinned_log_likelihood - thinned_chain_1_log_likelihoods[thinned_likely_log_likelihood_chain_1_index
        thinned_likely_omega_m_chain_1 = thinned_chain_1_Omega_M_values[i]
        thinned_likely_w_chain_1 = thinned_chain_1_w_values[i]
        thinned_likely_M_n_chain_1 = thinned_chain_1_M_n_values[i]
        thinned_likely_log_likelihood_chain_1 = thinned_chain_1_log_likelihoods[i]

print(f'The most likely thinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(thinned_like
print(f'The most likely thinned w value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(thinned_likely_w_c
print(f'The most likely thinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(thinned_likely_M
print(f'The most likely thinned log likelihood (Likelihood Indexing) for MCMC Chain 1 is: {np.round(thinned_lik
```

```
The most likely thinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: 0.276
The most likely thinned w value (Likelihood Indexing) for MCMC Chain 1 is: -0.996
The most likely thinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: 15.774
The most likely thinned log likelihood (Likelihood Indexing) for MCMC Chain 1 is: 118.737
```

In [19]:
```
# Chain 1 Thinned/Unthinned Percent Errors, Using Likelihood Indexing

likely_omega_m_error_thinned_unthinned_diff_chain_1 = ( np.abs(likely_omega_m_chain_1 - thinned_likely_omega_m_
likely_w_error_thinned_unthinned_diff_chain_1 = ( np.abs(likely_w_chain_1 - thinned_likely_w_chain_1) / thinned_
likely_M_n_error_thinned_unthinned_diff_chain_1 = ( np.abs(likely_M_n_chain_1 - thinned_likely_M_n_chain_1) / th
likely_log_likelihood_error_thinned_unthinned_diff_chain_1 = ( np.abs(likely_log_likelihood_chain_1 - thinned_li

print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for Omega_M: {np.round(like
print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for w: {np.round(likely_w_e
print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for M_n: {np.round(likely_M_
print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for log_likelihood: {np.rou
```

```
Percent error (Likelihood Indexing) between thinned and unthinned estimates for Omega_M: 2.733 %
Percent error (Likelihood Indexing) between thinned and unthinned estimates for w: -2.257 %
Percent error (Likelihood Indexing) between thinned and unthinned estimates for M_n: 0.006 %
Percent error (Likelihood Indexing) between thinned and unthinned estimates for log_likelihood: 0.003 %
```

In [20]:
```
# Chain 1 Unthinned Results, using Posteriors

# Unthinned Posteriors
posteriors = np.exp(chain_1_log_likelihoods - np.max(chain_1_log_likelihoods))
posteriors /= np.sum(posteriors)

# Parameter Estimations
Omega_M_estimate = np.sum(chain_1_Omega_M_values * posteriors)
w_estimate = np.sum(chain_1_w_values * posteriors)
M_n_estimate = np.sum(chain_1_M_n_values * posteriors)

# Parameter Uncertainties
Omega_M_std = np.sqrt(np.sum(posteriors * (chain_1_Omega_M_values - Omega_M_estimate) ** 2))
w_std = np.sqrt(np.sum(posteriors * (chain_1_w_values - w_estimate) ** 2))
M_n_std = np.sqrt(np.sum(posteriors * (chain_1_M_n_values - M_n_estimate) ** 2))

print(f"Estimated Unthinned Omega_M using Posteriors: {np.round(Omega_M_estimate, 3)} ± {np.round(Omega_M_std, 
print(f"Estimated Unthinned w using Posteriors: {np.round(w_estimate, 3)} ± {np.round(w_std, 3)}")
print(f"Estimated Unthinned M_n using Posteriors: {np.round(M_n_estimate, 3)} ± {np.round(M_n_std, 3)}")
```

```
Estimated Unthinned Omega_M using Posteriors: 0.278 ± 0.055
Estimated Unthinned w using Posteriors: -1.021 ± 0.146
Estimated Unthinned M_n using Posteriors: 15.774 ± 0.01
```

In [21]:
```
# Chain 1 Thinned Results, using Posteriors

# Thinned Posteriors
thinned_posteriors_chain_1 = np.exp(thinned_chain_1_log_likelihoods - np.max(thinned_chain_1_log_likelihoods))
thinned_posteriors_chain_1 /= np.sum(thinned_posteriors_chain_1)

# Parameter Estimations
thinned_Omega_M_estimate_chain_1 = np.sum(thinned_chain_1_Omega_M_values * thinned_posteriors_chain_1)
thinned_w_estimate_chain_1 = np.sum(thinned_chain_1_w_values * thinned_posteriors_chain_1)
thinned_M_n_estimate_chain_1 = np.sum(thinned_chain_1_M_n_values * thinned_posteriors_chain_1)

# Parameter Uncertainties
thinned_Omega_M_std = np.sqrt(np.sum(thinned_posteriors_chain_1 * (thinned_chain_1_Omega_M_values - thinned_Omeg
thinned_w_std = np.sqrt(np.sum(thinned_posteriors_chain_1 * (thinned_chain_1_w_values - thinned_w_estimate_chai
thinned_M_n_std = np.sqrt(np.sum(thinned_posteriors_chain_1 * (thinned_chain_1_M_n_values - thinned_M_n_estimat

print(f"Estimated Thinned Omega_M using Posteriors: {np.round(thinned_Omega_M_estimate_chain_1, 3)} ± {np.round
print(f"Estimated Thinned w using Posteriors: {np.round(thinned_w_estimate_chain_1, 3)} ± {np.round(thinned_w_s
print(f"Estimated Thinned M_n using Posteriors: {np.round(thinned_M_n_estimate_chain_1, 3)} ± {np.round(thinned_
```

```
Estimated Thinned Omega_M using Posteriors: 0.277 ± 0.056
Estimated Thinned w using Posteriors: -1.019 ± 0.15
Estimated Thinned M_n using Posteriors: 15.774 ± 0.011
```

In [22]:
```
# Chain 1 Thinned/Unthinned Percent Errors, Using Posteriors

posterior_likely_omega_m_error_thinned_unthinned_diff_chain_1 = ( np.abs(Omega_M_estimate - thinned_Omega_M_esti
posterior_likely_w_error_thinned_unthinned_diff_chain_1 = ( np.abs(w_estimate - thinned_w_estimate_chain_1) / th
posterior_likely_M_n_error_thinned_unthinned_diff_chain_1 = ( np.abs(M_n_estimate - thinned_M_n_estimate_chain_1

print(f'Percent error (with Posteriors) between thinned and unthinned estimates for Omega_M: {np.round(posterio
print(f'Percent error (with Posteriors) between thinned and unthinned estimates for w: {np.round(posterior_like
print(f'Percent error (with Posteriors) between thinned and unthinned estimates for M_n: {np.round(posterior_li

# Would an increase/decrease in these percent errors mean anything?
```

```
Percent error (with Posteriors) between thinned and unthinned estimates for Omega_M: 0.433 %
Percent error (with Posteriors) between thinned and unthinned estimates for w: -0.117 %
Percent error (with Posteriors) between thinned and unthinned estimates for M_n: 0.0 %
```

In [23]:
```
# Run multiple chains

def run_multiple_chains(num_chains, num_samples, burn_in_period, step_manual, Omega_M, w, M_n):
    all_chains_Omega_M = np.zeros((num_chains, num_samples-burn_in_period))
    all_chains_w = np.zeros((num_chains, num_samples-burn_in_period))
    all_chains_M_n = np.zeros((num_chains, num_samples-burn_in_period))
    all_chains_log_likelihoods = np.zeros((num_chains, num_samples-burn_in_period))
    accepted_percentages = np.zeros(num_chains)

    for i in range(num_chains):
        Omega_M_values, w_values, M_n_values, likelihoods, percent_accepted = accept_or_reject(num_samples, burn

        print(f'Chain {i+1} complete.')

        all_chains_Omega_M[i, :] = Omega_M_values
        all_chains_w[i, :] = w_values
        all_chains_M_n[i, :] = M_n_values
        all_chains_log_likelihoods[i, :] = likelihoods
        accepted_percentages[i] = percent_accepted

    # Compute R
    rhat_Omega_M = gelman_rubin(all_chains_Omega_M)
    rhat_w = gelman_rubin(all_chains_w)
    rhat_M_n = gelman_rubin(all_chains_M_n)

    return (all_chains_Omega_M, all_chains_w, all_chains_M_n, all_chains_log_likelihoods,
            rhat_Omega_M, rhat_w, rhat_M_n, accepted_percentages)


# Run multiple chains, checking convergence
num_chains = 7
all_chains_Omega_M, all_chains_w, all_chains_M_n, all_chains_log_likelihoods, r_Omega_M, r_w, r_M_n, accepted_p
run_multiple_chains(num_chains, num_samples, burn_in_period, fisher_matrix, start_Omega_M, start_w, start_M_n)
```

```
% of accepted values: 20.68 % @ 200000th iteration
% of accepted values: 20.892 % @ 250000th iteration
% of accepted values: 20.854 % @ 300000th iteration
% of accepted values: 21.017 % @ 350000th iteration
% of accepted values: 20.945 % @ 400000th iteration
% of accepted values: 21.046 % @ 450000th iteration
% of accepted values: 21.098 % @ 500000th iteration
% of accepted values: 21.071 % @ 550000th iteration
% of accepted values: 21.041 % @ 600000th iteration
% of accepted values: 21.071 % @ 650000th iteration
```

```
% of accepted values: 21.056 % @ 700000th iteration
% of accepted values: 21.032 % @ 750000th iteration
% of accepted values: 21.032 % @ 800000th iteration
% of accepted values: 20.965 % @ 850000th iteration
% of accepted values: 20.958 % @ 900000th iteration
% of accepted values: 20.952 % @ 950000th iteration
% of accepted values: 20.958 % @ 1000000th iteration
Chain 1 complete.
% of accepted values: 20.722 % @ 200000th iteration
% of accepted values: 20.769 % @ 250000th iteration
% of accepted values: 20.675 % @ 300000th iteration
% of accepted values: 20.787 % @ 350000th iteration
% of accepted values: 20.705 % @ 400000th iteration
% of accepted values: 20.776 % @ 450000th iteration
% of accepted values: 20.781 % @ 500000th iteration
% of accepted values: 20.801 % @ 550000th iteration
% of accepted values: 20.89 % @ 600000th iteration
% of accepted values: 20.856 % @ 650000th iteration
% of accepted values: 20.905 % @ 700000th iteration
% of accepted values: 20.928 % @ 750000th iteration
% of accepted values: 20.953 % @ 800000th iteration
% of accepted values: 20.935 % @ 850000th iteration
% of accepted values: 20.926 % @ 900000th iteration
% of accepted values: 20.935 % @ 950000th iteration
% of accepted values: 20.943 % @ 1000000th iteration
Chain 2 complete.
% of accepted values: 20.674 % @ 200000th iteration
% of accepted values: 20.696 % @ 250000th iteration
% of accepted values: 20.665 % @ 300000th iteration
% of accepted values: 20.788 % @ 350000th iteration
% of accepted values: 20.874 % @ 400000th iteration
% of accepted values: 20.905 % @ 450000th iteration
% of accepted values: 20.97 % @ 500000th iteration
% of accepted values: 20.977 % @ 550000th iteration
% of accepted values: 20.987 % @ 600000th iteration
% of accepted values: 21.002 % @ 650000th iteration
% of accepted values: 21.0 % @ 700000th iteration
% of accepted values: 21.004 % @ 750000th iteration
% of accepted values: 20.997 % @ 800000th iteration
% of accepted values: 20.971 % @ 850000th iteration
% of accepted values: 20.978 % @ 900000th iteration
% of accepted values: 20.96 % @ 950000th iteration
% of accepted values: 20.962 % @ 1000000th iteration
Chain 3 complete.
% of accepted values: 20.906 % @ 200000th iteration
% of accepted values: 20.907 % @ 250000th iteration
% of accepted values: 20.897 % @ 300000th iteration
% of accepted values: 20.954 % @ 350000th iteration
% of accepted values: 20.766 % @ 400000th iteration
% of accepted values: 20.818 % @ 450000th iteration
% of accepted values: 20.81 % @ 500000th iteration
% of accepted values: 20.874 % @ 550000th iteration
% of accepted values: 20.908 % @ 600000th iteration
% of accepted values: 20.885 % @ 650000th iteration
% of accepted values: 20.885 % @ 700000th iteration
% of accepted values: 20.828 % @ 750000th iteration
% of accepted values: 20.86 % @ 800000th iteration
% of accepted values: 20.892 % @ 850000th iteration
% of accepted values: 20.897 % @ 900000th iteration
% of accepted values: 20.891 % @ 950000th iteration
% of accepted values: 20.871 % @ 1000000th iteration
Chain 4 complete.
% of accepted values: 21.38 % @ 200000th iteration
% of accepted values: 21.032 % @ 250000th iteration
% of accepted values: 21.136 % @ 300000th iteration
% of accepted values: 21.176 % @ 350000th iteration
% of accepted values: 21.13 % @ 400000th iteration
% of accepted values: 21.15 % @ 450000th iteration
% of accepted values: 21.124 % @ 500000th iteration
% of accepted values: 21.166 % @ 550000th iteration
% of accepted values: 21.163 % @ 600000th iteration
% of accepted values: 21.133 % @ 650000th iteration
% of accepted values: 21.104 % @ 700000th iteration
% of accepted values: 21.12 % @ 750000th iteration
% of accepted values: 21.1 % @ 800000th iteration
% of accepted values: 21.113 % @ 850000th iteration
% of accepted values: 21.109 % @ 900000th iteration
% of accepted values: 21.116 % @ 950000th iteration
% of accepted values: 21.091 % @ 1000000th iteration
Chain 5 complete.
% of accepted values: 21.112 % @ 200000th iteration
% of accepted values: 20.99 % @ 250000th iteration
% of accepted values: 21.004 % @ 300000th iteration
```

```
% of accepted values: 20.968 % @ 350000th iteration
% of accepted values: 20.968 % @ 400000th iteration
% of accepted values: 20.959 % @ 450000th iteration
% of accepted values: 20.985 % @ 500000th iteration
% of accepted values: 21.034 % @ 550000th iteration
% of accepted values: 21.084 % @ 600000th iteration
% of accepted values: 21.064 % @ 650000th iteration
% of accepted values: 21.057 % @ 700000th iteration
% of accepted values: 21.038 % @ 750000th iteration
% of accepted values: 21.021 % @ 800000th iteration
% of accepted values: 21.064 % @ 850000th iteration
% of accepted values: 21.098 % @ 900000th iteration
% of accepted values: 21.054 % @ 950000th iteration
% of accepted values: 21.049 % @ 1000000th iteration
Chain 6 complete.
% of accepted values: 21.384 % @ 200000th iteration
% of accepted values: 21.104 % @ 250000th iteration
% of accepted values: 21.123 % @ 300000th iteration
% of accepted values: 21.092 % @ 350000th iteration
% of accepted values: 21.012 % @ 400000th iteration
% of accepted values: 21.108 % @ 450000th iteration
% of accepted values: 21.089 % @ 500000th iteration
% of accepted values: 21.082 % @ 550000th iteration
% of accepted values: 21.013 % @ 600000th iteration
% of accepted values: 21.012 % @ 650000th iteration
% of accepted values: 21.019 % @ 700000th iteration
% of accepted values: 21.009 % @ 750000th iteration
% of accepted values: 20.955 % @ 800000th iteration
% of accepted values: 20.932 % @ 850000th iteration
% of accepted values: 20.946 % @ 900000th iteration
% of accepted values: 20.959 % @ 950000th iteration
% of accepted values: 20.938 % @ 1000000th iteration
Chain 7 complete.
Accepted value percentage for chain 1: 20.957529411764707 %
Accepted value percentage for chain 2: 20.942588235294117 %
Accepted value percentage for chain 3: 20.962235294117647 %
Accepted value percentage for chain 4: 20.87070588235294 %
Accepted value percentage for chain 5: 21.091058823529412 %
Accepted value percentage for chain 6: 21.049294117647058 %
Accepted value percentage for chain 7: 20.938470588235294 %
Gelman-Rubin R for Omega_M: 1.001
Gelman-Rubin R for w: 1.001
Gelman-Rubin R for M_n: 1.0
```

In [150…

```python
# Accepted value %s for each chain
for i, accepted_percent in enumerate(accepted_percentages):
    print(f"Accepted value % for chain {i+1}: {accepted_percent} %")

# Average accepted value %s
print(f'Average accepted value % for all chains: {np.mean(accepted_percentages)} %')

# Gelman-Rubin R stats
print('')
print(f"Gelman-Rubin R for Omega_M: {np.round(r_Omega_M, 5)}")
print(f"Gelman-Rubin R for w: {np.round(r_w, 5)}")
print(f"Gelman-Rubin R for M_n: {np.round(r_M_n, 5)}")
```

```
Accepted value % for chain 1: 20.957529411764707 %
Accepted value % for chain 2: 20.942588235294117 %
Accepted value % for chain 3: 20.962235294117647 %
Accepted value % for chain 4: 20.87070588235294 %
Accepted value % for chain 5: 21.091058823529412 %
Accepted value % for chain 6: 21.049294117647058 %
Accepted value % for chain 7: 20.938470588235294 %
Average accepted value % for all chains: 20.97312605042017 %

Gelman-Rubin R for Omega_M: 1.00079
Gelman-Rubin R for w: 1.00094
Gelman-Rubin R for M_n: 1.00043
```

In [86]:

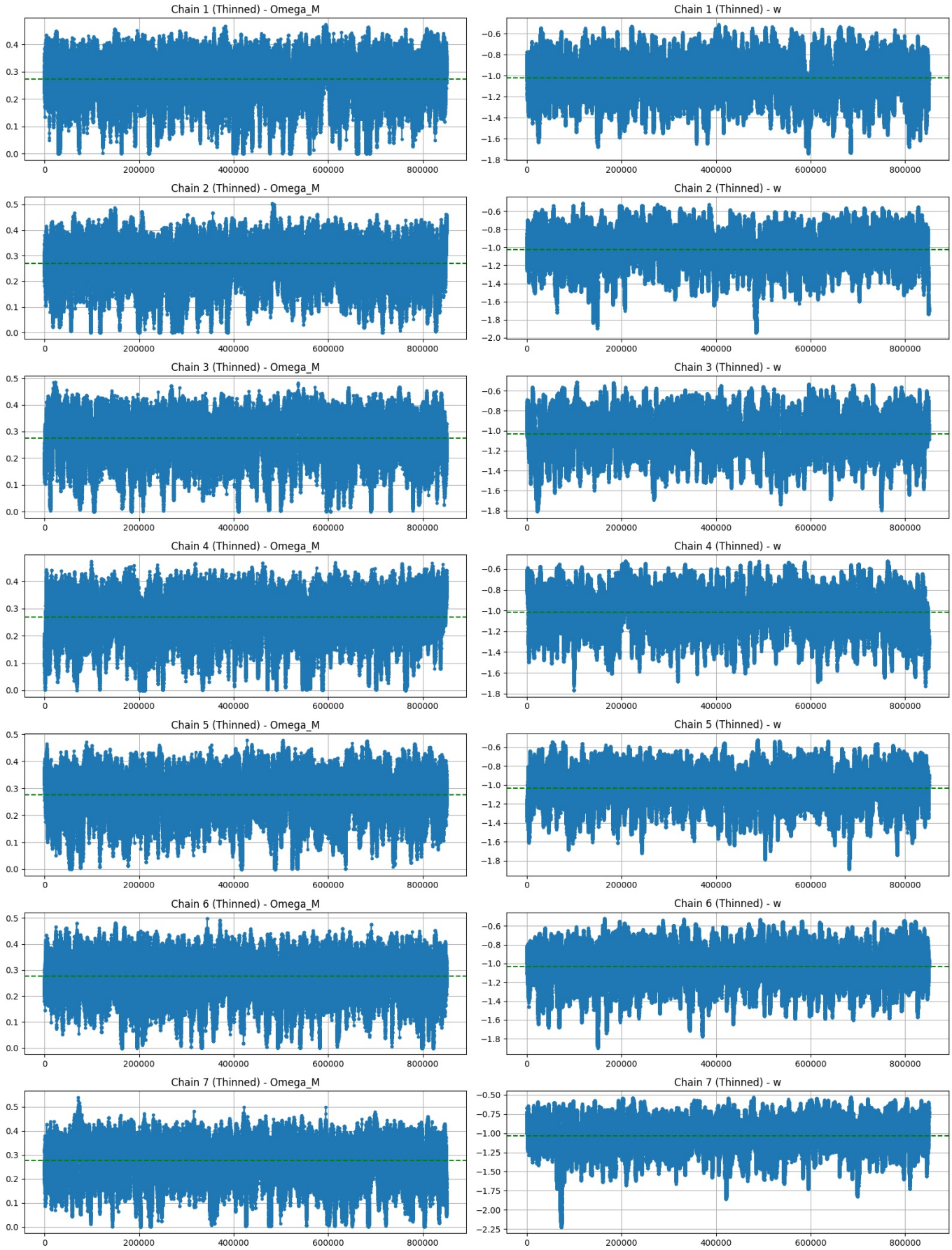
```python
# Thinning All Chains

thinned_all_chains_Omega_M_values = all_chains_Omega_M[:, ::selected_thinning_value]
thinned_all_chains_w_values = all_chains_w[:, ::selected_thinning_value]
thinned_all_chains_M_n_values = all_chains_M_n[:, ::selected_thinning_value]
thinned_all_chains_log_likelihood_values = all_chains_log_likelihoods[:, ::selected_thinning_value]
```

In [87]:

```python
# Unthinned Averages

fig, axs = plt.subplots(num_chains, 2, figsize=(16, num_chains * 3))

mean_values_Omega_M = np.zeros(num_chains)
std_values_Omega_M = np.zeros(num_chains)
mean_values_w = np.zeros(num_chains)
```

```
std_values_w = np.zeros(num_chains)
mean_values_M_n = np.zeros(num_chains)
std_values_M_n = np.zeros(num_chains)

# Plotting
for i in range(len(all_chains_Omega_M)):

    # Omega_M
    axs[i, 0].plot(all_chains_Omega_M[i], marker='.')
    mean_value_Omega_M = np.mean(all_chains_Omega_M[i])
    std_value_Omega_M = np.std(all_chains_Omega_M[i])
    axs[i, 0].axhline(y=mean_value_Omega_M, color='green', linestyle='--', label='Overall Mean')
    axs[i, 0].set_title(f'Chain {i+1} (Unthinned) - Omega_M')
    axs[i, 0].grid()

    # w
    axs[i, 1].plot(all_chains_w[i], marker='.')
    mean_value_w = np.mean(all_chains_w[i])
    std_value_w = np.std(all_chains_w[i])
    axs[i, 1].axhline(y=mean_value_w, color='green', linestyle='--', label='Overall Mean')
    axs[i, 1].set_title(f'Chain {i+1} (Thinned) - w')
    axs[i, 1].grid()

    # M_n
    mean_value_M_n = np.mean(all_chains_M_n[i])
    std_value_M_n = np.std(all_chains_M_n[i])

    # Print mean and standard deviation for each chain
    print(f'Chain {i+1} Mean Omega_M value: {np.round(mean_value_Omega_M, 3)}, ± {np.round(std_value_Omega_M, 3
    print(f'Chain {i+1} Mean w value: {np.round(mean_value_w, 3)}, ± {np.round(std_value_w, 3)}')
    print('')

    # Store values in arrays
    mean_values_Omega_M[i] = mean_value_Omega_M
    std_values_Omega_M[i] = std_value_Omega_M

    mean_values_w[i] = mean_value_w
    std_values_w[i] = std_value_w

    mean_values_M_n[i] = mean_value_M_n
    std_values_M_n[i] = std_value_M_n

# Mean and Std.
mean_mean_value_Omega_M = np.mean(mean_values_Omega_M)
mean_mean_value_w = np.mean(mean_values_w)
mean_mean_value_M_n = np.mean(mean_values_M_n)

overall_std_Omega_M = np.mean(std_values_Omega_M)
overall_std_w = np.mean(std_values_w)
overall_std_M_n = np.mean(std_values_M_n)

# Overall means and stds
print('')
print(f'Mean Omega_M value across {num_chains} unthinned chains: {np.round(mean_mean_value_Omega_M, 3)} ± {np.r
print(f'Mean w value across {num_chains} unthinned chains: {np.round(mean_mean_value_w, 3)} ± {np.round(overall_

plt.tight_layout()
plt.show()
```

```
Chain 1 Mean Omega_M value: 0.272, ± 0.078
Chain 1 Mean w value: -1.022, ± 0.196

Chain 2 Mean Omega_M value: 0.271, ± 0.079
Chain 2 Mean w value: -1.021, ± 0.208

Chain 3 Mean Omega_M value: 0.276, ± 0.078
Chain 3 Mean w value: -1.035, ± 0.206

Chain 4 Mean Omega_M value: 0.27, ± 0.078
Chain 4 Mean w value: -1.014, ± 0.194

Chain 5 Mean Omega_M value: 0.276, ± 0.074
Chain 5 Mean w value: -1.032, ± 0.198

Chain 6 Mean Omega_M value: 0.277, ± 0.074
Chain 6 Mean w value: -1.033, ± 0.197

Chain 7 Mean Omega_M value: 0.277, ± 0.078
Chain 7 Mean w value: -1.038, ± 0.216


Mean Omega_M value across 7 thinned chains: 0.274 ± 0.077
Mean w value across 7 thinned chains: -1.028 ± 0.202
```

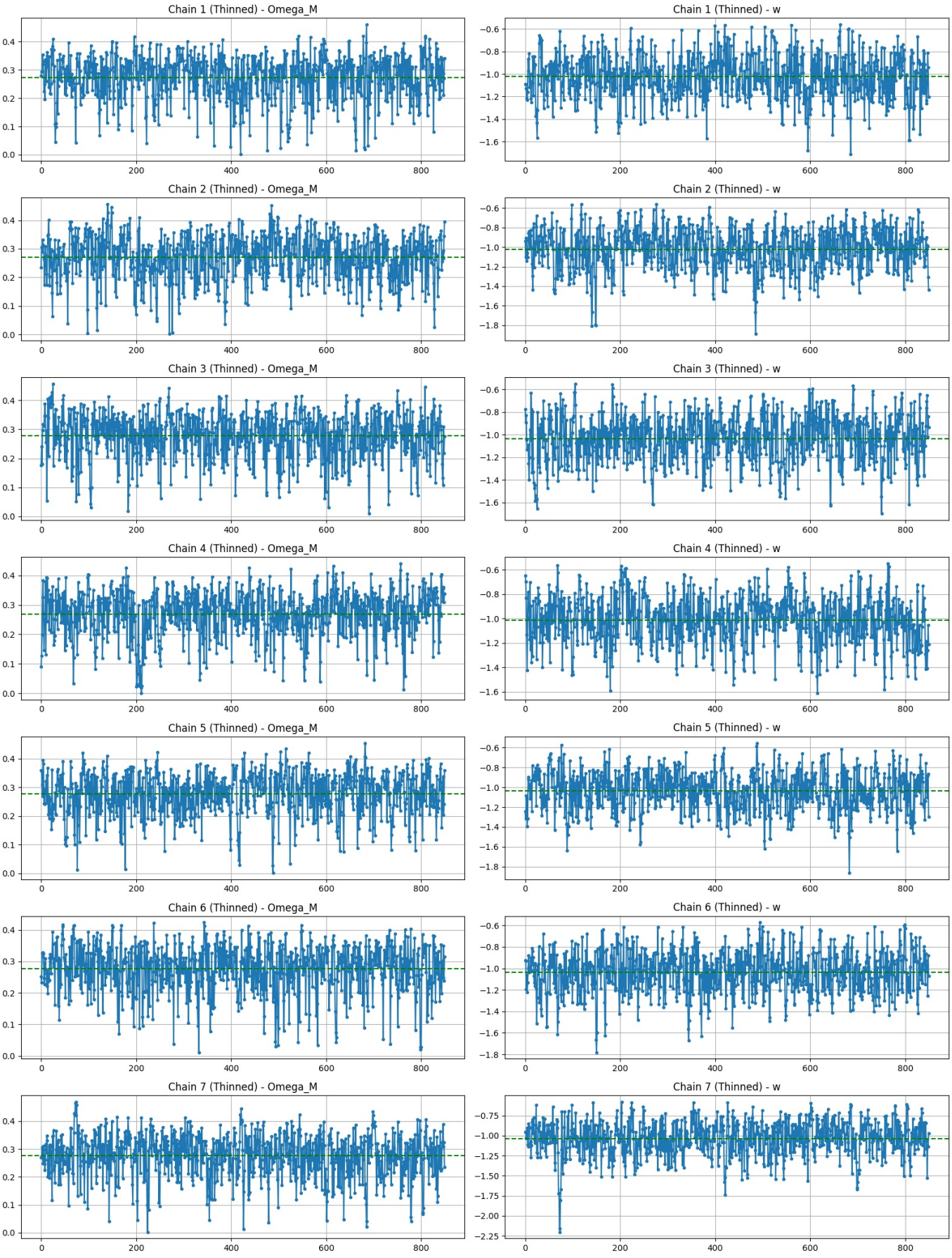


In [125…

```
# Thinned Averages

fig, axs = plt.subplots(num_chains, 2, figsize=(16, num_chains * 3))

thinned_mean_values_Omega_M = np.zeros(num_chains)
thinned_std_values_Omega_M = np.zeros(num_chains)
thinned_mean_values_w = np.zeros(num_chains)
thinned_std_values_w = np.zeros(num_chains)
thinned_mean_values_M_n = np.zeros(num_chains)
thinned_std_values_M_n = np.zeros(num_chains)

# Plotting
```

```python
for i in range(len(thinned_all_chains_Omega_M_values)):

    # Omega_M
    axs[i, 0].plot(thinned_all_chains_Omega_M_values[i], marker='.')
    thinned_mean_value_Omega_M = np.mean(thinned_all_chains_Omega_M_values[i])
    thinned_std_value_Omega_M = np.std(thinned_all_chains_Omega_M_values[i])
    axs[i, 0].axhline(y=thinned_mean_value_Omega_M, color='green', linestyle='--', label='Overall Mean')
    axs[i, 0].set_title(f'Chain {i+1} (Thinned) - Omega_M')
    axs[i, 0].grid()

    # w
    axs[i, 1].plot(thinned_all_chains_w_values[i], marker='.')
    thinned_mean_value_w = np.mean(thinned_all_chains_w_values[i])
    thinned_std_value_w = np.std(thinned_all_chains_w_values[i])
    axs[i, 1].axhline(y=thinned_mean_value_w, color='green', linestyle='--', label='Overall Mean')
    axs[i, 1].set_title(f'Chain {i+1} (Thinned) - w')
    axs[i, 1].grid()

    # M_n
    thinned_mean_value_M_n = np.mean(thinned_all_chains_M_n_values[i])
    thinned_std_value_M_n = np.std(thinned_all_chains_M_n_values[i])

    # Print mean and standard deviation for each chain
    print(f'Chain {i+1} Mean Omega_M value: {np.round(thinned_mean_value_Omega_M, 3)}, ± {np.round(thinned_std_
    print(f'Chain {i+1} Mean w value: {np.round(thinned_mean_value_w, 3)}, ± {np.round(thinned_std_value_w, 3)}
    print('')

    # Store values in arrays
    thinned_mean_values_Omega_M[i] = thinned_mean_value_Omega_M
    thinned_std_values_Omega_M[i] = thinned_std_value_Omega_M

    thinned_mean_values_w[i] = thinned_mean_value_w
    thinned_std_values_w[i] = thinned_std_value_w

    thinned_mean_values_M_n[i] = thinned_mean_value_M_n
    thinned_std_values_M_n[i] = thinned_std_value_M_n

# Mean and Std.
thinned_mean_mean_value_Omega_M = np.mean(thinned_mean_values_Omega_M)
thinned_mean_mean_value_w = np.mean(thinned_mean_values_w)
thinned_mean_mean_value_M_n = np.mean(thinned_mean_values_M_n)

thinned_overall_std_Omega_M = np.mean(thinned_std_values_Omega_M)
thinned_overall_std_w = np.mean(thinned_std_values_w)
thinned_overall_std_M_n = np.mean(thinned_std_values_M_n)

# Overall means and stds
print('')
print(f'Mean Omega_M value across {num_chains} thinned chains: {np.round(thinned_mean_mean_value_Omega_M, 3)} ±
print(f'Mean w value across {num_chains} thinned chains: {np.round(thinned_mean_mean_value_w, 3)} ± {np.round(t

plt.tight_layout()
plt.show()
```

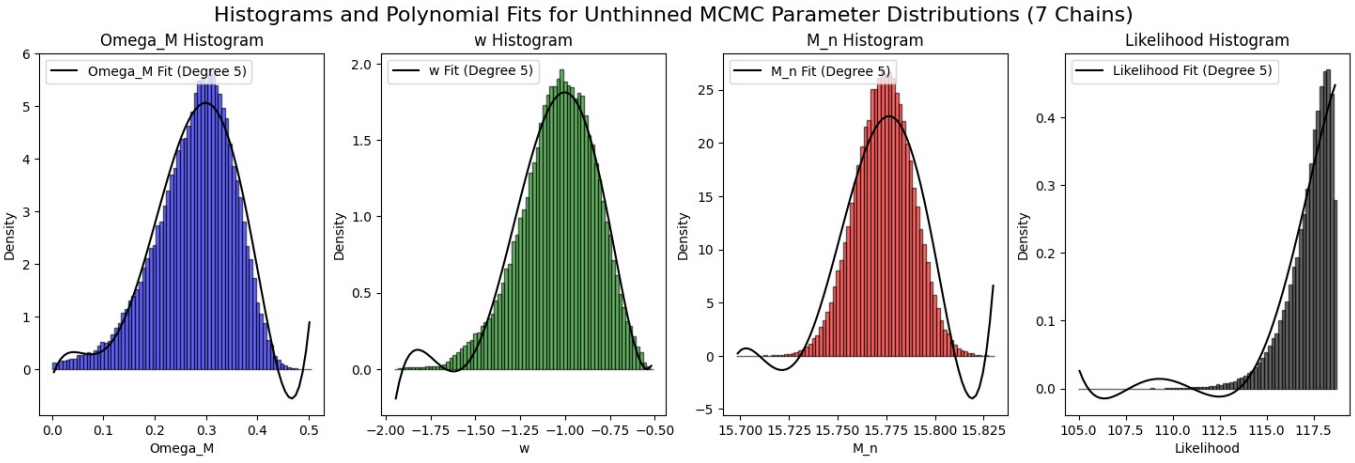
```
Chain 1 Mean Omega_M value: 0.273, ± 0.078
Chain 1 Mean w value: -1.023, ± 0.194

Chain 2 Mean Omega_M value: 0.271, ± 0.078
Chain 2 Mean w value: -1.022, ± 0.208

Chain 3 Mean Omega_M value: 0.278, ± 0.078
Chain 3 Mean w value: -1.035, ± 0.207

Chain 4 Mean Omega_M value: 0.269, ± 0.078
Chain 4 Mean w value: -1.012, ± 0.194

Chain 5 Mean Omega_M value: 0.277, ± 0.074
Chain 5 Mean w value: -1.034, ± 0.194

Chain 6 Mean Omega_M value: 0.277, ± 0.075
Chain 6 Mean w value: -1.034, ± 0.196

Chain 7 Mean Omega_M value: 0.276, ± 0.077
Chain 7 Mean w value: -1.034, ± 0.214


Mean Omega_M value across 7 thinned chains: 0.274 ± 0.077
Mean w value across 7 thinned chains: -1.028 ± 0.201
```



In [126…

```
# Unthinned Histograms All Chains

def plot_histograms_with_fit(degree, degree_max, params, chain_1_all_values):

    # Begin plot
    fig, axes = plt.subplots(1, len(params), figsize=(18, 5))
    num_bins = 75
```

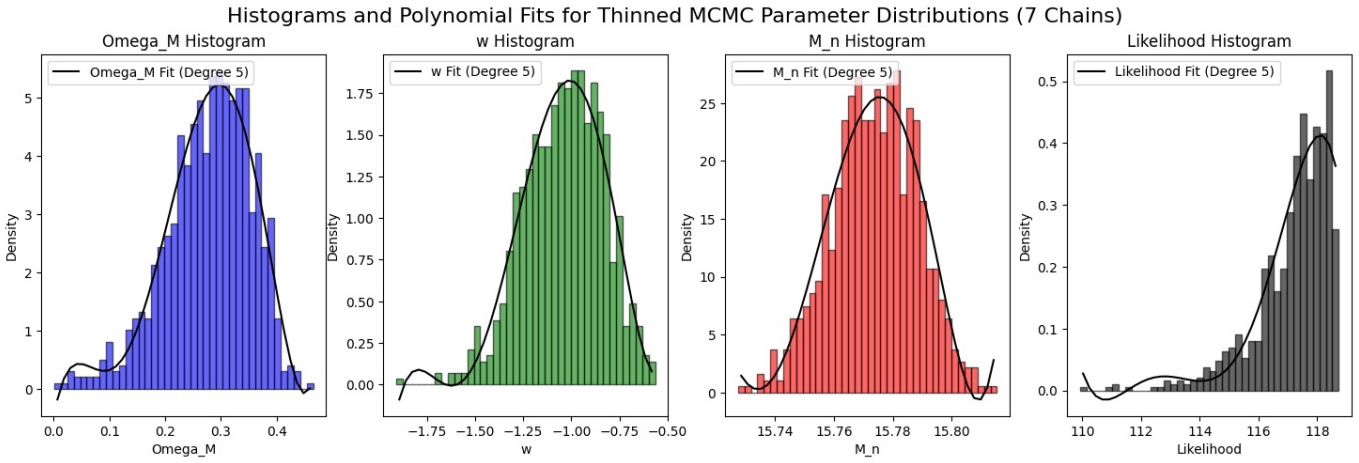
```
    colors = ['blue', 'green', 'red', 'black']

    def fit_polynomial_and_find_peak(values, ax, poly_degree, degree_max, param_name, color):
        # Histogram data
        counts, bin_edges = np.histogram(values, bins=num_bins, density=True)
        bin_centers = (bin_edges[:-1] + bin_edges[1:]) / 2

        # Prepare array for peak values and fit polynomials
        peak_value_array = np.zeros(degree_max + 1)

        # Fit polynomial and find peaks for each degree
        for i in range(1, degree_max + 1):
            p = Polynomial.fit(bin_centers, counts, deg=i)
            fitted_values = p(bin_centers)
            peak_index = np.argmax(fitted_values)
            peak_value = bin_centers[peak_index]
            peak_value_array[i] = peak_value

        # Peak value Estimations
        avg_peak_value = np.mean(peak_value_array[2:])
        std_peak_value = np.std(peak_value_array[2:])

        # Fit and plot the polynomial of the specified degree (poly_degree) for visualization
        selected_poly = Polynomial.fit(bin_centers, counts, deg=poly_degree)
        selected_fitted_values = selected_poly(bin_centers)

        # Plot histogram and the selected polynomial fit
        ax.hist(values, bins=num_bins, color=color, alpha=0.6, edgecolor='black', density=True)
        ax.plot(bin_centers, selected_fitted_values, color='black', label=f'{param_name} Fit (Degree {poly_degr
        ax.set_title(f'{param_name} Histogram')
        ax.set_xlabel(param_name)
        ax.set_ylabel('Density')
        ax.legend(loc='upper left')

        return avg_peak_value, std_peak_value

    # Plot calls
    converged_values = np.zeros(len(params))
    converged_values_errors = np.zeros(len(params))

    for i, param in enumerate(params):
        converged_values[i], converged_values_errors[i] = fit_polynomial_and_find_peak(chain_1_all_values[i], a

    fig.suptitle(f'Histograms and Polynomial Fits for Unthinned MCMC Parameter Distributions ({num_chains} Chai

    plt.show()

    return converged_values, converged_values_errors

# Plot histograms and fit polynomial curves for the 3 parameters
poly_degree = 5
max_degree = 12
params = ['Omega_M', 'w', 'M_n', 'Likelihood']
chain_1_all_values = np.array([chain_1_Omega_M_values, chain_1_w_values, chain_1_M_n_values, chain_1_log_likeli

conv_vals, conv_vals_errs = plot_histograms_with_fit(poly_degree, max_degree, params, chain_1_all_values)
```



In [127…

```
# Thinned Histograms All Chains

def plot_histograms_with_fit(degree, degree_max, params, chain_1_all_values):

    # Begin plot
    fig, axes = plt.subplots(1, len(params), figsize=(18, 5))
    num_bins = 40
    colors = ['blue', 'green', 'red', 'black']
```

```
    def fit_polynomial_and_find_peak(values, ax, poly_degree, degree_max, param_name, color):
        # Histogram data
        counts, bin_edges = np.histogram(values, bins=num_bins, density=True)
        bin_centers = (bin_edges[:-1] + bin_edges[1:]) / 2

        # Prepare array for peak values and fit polynomials
        peak_value_array = np.zeros(degree_max + 1)

        # Fit polynomial and find peaks for each degree
        for i in range(1, degree_max + 1):
            p = Polynomial.fit(bin_centers, counts, deg=i)
            fitted_values = p(bin_centers)
            peak_index = np.argmax(fitted_values)
            peak_value = bin_centers[peak_index]
            peak_value_array[i] = peak_value

        # Peak value Estimations
        avg_peak_value = np.mean(peak_value_array[2:])
        std_peak_value = np.std(peak_value_array[2:])

        # Fit and plot the polynomial of the specified degree (poly_degree) for visualization
        selected_poly = Polynomial.fit(bin_centers, counts, deg=poly_degree)
        selected_fitted_values = selected_poly(bin_centers)

        # Plot histogram and the selected polynomial fit
        ax.hist(values, bins=num_bins, color=color, alpha=0.6, edgecolor='black', density=True)
        ax.plot(bin_centers, selected_fitted_values, color='black', label=f'{param_name} Fit (Degree {poly_degr
        ax.set_title(f'{param_name} Histogram')
        ax.set_xlabel(param_name)
        ax.set_ylabel('Density')
        ax.legend(loc='upper left')

        return avg_peak_value, std_peak_value

    # Plot calls
    converged_values = np.zeros(len(params))
    converged_values_errors = np.zeros(len(params))

    for i, param in enumerate(params):
        converged_values[i], converged_values_errors[i] = fit_polynomial_and_find_peak(chain_1_all_values[i], a

    fig.suptitle(f'Histograms and Polynomial Fits for Thinned MCMC Parameter Distributions ({num_chains} Chains

    plt.show()

    return converged_values, converged_values_errors

# Plot histograms and fit polynomial curves for the 3 parameters
poly_degree = 5
max_degree = 12
params = ['Omega_M', 'w', 'M_n', 'Likelihood']
thinned_chain_1_all_values = np.array([thinned_chain_1_Omega_M_values, thinned_chain_1_w_values, thinned_chain_

thinned_conv_vals, thinned_conv_vals_errs = plot_histograms_with_fit(poly_degree, max_degree, params, thinned_cl
```



In [128…

```
# All Chains Unthinned Results, Raw Samples

print(f'Mean Omega_M value (Raw Samples) across {num_chains} unthinned chains: {np.round(mean_mean_value_Omega_
print(f'Mean w value (Raw Samples) across {num_chains} unthinned chains: {np.round(mean_mean_value_w, 3)}')
print(f'Mean M_n value (Raw Samples) across {num_chains} unthinned chains: {np.round(mean_mean_value_M_n, 3)}')
```

```
Mean Omega_M value (Raw Samples) across 7 unthinned chains: 0.274
Mean w value (Raw Samples) across 7 unthinned chains: -1.028
Mean M_n value (Raw Samples) across 7 unthinned chains: 15.774
```

In [129…
```
# All Chains Thinned Results, Raw Samples

print(f'Mean Omega_M (Raw Samples) value across {num_chains} thinned chains: {np.round(thinned_mean_mean_value_
print(f'Mean w value (Raw Samples) across {num_chains} thinned chains: {np.round(thinned_mean_mean_value_w, 3)}
print(f'Mean M_n value (Raw Samples) across {num_chains} thinned chains: {np.round(thinned_mean_mean_value_M_n,
```

```
Mean Omega_M (Raw Samples) value across 7 thinned chains: 0.274 ± 0.077
Mean w value (Raw Samples) across 7 thinned chains: -1.028 ± 0.201
Mean M_n value (Raw Samples) across 7 thinned chains: 15.774 ± 0.015
```

In [130…
```
# All Chains Thinned/Unthinned Percent Errors, Raw Samples

likely_raw_sample_error_thinned_unthinned_diff_all_chains_Omega_M = ( np.abs(mean_mean_value_Omega_M - thinned_
likely_raw_sample_error_thinned_unthinned_diff_all_chains_w = ( np.abs(mean_mean_value_w - thinned_mean_mean_va
likely_raw_sample_error_thinned_unthinned_diff_all_chains_M_n = ( np.abs(mean_mean_value_M_n - thinned_mean_mea

print(f'Percent error (Raw Samples) between thinned and unthinned estimates for Omega_M: {np.round(likely_raw_s
print(f'Percent error (Raw Samples) between thinned and unthinned estimates for w: {np.round(likely_raw_sample_
print(f'Percent error (Raw Samples) between thinned and unthinned estimates for M_n: {np.round(likely_raw_sampl
```

```
Percent error (Raw Samples) between thinned and unthinned estimates for Omega_M: 0.057 %
Percent error (Raw Samples) between thinned and unthinned estimates for w: -0.008 %
Percent error (Raw Samples) between thinned and unthinned estimates for M_n: 0.001 %
```

In [131…
```
# All Chains Unthinned Results, Polynomial Averaging

for i in range(len(conv_vals)):
    print(f"Estimated {params[i]} (Polynomial Averaging): {np.round(conv_vals[i], 3)} ± {np.round(conv_vals_err
```

```
Estimated Omega_M (Polynomial Averaging): 0.297 ± 0.013
Estimated w (Polynomial Averaging): -1.0 ± 0.027
Estimated M_n (Polynomial Averaging): 15.775 ± 0.003
Estimated Likelihood (Polynomial Averaging): 118.315 ± 0.258
```

In [132…
```
# All Chains Thinned Results, Polynomial Averaging

for i in range(len(thinned_conv_vals)):
    print(f"Estimated {params[i]} (Polynomial Averaging): {np.round(thinned_conv_vals[i], 3)} ± {np.round(thinn
```

```
Estimated Omega_M (Polynomial Averaging): 0.299 ± 0.013
Estimated w (Polynomial Averaging): -0.985 ± 0.029
Estimated M_n (Polynomial Averaging): 15.774 ± 0.001
Estimated Likelihood (Polynomial Averaging): 118.286 ± 0.237
```

In [133…
```
# All Chains Thinned/Unthinned Percent Errors, Polynomial Averaging

likely_error_thinned_unthinned_diff_all_chains = np.zeros(len(params))

for i in range(len(params)):
    likely_error_thinned_unthinned_diff_all_chains[i] = ( np.abs(conv_vals[i] - thinned_conv_vals[i]) / thinned_

    print(f'Percent error (Polynomial Averaging) between thinned and unthinned estimates for {params[i]}: {np.r
```

```
Percent error (Polynomial Averaging) between thinned and unthinned estimates for Omega_M: 0.884 %
Percent error (Polynomial Averaging) between thinned and unthinned estimates for w: -1.539 %
Percent error (Polynomial Averaging) between thinned and unthinned estimates for M_n: 0.003 %
Percent error (Polynomial Averaging) between thinned and unthinned estimates for Likelihood: 0.024 %
```

In [134…
```
# All Chains Unthinned Results, Likelihood Indexing

log_likelihood_flat = all_chains_log_likelihoods.flatten()
log_likelihood_all_chains_index = np.argmax(log_likelihood_flat)
max_log_likelihood_value = log_likelihood_flat[log_likelihood_all_chains_index]

tolerance_thinned = 0.1

for i in range(all_chains_log_likelihoods.shape[0]):
    for j in range(all_chains_log_likelihoods.shape[1]):
        if abs(all_chains_log_likelihoods[i, j] - max_log_likelihood_value) < tolerance_thinned:
            likely_omega_m_all_chains = all_chains_Omega_M[i, j]
            likely_w_all_chains = all_chains_w[i, j]
            likely_M_n_all_chains = all_chains_M_n[i, j]
            likely_log_likelihood_all_chains = all_chains_log_likelihoods[i, j]

print(f'The most likely unthinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_ome
print(f'The most likely unthinned w value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_w_all_cha
print(f'The most likely unthinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_M_n_all
print(f'The most likely unthinned log likelihood (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_lo
```

```
The most likely unthinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: 0.26
The most likely unthinned w value (Likelihood Indexing) for MCMC Chain 1 is: -0.968
The most likely unthinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: 15.775
The most likely unthinned log likelihood (Likelihood Indexing) for MCMC Chain 1 is: 118.681
```

In [135…
```
# All Chains Thinned Results, Likelihood Indexing
```

```
thinned_log_likelihood_flat = thinned_all_chains_log_likelihood_values.flatten()
thinned_log_likelihood_all_chains_index = np.argmax(thinned_log_likelihood_flat)
max_log_likelihood_value = thinned_log_likelihood_flat[thinned_log_likelihood_all_chains_index]

tolerance_thinned = 0.1

for i in range(thinned_all_chains_log_likelihood_values.shape[0]):
    for j in range(thinned_all_chains_log_likelihood_values.shape[1]):
        if abs(thinned_all_chains_log_likelihood_values[i, j] - max_log_likelihood_value) < tolerance_thinned:
            likely_thinned_omega_m_all_chains = thinned_all_chains_Omega_M_values[i, j]
            likely_thinned_w_all_chains = thinned_all_chains_w_values[i, j]
            likely_thinned_M_n_all_chains = thinned_all_chains_M_n_values[i, j]
            likely_thinned_log_likelihood_all_chains = thinned_all_chains_log_likelihood_values[i, j]

print(f'The most likely thinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_thinn
print(f'The most likely thinned w value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_thinned_w_a
print(f'The most likely thinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likely_thinned_M_
print(f'The most likely thinned log likelihood value (Likelihood Indexing) for MCMC Chain 1 is: {np.round(likel
```

```
The most likely thinned Omega_M value (Likelihood Indexing) for MCMC Chain 1 is: 0.299
The most likely thinned w value (Likelihood Indexing) for MCMC Chain 1 is: -1.061
The most likely thinned M_n value (Likelihood Indexing) for MCMC Chain 1 is: 15.77
The most likely thinned log likelihood value (Likelihood Indexing) for MCMC Chain 1 is: 118.665
```

In [136…

```
# All Chains Thinned/Unthinned Percent Errors, Likelihood Indexing

likely_omega_m_error_thinned_unthinned_diff_all_chains = ( np.abs(likely_omega_m_all_chains - likely_thinned_om
likely_w_error_thinned_unthinned_diff_all_chains = ( np.abs(likely_w_chain_1 - likely_thinned_w_all_chains) / l
likely_M_n_error_thinned_unthinned_diff_all_chains = ( np.abs(likely_M_n_chain_1 - likely_thinned_M_n_all_chain
likely_log_likelihood_error_thinned_unthinned_diff_all_chains = ( np.abs(likely_log_likelihood_all_chains - lik

print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for Omega_M: {np.round(like
print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for w: {np.round(likely_w_e
print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for M_n: {np.round(likely_M
print(f'Percent error (Likelihood Indexing) between thinned and unthinned estimates for log_likelihood: {np.rou
```

```
Percent error (Likelihood Indexing) between thinned and unthinned estimates for Omega_M: 13.024 %
Percent error (Likelihood Indexing) between thinned and unthinned estimates for w: -3.985 %
Percent error (Likelihood Indexing) between thinned and unthinned estimates for M_n: 0.024 %
Percent error (Likelihood Indexing) between thinned and unthinned estimates for log_likelihood: 0.014 %
```

In [137…

```
# All Chains Unthinned Results, Posteriors

# Unthinned Posteriors
posteriors_all_chains = np.exp(all_chains_log_likelihoods - np.max(all_chains_log_likelihoods))
posteriors_all_chains /= np.sum(posteriors_all_chains)

# Parameter Estimations
Omega_M_estimate_all_chains = np.sum(all_chains_Omega_M * posteriors_all_chains)
w_estimate_all_chains = np.sum(all_chains_w * posteriors_all_chains)
M_n_estimate_all_chains = np.sum(all_chains_M_n * posteriors_all_chains)

# Parameter Uncertainties
Omega_M_std_all_chains = np.sqrt(np.sum(posteriors_all_chains * (all_chains_Omega_M - Omega_M_estimate_all_chai
w_std_all_chains = np.sqrt(np.sum(posteriors_all_chains * (all_chains_w - w_estimate_all_chains) ** 2))
M_n_std_all_chains = np.sqrt(np.sum(posteriors_all_chains * (all_chains_M_n - M_n_estimate_all_chains) ** 2))

print(f"Estimated Omega_M (Posterior): {np.round(Omega_M_estimate_all_chains, 3)} ± {np.round(Omega_M_std_all_c
print(f"Estimated w (Posterior): {np.round(w_estimate_all_chains, 3)} ± {np.round(w_std_all_chains, 3)}")
print(f"Estimated M_n (Posterior): {np.round(M_n_estimate_all_chains, 3)} ± {np.round(M_n_std_all_chains, 3)}")
```

```
Estimated Omega_M (Posterior): 0.277 ± 0.055
Estimated w (Posterior): -1.018 ± 0.144
Estimated M_n (Posterior): 15.774 ± 0.01
```

In [214…

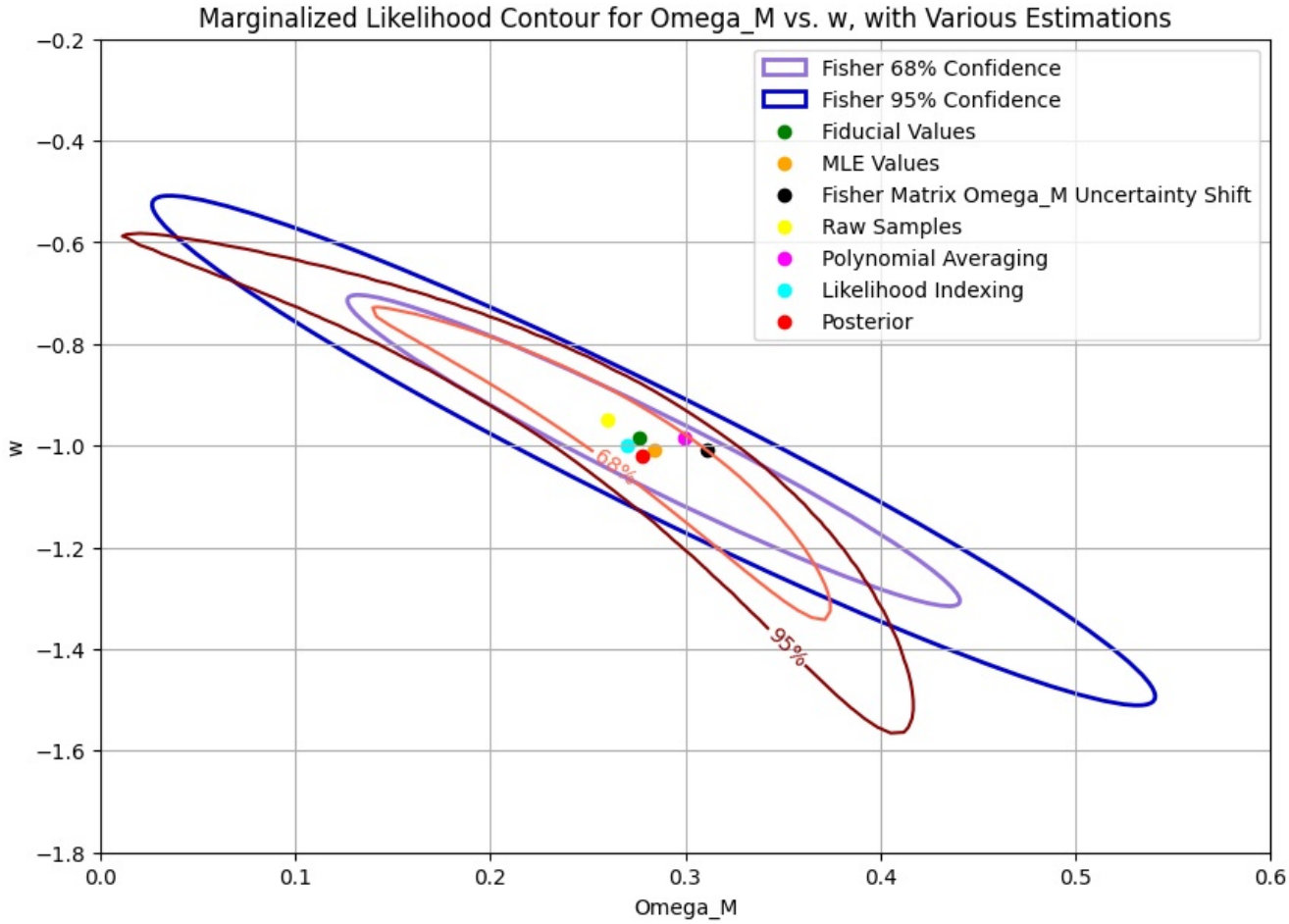
```
# All Chains Thinned Results, Posteriors

# Unthinned Posteriors
thinned_posteriors_all_chains = np.exp(thinned_all_chains_log_likelihood_values - np.max(thinned_all_chains_log
thinned_posteriors_all_chains /= np.sum(thinned_posteriors_all_chains)

# Parameter Estimations
thinned_Omega_M_estimate_all_chains = np.sum(thinned_all_chains_Omega_M_values * thinned_posteriors_all_chains)
thinned_w_estimate_all_chains = np.sum(thinned_all_chains_w_values * thinned_posteriors_all_chains)
thinned_M_n_estimate_all_chains = np.sum(thinned_all_chains_M_n_values * thinned_posteriors_all_chains)

# Parameter Uncertainties
thinned_Omega_M_std_all_chains = np.sqrt(np.sum(thinned_posteriors_all_chains * (thinned_all_chains_Omega_M_valu
thinned_w_std_all_chains = np.sqrt(np.sum(thinned_posteriors_all_chains * (thinned_all_chains_w_values - thinne
thinned_M_n_std_all_chains = np.sqrt(np.sum(thinned_posteriors_all_chains * (thinned_all_chains_M_n_values - th

print(f"Estimated Omega_M (Posterior): {np.round(thinned_Omega_M_estimate_all_chains, 3)} ± {np.round(thinned_O
print(f"Estimated w (Posterior): {np.round(thinned_w_estimate_all_chains, 3)} ± {np.round(thinned_w_std_all_cha
print(f"Estimated M_n (Posterior): {np.round(thinned_M_n_estimate_all_chains, 3)} ± {np.round(thinned_M_n_std_a
```

```
Estimated Omega_M (Posterior): 0.278 ± 0.055
Estimated w (Posterior): -1.019 ± 0.144
Estimated M_n (Posterior): 15.774 ± 0.01
```

In [215…
```python
# All Chains Thinned/Unthinned Percent Errors, Using Posteriors

posterior_likely_omega_m_error_thinned_unthinned_diff_all_chains = ( np.abs(Omega_M_estimate_all_chains - thinn
posterior_likely_w_error_thinned_unthinned_diff_all_chains = ( np.abs(w_estimate_all_chains - thinned_w_estimat
posterior_likely_M_n_error_thinned_unthinned_diff_all_chains = ( np.abs(M_n_estimate_all_chains - thinned_M_n_e

print(f'Percent error (with Posteriors) between thinned and unthinned estimates for Omega_M: {np.round(posterio
print(f'Percent error (with Posteriors) between thinned and unthinned estimates for w: {np.round(posterior_like
print(f'Percent error (with Posteriors) between thinned and unthinned estimates for M_n: {np.round(posterior_li

# Would an increase/decrease in these percent errors mean anything?
```

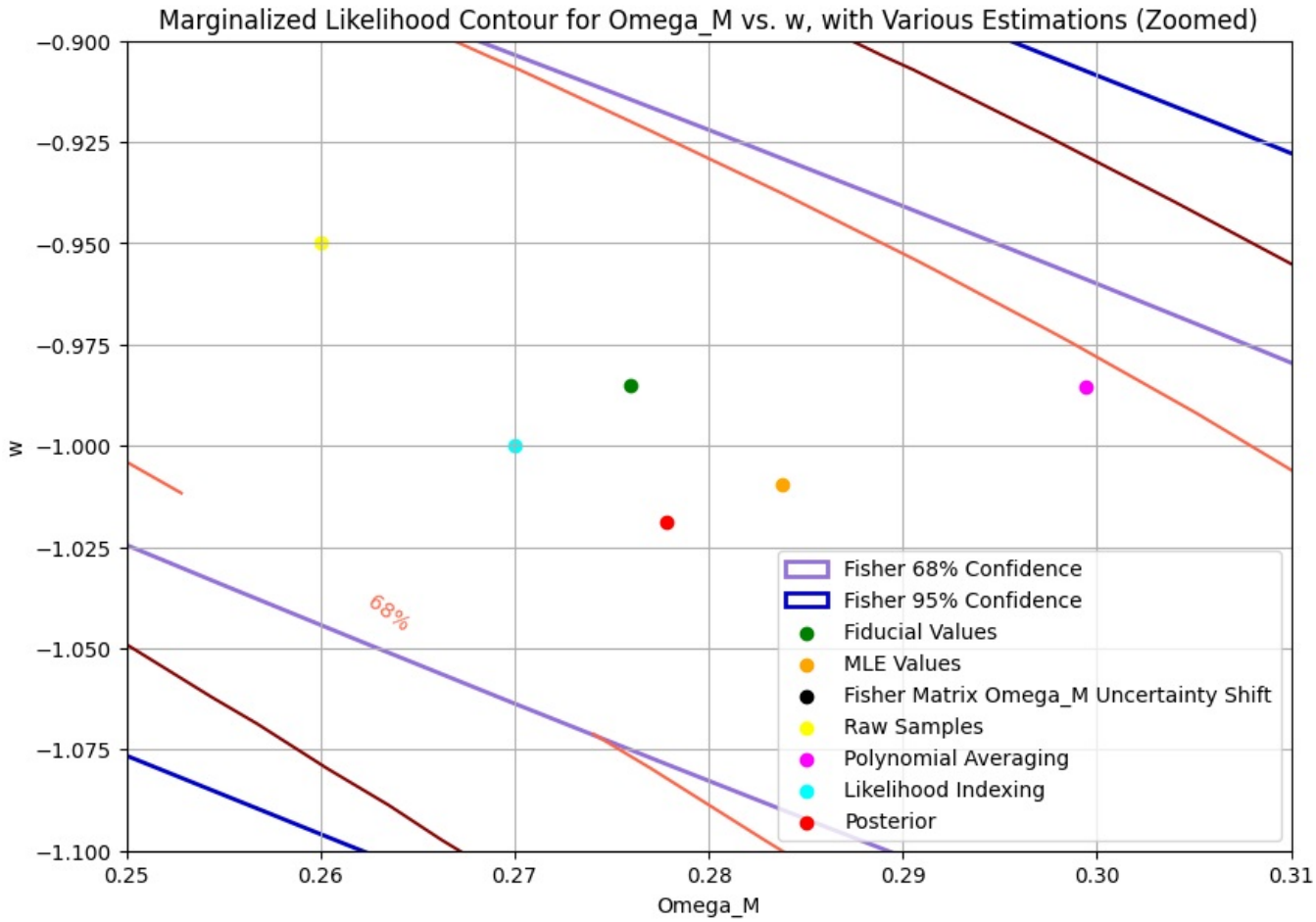
```
Percent error (with Posteriors) between thinned and unthinned estimates for Omega_M: 0.213 %
Percent error (with Posteriors) between thinned and unthinned estimates for w: -0.083 %
Percent error (with Posteriors) between thinned and unthinned estimates for M_n: 0.0 %
```

In [216…
```python
# Plot with all analyses and estimations

# Contour plot
plt.figure(figsize=(10, 7))
ax = plt.gca()
cp = plt.contour(omega_m_grid1_slice, w_grid1_slice, marg_likelihoods_w_m_mn, levels=[level_95_w_m_mn, level_68
plt.clabel(cp, inline=True, fontsize=10, fmt={level_68_w_m_mn: '68%', level_95_w_m_mn: '95%'})

# Plot Fisher confidence ellipses
boundary_right_sig1, boundary_right_sig2 = plot_fisher_confidence_ellipse(ax, fisher_at_most_likely, (most_like
plot_fisher_confidence_ellipse(ax, fisher_at_most_likely, (most_likely_omega_m1, most_likely_w1), 6.17)

# Plot Estimations
plt.scatter(fixed_omega_m, fixed_w, color='green', label='Fiducial Values')
plt.scatter(most_likely_omega_m1, most_likely_w1, color='orange', label='MLE Values')
plt.scatter(boundary_right_sig1, most_likely_w1, color='black', label='Fisher Matrix Omega_M Uncertainty Shift'
plt.scatter(thinned_mean_mean_value_Omega_M, thinned_mean_mean_value_w, color='yellow', label='Raw Samples')
plt.scatter(thinned_conv_vals[0], thinned_conv_vals[1], color='magenta', label='Polynomial Averaging')
plt.scatter(likely_thinned_omega_m_all_chains, likely_thinned_w_all_chains, color='cyan', label='Likelihood Ind
plt.scatter(thinned_Omega_M_estimate_all_chains, thinned_w_estimate_all_chains, color='red', label='Posterior')

# Plot
plt.title('Marginalized Likelihood Contour for Omega_M vs. w, with Various Estimations')
plt.xlabel('Omega_M')
plt.ylabel('w')
plt.xlim(0, 0.6)
plt.ylim(-1.8, -0.2)
plt.legend(loc='upper right')
plt.grid(True)
plt.show()
```



In [217…

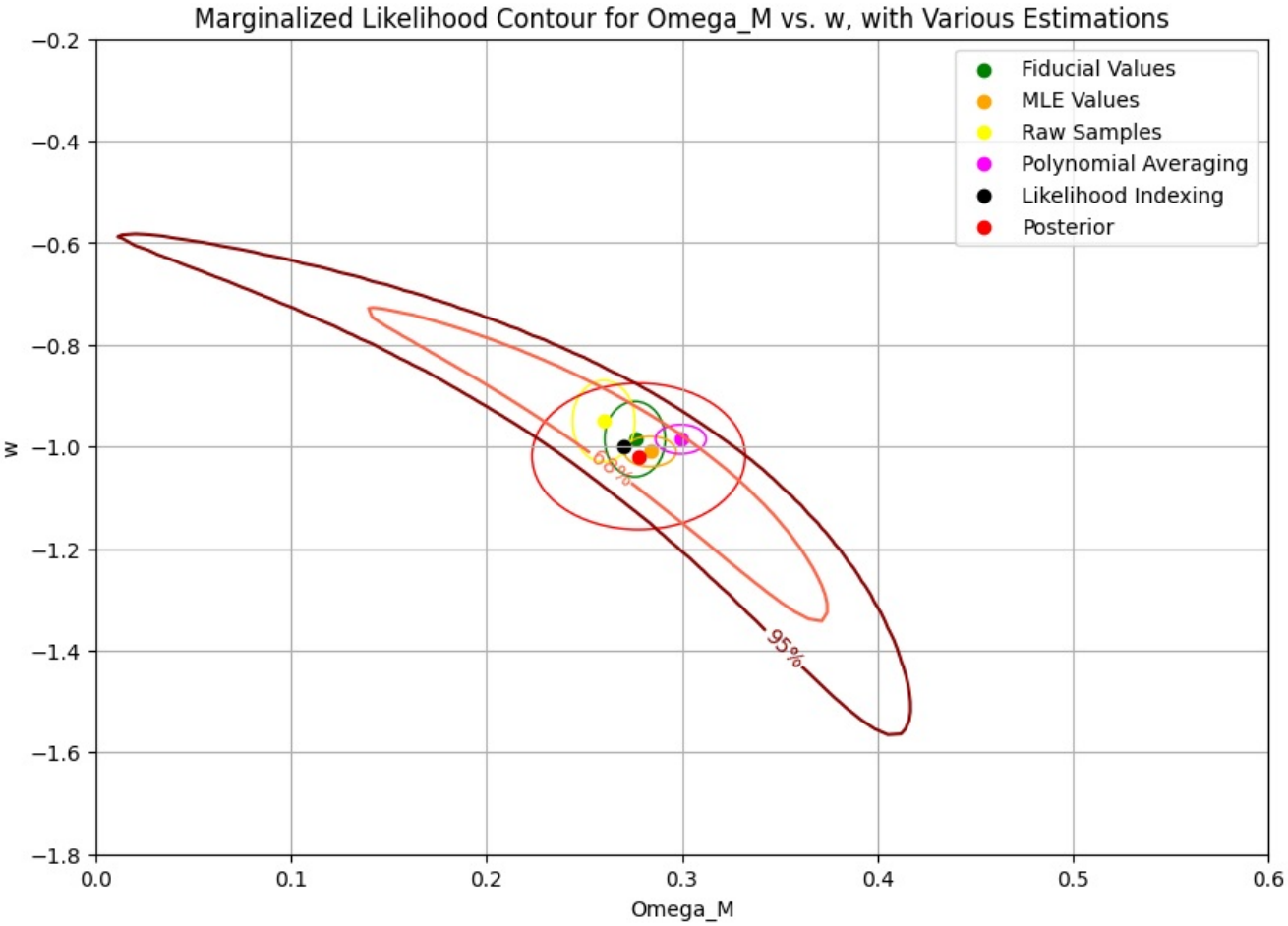
```python
# Plot with all analyses and estimations (Zoomed in)

# Contour plot
plt.figure(figsize=(10, 7))
ax = plt.gca()
cp = plt.contour(omega_m_grid1_slice, w_grid1_slice, marg_likelihoods_w_m_mn, levels=[level_95_w_m_mn, level_68_
plt.clabel(cp, inline=True, fontsize=10, fmt={level_68_w_m_mn: '68%', level_95_w_m_mn: '95%'})

# Plot Fisher confidence ellipses
boundary_right_sig1, boundary_right_sig2 = plot_fisher_confidence_ellipse(ax, fisher_at_most_likely, (most_like
plot_fisher_confidence_ellipse(ax, fisher_at_most_likely, (most_likely_omega_m1, most_likely_w1), 6.17)

# Plot Estimations
plt.scatter(fixed_omega_m, fixed_w, color='green', label='Fiducial Values')
plt.scatter(most_likely_omega_m1, most_likely_w1, color='orange', label='MLE Values')
plt.scatter(boundary_right_sig1, most_likely_w1, color='black', label='Fisher Matrix Omega_M Uncertainty Shift'
plt.scatter(thinned_mean_mean_value_Omega_M, thinned_mean_mean_value_w, color='yellow', label='Raw Samples')
plt.scatter(thinned_conv_vals[0], thinned_conv_vals[1], color='magenta', label='Polynomial Averaging')
plt.scatter(likely_thinned_omega_m_all_chains, likely_thinned_w_all_chains, color='cyan', label='Likelihood Ind
plt.scatter(thinned_Omega_M_estimate_all_chains, thinned_w_estimate_all_chains, color='red', label='Posterior')

# Plot
plt.title('Marginalized Likelihood Contour for Omega_M vs. w, with Various Estimations (Zoomed)')
plt.xlabel('Omega_M')
plt.ylabel('w')
plt.xlim(0.25, 0.31)
plt.ylim(-1.1, -0.9)
plt.legend(loc='lower right')
plt.grid(True)
plt.show()
```



In [ ]:
```
# Std. Estimation Methods Arrays

thinned_Omega_M_estimation_stds = np.array([ \
    0.0155,                          # Fiducial Value
    0.013232117398141879,            # MLE Value
    thinned_overall_std_Omega_M,     # Raw Samples
    thinned_conv_vals_errs[0],       # Polynomial Averaging
    0,                               # Likelihood Indexing
    thinned_Omega_M_std_all_chains   # Posteriors
])

thinned_w_estimation_stds = np.array([
    0.074,                       # Fiducial Value
    0.02973710624537613,         # MLE Value
    thinned_overall_std_w,       # Raw Samples
    thinned_conv_vals_errs[1],   # Polynomial Averaging
    0,                           # Likelihood Indexing
    thinned_w_std_all_chains     # Posteriors
])

thinned_M_n_estimation_stds = np.array([
    0,                               # Fiducial Value
    0.00714790672990374,             # MLE Value
    thinned_overall_std_M_n,         # Raw Samples
    thinned_conv_vals_errs[2],       # Polynomial Averaging
    0,                               # Likelihood Indexing
    thinned_M_n_std_all_chains       # Posteriors
])
```

In [247…
```
# Plot with all analyses, estimations, and uncertainties

# Contour plot
plt.figure(figsize=(10, 7))
ax = plt.gca()
cp = plt.contour(omega_m_grid1_slice, w_grid1_slice, marg_likelihoods_w_m_mn, levels=[level_95_w_m_mn, level_68_
plt.clabel(cp, inline=True, fontsize=10, fmt={level_68_w_m_mn: '68%', level_95_w_m_mn: '95%'})

estimation_points = [
    (fixed_omega_m, fixed_w),
    (most_likely_omega_m1, most_likely_w1),
    (thinned_mean_mean_value_Omega_M, thinned_mean_mean_value_w),
    (thinned_conv_vals[0], thinned_conv_vals[1]),
    (likely_thinned_omega_m_all_chains, likely_thinned_w_all_chains),
    (thinned_Omega_M_estimate_all_chains, thinned_w_estimate_all_chains)
]
```

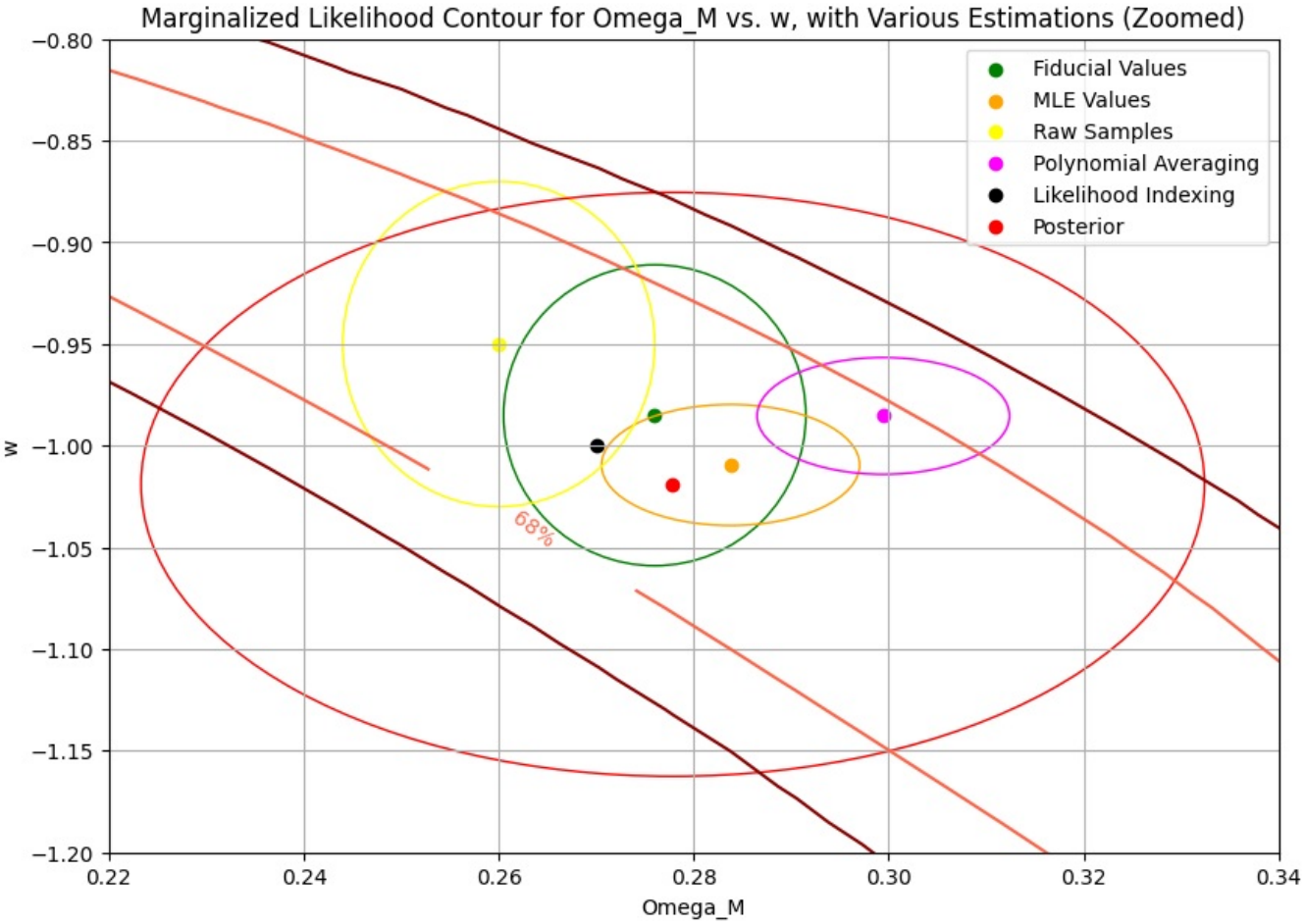
```
colors = ['green', 'orange', 'yellow', 'magenta', 'black', 'red']
labels = ['Fiducial Values', 'MLE Values', 'Raw Samples', 'Polynomial Averaging', 'Likelihood Indexing', 'Poste

for i, ((x, y), color, label) in enumerate(zip(estimation_points, colors, labels)):
    std_x = thinned_Omega_M_estimation_stds[i]
    std_y = thinned_w_estimation_stds[i]
    ellipse = Ellipse((x, y), 2*std_x, 2*std_y, edgecolor=color, facecolor='none')
    ax.add_patch(ellipse)
    plt.scatter(x, y, color=color, label=label)

# Plot
plt.title('Marginalized Likelihood Contour for Omega_M vs. w, with Various Estimations')
plt.xlabel('Omega_M')
plt.ylabel('w')
plt.xlim(0, 0.6)
plt.ylim(-1.8, -0.2)
plt.legend(loc='upper right')
plt.grid(True)
plt.show()
```



In [248…

```
# Plot with all analyses, estimations, and uncertainties (Zoomed in)

# Contour plot
plt.figure(figsize=(10, 7))
ax = plt.gca()
cp = plt.contour(omega_m_grid1_slice, w_grid1_slice, marg_likelihoods_w_m_mn, levels=[level_95_w_m_mn, level_68_
plt.clabel(cp, inline=True, fontsize=10, fmt={level_68_w_m_mn: '68%', level_95_w_m_mn: '95%'})

estimation_points = [
    (fixed_omega_m, fixed_w),
    (most_likely_omega_m1, most_likely_w1),
    (thinned_mean_mean_value_Omega_M, thinned_mean_mean_value_w),
    (thinned_conv_vals[0], thinned_conv_vals[1]),
    (likely_thinned_omega_m_all_chains, likely_thinned_w_all_chains),
    (thinned_Omega_M_estimate_all_chains, thinned_w_estimate_all_chains)
]

colors = ['green', 'orange', 'yellow', 'magenta', 'black', 'red']
labels = ['Fiducial Values', 'MLE Values', 'Raw Samples', 'Polynomial Averaging', 'Likelihood Indexing', 'Poste

for i, ((x, y), color, label) in enumerate(zip(estimation_points, colors, labels)):
    std_x = thinned_Omega_M_estimation_stds[i]
    std_y = thinned_w_estimation_stds[i]
    ellipse = Ellipse((x, y), 2*std_x, 2*std_y, edgecolor=color, facecolor='none')
    ax.add_patch(ellipse)
```

```
    plt.scatter(x, y, color=color, label=label)

# Plot
plt.title('Marginalized Likelihood Contour for Omega_M vs. w, with Various Estimations (Zoomed)')
plt.xlabel('Omega_M')
plt.ylabel('w')
plt.xlim(0.22, 0.34)
plt.ylim(-1.2, -0.8)
plt.legend(loc='upper right')
plt.grid(True)
plt.show()
```



In [ ]:
```
# fmin scipy optimize to get MLE (try different guesses)
# compare with MCMC
```

In [231…
```
# Dataframe of Unthinned Estimated Values and % Errors

# Values
unthinned_omega_m_estimates = np.array([
    fixed_omega_m,                # Fiducial Value
    most_likely_omega_m1,         # MLE Value
    mean_mean_value_Omega_M,      # Raw Samples
    conv_vals[0],                 # Polynomial Averaging
    likely_omega_m_all_chains,    # Likelihood Indexing
    Omega_M_estimate_all_chains   # Posteriors
])

unthinned_w_estimates = np.array([
    fixed_w,                # Fiducial Value
    most_likely_w1,         # MLE Value
    mean_mean_value_w,      # Raw Samples
    conv_vals[1],           # Polynomial Averaging
    likely_w_all_chains,    # Likelihood Indexing
    w_estimate_all_chains   # Posteriors
])

# DataFrame
df_unthinned_estimates = pd.DataFrame({
    "Method": ["Fiducial Value", "MLE Value", "Raw Samples", "Polynomial Averaging", "Likelihood Indexing", "Pos
    "Unthinned Omega_M Estimates": unthinned_omega_m_estimates,
    "Unthinned w Estimates": unthinned_w_estimates,
})

# Formatting
df_unthinned_estimates_sorted = df_unthinned_estimates.sort_values(by="Unthinned Omega_M Estimates", ascending=
```

```
for col in df_unthinned_estimates_sorted.columns:
    if col != "Method":
        df_unthinned_estimates_sorted[col] = df_unthinned_estimates_sorted[col].apply(lambda x: f'{x:.3f}')

html = df_unthinned_estimates_sorted.to_html(index=False)

display(HTML(html))
```

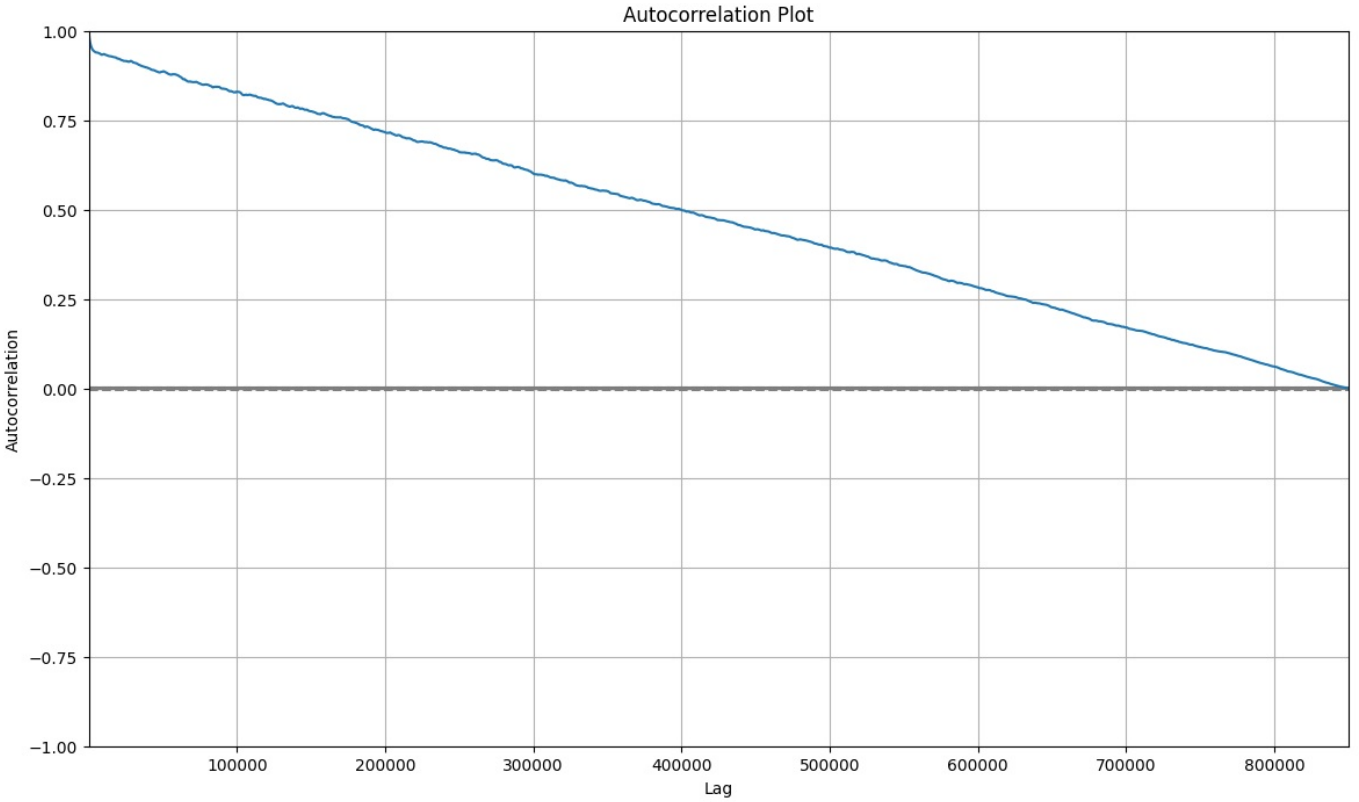
Method	Unthinned Omega_M Estimates	Unthinned w Estimates
Polynomial Averaging	0.297	-1.000
MLE Value	0.284	-1.009
Posteriors	0.277	-1.018
Fiducial Value	0.276	-0.985
Raw Samples	0.274	-1.028
Likelihood Indexing	0.260	-0.968

In [258…

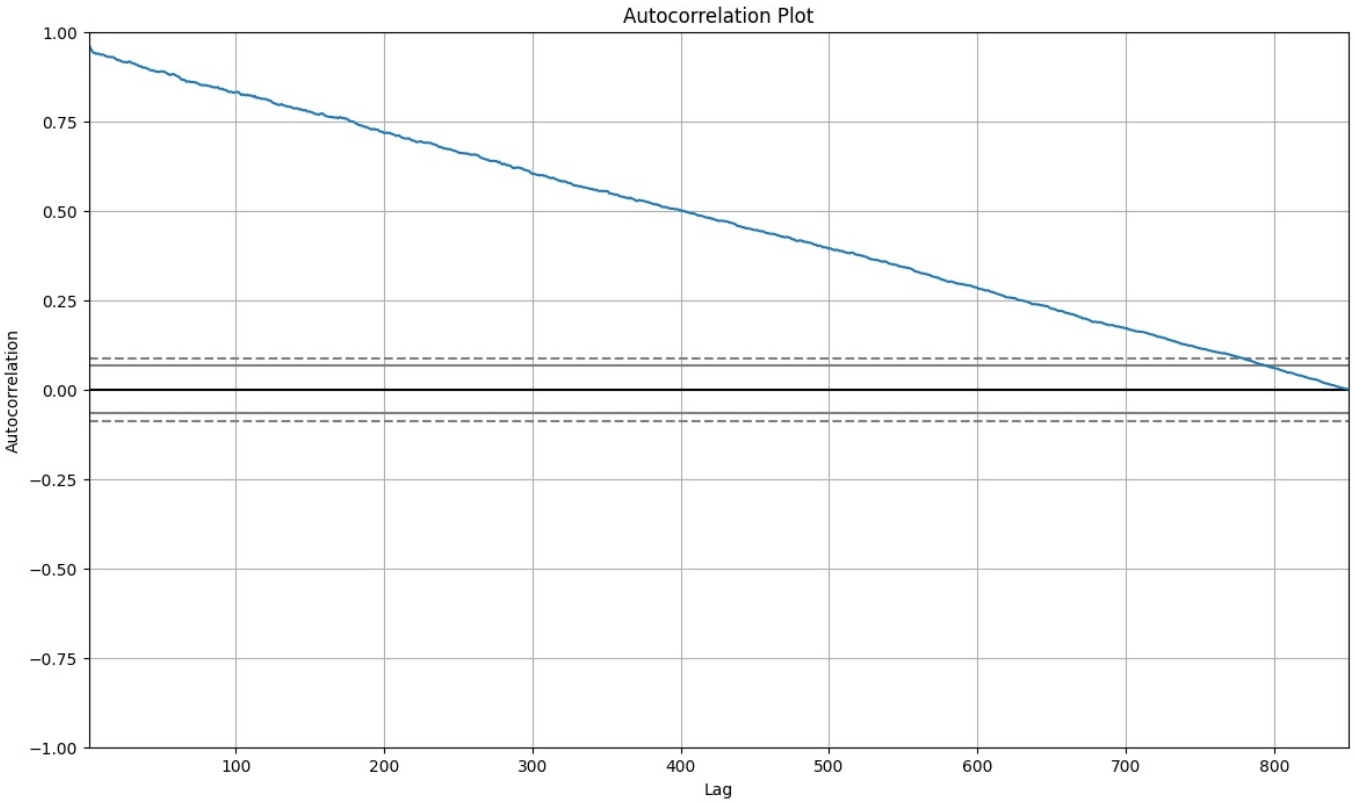
```
# Dataframe of Thinned Estimated Values and % Errors

# Values
thinned_omega_m_estimates = np.array([
    fixed_omega_m,                          # Fiducial Value
    most_likely_omega_m1,                   # MLE Value
    thinned_mean_mean_value_Omega_M,        # Raw Samples
    thinned_conv_vals[0],                   # Polynomial Averaging
    likely_thinned_omega_m_all_chains,      # Likelihood Indexing
    thinned_Omega_M_estimate_all_chains     # Posteriors
])

thinned_w_estimates = np.array([
    fixed_w,                          # Fiducial Value
    most_likely_w1,                   # MLE Value
    thinned_mean_mean_value_w,        # Raw Samples
    thinned_conv_vals[1],             # Polynomial Averaging
    likely_thinned_w_all_chains,      # Likelihood Indexing
    thinned_w_estimate_all_chains     # Posteriors
])

percent_error_omega_m = np.array([
    0,                                                                  # Fiducial Value
    0,                                                                  # MLE Value
    likely_raw_sample_error_thinned_unthinned_diff_all_chains_Omega_M,  # Raw Samples
    likely_error_thinned_unthinned_diff_all_chains[0],                  # Polynomial Averaging
    likely_omega_m_error_thinned_unthinned_diff_all_chains,             # Likelihood Indexing
    posterior_likely_omega_m_error_thinned_unthinned_diff_all_chains    # Posteriors
])

percent_error_w = np.array([
    0,                                                          # Fiducial Value
    0,                                                          # MLE Value
    likely_raw_sample_error_thinned_unthinned_diff_all_chains_w,    # Raw Samples
    likely_error_thinned_unthinned_diff_all_chains[1],          # Polynomial Averaging
    likely_w_error_thinned_unthinned_diff_all_chains,           # Likelihood Indexing
    posterior_likely_w_error_thinned_unthinned_diff_all_chains      # Posteriors
])

# Thinned
df_thinned_estimates = pd.DataFrame({
    "Method": ["Fiducial Value", "MLE Value", "Raw Samples", "Polynomial Averaging", "Likelihood Indexing", "Pos
    "Thinned Omega_M Est.": thinned_omega_m_estimates,
    "Thinned Omega_M Uncert.": thinned_Omega_M_estimation_stds,
    "Thinned w Est.": thinned_w_estimates,
    "Thinned w Uncert.": thinned_w_estimation_stds,
    "% " + "Error Thinned/Unthinned Omega_M": percent_error_omega_m,
    "% " + "Error Thinned/Unthinned w": percent_error_w
})

# Formatting
df_thinned_estimates_sorted = df_thinned_estimates.sort_values(by="Thinned Omega_M Est.", ascending=False).rese

for col in df_thinned_estimates_sorted.columns:
    if col != "Method":
        if "%" in col:
            df_thinned_estimates_sorted[col] = df_thinned_estimates_sorted[col].apply(lambda x: f'{x:.3f} %')
        else:
            df_thinned_estimates_sorted[col] = df_thinned_estimates_sorted[col].apply(lambda x: f'{x:.3f}')

html1 = df_thinned_estimates_sorted.to_html(index=False)

display(HTML(html1))
```

Method	Thinned Omega_M Est.	Thinned Omega_M Uncert.	Thinned w Est.	Thinned w Uncert.	% Error Thinned/Unthinned Omega_M	% Error Thinned/Unthinned w
Polynomial Averaging	0.299	0.013	-0.985	0.029	0.884 %	-1.539 %
MLE Value	0.284	0.013	-1.009	0.030	0.000 %	0.000 %
Posteriors	0.278	0.055	-1.019	0.144	0.213 %	-0.083 %
Fiducial Value	0.276	0.015	-0.985	0.074	0.000 %	0.000 %
Likelihood Indexing	0.270	0.000	-1.000	0.000	13.024 %	-3.985 %
Raw Samples	0.260	0.016	-0.950	0.080	0.057 %	-0.008 %

In [195…
```
# Unthinned Autocorrelation Plot

data_ac = np.vstack((chain_1_Omega_M_values, chain_1_w_values)).T
df_autocorr = pd.DataFrame(data=data_ac, columns=["$\\Omega_M$", 'w'])

fig = plt.figure(figsize = (14, 8))
plt.title("Autocorrelation Plot")
plt.xlabel("Autocorrelation")
plt.ylabel("Lag")
plt.ylim(-0.25, 1)
autocorrelation_plot(df_autocorr[:])
plt.show()
```

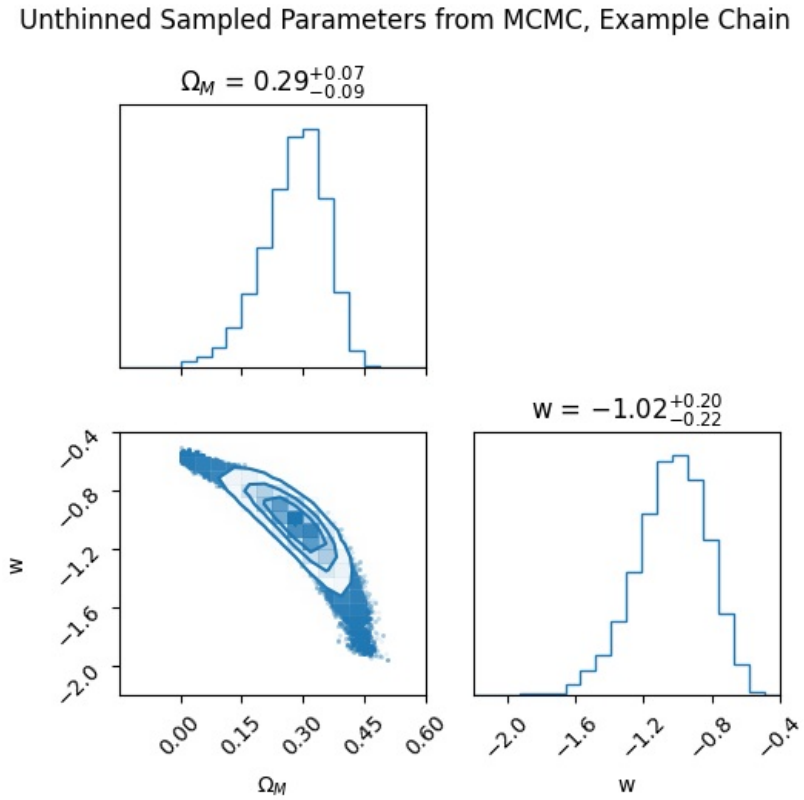


In [196…
```
# Thinned Autocorrelation Plot

thinned_data_ac = np.vstack((thinned_chain_1_Omega_M_values, thinned_chain_1_w_values)).T
df_autocorr = pd.DataFrame(data=thinned_data_ac, columns=["$\\Omega_M$", 'w'])

fig = plt.figure(figsize = (14, 8))
plt.title("Autocorrelation Plot")
plt.xlabel("Autocorrelation")
plt.ylabel("Lag")
plt.ylim(-0.25, 1)
autocorrelation_plot(df_autocorr[:])
plt.show()

# If the autocorrelation plot goes to 0 at max iteration, does that mean that the # of samples is too low?
```



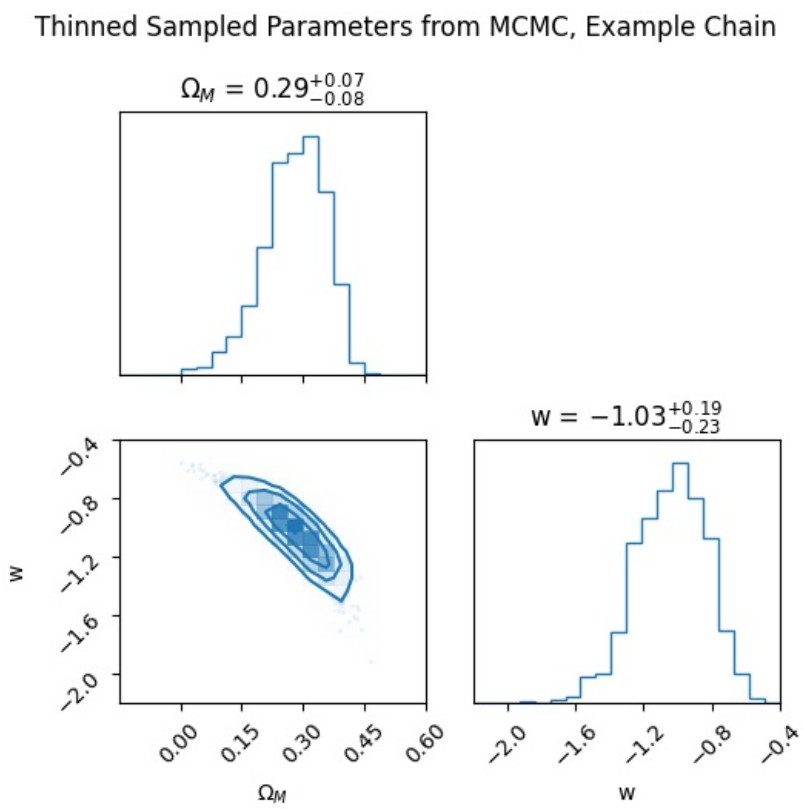
In [203…
```
# Unthinned Corner Plot Sample Chain

data_corner = np.vstack((chain_1_Omega_M_values, chain_1_w_values)).T
fig = corner.corner(data_corner, color='C0', smooth=0.7, labels=['$\\Omega_M$', 'w'], range=[[-0.15, 0.6], [-2.1
fig.suptitle("Unthinned Sampled Parameters from MCMC, Example Chain")
plt.tight_layout()
plt.show()
```

```
<Figure size 1400x800 with 0 Axes>
```



In [198…
```
# Thinned Corner Plot Sample Chain

thinned_data_corner = np.vstack((thinned_chain_1_Omega_M_values, thinned_chain_1_w_values)).T
fig = corner.corner(thinned_data_corner, color='C0', smooth=0.7, labels=['$\\Omega_M$', 'w'], range=[[-0.15, 0.6
fig.suptitle("Thinned Sampled Parameters from MCMC, Example Chain")
plt.tight_layout()
plt.show()
```

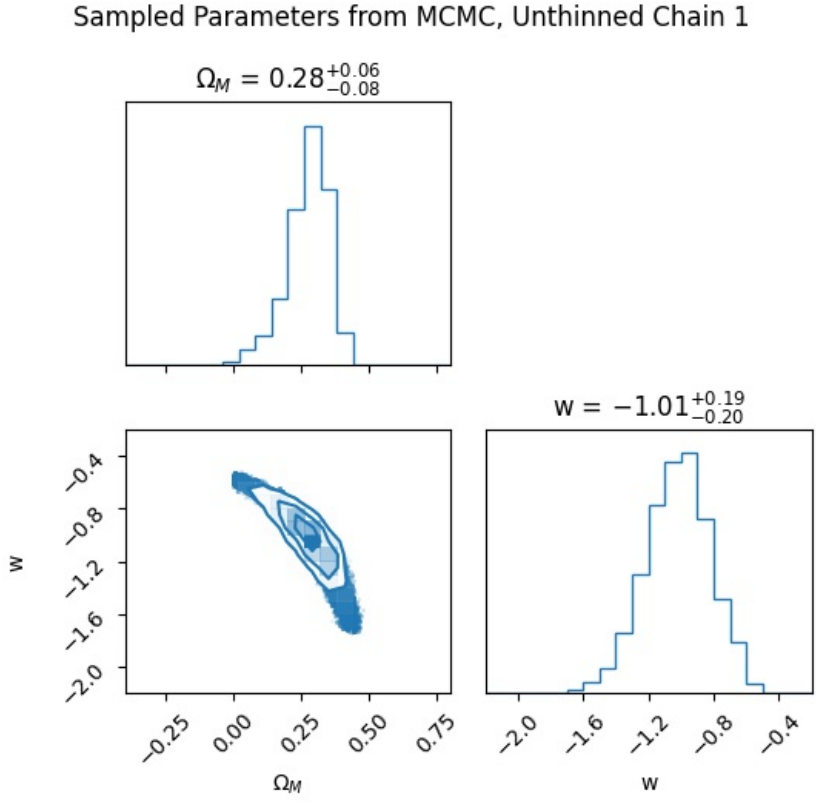
```
<Figure size 1400x800 with 0 Axes>
```



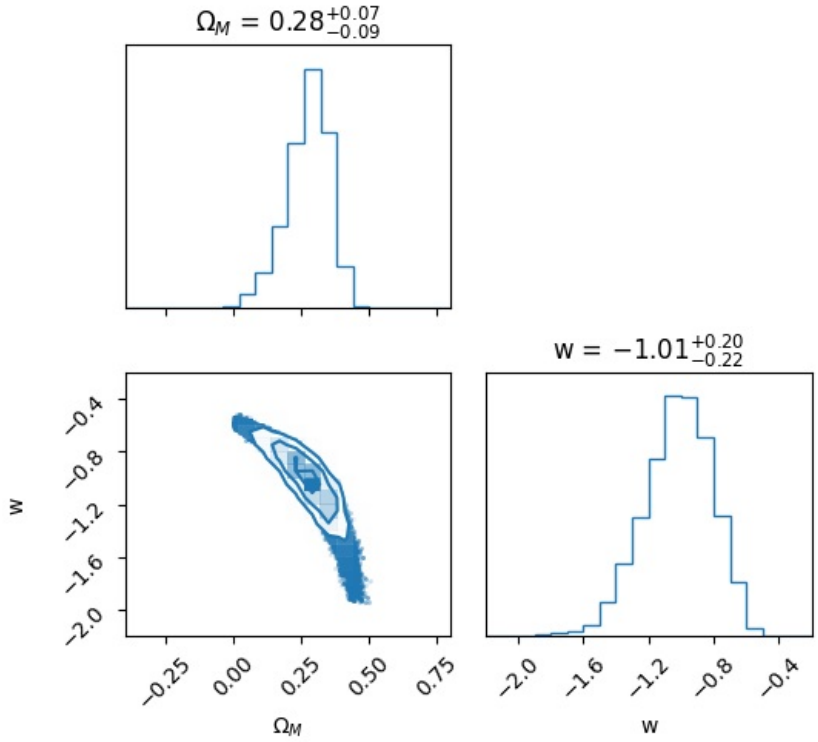
In [199…

```
# Unthinned Corner Plot All Chains

for i in range(num_chains):
    chain_Omega_M_values = all_chains_Omega_M[i, :]
    chain_w_values = all_chains_w[i, :]
    data_corner = np.vstack((chain_Omega_M_values, chain_w_values)).T

    figure = corner.corner(data_corner, color='C0', smooth=0.5, labels=['$\\Omega_M$', 'w'],
                           range=[[-0.4, 0.8], [-2.2, -0.2]], show_titles=True,
                           hist_kwargs={'density': True})

    figure.suptitle(f"Sampled Parameters from MCMC, Unthinned Chain {i + 1}")
    plt.tight_layout()
    plt.show()
```

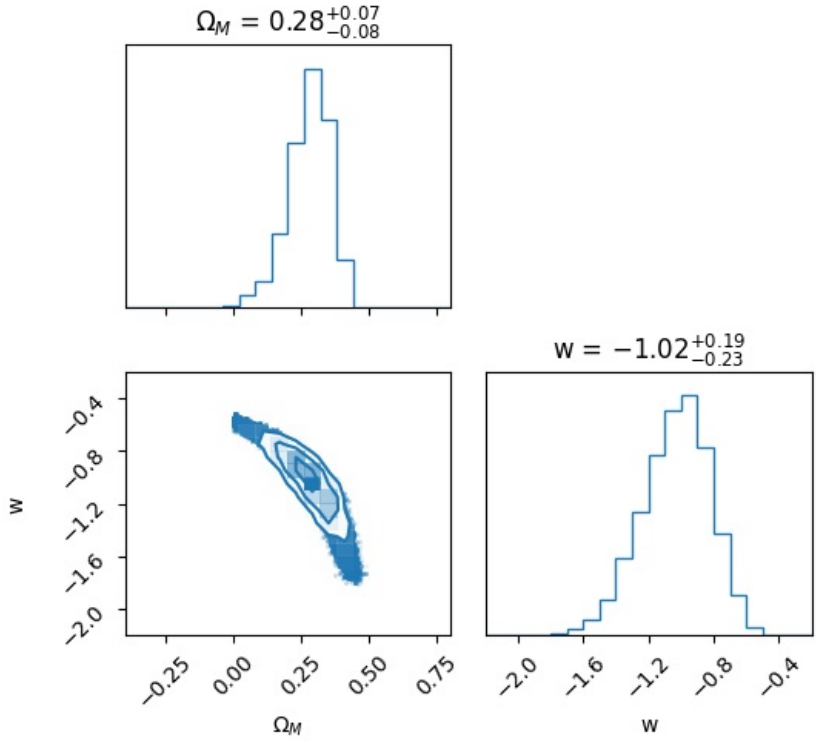
```
<Figure size 1400x800 with 0 Axes>
```



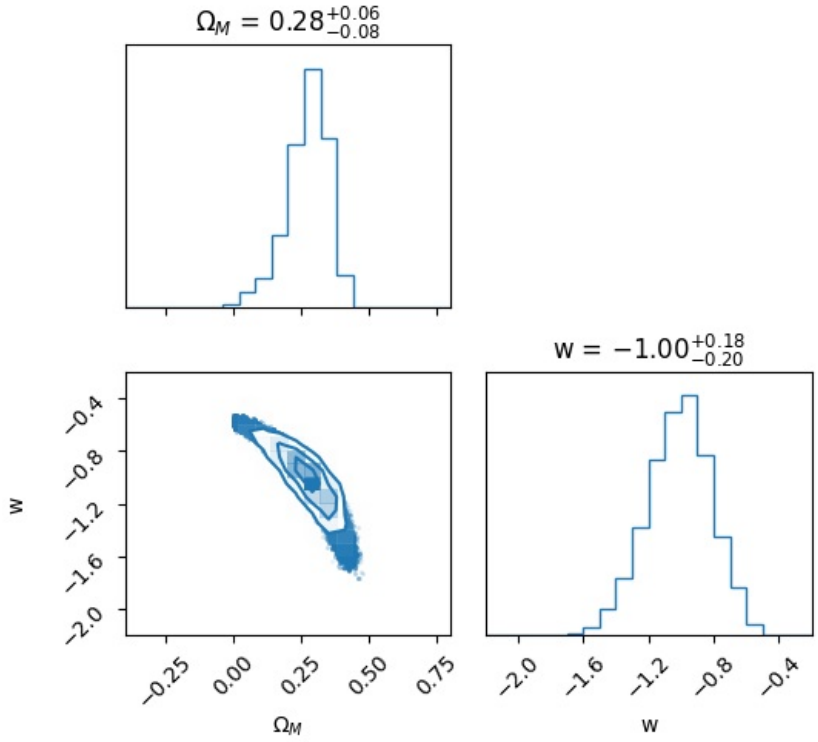
Sampled Parameters from MCMC, Unthinned Chain 2



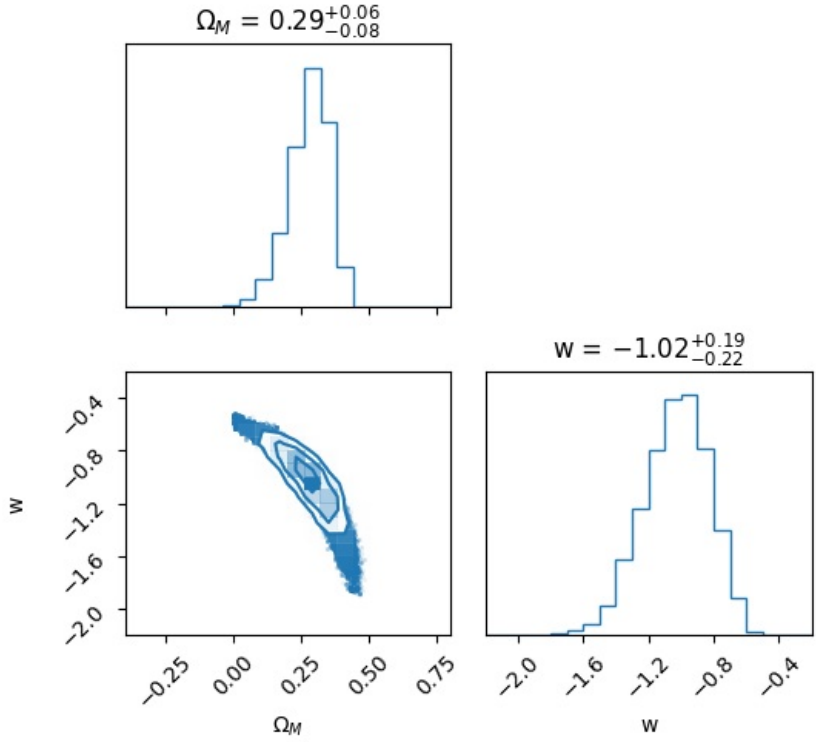
Sampled Parameters from MCMC, Unthinned Chain 3

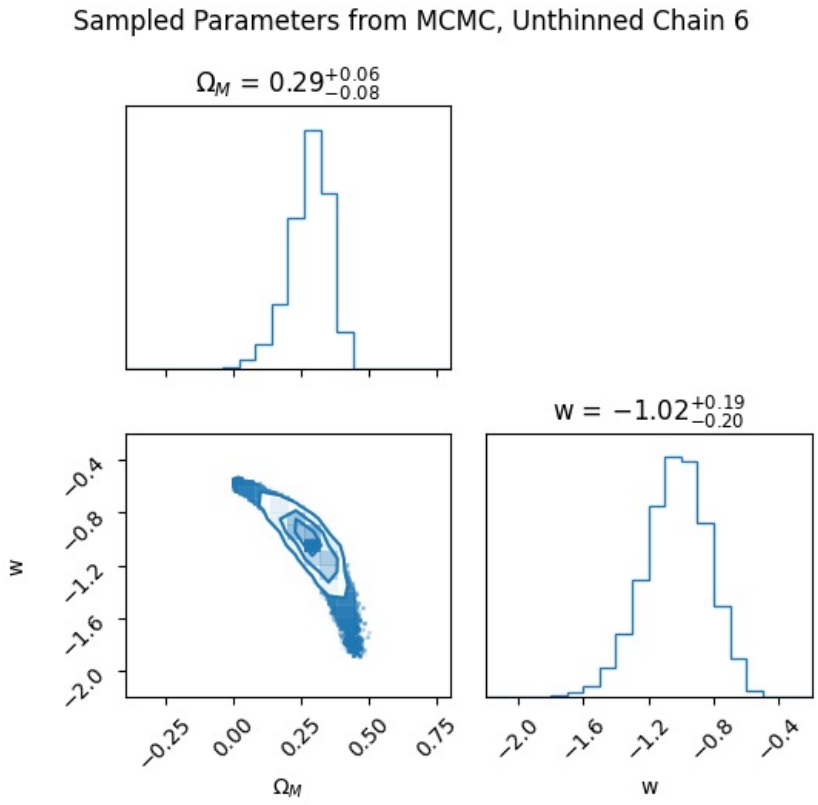


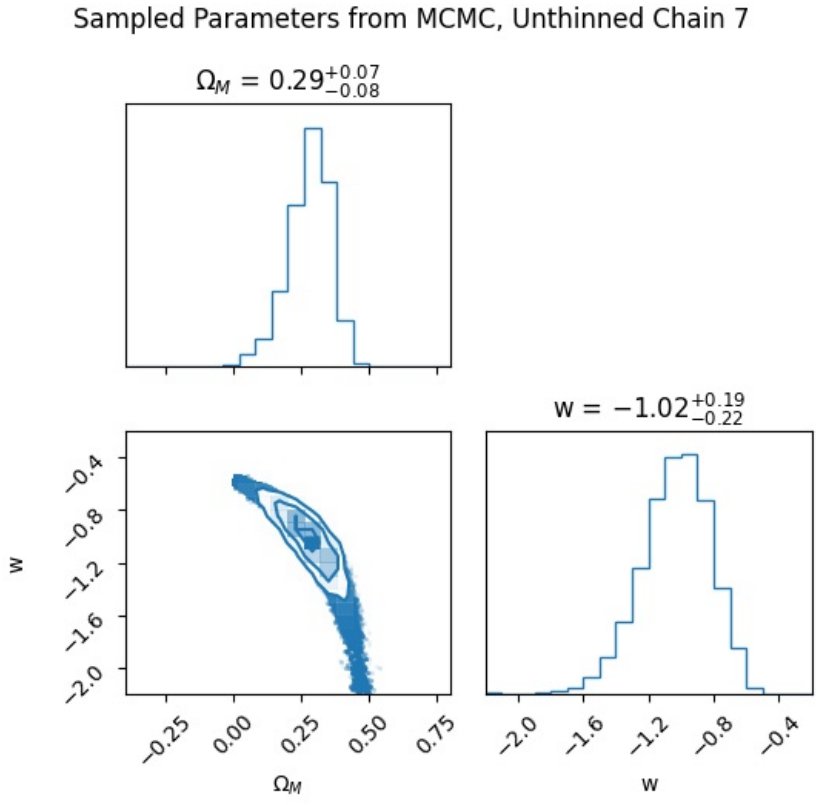
Sampled Parameters from MCMC, Unthinned Chain 4



Sampled Parameters from MCMC, Unthinned Chain 5





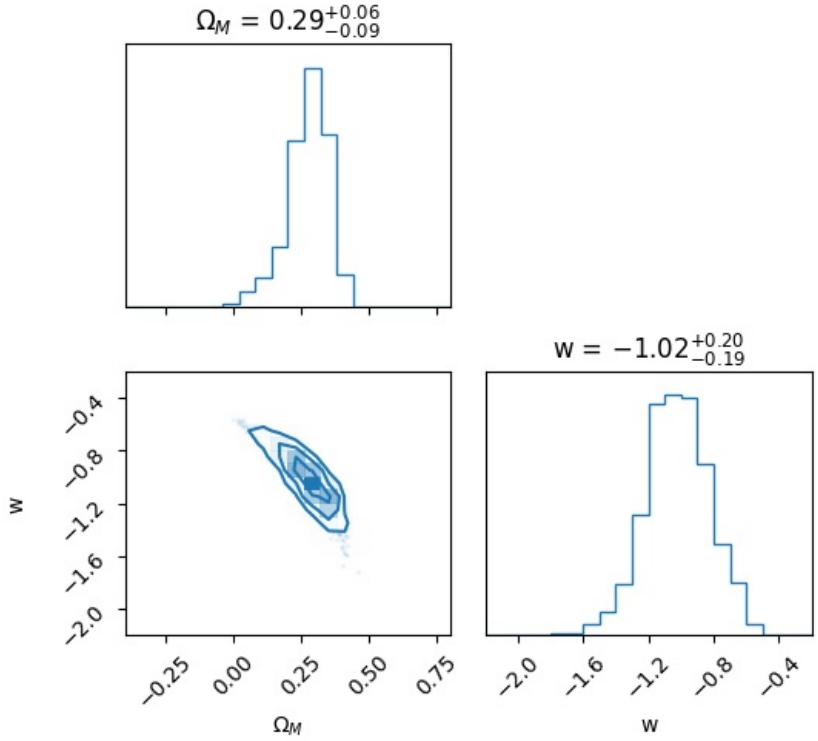


In [200…

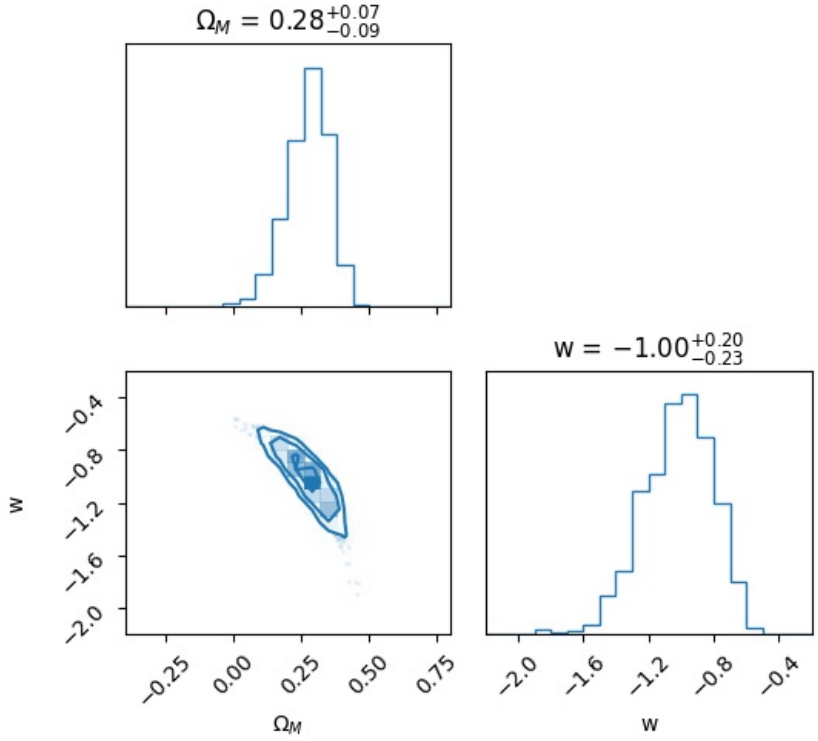
```python
# Thinned Corner Plot All Chains

for i in range(num_chains):
    thinned_chain_Omega_M_values = thinned_all_chains_Omega_M_values[i, :]
    thinned_chain_w_values = thinned_all_chains_w_values[i, :]
    thinned_data_corner = np.vstack((thinned_chain_Omega_M_values, thinned_chain_w_values)).T

    figure = corner.corner(thinned_data_corner, color='C0', smooth=0.5, labels=['$\\Omega_M$', 'w'],
                           range=[[-0.4, 0.8], [-2.2, -0.2]], show_titles=True,
                           hist_kwargs={'density': True})

    figure.suptitle(f"Sampled Parameters from MCMC, Thinned Chain {i + 1}")
    plt.tight_layout()
    plt.show()
```

```
<Figure size 1400x800 with 0 Axes>
```

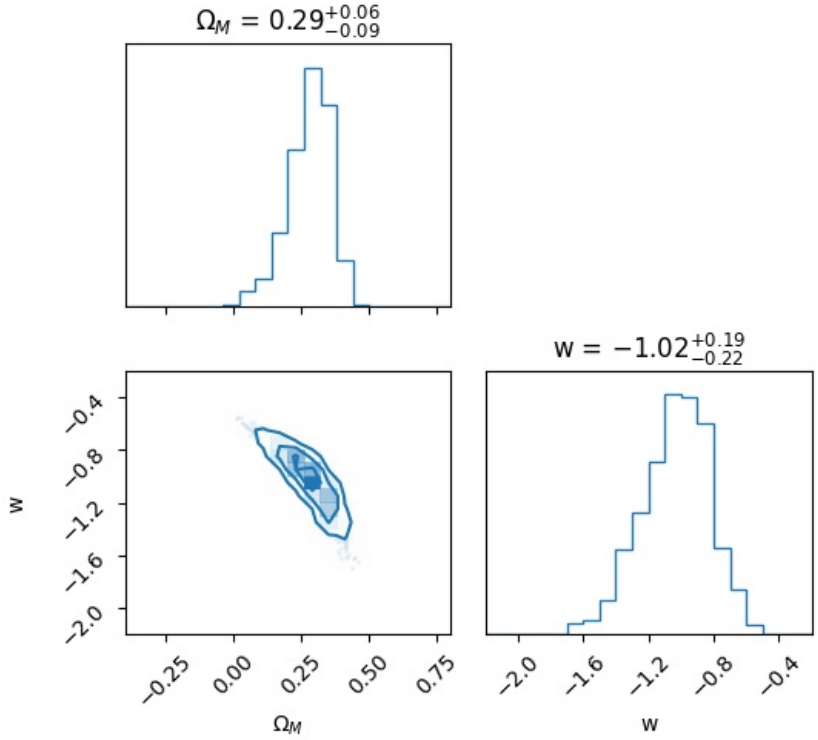
Sampled Parameters from MCMC, Thinned Chain 1



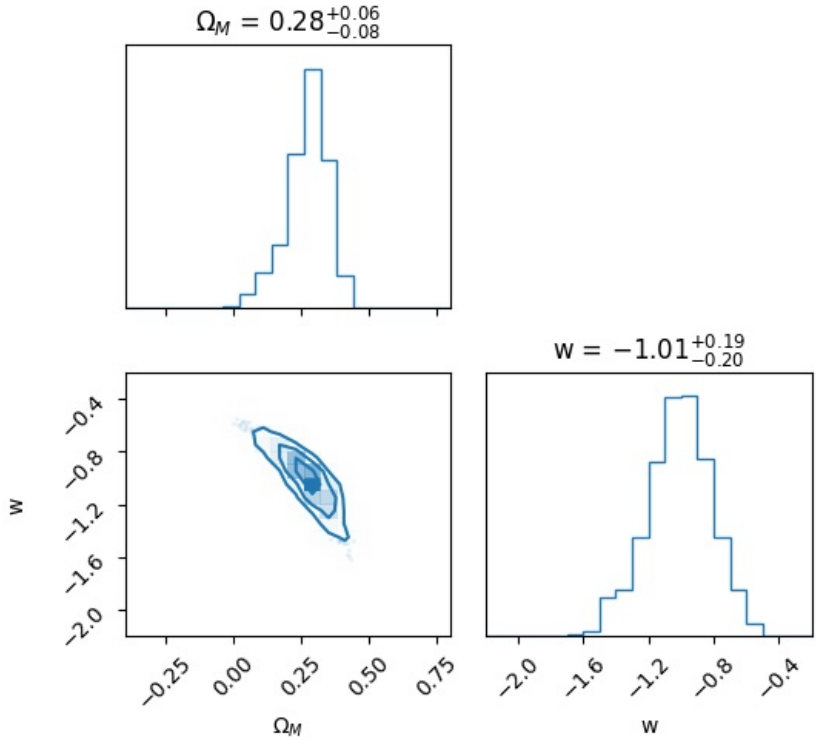
Sampled Parameters from MCMC, Thinned Chain 2



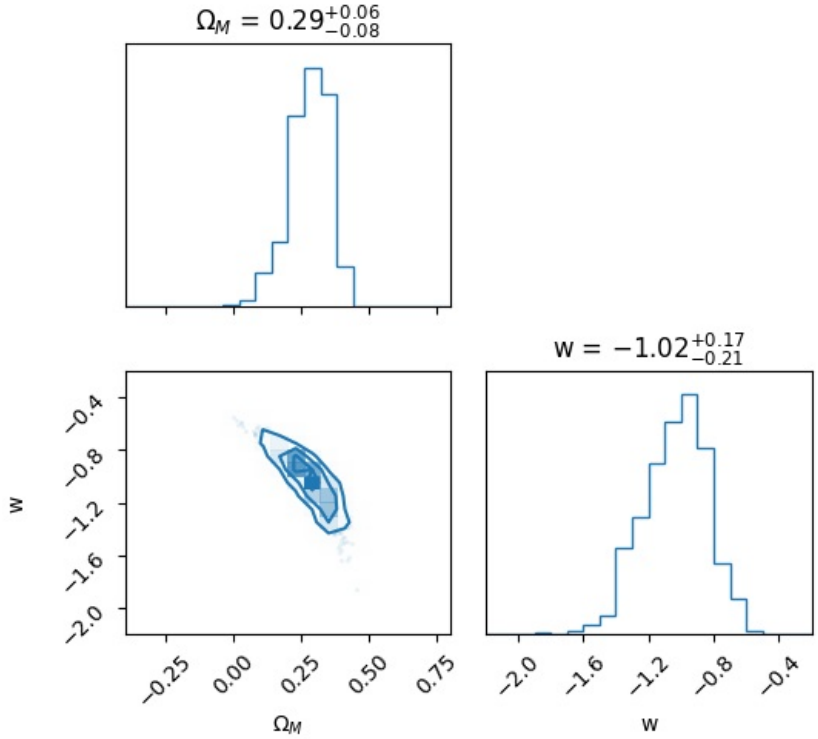
Sampled Parameters from MCMC, Thinned Chain 3



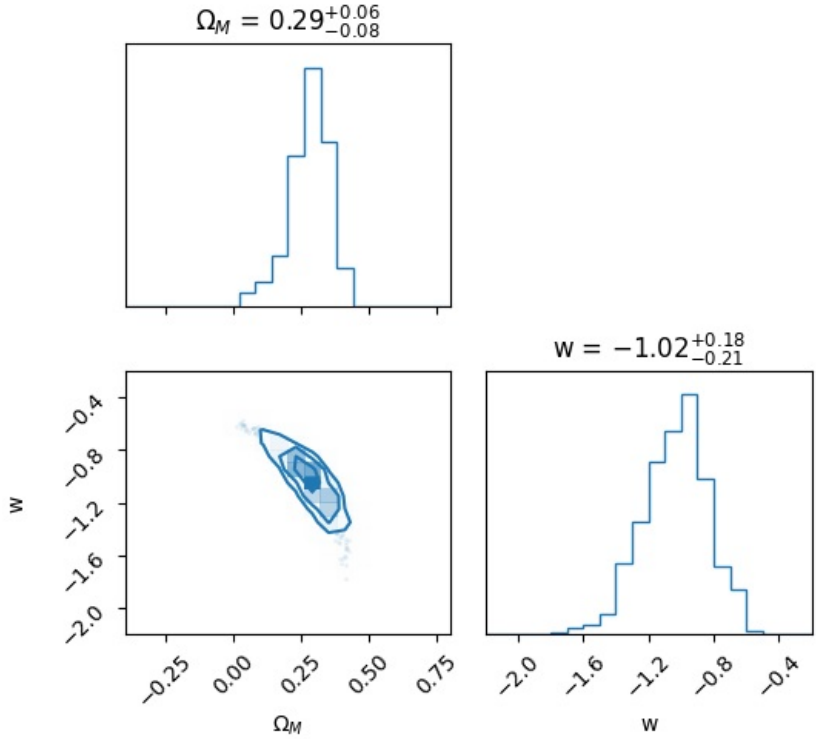
Sampled Parameters from MCMC, Thinned Chain 4

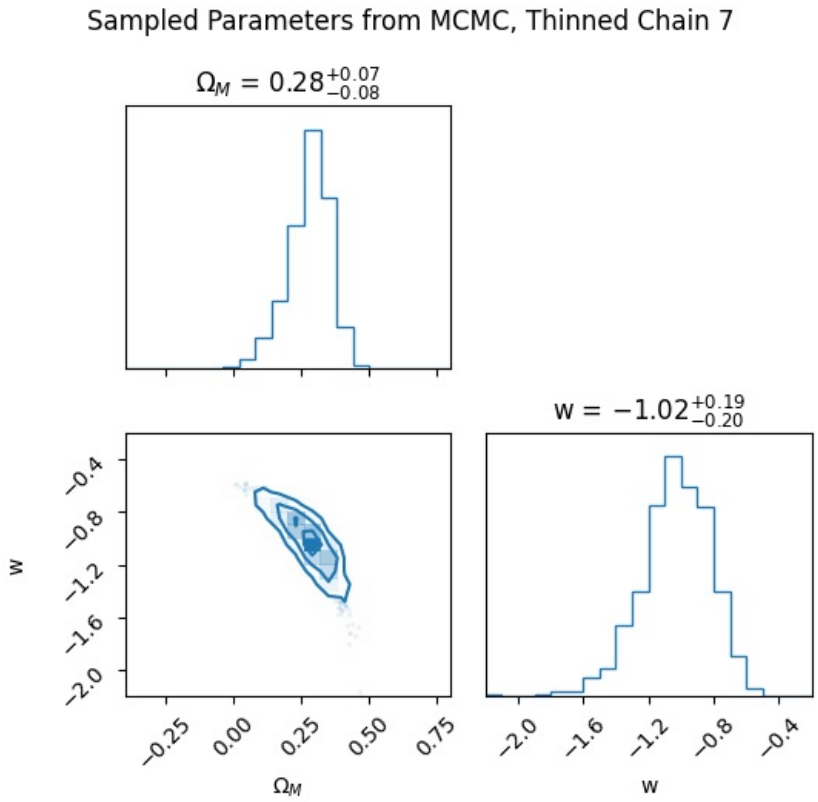


Sampled Parameters from MCMC, Thinned Chain 5



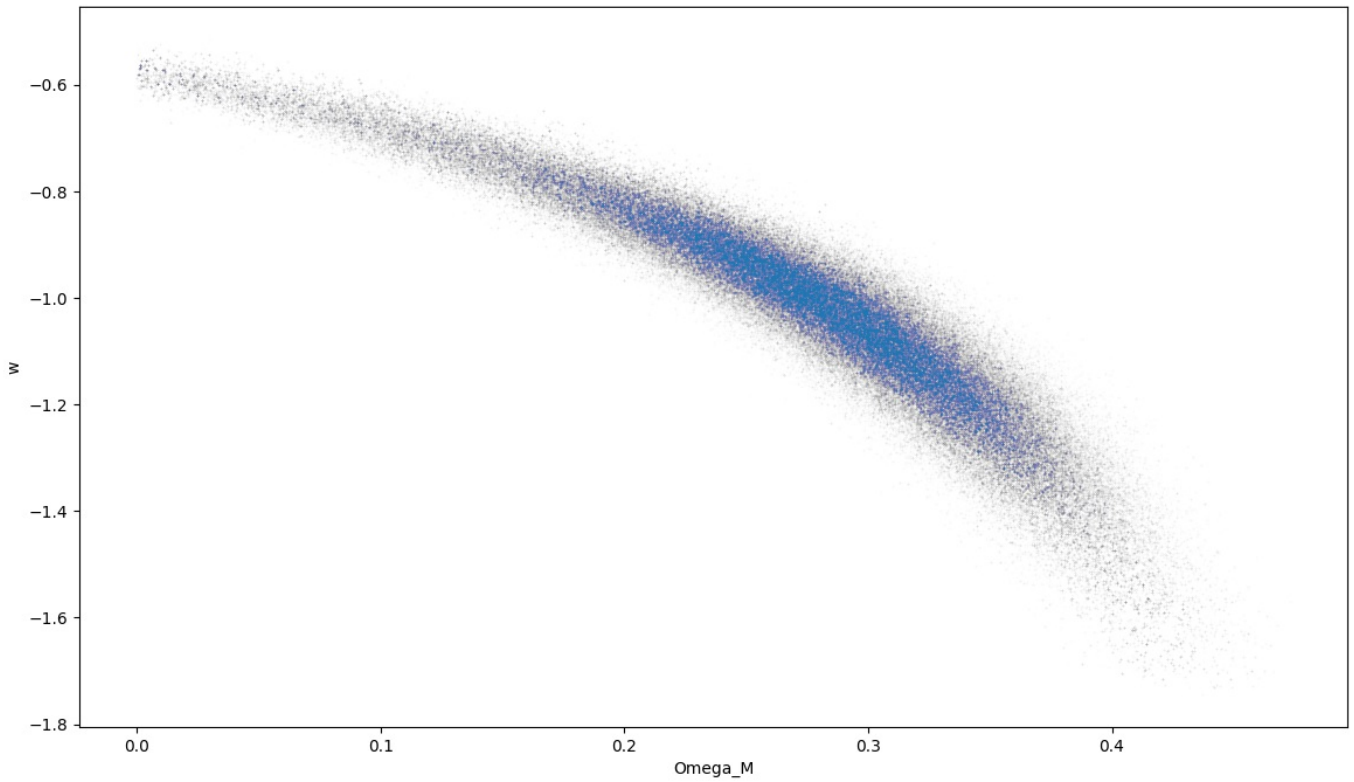
Sampled Parameters from MCMC, Thinned Chain 6





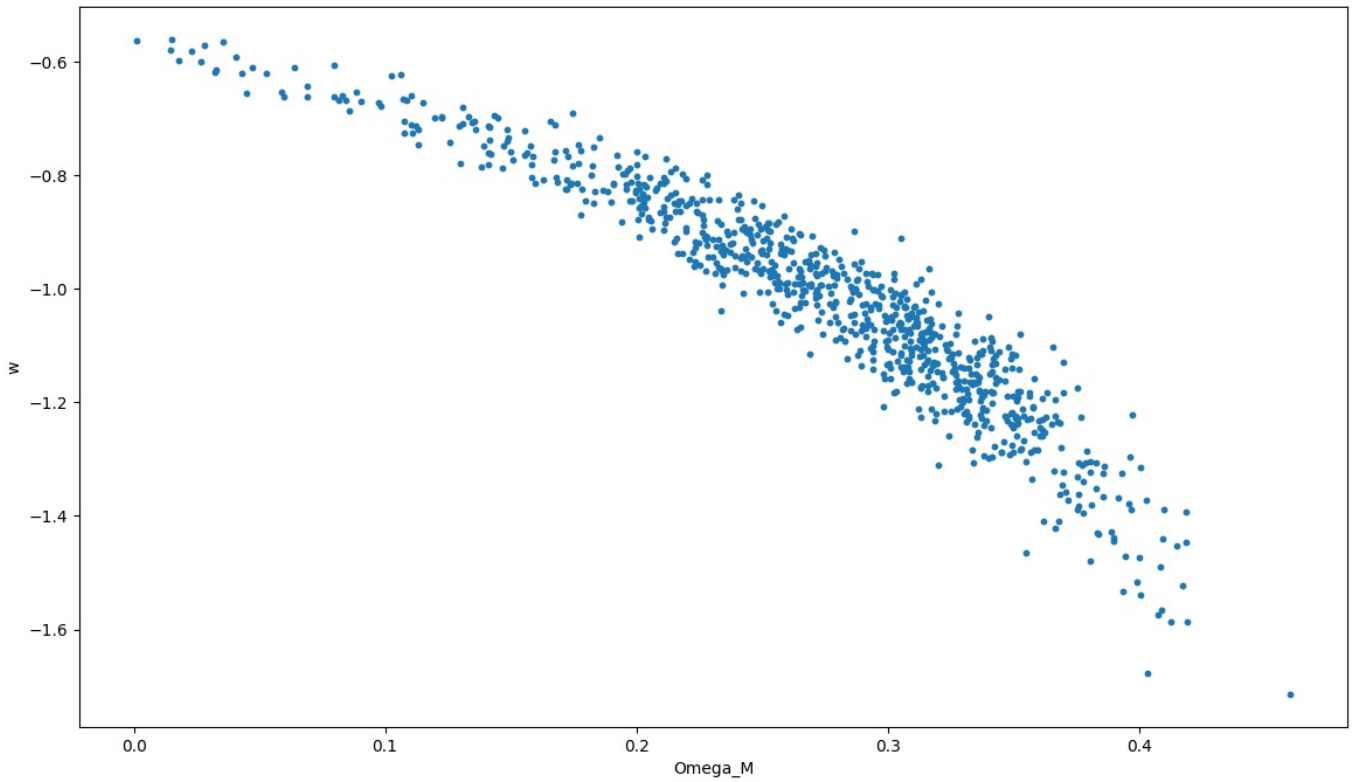
In [208…

```
# Unthinned Scatter Plot All Chains

fig = plt.figure(figsize = (14, 8))
plt.scatter(all_chains_Omega_M[0,:], all_chains_w[0,:], s = 0.1, alpha = 0.015)
plt.xlabel('Omega_M')
plt.ylabel('w')
plt.show()
```



In [209…

```
# Thinned Scatter Plot All Chains

fig = plt.figure(figsize = (14, 8))
plt.scatter(thinned_all_chains_Omega_M_values[0,:], thinned_all_chains_w_values[0,:], s = 10, alpha = 1)
plt.xlabel('Omega_M')
plt.ylabel('w')
plt.show()
```



In [116…

```
# Model
# with pm.Model() as model:
#     Omega_M = pm.Uniform('Omega_M', lower=0, upper=1)
#     Omega_DE = 1 - Omega_M

#     step = pm.Metropolis()

#     trace = pm.sample(50000, step=step, initvals = {'Omega_M': 0.28}, return_inferencedata=False, tune=5000, 


# Plot

# fig, axes = plt.subplots(nrows=1, ncols=2, figsize=(10, 5))
```

```
# az.plot_trace(trace, var_names=['Omega_M', 'w'], ax=axes)
# plt.show()

# Posterior predictive checks
# az.plot_posterior(data=trace,var_names=["Omega_M"],figsize=(16,4))
# plt.show()

# Summary of findings
# summary = az.summary(trace, round_to=2)
# print(summary)


# Omega_M_samples, w_samples, M_n_samples = parameter_arrays(samples, step_manual, start_Omega_M, start_w, star
# print(Omega_M_samples[990:])
# print(w_samples[990:])
# print(M_n_samples[990:])

# def parameter_arrays(samples, step, new_Omega_M, new_w, new_M_n):
#     Omega_M_vals = []
#     w_vals = []
#     M_n_vals = []

#     for _ in range(samples):
#         # Propose new values
#         new_Omega_M = new_Omega_M + np.random.normal(scale=step)
#         new_w = new_w + np.random.normal(scale=step)
#         new_M_n = new_M_n + np.random.normal(scale=step)

#         Omega_M_vals = np.append(Omega_M_vals, new_Omega_M)
#         w_vals = np.append(w_vals, new_w)
#         M_n_vals = np.append(M_n_vals, new_M_n)

#     return Omega_M_vals, w_vals, M_n_vals
```

In [747…

```
# Select a proposal function that defines how to step across data points (decides on what grounds to reject/acc
# Pick a starting point, use proposal function to move to the next point/stay at current point
# Repeat until model convergence

# Burn-in stage - Decide how many of the initially selected data point acceptances/rejections to omit from infe
# PYMC example - trace3 = pm.sample(#samples = N, step=[H388],tune=5000,chains=1) (tune out first 5000 values)
# If a different posterior analysis is observed after running the model, use a larger burn-in period, or change
# Chain should start at expected peak of model landscape (i.e start = pm.find_MAP(fmin=scipy.optimize.fmin_powe

# Ensure that the model is efficient, and doesn't accept/reject values disproportionately
# Possibility of reparameterizing model (i.e multiplying/dividing one parameter by the other, and leaving the o

# Ensure "mixing" of chain
# Example - thinning the chain by removing every N value (N > 10 not necessary)
# Example - running multiple chain, comparing convergence between each one, and determining the general cause b
```

In [748…

```
# FIM Case 2

def fisher_matrix_2(z, m_err, Omega_M, Omega_DE, M_n, Omega_K):
    # Define parameters
    params = {'Omega_M': Omega_M, 'Omega_DE': Omega_DE, 'w': w, 'M_n': M_n, 'Omega_K': Omega_K}
    vary_params_flat = ['Omega_M', 'Omega_DE', 'Omega_K', 'M_n']
    param_keys = list(vary_params_flat)
    n_params = len(param_keys)
    fisher = np.zeros((n_params, n_params))
    h = 1e-5

    for i in range(n_params):
        for j in range(n_params):
            base_params = params.copy()

            # Modify parameters i and j
            base_params[param_keys[i]] += h
            m_th1 = m_th(z, **base_params)

            base_params[param_keys[i]] -= 2 * h
            m_th2 = m_th(z, **base_params)

            # Reset to original
            base_params[param_keys[i]] += h

            # Central difference for partial derivatives
            dm_dpi = (m_th1 - m_th2) / (2 * h)

            if i == j:  # Diagonal elements, derivative wrt same parameter
                fisher[i, j] = np.sum((dm_dpi**2) / (m_err**2))
            else:  # Off-diagonal elements, mixed partial derivatives
                base_params[param_keys[j]] += h
                m_th1 = m_th(z, **base_params)
```

```
                base_params[param_keys[j]] -= 2 * h
                m_th2 = m_th(z, **base_params)

                dm_dpj = (m_th1 - m_th2) / (2 * h)
                fisher[i, j] = np.sum((dm_dpi * dm_dpj) / (m_err**2))

    return fisher

F_flat = fisher_matrix_2(z, m_err, Omega_M, Omega_DE, M_n, Omega_K)

print("Fisher Matrix (Flat Universe):")
print(F_flat)
```

```
Fisher Matrix (Flat Universe):
[[ 38622.96362906  25691.94522167  41583.98284445 -26881.4705564 ]
 [ 25691.94522196  18692.43960121  28994.78991589 -18959.0305462 ]
 [ 41583.98284445  28994.78991557  45888.27340232 -29838.92600466]
 [-26881.4705564  -18959.0305463  -29838.92600466  19434.0772081 ]]
```