Bounds on Energy-Harvesting Wireless Transmission over a Fast Fading Channel
Final Project: EE 376D

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1 Summary

This paper concerns optimal policies for a finite-battery energy-harvesting wireless transmitter with Bernoulli power arrivals and an i.i.d. fast-fading channel. The paper proposes a policy for energy harvesting over such a channel. It develops a novel recursive framework for discovering upper bounds for energy-harvesting throughput under a given fading distribution, and applies it to the case of constant and Bernoulli channel state distributions.

2 Statement of Problem and Proposed Solution

We seek to maximize the following expression over all g_t.

$\mathcal{J}(g_t) = \frac{1}{\mathbb{E}L}\mathbb{E}\left[\sum_{t=1}^{L} \frac{1}{2}\log(1+\gamma_t g_t)\right] = \lim_{n\to\infty} \mathbb{E}\left[\sum_{t=1}^{n-1} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)\right]$,

where $\sum_{t=1}^{\infty} g_t \leq \bar{B}$ everywhere, $L \sim \text{Geom}(p)$, γ_t are i.i.d. non-negative random variables. g must satisfy $0 \leq g_t \leq b_t$, where b_t is the remaining battery life (i.e. $b_t = \bar{B} - \sum_{i=1}^{t-1} g_i$), and is only dependent on the current channel and battery state. Thus, we can rewrite g_t as $g(b_t, \gamma_t)$.

I will investigate the near-optimality of the following simple policy:

$$g(b_t, \gamma_t) = b_t r_{\gamma_t} \tag{1}$$

Here $r_\gamma = \frac{b_t p \gamma}{p\gamma + (1-p)\bar{\gamma}}$. We establish near-optimality with reference to an optimal throughput, defined as $\Theta := \sup_{g_t} \mathcal{J}(g_t)$.

3 Establishing a Pseudo-Bound

Although there are provable exceptions (which can be found for many binary channels by letting $p \to 0$), the expression $\hat{\Theta} := \mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_\gamma)\right]$ behaves similarly to an upper bound for $\mathcal{J}(g_t)$.

3.1 Multiplicative Gap to Pseudo-Bound

Lemma 3.1. $\frac{1}{2}\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \le \mathcal{J}(g_t)$.

Proof.

$$
\begin{aligned}
\mathcal{J}(g_t) &= \frac{1}{\mathbb{E}L}\mathbb{E}\left[\sum_{t=1}^{L}\frac{1}{2}\log(1+\gamma_t g_t)\right] = \frac{1}{\mathbb{E}L}\mathbb{E}\left[\sum_{t=1}^{L}\frac{1}{2}\log(1+\gamma_t\bar{B}r_{\gamma_t}\prod_{i=1}^{t-1}(1-r_{\gamma_i})))\right] \\
&\ge p\sum_{k=1}^{\infty}p(1-p)^{k-1}\sum_{t=1}^{k}\mathbb{E}\left[\prod_{i=1}^{t-1}(1-r_{\gamma_i})\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \\
&= \sum_{k=1}^{\infty}p^2(1-p)^{k-1}\sum_{t=1}^{k}\prod_{i=1}^{t-1}(1-\mathbb{E}r_{\gamma_i})\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \\
&\overset{(a)}{\ge} \sum_{k=1}^{\infty}p^2(1-p)^{k-1}\sum_{t=1}^{k}(1-p)^{t-1}\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \\
&= \sum_{k=1}^{\infty}p^2(1-p)^{k-1}\frac{1-(1-p)^k}{p}\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \\
&= \frac{p}{1-p}\sum_{k=1}^{\infty}((1-p)^k-(1-2p+p^2)^k)\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \\
&= \frac{p}{1-p}\left(\frac{1}{p}-\frac{1}{2p-p^2}\right)\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \\
&= \frac{1}{1-p}\left(1-\frac{1}{2-p}\right)\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \\
&= \frac{1}{2-p}\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] \ge \frac{1}{2}\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right]
\end{aligned}
\tag{2}
$$

(a) is true since r_{γ_i} is concave in γ_i, allowing Jensen's inequality to be applied:

$$
\mathbb{E}r_{\gamma_i} = \mathbb{E}\left[\frac{p\gamma_i}{p\gamma_i+(1-p)\bar{\gamma}_i}\right] \le \frac{p\mathbb{E}\gamma}{p\mathbb{E}\gamma+(1-p)\bar{\gamma}} = \frac{p\bar{\gamma}}{p\bar{\gamma}+(1-p)\bar{\gamma}} = p \tag{3}
$$

□

3.2 Additive Gap to Pseudo-Bound

Lemma 3.2. $\mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] - \frac{1}{2} \le \mathcal{J}(g_t)$.

Proof.

$$
\begin{aligned}
\mathcal{J}(g_t) &= \frac{1}{\mathbb{E}L}\mathbb{E}\left[\sum_{t=1}^{L}\frac{1}{2}\log(1+\gamma_t g_t)\right] = \frac{1}{\mathbb{E}L}\mathbb{E}\left[\sum_{t=1}^{L}\frac{1}{2}\log(1+\gamma_t \bar{B} r_{\gamma_t}\prod_{i=1}^{t-1}(1-r_{\gamma_i})))\right] \\
&\geq \frac{1}{\mathbb{E}L}\mathbb{E}\left[\sum_{t=1}^{L}\frac{1}{2}\log(1+\gamma_t \bar{B} r_{\gamma_t})) + \sum_{i=1}^{t-1}\frac{1}{2}\log(1-r_{\gamma_i})\right] \\
&= \frac{1}{\mathbb{E}L}\mathbb{E}\left[L\mathbb{E}\left[\frac{1}{2}\log(1+\gamma_t \bar{B} r_{\gamma_t}))\right] + \frac{L(L-1)}{2}\mathbb{E}\left[\frac{1}{2}\log(1-r_\gamma)\right]\right] \\
&= \mathbb{E}\left[\frac{1}{2}\log(1+\gamma_t \bar{B} r_{\gamma_t}))\right] + \frac{\mathbb{E}[L(L-1)]}{2\mathbb{E}[L]}\mathbb{E}\left[\frac{1}{2}\log(1-r_\gamma)\right] \\
&\overset{(a)}{\geq} \mathbb{E}\left[\frac{1}{2}\log(1+\gamma_t \bar{B} r_{\gamma_t}))\right] + \frac{\mathbb{E}[L(L-1)]}{2\mathbb{E}[L]}\frac{1}{2}\log(1-p) \\
&= \mathbb{E}\left[\frac{1}{2}\log(1+\gamma_t \bar{B} r_{\gamma_t}))\right] - \frac{1-p}{2p}\log(1/(1-p)) \\
&\geq \mathbb{E}\left[\frac{1}{2}\log(1+\gamma_t \bar{B} r_{\gamma_t}))\right] - \frac{1}{2}
\end{aligned}
\tag{4}
$$

Explanation for (a) is given in previous proof. $\square$

4 Recurrence Method for Finding Bounds

This method gives us the ability to find upper bounds to the optimal throughput without using infinite series.

Theorem 4.1. *Let $\tilde{\Theta}(b)$ be any positive function of b. If $\max_{0\leq z(\gamma)\leq b}\mathbb{E}\left[p\frac{1}{2}\log(1+\gamma z(\gamma)) + (1-p)\tilde{\Theta}(b-z(\gamma))\right] \leq \tilde{\Theta}(b)$ for all b, then $\Theta \leq \tilde{\Theta}$.*

Proof. We are given that $\max_{0\leq z(\gamma)\leq b}\mathbb{E}\left[p\frac{1}{2}\log(1+\gamma_t z(\gamma_t)) + (1-p)\tilde{\Theta}(b-z(\gamma_t))\right] \leq \tilde{\Theta}(b)$ for all b.

Now let us consider any policy g_t. We can show inductively that for any $0 \leq k \leq n$, the following statement is true:

$$
\mathbb{E}\left[\sum_{t=k}^{n} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)\right] \leq (1-p)^{k-1}\mathbb{E}\left[\tilde{\Theta}(b_k)\right] \tag{5}
$$

Our base case is $k = n$. Thus, the above statement is equivalent to

$$
\mathbb{E}\left[p(1-p)^{n-1}\frac{1}{2}\log(1+\gamma_n g_n)\right] \leq (1-p)^{n-1}\mathbb{E}\left[\tilde{\Theta}(b_n)\right] \tag{6}
$$

which follows directly from our presupposition since

$$
\begin{aligned}
\mathbb{E}\left[p(1-p)^{n-1}\frac{1}{2}\log(1+\gamma_n g_n)\right] &\le (1-p)^{n-1}\mathbb{E}\left[p\frac{1}{2}\log(1+\gamma_n g_n)+(1-p)\tilde{\Theta}(b-g_n)\right] \\
&\le \max_{0\le z(\gamma_n)\le b_n}\mathbb{E}\left[p\frac{1}{2}\log(1+\gamma z(\gamma_n))+(1-p)\tilde{\Theta}(b_n-z(\gamma_n))\right] \\
&\le (1-p)^{n-1}\mathbb{E}\left[\tilde{\Theta}(b_n)\right]
\end{aligned} \tag{7}
$$

Now we proceed inductively by showing that $k \leftarrow k$ satisfies the equation whenever $k \leftarrow k+1$ does. Note that $b_k = \bar{B} - \sum_{i=1}^{k}\gamma_i$, so b_k is dependent on $\gamma_1, \ldots \gamma_k$.

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{t=k}^{n} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)\right] = \int \mathbb{E}\left[\sum_{t=k}^{n} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)|\gamma_k=\gamma\right] dp_{\gamma_k}(\gamma) \\
&= \int \mathbb{E}\left[p(1-p)^{k-1}\frac{1}{2}\log(1+\gamma_k g_k) + \sum_{t=k+1}^{n} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)|\gamma_k=\gamma\right] dp_{\gamma_k}(\gamma) \\
&= \mathbb{E}\left[p(1-p)^{k-1}\frac{1}{2}\log(1+\gamma_k g_k)\right] + \int \mathbb{E}\left[\sum_{t=k+1}^{n} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)|\gamma_k=\gamma\right] dp_{\gamma_k}(\gamma) \\
&\le \mathbb{E}\left[p(1-p)^{k-1}\frac{1}{2}\log(1+\gamma_k g_k)\right] + (1-p)^k \int \mathbb{E}\left[\tilde{\Theta}(b_k-g_k)|\gamma_k=\gamma\right] dp_{\gamma_k}(\gamma) \\
&= (1-p)^{k-1}\mathbb{E}\left[p\frac{1}{2}\log(1+\gamma_k g_k)+(1-p)\tilde{\Theta}(b_k-g_k)\right] \\
&= (1-p)^{k-1}\max_{z(\gamma_k)}\mathbb{E}\left[p\frac{1}{2}\log(1+\gamma_k z(\gamma_k))+(1-p)\tilde{\Theta}(b_k-z(\gamma_k))\right] \\
&\le (1-p)^{k-1}\mathbb{E}\left[\tilde{\Theta}(b_k)\right]
\end{aligned} \tag{8}
$$

Having proven that (2) is true, we can set $k \leftarrow 1$ to see that for any n,

$$
\mathbb{E}\left[\sum_{t=1}^{n} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)\right] \le \mathbb{E}\left[\tilde{\Theta}(b_1)\right] = \tilde{\Theta}(\bar{B}) \tag{9}
$$

It follows directly that $\mathcal{J}(g_t) = \lim_{n\to\infty}\mathbb{E}\left[\sum_{t=1}^{n} p(1-p)^{t-1}\frac{1}{2}\log(1+\gamma_t g_t)\right] \le \tilde{\Theta}(\bar{B})$. Since $\mathcal{J}(g_t)$ is arbitrary, $\Theta = \sup_{g_t}\mathcal{J}(g_t) \le \tilde{\Theta}$.

□

5 No Fading

We can also use the recurrence method to show that (1) is a near-optimal policy for the case of $\gamma_t = c$ where c is a constant, by verifying that $\tilde{\Theta}$ is an upper bound, and therefore the additive and multiplicative gaps of Section 3 in fact represent gaps to optimality. This is another way of arriving at the same conclusion as (Shaviv 2015).

Theorem 5.1. *When $\gamma_t = c$ for some constant c, $\Theta \leq \hat{\Theta}$.*

Proof. Using the recurrence method, it suffices to prove that for all b, $\max_{0 \leq z(\gamma) \leq b} \mathbb{E}\left[p\frac{1}{2}\log(1+\gamma z(\gamma)) + (1-p)\tilde{\Theta}(b - z(\gamma))\right] \leq \tilde{\Theta}(b)$. Since γ is constant, we can rewrite this as:

$$\max_{0 \leq z \leq b}\left(p\frac{1}{2}\log(1+cz) + (1-p)\frac{1}{2}\log(1+cp(b-z))\right) \leq \frac{1}{2}\log(1+cbp)$$

It is readily verified using basic calculus that to achieve the optimal value

$$z = \begin{cases} b & p(b+c^{-1}) > b \\ p(b+c^{-1}) & p(b+c^{-1}) \leq b \end{cases} \tag{10}$$

We can consider both cases separately. In the case that $z = b$, the equation we are trying to prove reduces to $p\log(1+cb) \leq \log(1+pcb)$, which is true based on the identity $a\log(1+x) \leq \log(1+ax)$ for $0 \leq a \leq 1$.

We consider the remaining case, where $p(b+c^{-1}) \leq b$ and $z = p(b+c^{-1})$.

$$\begin{aligned}
&\max_{0 \leq z \leq b}\left[p\frac{1}{2}\log(1+cz) + (1-p)\frac{1}{2}\log(1+cp(b-z))\right] \\
&= \frac{1}{2}\left[p\log(1+cp(b+c^{-1})) + (1-p)\log(1+cp(b-p(b+c^{-1})))\right] \\
&= \frac{1}{2}\left[p\log(1+p+cpb) + (1-p)\log(1+p((1-p)bc-p))\right] \\
&= \frac{1}{2}\left[p\log(1+p+cpb) + (1-p)\log(1-p^2+p(1-p)bc)\right] \\
&= \frac{1}{2}\left[p\log(1+p+cpb) + (1-p)\log(1+p+cpb) + (1-p)\log(1-p)\right] \\
&= \frac{1}{2}\left[\log(1+p+\frac{p^2}{1-p}+cpb-\frac{p^2}{1-p}) + (1-p)\log(1-p)\right]
\end{aligned} \tag{11}$$

Here, we can define $k := cpb - \frac{p^2}{1-p}$. The condition $p(b+c^{-1}) \leq b$ implies $k \geq 0$.

$$
\begin{aligned}
&= \frac{1}{2}\left[\log\left(1+p+\frac{p^2}{1-p}+k\right)+(1-p)\log(1-p)\right] \\
&= \frac{1}{2}\log\left(\left(\frac{1}{1-p}\right)(1-p)^{(1-p)}+k(1-p)^{(1-p)}\right) \\
&\leq \frac{1}{2}\log\left((1-p)^{-p}+k\right) \qquad (12) \\
&\overset{(b)}{\leq} \frac{1}{2}\log\left(1+\frac{p^2}{1-p}+k\right) = \frac{1}{2}\log\left(1+\frac{p^2}{1-p}+cpb-\frac{p^2}{1-p}\right) \\
&= \frac{1}{2}\log\left(1+cpb\right)
\end{aligned}
$$

(b) is based on the inequality $(1-p)^{-p} \leq 1+\frac{p^2}{1-p}$.
Having demonstrated the inequality for both cases, we are done.

□

6 Bernoulli Fading

Similarly, we can also use the recurrence method in a straightforward manner to show that (1) is a near-optimal policy for the case of $\gamma_t = \begin{cases} c & \text{w.p. } q \\ 0 & \text{w.p. } 1-q \end{cases}$.

Theorem 6.1. *When $\gamma_t = \begin{cases} c & w.p.\ q \\ 0 & w.p.\ 1-q \end{cases}$ for some constant c, $\Theta \leq \hat{\Theta}$.*

Proof. We begin by noting that $\hat{\Theta} = \mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_{\gamma_t})\right] = q\frac{1}{2}\log(1+c\bar{B}r_c)$. We thus proceed:

$$
\begin{aligned}
&\max_{0\le z(\gamma)\le b} \mathbb{E}\left[p\frac{1}{2}\log(1+\gamma z(\gamma)) + (1-p)\hat{\Theta}(b-z(\gamma))\right] \\
&= \max_{0\le z_0, z_c\le b}\left(q\left[p\frac{1}{2}\log(1+cz_c) + (1-p)\hat{\Theta}(b-z_c)\right] + (1-q)(1-p)\hat{\Theta}(b-z_0)\right) \\
&= \max_{0\le z_c\le b} q\left[p\frac{1}{2}\log(1+cz_c) + (1-p)q\frac{1}{2}\log(1+c(b-z_c)r_c)\right] \\
&\quad + \max_{0\le z_0\le b}(1-q)(1-p)q\frac{1}{2}\log(1+c(b-z_0)r_c) \\
&= q(p+q(1-p))\max_{0\le z_c\le b}\left(\frac{p}{p+q(1-p)}\frac{1}{2}\log(1+cz_c) + \frac{q(1-p)}{p+q(1-p)}\frac{1}{2}\log(1+c(b-z_c)r_c)\right) \\
&\quad + \frac{1}{2}q(1-q)(1-p)\log(1+cbr_c) \\
&= q(p+q(1-p))\max_{0\le z_c\le b}\left(r_c\frac{1}{2}\log(1+cz_c) + (1-r_c)\frac{1}{2}\log(1+c(b-z_c)r_c)\right) \\
&\quad + q(1-q)(1-p)\frac{1}{2}\log(1+cbr_c) \\
&\overset{(c)}{\le} q(p+q(1-p))\frac{1}{2}\log(1+cbr_c) + \frac{1}{2}q(1-q)(1-p)\log(1+cbr_c) \\
&= q\frac{1}{2}\log(1+cbr_c) = \hat{\Theta}(b)
\end{aligned}
\tag{13}
$$

(c) follows from the recurrence relation from the constant channel case (see Thm. 5.1), with r_c substituted in place of p. □

7 Discussion

As a next step, I suggest using the recurrence method to find true upper bounds for the proposed policy in Section 2, and hopefully uncover additive and multiplicative gaps between true upper bounds and the pseudo-bound. For example, I have performed extensive simulations (on the order of 10^7 total) of a variety of randomly-generated finite channel distributions that suggest that $\tilde{\Theta} := \mathbb{E}\left[\frac{1}{2}\log(1+\gamma\bar{B}r_\gamma) + \frac{1}{2}\log\left(p+(1-p)\frac{\bar{\gamma}}{\gamma}\right)\right]$ satisfies the recurrence relation specified in Section 4, and therefore is an upper bound. Note that the gap from the pseudo-bound to our new proposed bound is strictly less than $\frac{1}{2}\log(\mathbb{E}\gamma\mathbb{E}\gamma^{-1})$. Thus, for families of distributions where this quantity is bounded universally, such as Rayleigh and Rician distributions, for which the quantity is less than $\frac{1}{2}\log(\pi)$, we have something like a universal additive gap.

It is surprising that the pseudo-bound from Section 3 is not universal, since it works with the canonically extreme cases of Bernoulli and constant channels, as we have demonstrated. A major issue that was encountered was that our proposed policy, and many policies in a similar form that generalize the Bernoulli and constant cases, are not concave in γ_t. Perhaps an improved policy would be able to take this into account and improve upon our proposed policy.

The recursive framework outlined in Section 4 can be broadly useful as a tool to find upper bounds for throughput where other methods fail. It does not assume any particular policy, and thus is flexible enough to serve as both a tool for proving further results related to energy harvesting and for performing exploratory simulations.

8 Bibliography

Shaviv, Dor, and Ayfer Özgür. “Universally near-optimal online power control for energy harvesting nodes.” arXiv preprint arXiv:1511.00353 (2015).