

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.4 PRIORITY QUEUES

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- ▶ *API and elementary implementations*
- ▶ *binary heaps*
- ▶ *heapsort*
- ▶ *event-driven simulation*

# Algorithms

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- ▶ *event-driven simulation*

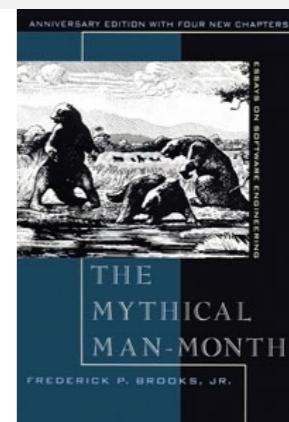
# Collections

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A **collection** is a data type that stores a group of items.

data type	core operations	data structure
<b>stack</b>	PUSH, POP	<i>linked list, resizing array</i>
<b>queue</b>	ENQUEUE, DEQUEUE	<i>linked list, resizing array</i>
<b>priority queue</b>	INSERT, DELETE-MAX	<i>binary heap</i>
<b>symbol table</b>	PUT, GET, DELETE	<i>binary search tree, hash table</i>
<b>set</b>	ADD, CONTAINS, DELETE	<i>binary search tree, hash table</i>

“Show me your code and conceal your data structures, and I shall continue to be mystified. Show me your data structures, and I won’t usually need your code; it’ll be obvious.” — Fred Brooks



# Priority queue

---

Collections. Insert and delete items. Which item to delete?

Stack. Remove the item most recently added.

Queue. Remove the item least recently added.

Randomized queue. Remove a random item.

Priority queue. Remove the **largest** (or **smallest**) item.

Generalizes: stack, queue, randomized queue.

<i>operation</i>	<i>argument</i>	<i>return value</i>
<i>insert</i>	P	
<i>insert</i>	Q	
<i>insert</i>	E	
<i>remove max</i>		Q
<i>insert</i>	X	
<i>insert</i>	A	
<i>insert</i>	M	
<i>remove max</i>		X
<i>insert</i>	P	
<i>insert</i>	L	
<i>insert</i>	E	
<i>remove max</i>		P

# Priority queue API

**Requirement.** Items are generic; they must also be Comparable.

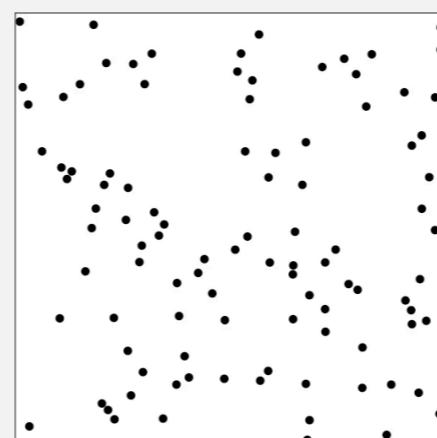
public class MaxPQ<Key extends Comparable<Key>>		Key must be Comparable (bounded type parameter)
MaxPQ()		<i>create an empty priority queue</i>
MaxPQ(Key[] a)		<i>create a priority queue with given keys</i>
void insert(Key v)		<i>insert a key into the priority queue</i>
Key delMax()		<i>return and remove a largest key</i>
boolean isEmpty()		<i>is the priority queue empty?</i>
Key max()		<i>return a largest key</i>
int size()		<i>number of entries in the priority queue</i>

**Note.** Duplicate keys allowed; delMax() picks any maximum key.

# Priority queue: applications

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- Event-driven simulation. [ customers in a line, colliding particles ]
- Numerical computation. [ reducing roundoff error ]
- Discrete optimization. [ bin packing, scheduling ]
- Artificial intelligence. [ A\* search ]
- Computer networks. [ web cache ]
- Operating systems. [ load balancing, interrupt handling ]
- Data compression. [ Huffman codes ]
- Graph searching. [ Dijkstra's algorithm, Prim's algorithm ]
- Number theory. [ sum of powers ]
- Spam filtering. [ Bayesian spam filter ]
- Statistics. [ online median in data stream ]



8	4	7
1	5	6
3	2	

# Priority queue: client example

**Challenge.** Find the largest  $m$  items in a stream of  $n$  items.

- Fraud detection: isolate \$\$ transactions.
- NSA monitoring: flag most suspicious documents.

$n$  huge,  $m$  large

**Constraint.** Not enough memory to store  $n$  items.

use a min-oriented pq

```
MinPQ<Transaction> pq = new MinPQ<Transaction>();  
while (StdIn.hasNextLine())  
{  
    String line = StdIn.readLine();  
    Transaction transaction = new Transaction(line);  
    pq.insert(transaction);  
    if (pq.size() > m)  
        pq.delMin();    ← pq now contains  
}                                largest m items
```

Transaction data  
type is Comparable  
(ordered by \$\$)

# Priority queue: client example

---

Challenge. Find the largest  $m$  items in a stream of  $n$  items.

implementation	time	space
sort	$n \log n$	$n$
elementary PQ	$m n$	$m$
binary heap	$n \log m$	$m$
best in theory	$n$	$m$

order of growth of finding the largest  $m$  in a stream of  $n$  items

# Priority queue: unordered and ordered array implementation

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<i>operation</i>	<i>argument</i>	<i>return value</i>	<i>size</i>	<i>contents (unordered)</i>	<i>contents (ordered)</i>
<i>insert</i>	P		1	P	P
<i>insert</i>	Q		2	P Q	P Q
<i>insert</i>	E		3	P Q E	E P Q
<i>remove max</i>		Q	2	P E	E P
<i>insert</i>	X		3	P E X	E P X
<i>insert</i>	A		4	P E X A	A E P X
<i>insert</i>	M		5	P E X A M	A E M P X
<i>remove max</i>		X	4	P E M A	A E M P
<i>insert</i>	P		5	P E M A P	A E M P P
<i>insert</i>	L		6	P E M A P L	A E L M P P
<i>insert</i>	E		7	P E M A P L E	A E E L M P P
<i>remove max</i>		P	6	E M A P L E	A E E L M P P

A sequence of operations on a priority queue

# Priority queue: implementations cost summary

---

Challenge. Implement **all** operations efficiently.

implementation	insert	del max	max
<b>unordered array</b>	1	$n$	$n$
<b>ordered array</b>	$n$	1	1
<b>goal</b>	$\log n$	$\log n$	$\log n$

order of growth of running time for priority queue with  $n$  items

Solution. Partially-ordered array.

# Algorithms

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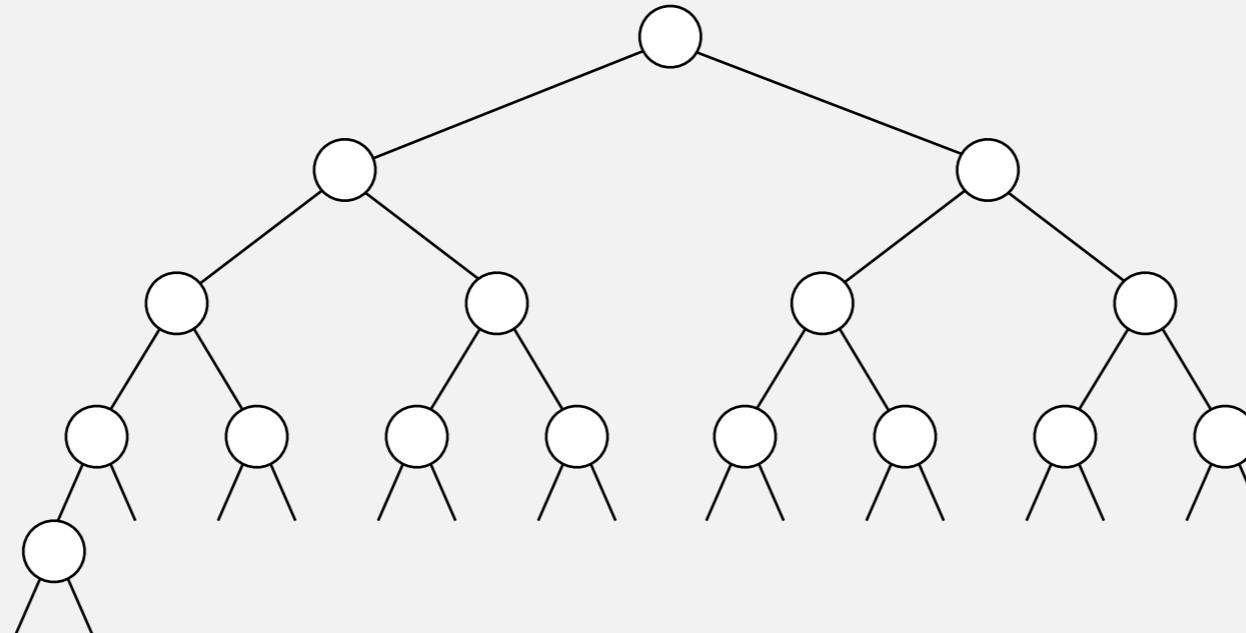
- ▶ *API and elementary implementations*
- ▶ ***binary heaps***
- ▶ *heapsort*
- ▶ *event-driven simulation*

# Complete binary tree

---

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



complete binary tree with  $n = 16$  nodes (height = 4)

Property. Height of complete binary tree with  $n$  nodes is  $\lfloor \lg n \rfloor$ .

Pf. Height increases only when  $n$  is a power of 2.

# A complete binary tree in nature

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Hyphaene Compressa - Doum Palm

© Shlomit Pinter

# Binary heap: representation

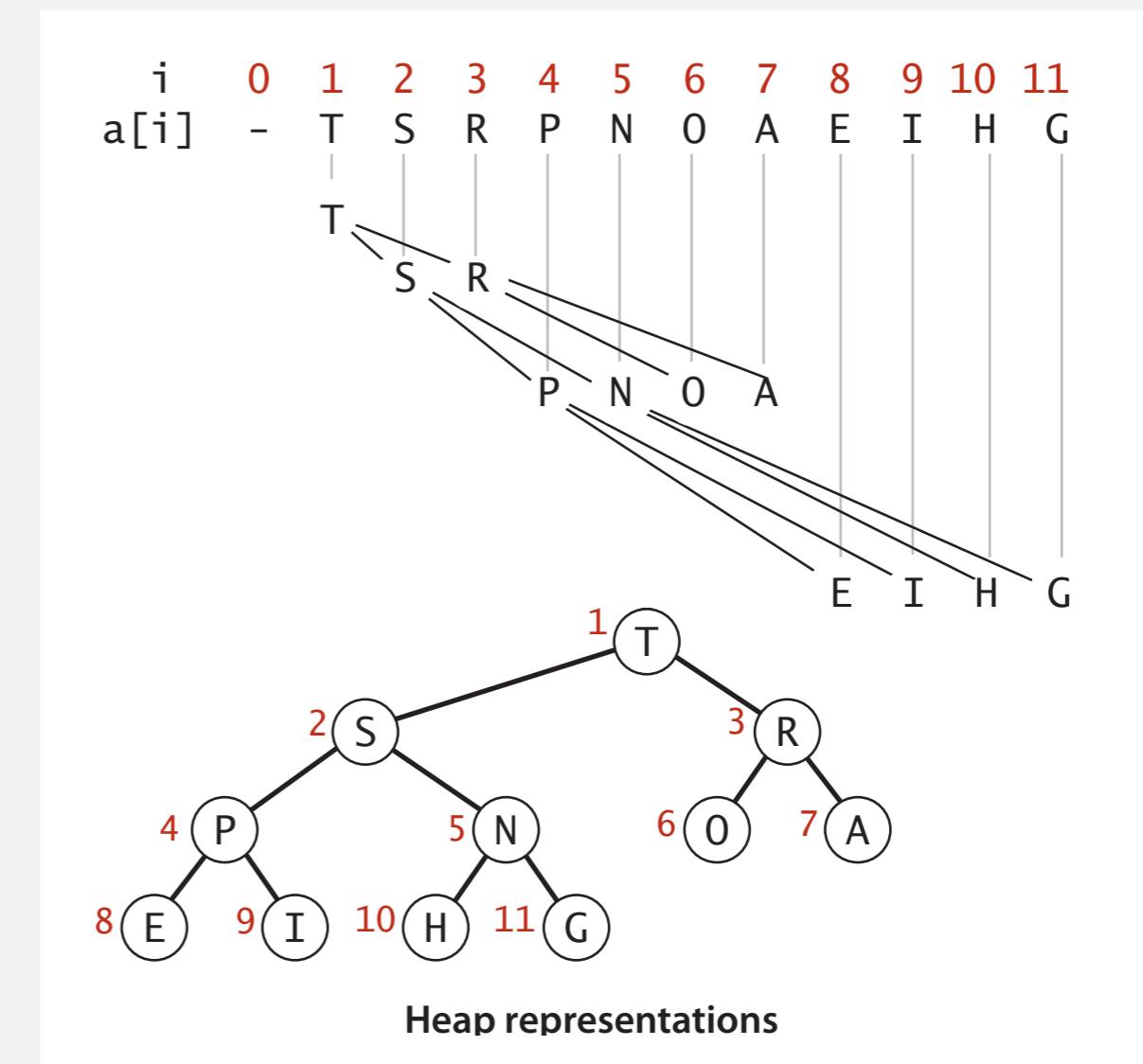
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- Parent's key no smaller than children's keys.

Array representation.

- Indices start at 1.
- Take nodes in **level** order.
- No explicit links needed!

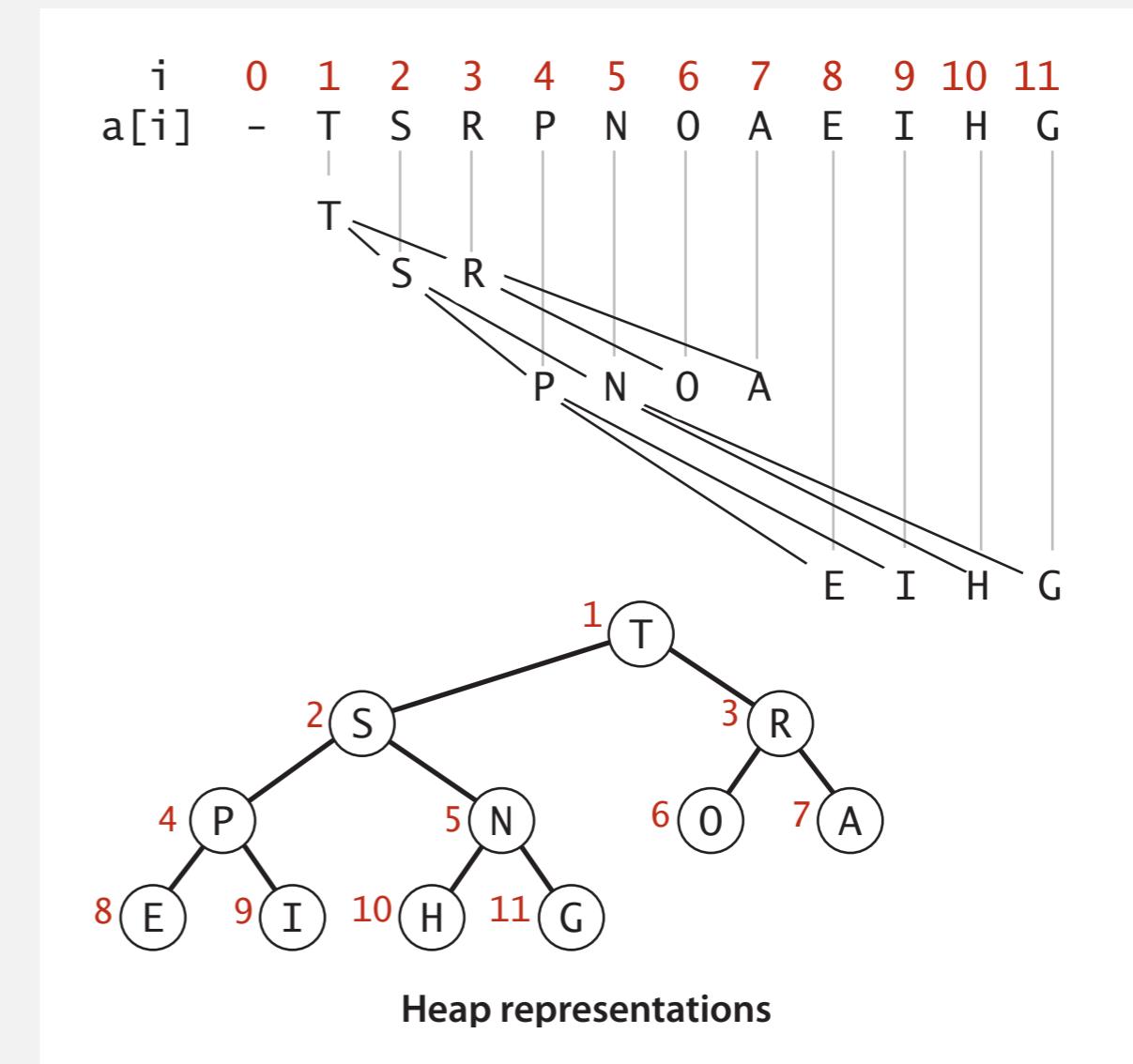


# Binary heap: properties

Proposition. Largest key is  $a[1]$ , which is root of binary tree.

Proposition. Can use array indices to move through tree.

- Parent of node at  $k$  is at  $k/2$ .
- Children of node at  $k$  are at  $2k$  and  $2k+1$ .



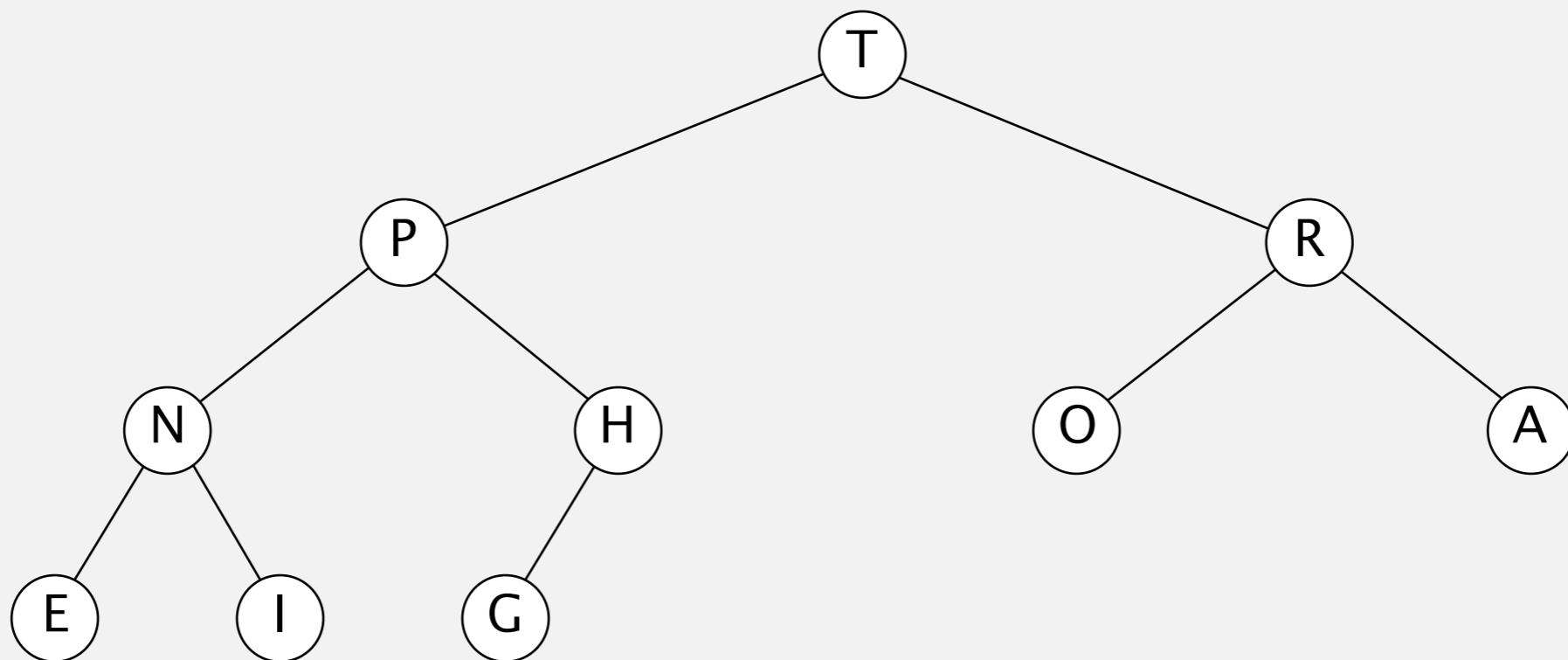
# Binary heap demo

---

**Insert.** Add node at end, then swim it up.

**Remove the maximum.** Exchange root with node at end, then sink it down.

heap ordered



T	P	R	N	H	O	A	E	I	G
---	---	---	---	---	---	---	---	---	---

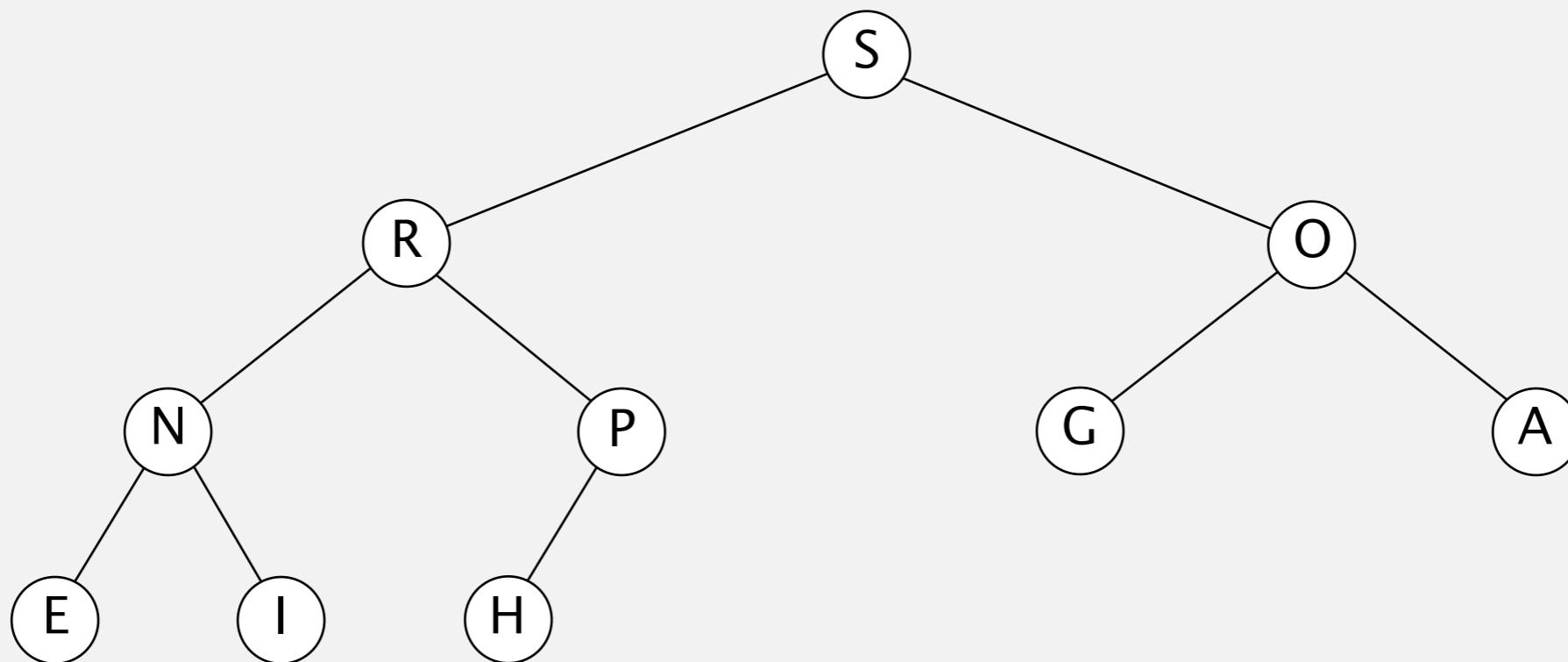
# Binary heap demo

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**Insert.** Add node at end, then swim it up.

**Remove the maximum.** Exchange root with node at end, then sink it down.

**heap ordered**



S	R	O	N	P	G	A	E	I	H	
---	---	---	---	---	---	---	---	---	---	--

# Binary heap: promotion

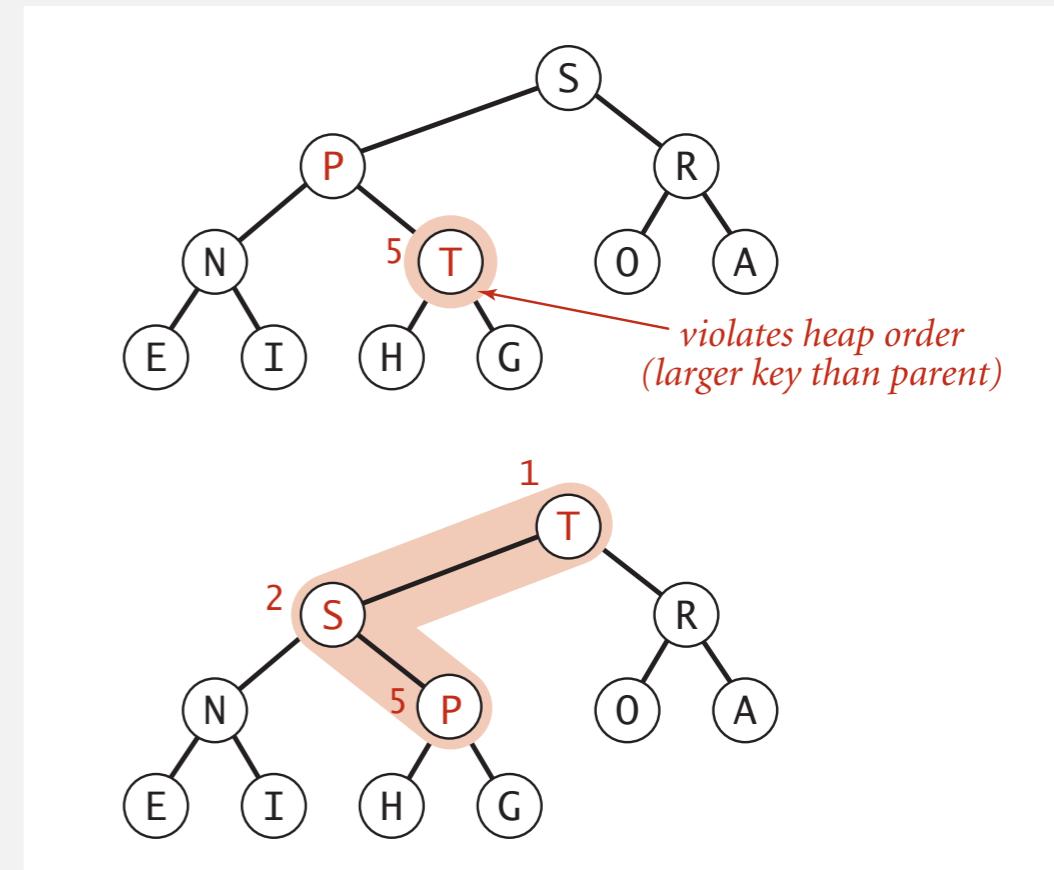
Scenario. A key becomes **larger** than its parent's key.

To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
    while (k > 1 && less(k/2, k))
    {
        exch(k, k/2);
        k = k/2;
    }
}
```

parent of node at k is at k/2



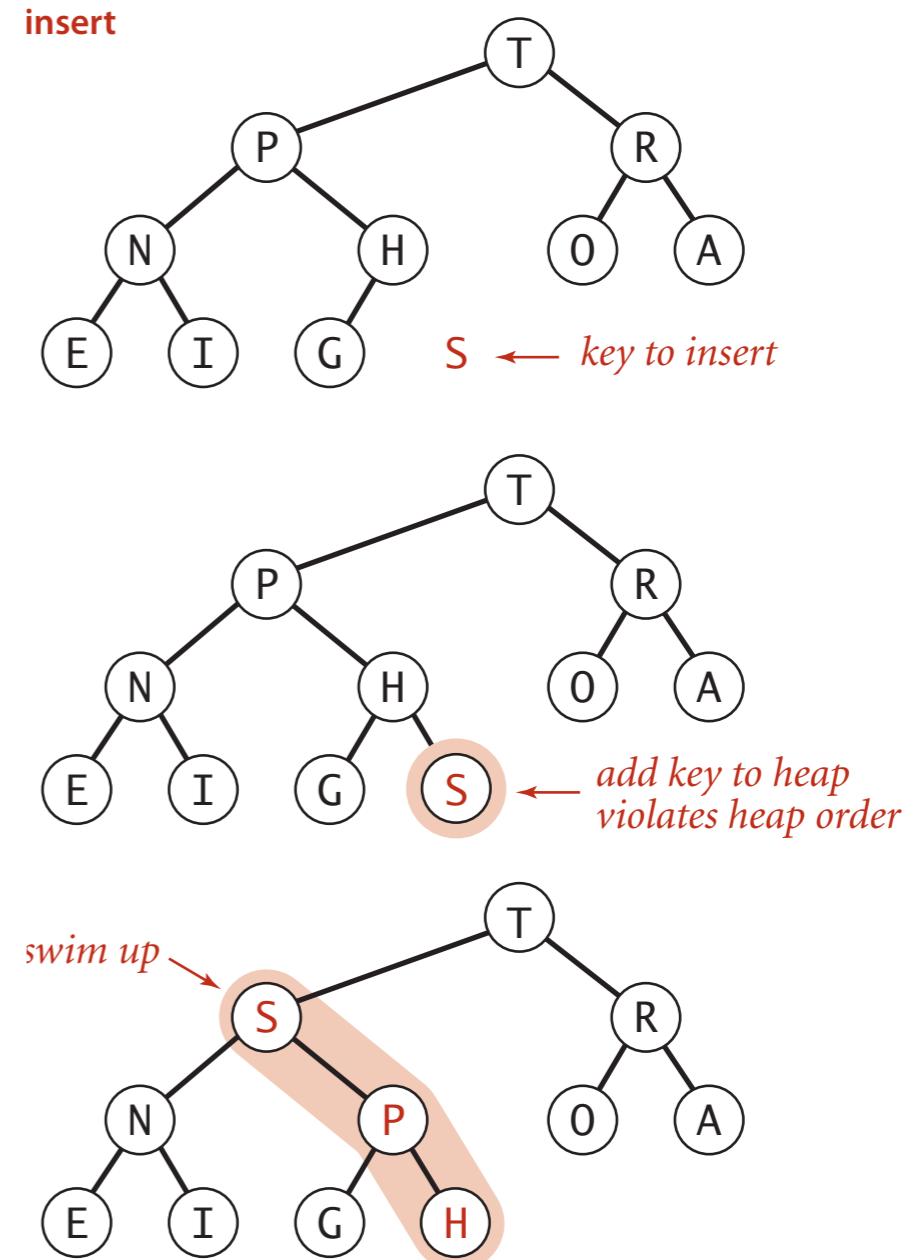
Peter principle. Node promoted to level of incompetence.

# Binary heap: insertion

**Insert.** Add node at end, then swim it up.

**Cost.** At most  $1 + \lg n$  compares.

```
public void insert(Key x)
{
    pq[++n] = x;
    swim(n);
}
```



# Binary heap: demotion

Scenario. A key becomes **smaller** than one (or both) of its children's.

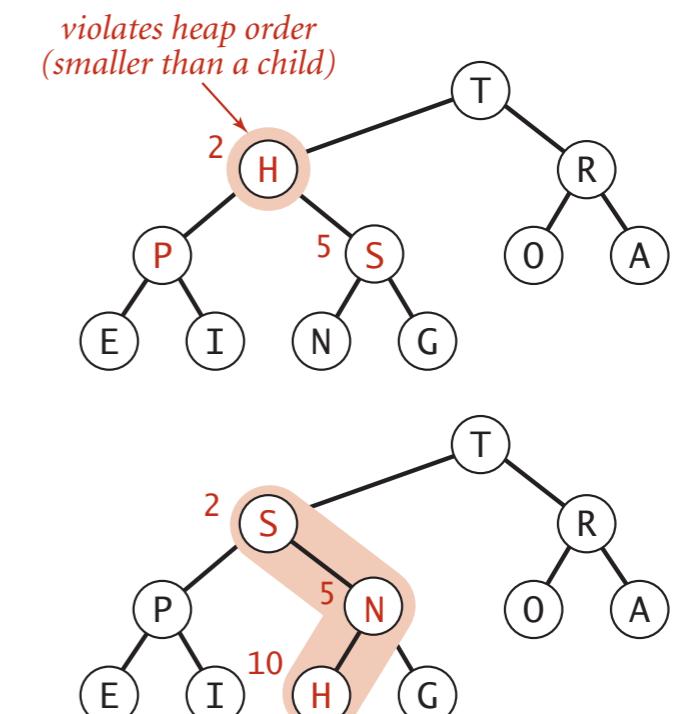
To eliminate the violation:

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

why not smaller child?  
↓

```
private void sink(int k)
{
    while (2*k <= n)
    {
        int j = 2*k;
        if (j < n && less(j, j+1)) j++;
        if (!less(k, j)) break;
        exch(k, j);
        k = j;
    }
}
```

children of node at k  
are  $2 \cdot k$  and  $2 \cdot k + 1$



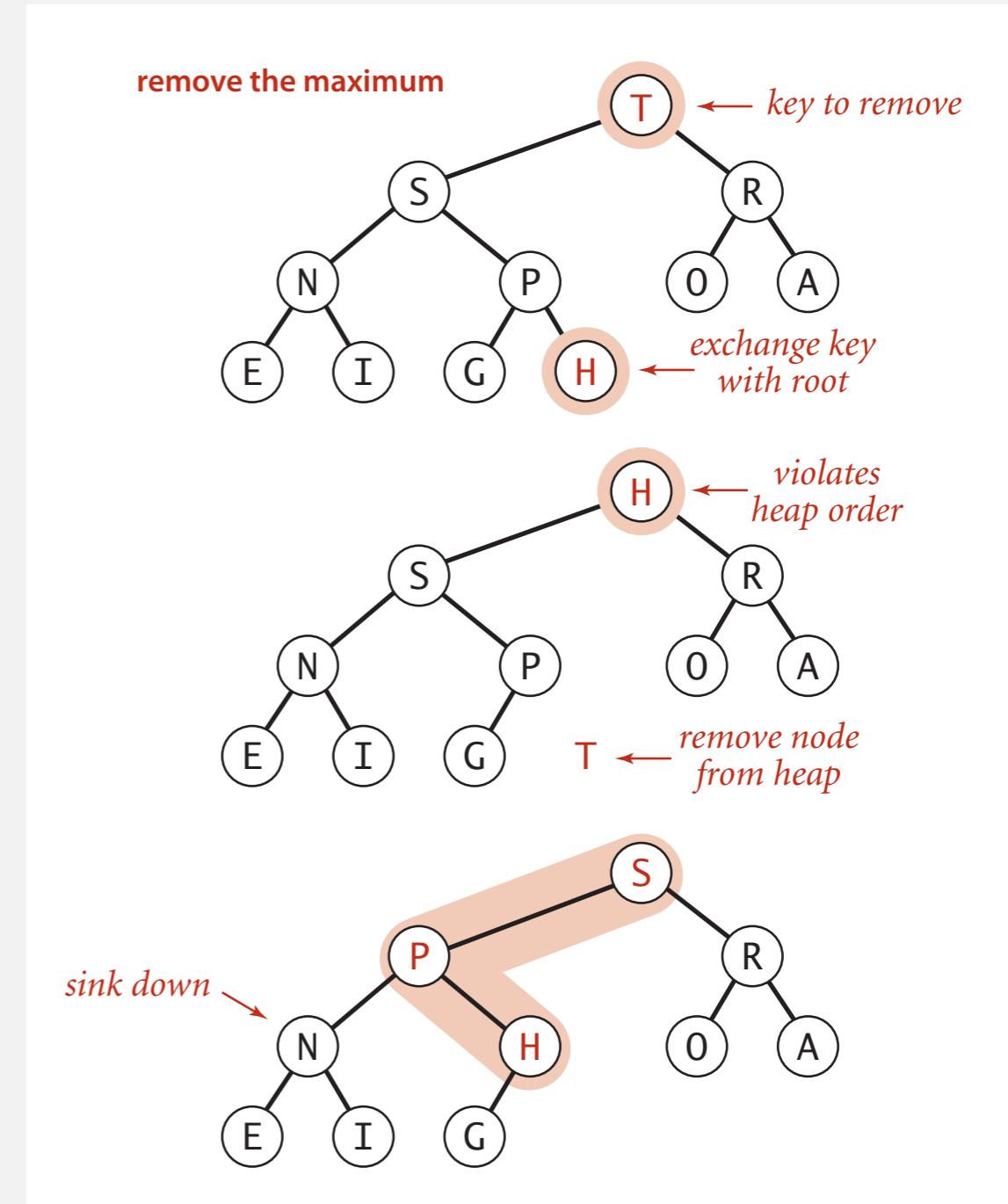
Power struggle. Better subordinate promoted.

# Binary heap: delete the maximum

Delete max. Exchange root with node at end, then sink it down.

Cost. At most  $2 \lg n$  compares.

```
public Key delMax()
{
    Key max = pq[1];
    exch(1, n--);
    sink(1);
    pq[n+1] = null; ← prevent loitering
    return max;
}
```



# Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
    private Key[] pq;
    private int n;

    public MaxPQ(int capacity)
    {   pq = (Key[]) new Comparable[capacity+1]; }

    public boolean isEmpty()
    {   return n == 0; }
    public void insert(Key key) // see previous code
    public Key delMax() // see previous code

    private void swim(int k) // see previous code
    private void sink(int k) // see previous code

    private boolean less(int i, int j)
    {   return pq[i].compareTo(pq[j]) < 0; }
    private void exch(int i, int j)
    {   Key t = pq[i]; pq[i] = pq[j]; pq[j] = t; }

}
```

fixed capacity  
(for simplicity)

PQ ops

heap helper functions

array helper functions

# Priority queue: implementations cost summary

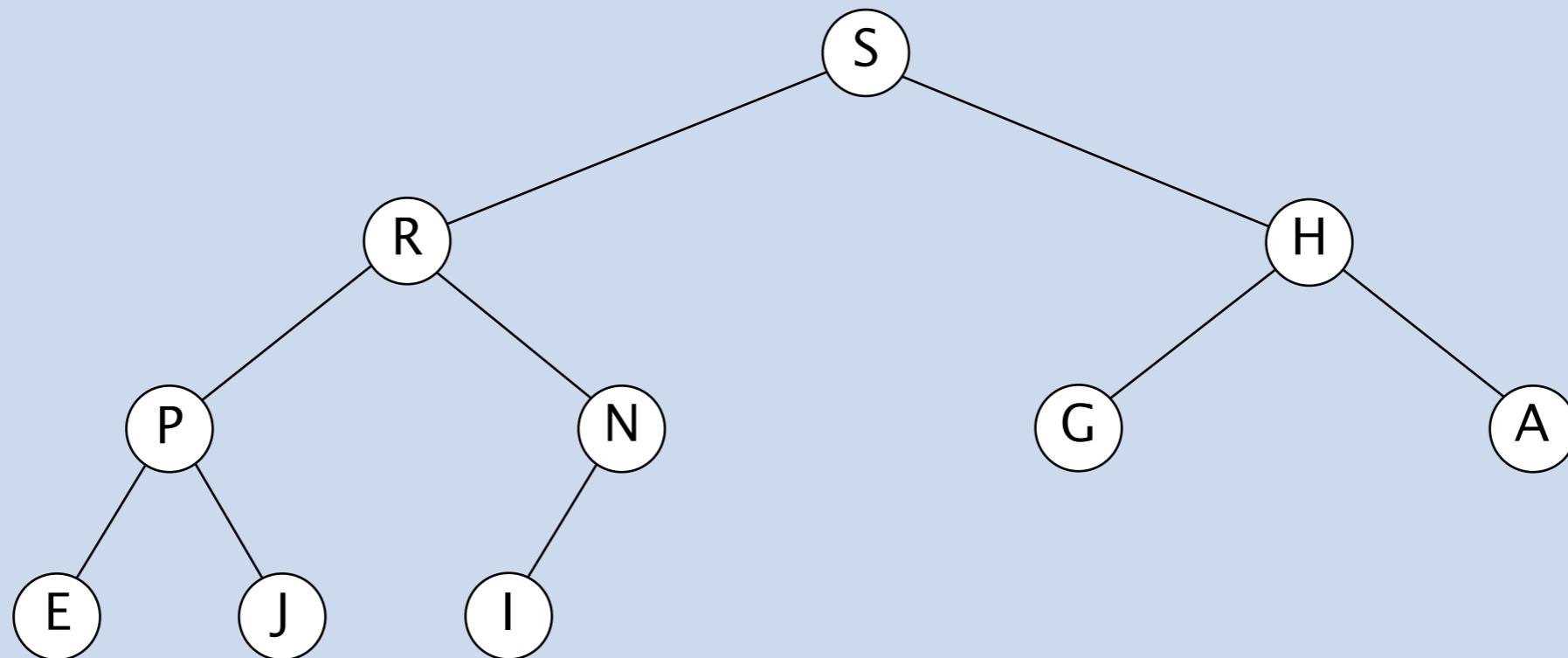
---

implementation	insert	del max	max
<b>unordered array</b>	1	$n$	$n$
<b>ordered array</b>	$n$	1	1
<b>binary heap</b>	$\log n$	$\log n$	1

order-of-growth of running time for priority queue with  $n$  items

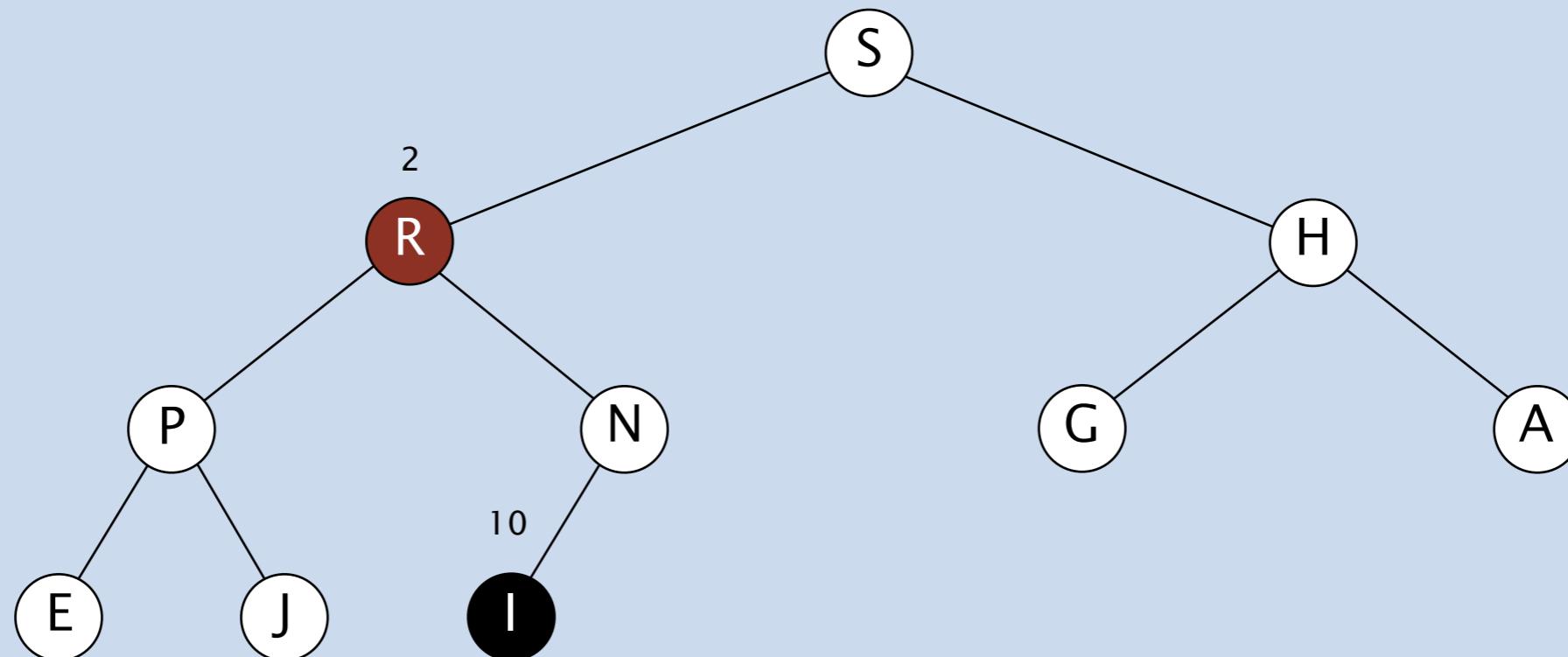
# DELETE-RANDOM FROM A BINARY HEAP

Goal. Delete a random key from a binary heap in logarithmic time.



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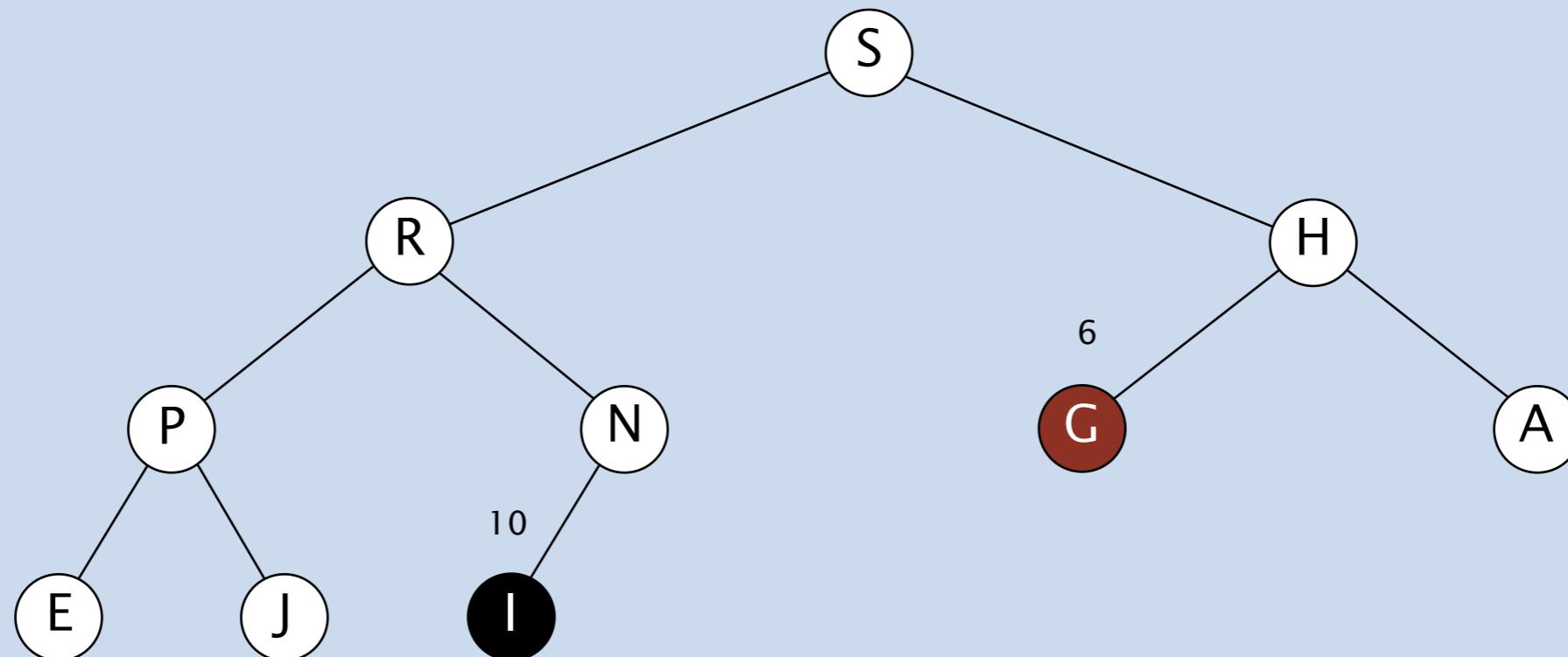


Solution.

- Pick a random index  $r$  between 1 and  $n$ .
- Perform  $\text{exch}(r, n--)$ .
- Perform either **sink( $r$ )** or **swim( $r$ )**.

# DELETE-RANDOM FROM A BINARY HEAP

Goal. Delete a random key from a binary heap in logarithmic time.



Solution.

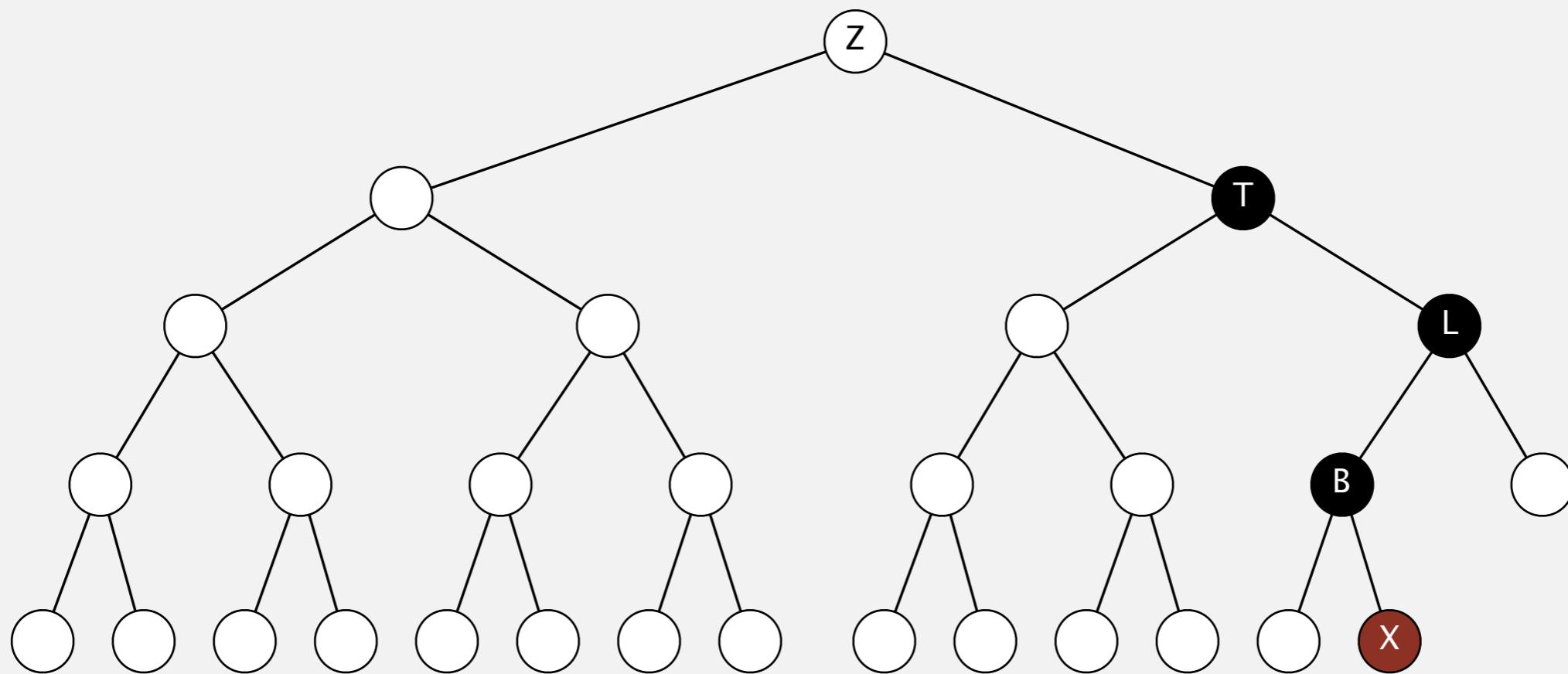
- Pick a random index  $r$  between 1 and  $n$ .
- Perform  $\text{exch}(r, n--)$ .
- Perform either  $\text{sink}(r)$  or  $\text{swim}(r)$ .

# Binary heap: practical improvements

---

Do "half-exchanges" in sink and swim.

- Reduces number of array accesses.
- Worth doing.



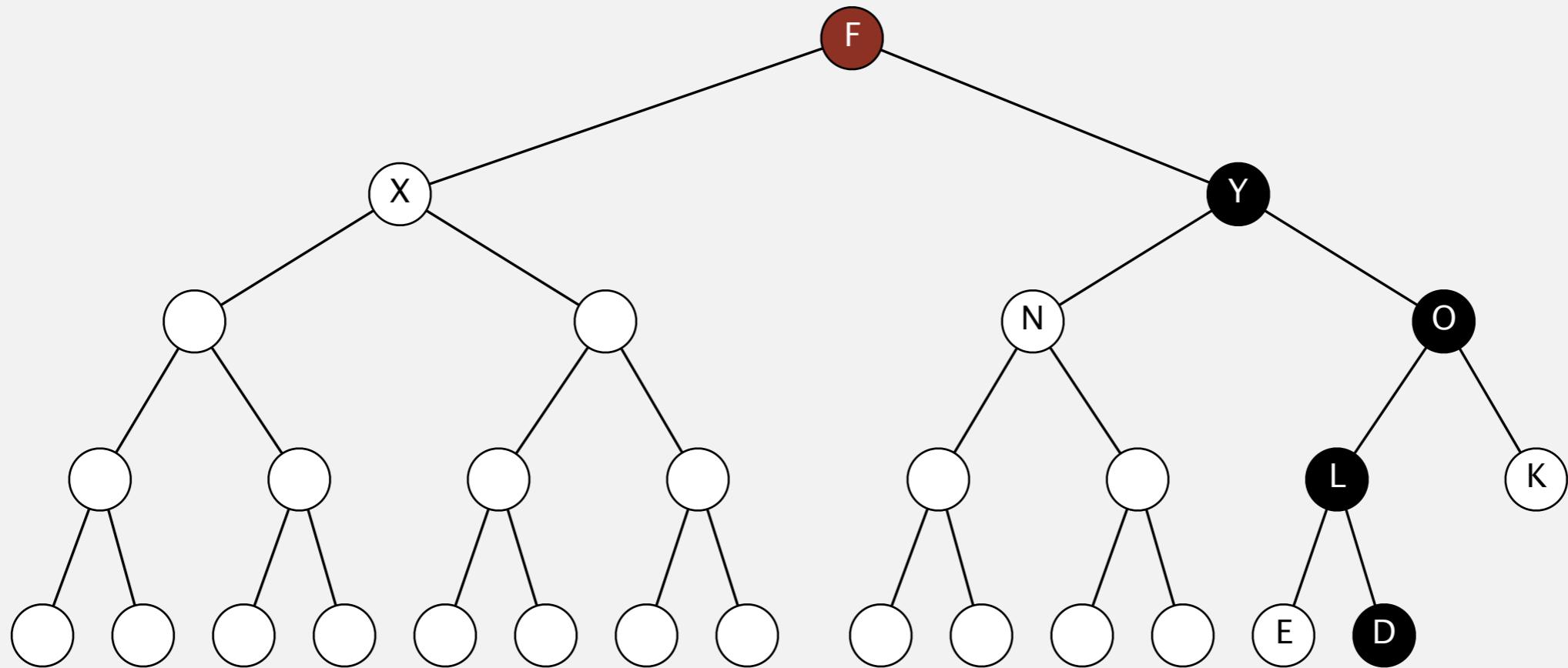
# Binary heap: practical improvements

Floyd's "bounce" heuristic.

- Sink key at root all the way to bottom. ← only 1 compare per node
- Swim key back up. ← some extra compares and exchanges
- Overall, fewer compares; more exchanges.



R. W. Floyd  
1978 Turing award



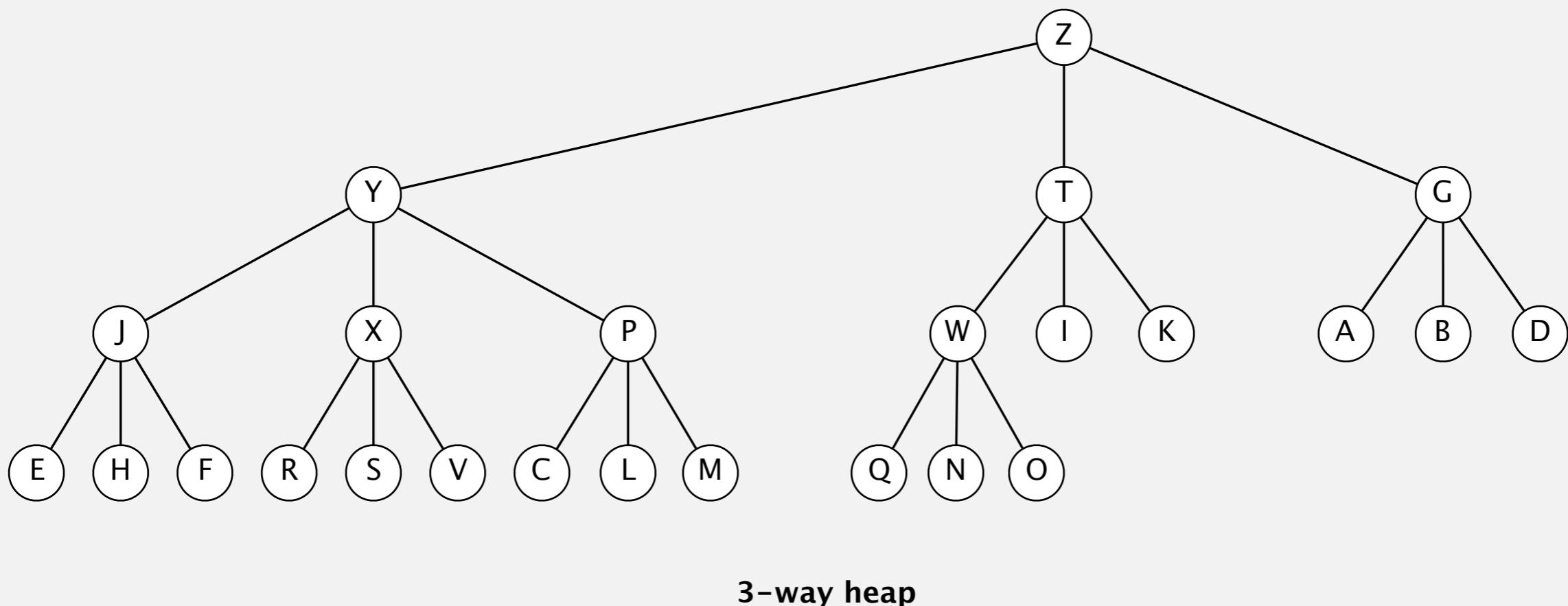
# Binary heap: practical improvements

---

## Multiway heaps.

- Complete  $d$ -way tree.
- Parent's key no smaller than its children's keys.

Fact. Height of complete  $d$ -way tree on  $n$  nodes is  $\sim \log_d n$ .



## Priority queues: quiz 1

---

How many compares (in the worst case) to **insert** in a  $d$ -way heap?

- A.  $\sim \log_2 n$
- B.  $\sim \log_d n$
- C.  $\sim d \log_2 n$
- D.  $\sim d \log_d n$
- E. *I don't know.*

## Priority queues: quiz 2

---

How many compares (in the worst case) to **delete-max** in a  $d$ -way heap?

- A.  $\sim \log_2 n$
- B.  $\sim \log_d n$
- C.  $\sim d \log_2 n$
- D.  $\sim d \log_d n$
- E. *I don't know.*

# Priority queue: implementation cost summary

implementation	insert	del max	max
<b>unordered array</b>	1	$n$	$n$
<b>ordered array</b>	$n$	1	1
<b>binary heap</b>	$\log n$	$\log n$	1
<b>d-ary heap</b>	$\log_d n$	$d \log_d n$	1
Fibonacci	1	$\log n^\dagger$	1
Brodal queue	1	$\log n$	1
impossible	1	1	1

← sweet spot:  $d = 4$

← why impossible?

† amortized

order-of-growth of running time for priority queue with  $n$  items

# Binary heap: considerations

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## Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: add no-arg constructor and use resizing array.

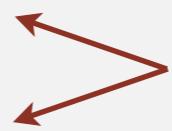
leads to log n  
amortized time per op  
(how to make worst case?)

## Minimum-oriented priority queue.

- Replace less() with greater().
- Implement greater().

## Other operations.

- Remove an arbitrary item.
- Change the priority of an item.



can implement efficiently with sink() and swim()  
[ stay tuned for Prim/Dijkstra ]

## Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

# Immutability: implementing in Java

**Data type.** Set of values and operations on those values.

**Immutable data type.** Can't change the data type value once created.

```
public class Vector {  
    private final int n;  
    private final double[] data;  
  
    public Vector(double[] data) {  
        this.n = data.length;  
        this.data = new double[n];  
        for (int i = 0; i < n; i++)  
            this.data[i] = data[i];  
    }  
    :  
}
```

instance variables private and final  
(neither necessary nor sufficient,  
but good programming practice)

defensive copy of mutable  
instance variables

instance methods don't  
change instance variables

**Immutable.** String, Integer, Double, Color, Vector, Transaction, Point2D.  
**Mutable.** StringBuilder, Stack, Counter, Java array.

# Immutability: properties

---

**Data type.** Set of values and operations on those values.

**Immutable data type.** Can't change the data type value once created.

## Advantages.

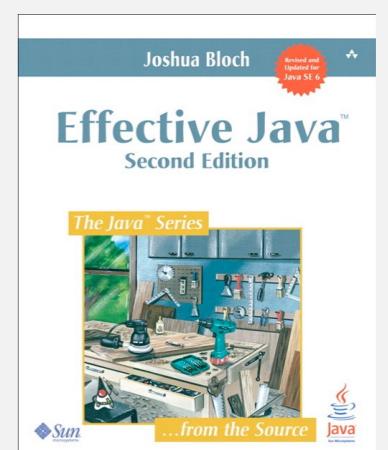
- Simplifies debugging.
- Simplifies concurrent programming.
- More secure in presence of hostile code.
- Safe to use as key in priority queue or symbol table.



**Disadvantage.** Must create new object for each data type value.

*“Classes should be immutable unless there's a very good reason to make them mutable.... If a class cannot be made immutable, you should still limit its mutability as much as possible.”*

— Joshua Bloch (Java architect)



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## 2.4 PRIORITY QUEUES

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- ▶ *API and elementary implementations*
- ▶ *binary heaps*
- ▶ ***heapsort***
- ▶ *event-driven simulation*

## Priority queues: quiz 3

---

What is the name of this sorting algorithm?

```
public void sort(String[] a)
{
    int n = a.length;
    MaxPQ<String> pq = new MaxPQ<String>();
    for (int i = 0; i < n; i++)
        pq.insert(a[i]);
    for (int i = n-1; i >= 0; i--)
        a[i] = pq.delMax();
}
```

- A. Insertion sort.
- B. Mergesort.
- C. Quicksort.
- D. *None of the above.*
- E. *I don't know.*

## Priority queues: quiz 4

---

What are its properties?

```
public void sort(String[] a)
{
    int n = a.length;
    MaxPQ<String> pq = new MaxPQ<String>();
    for (int i = 0; i < n; i++)
        pq.insert(a[i]);
    for (int i = n-1; i >= 0; i--)
        a[i] = pq.delMax();
}
```

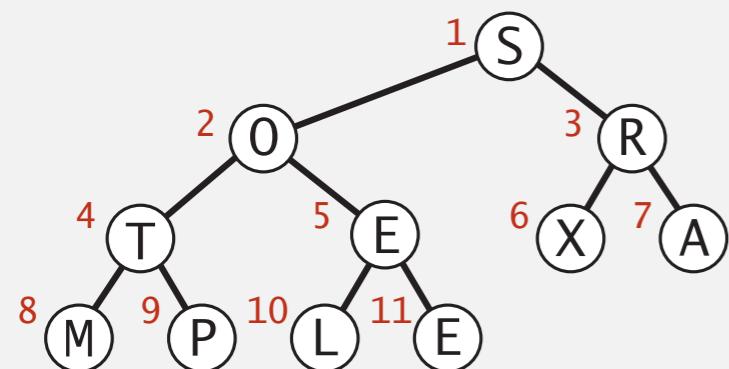
- A.  $n \log n$  compares in the worst case.
- B. In-place.
- C. Stable.
- D. All of the above.
- E. I don't know.

# Heapsort

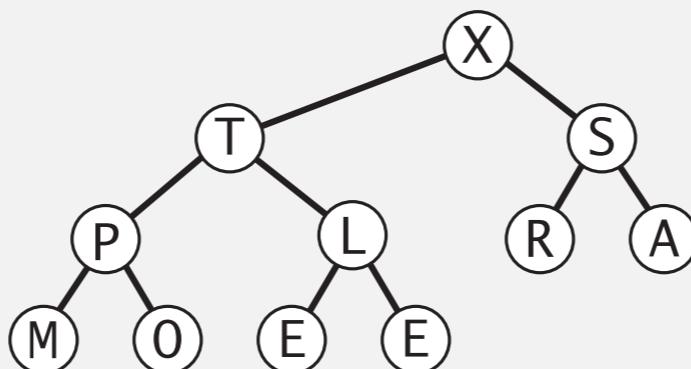
Basic plan for in-place sort.

- View input array as a complete binary tree.
- Heap construction: build a max-heap with all  $n$  keys.
- Sortdown: repeatedly remove the maximum key.

keys in arbitrary order



build max heap  
(in place)



sorted result  
(in place)



1	2	3	4	5	6	7	8	9	10	11
S	O	R	T	E	X	A	M	P	L	E

1	2	3	4	5	6	7	8	9	10	11
X	T	S	P	L	R	A	M	O	E	E

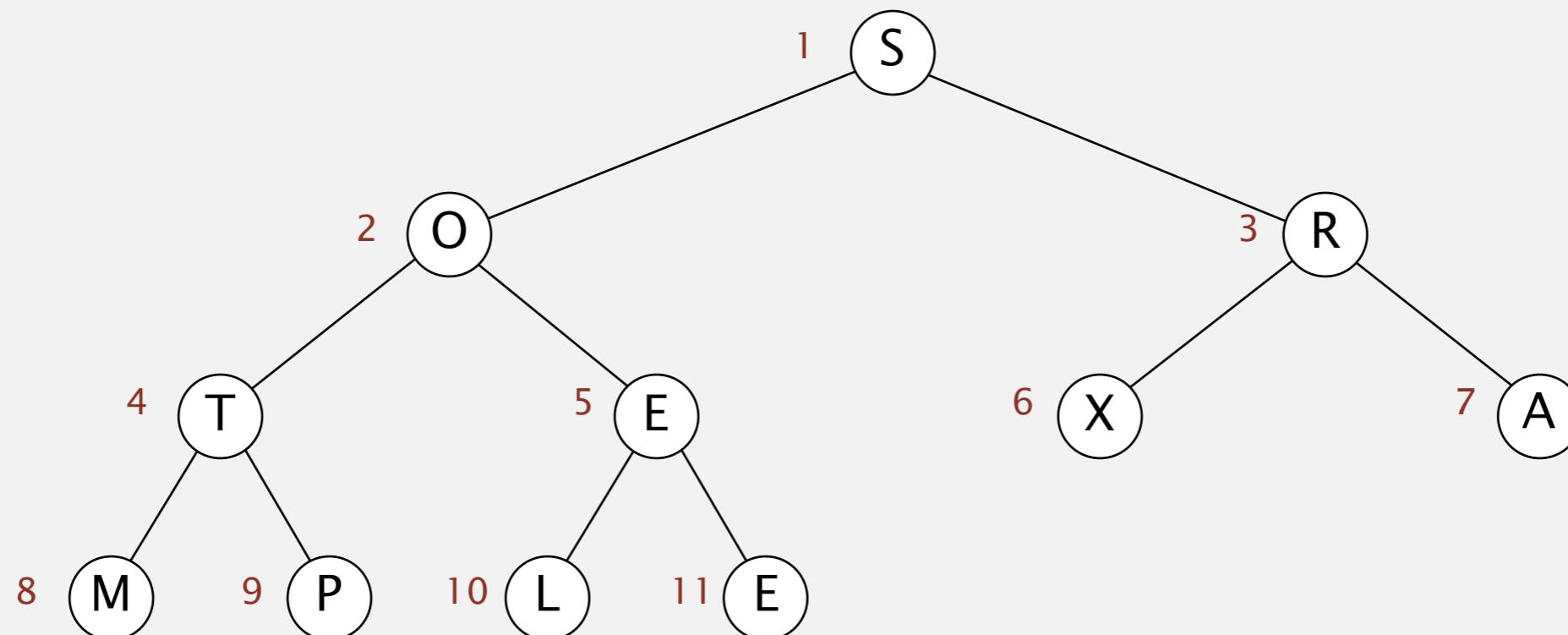
1	2	3	4	5	6	7	8	9	10	11
A	E	E	L	M	O	P	R	S	T	X

# Heapsort demo

Heap construction. Build max heap using bottom-up method.

we assume array entries are indexed 1 to n

array in arbitrary order



S	O	R	T	E	X	A	M	P	L	E
1	2	3	4	5	6	7	8	9	10	11

# Heapsort demo

---

**Sortdown.** Repeatedly delete the largest remaining item.

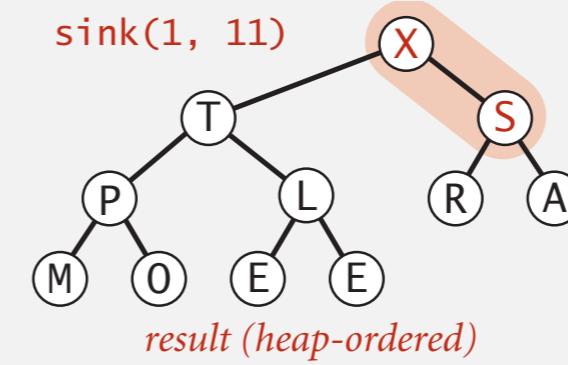
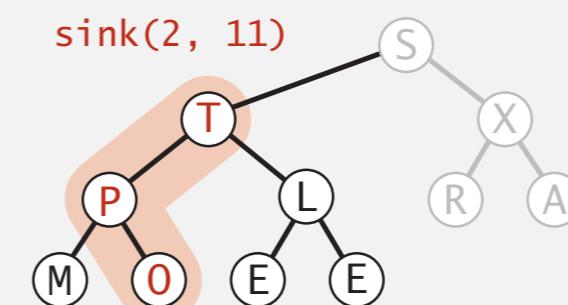
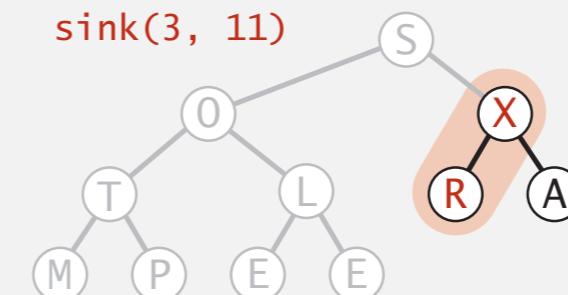
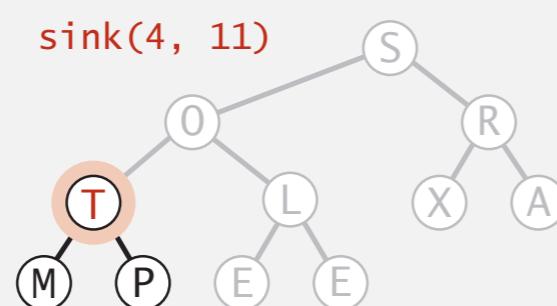
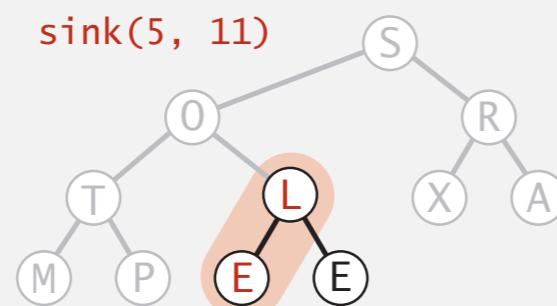
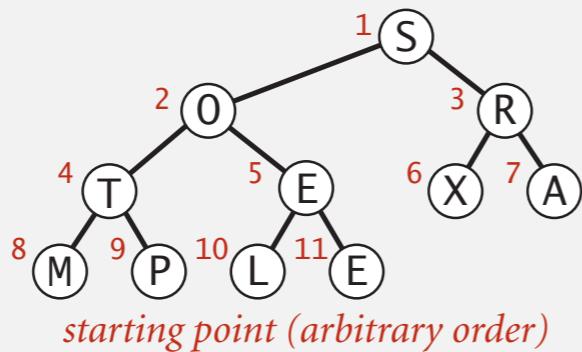
**array in sorted order**



# Heapsort: heap construction

First pass. Build heap using bottom-up method.

```
for (int k = n/2; k >= 1; k--)  
    sink(a, k, n);
```

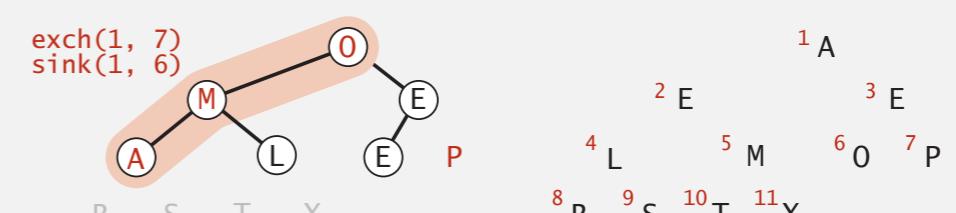
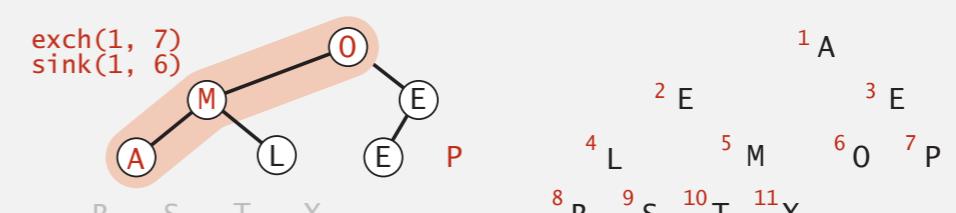
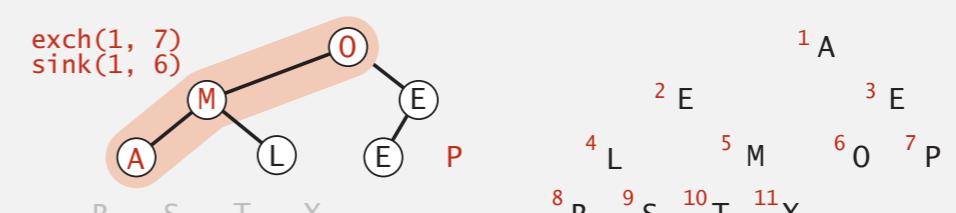
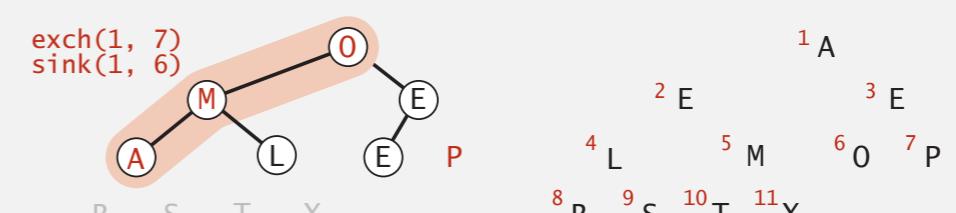
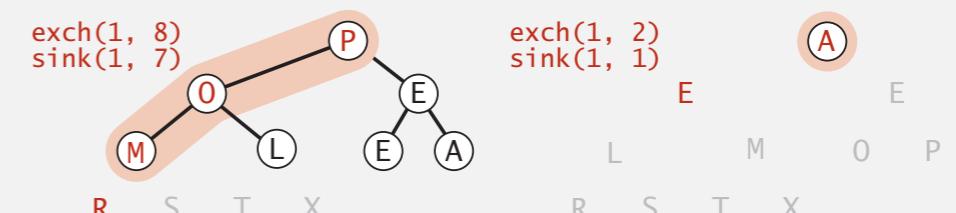
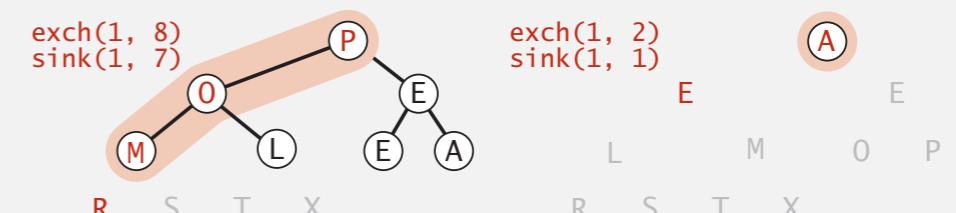
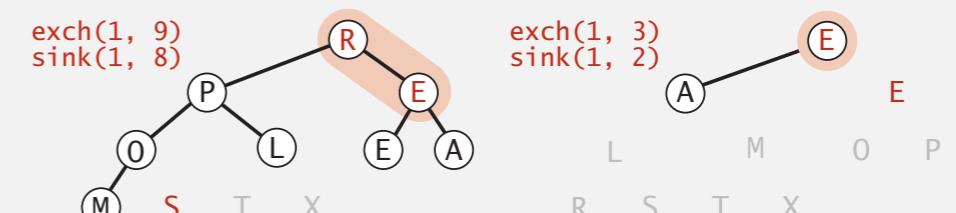
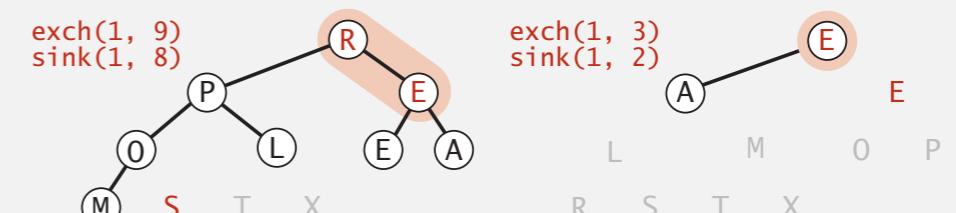
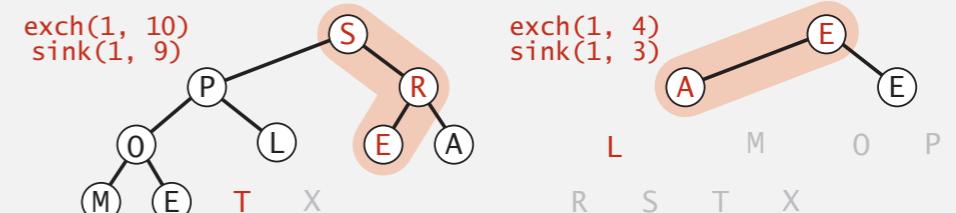
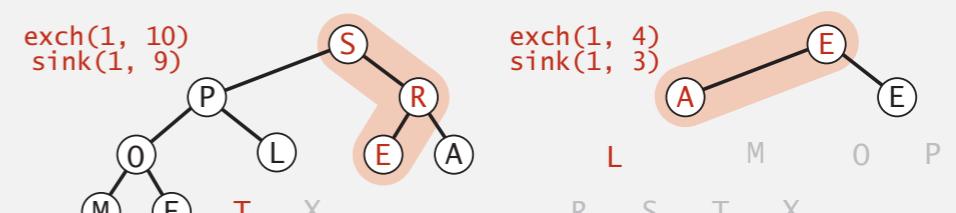
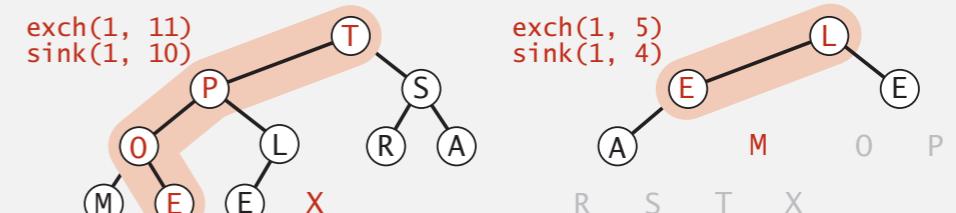
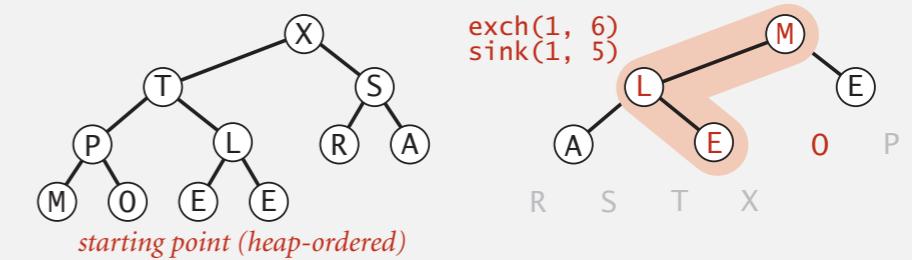


# Heapsort: sortdown

## Second pass.

- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (n > 1)
{
    exch(a, 1, n--);
    sink(a, 1, n);
}
```



# Heapsort: Java implementation

```
public class Heap
{
    public static void sort(Comparable[] a)
    {
        int n = a.length;
        for (int k = n/2; k >= 1; k--)
            sink(a, k, n);
        while (n > 1)
        {
            exch(a, 1, n);
            sink(a, 1, --n);
        }
    }

    private static void sink(Comparable[] a, int k, int n)
    { /* as before */ }

    private static boolean less(Comparable[] a, int i, int j)
    { /* as before */ }

    private static void exch(Object[] a, int i, int j)
    { /* as before */ }
}

but make static (and pass arguments)

private static void sink(Comparable[] a, int k, int n)
{ /* as before */ }

but convert from 1-based
indexing to 0-base indexing
```

# Heapsort: trace

---

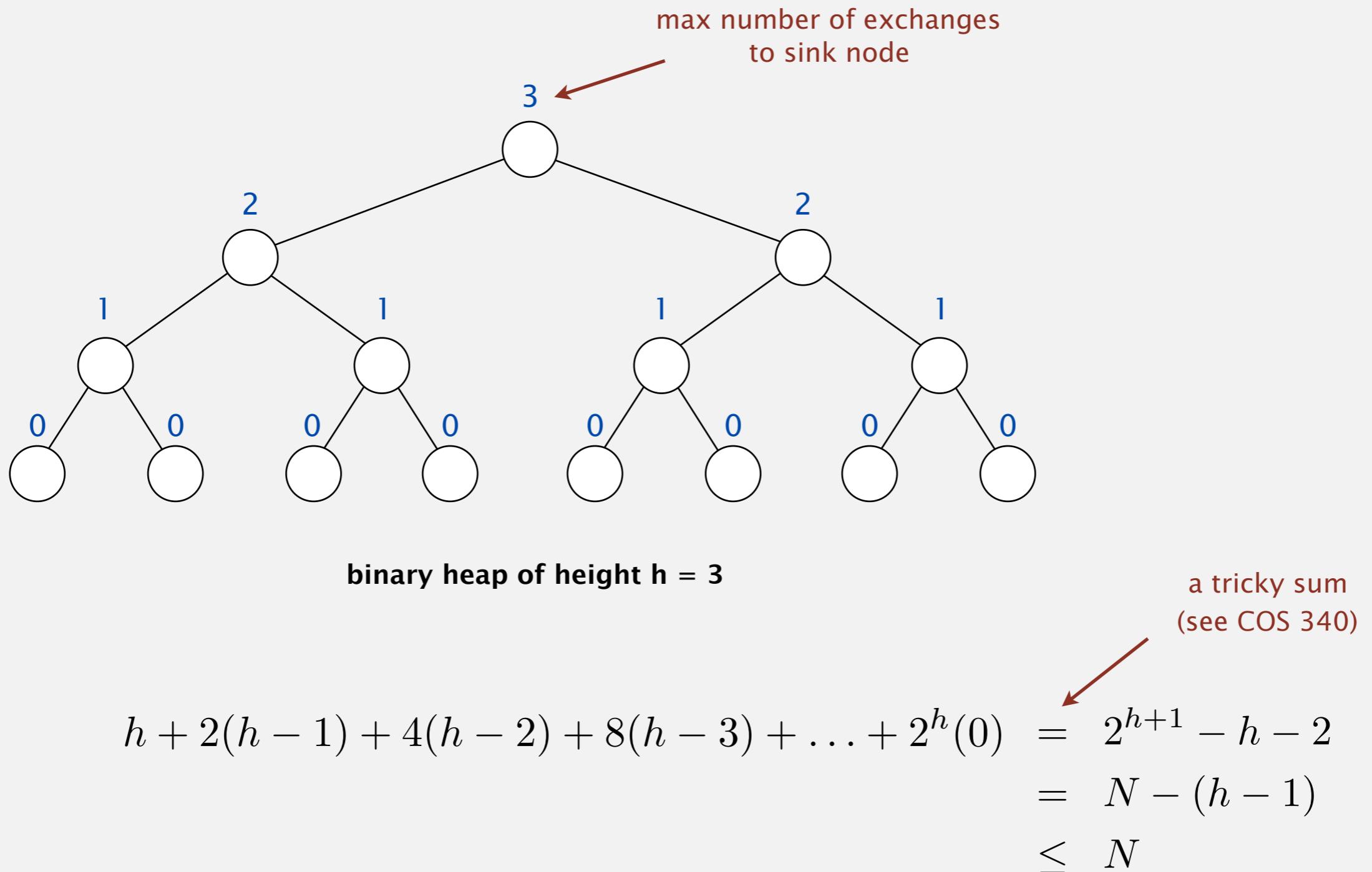
		a[i]											
N	k	0	1	2	3	4	5	6	7	8	9	10	11
<i>initial values</i>		S	0	R	T	E	X	A	M	P	L	E	
11	5	S	0	R	T	L	X	A	M	P	E	E	
11	4	S	0	R	T	L	X	A	M	P	E	E	
11	3	S	0	X	T	L	R	A	M	P	E	E	
11	2	S	T	X	P	L	R	A	M	O	E	E	
11	1	X	T	S	P	L	R	A	M	O	E	E	
<i>heap-ordered</i>		X	T	S	P	L	R	A	M	O	E	E	
10	1	T	P	S	O	L	R	A	M	E	E	X	
9	1	S	P	R	O	L	E	A	M	E	T	X	
8	1	R	P	E	O	L	E	A	M	S	T	X	
7	1	P	O	E	M	L	E	A	R	S	T	X	
6	1	O	M	E	A	L	E	P	R	S	T	X	
5	1	M	L	E	A	E	O	P	R	S	T	X	
4	1	L	E	E	A	M	O	P	R	S	T	X	
3	1	E	A	E	L	M	O	P	R	S	T	X	
2	1	E	A	E	L	M	O	P	R	S	T	X	
1	1	A	E	E	L	M	O	P	R	S	T	X	
<i>sorted result</i>		A	E	E	L	M	O	P	R	S	T	X	

Heapsort trace (array contents just after each sink)

# Heapsort: mathematical analysis

Proposition. Heap construction makes  $\leq n$  exchanges and  $\leq 2n$  compares.

Pf sketch. [assume  $n = 2^{h+1} - 1$ ]



# Heapsort: mathematical analysis

---

Proposition. Heap construction uses  $\leq 2n$  compares and  $\leq n$  exchanges.

Proposition. Heapsort uses  $\leq 2n \lg n$  compares and exchanges.

↑  
algorithm can be improved to  $\sim n \lg n$   
(but no such variant is known to be practical)

Significance. In-place sorting algorithm with  $n \log n$  worst-case.

- Mergesort: no, linear extra space.      ← in-place merge possible, not practical
- Quicksort: no, quadratic time in worst case.      ←  $n \log n$  worst-case quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache.
- Not stable.

↑  
can be improved using  
advanced caching tricks

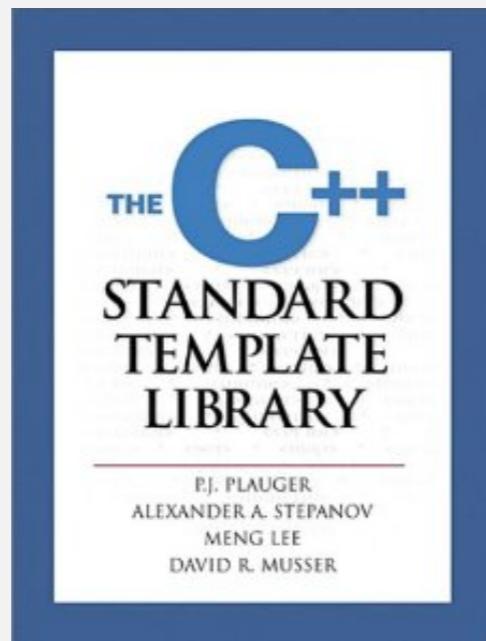
# Introsort

---

Goal. As fast as quicksort in practice;  $n \log n$  worst case, in place.

## Introsort.

- Run quicksort.
- Cutoff to heapsort if stack depth exceeds  $2 \lg n$ .
- Cutoff to insertion sort for  $n = 16$ .



### Introspective Sorting and Selection Algorithms

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#### Abstract

Quicksort is the preferred in-place sorting algorithm in many contexts, since its average computing time on uniformly distributed inputs is  $\Theta(N \log N)$  and it is in fact faster than most other sorting algorithms on most inputs. Its drawback is that its worst-case time bound is  $\Theta(N^2)$ . Previous attempts to protect against the worst case by improving the way quicksort chooses pivot elements for partitioning have increased the average computing time too much—one might as well use heapsort, which has a  $\Theta(N \log N)$  worst-case time bound but is on the average 2 to 5 times slower than quicksort. A similar dilemma exists with selection algorithms (for finding the  $i$ -th largest element) based on partitioning. This paper describes a simple solution to this dilemma: limit the depth of partitioning, and for subproblems that exceed the limit switch to another algorithm with a better worst-case bound. Using heapsort as the “stopper” yields a sorting algorithm that is just as fast as quicksort in the average case but also has an  $\Theta(N \log N)$  worst case time bound. For selection, a hybrid of Hoare’s FIND algorithm, which is linear on average but quadratic in the worst case, and the Blum-Floyd-Pratt-Rivest-Tarjan algorithm is as fast as Hoare’s algorithm in practice, yet has a linear worst-case time bound. Also discussed are issues of implementing the new algorithms as generic algorithms and accurately measuring their performance in the framework of the C++ Standard Template Library.

In the wild. C++ STL, Microsoft .NET Framework.

# Sorting algorithms: summary

---

	inplace?	stable?	best	average	worst	remarks
selection	✓		$\frac{1}{2} n^2$	$\frac{1}{2} n^2$	$\frac{1}{2} n^2$	$n$ exchanges
insertion	✓	✓	$n$	$\frac{1}{4} n^2$	$\frac{1}{2} n^2$	use for small $n$ or partially ordered
shell	✓		$n \log_3 n$	?	$c n^{3/2}$	tight code; subquadratic
merge		✓	$\frac{1}{2} n \lg n$	$n \lg n$	$n \lg n$	$n \log n$ guarantee; stable
timsort		✓	$n$	$n \lg n$	$n \lg n$	improves mergesort when preexisting order
quick	✓		$n \lg n$	$2 n \ln n$	$\frac{1}{2} n^2$	$n \log n$ probabilistic guarantee; fastest in practice
3-way quick	✓		$n$	$2 n \ln n$	$\frac{1}{2} n^2$	improves quicksort when duplicate keys
heap	✓		$3 n$	$2 n \lg n$	$2 n \lg n$	$n \log n$ guarantee; in-place
?	✓	✓	$n$	$n \lg n$	$n \lg n$	holy sorting grail

# Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 2.4 PRIORITY QUEUES

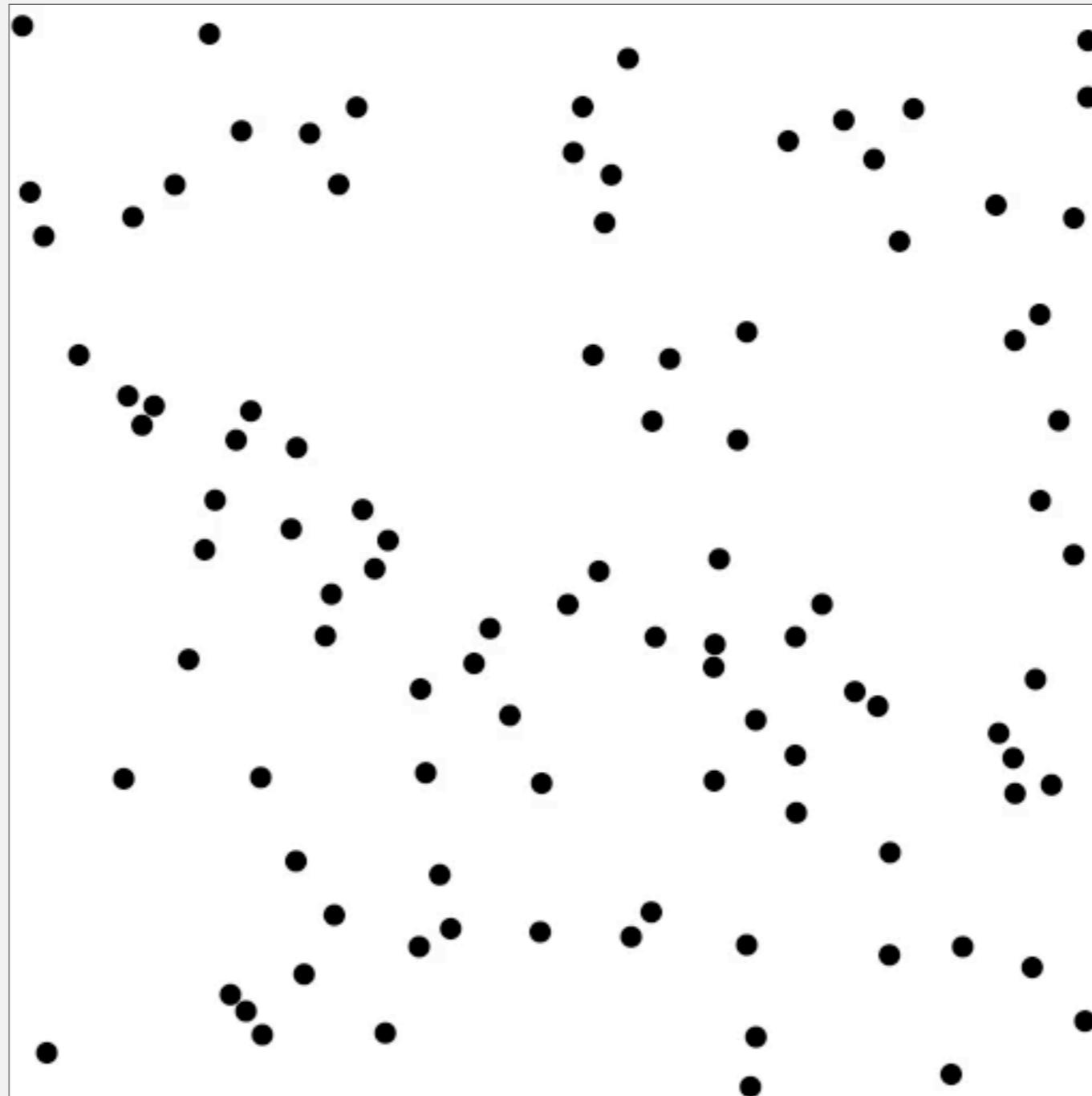
---

- ▶ *API and elementary implementations*
- ▶ *binary heaps*
- ▶ *heapsort*
- ▶ ***event-driven simulation***

# Molecular dynamics simulation of hard discs

---

**Goal.** Simulate the motion of  $n$  moving particles that behave according to the laws of elastic collision.



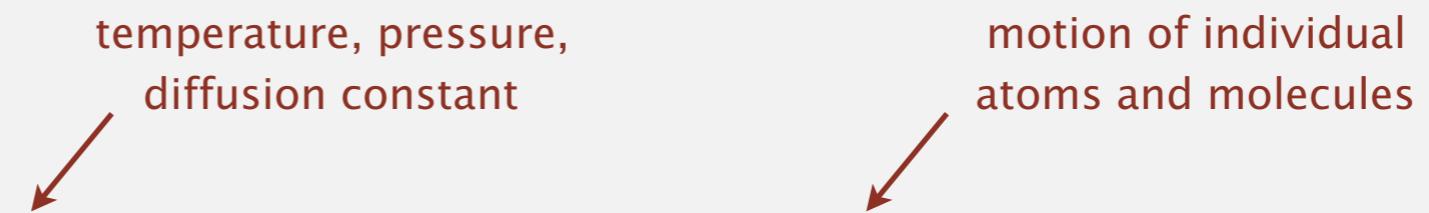
# Molecular dynamics simulation of hard discs

---

**Goal.** Simulate the motion of  $n$  moving particles that behave according to the laws of elastic collision.

## Hard disc model.

- Moving particles interact via elastic collisions with each other and walls.
- Each particle is a disc with known position, velocity, mass, and radius.
- No other forces.



**Significance.** Relates macroscopic observables to microscopic dynamics.

- Maxwell-Boltzmann: distribution of speeds as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

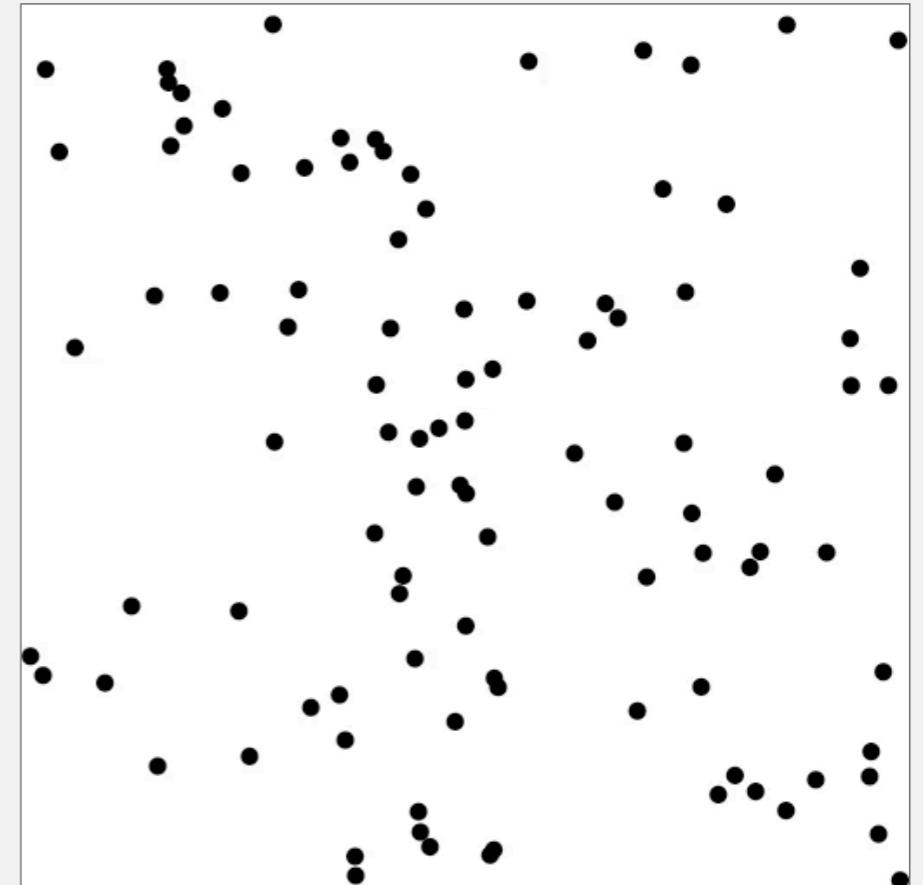
## Warmup: bouncing balls

Time-driven simulation.  $n$  bouncing balls in the unit square.

```
public class BouncingBalls
{
    public static void main(String[] args)
    {
        int n = Integer.parseInt(args[0]);
        Ball[] balls = new Ball[n];
        for (int i = 0; i < n; i++)
            balls[i] = new Ball();
        while(true)
        {
            StdDraw.clear();
            for (int i = 0; i < n; i++)
            {
                balls[i].move(0.5);
                balls[i].draw();
            }
            StdDraw.show(50);
        }
    }
}
```

main simulation loop

```
% java BouncingBalls 100
```



## Warmup: bouncing balls

```
public class Ball
{
    private double rx, ry;          // position
    private double vx, vy;          // velocity
    private final double radius;    // radius
    public Ball(...)
    { /* initialize position and velocity */ }

    public void move(double dt)
    {
        if ((rx + vx*dt < radius) || (rx + vx*dt > 1.0 - radius)) { vx = -vx; }
        if ((ry + vy*dt < radius) || (ry + vy*dt > 1.0 - radius)) { vy = -vy; }
        rx = rx + vx*dt;
        ry = ry + vy*dt;
    }

    public void draw()
    { StdDraw.filledCircle(rx, ry, radius); }

}
```

check for collision with walls



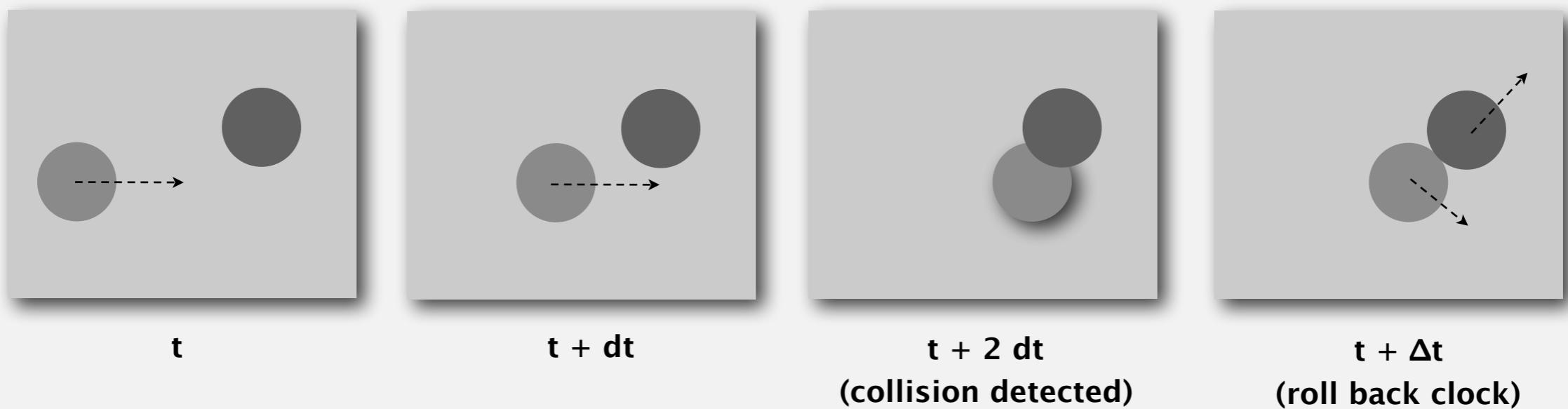
Missing. Check for balls colliding with each other.

- Physics problems: when? what effect?
- CS problems: which object does the check? too many checks?

# Time-driven simulation

---

- Discretize time in quanta of size  $dt$ .
- Update the position of each particle after every  $dt$  units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.

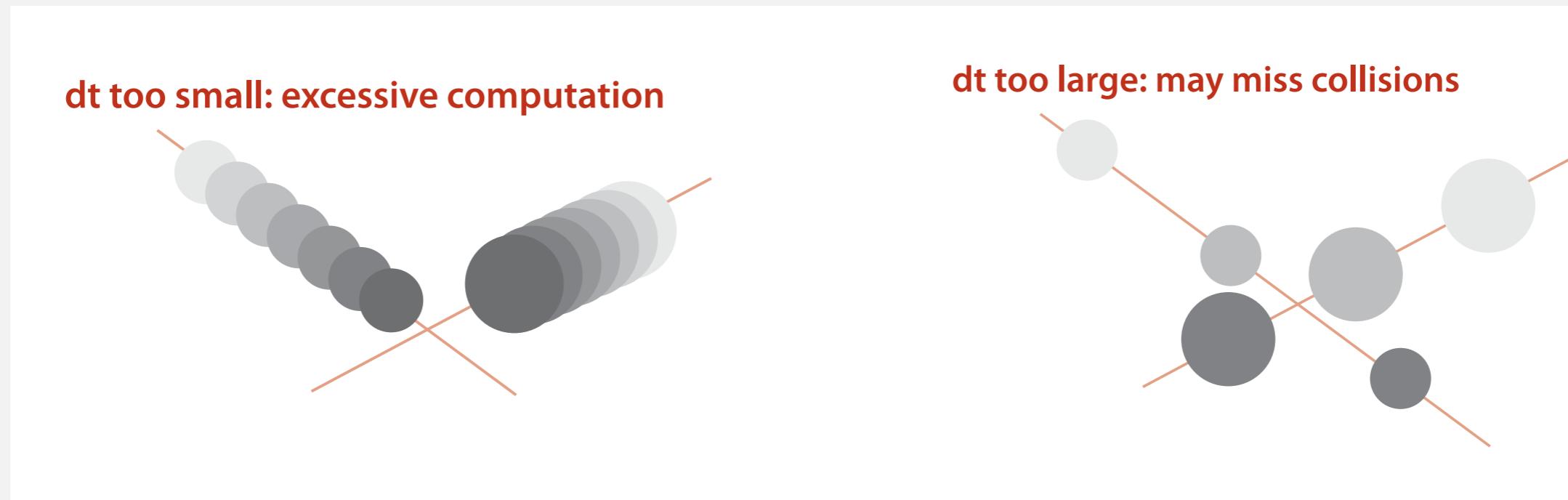


# Time-driven simulation

---

## Main drawbacks.

- $\sim n^2 / 2$  overlap checks per time quantum.
- Simulation is too slow if  $dt$  is very small.
- May miss collisions if  $dt$  is too large.  
(if colliding particles fail to overlap when we are looking)



# Event-driven simulation

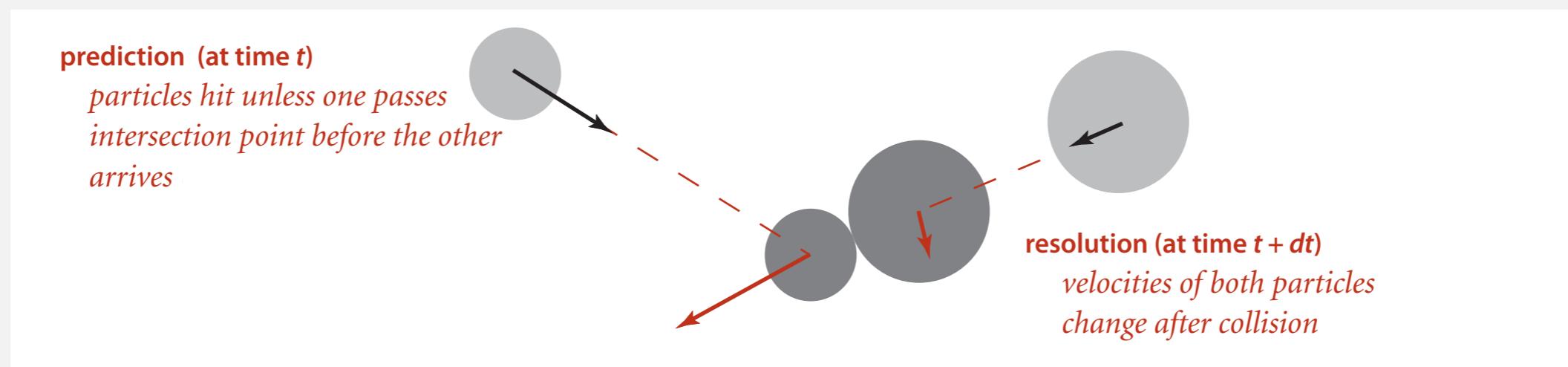
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Change state only when something interesting happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain **PQ** of collision events, prioritized by time.
- Delete min = get next collision.

**Collision prediction.** Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

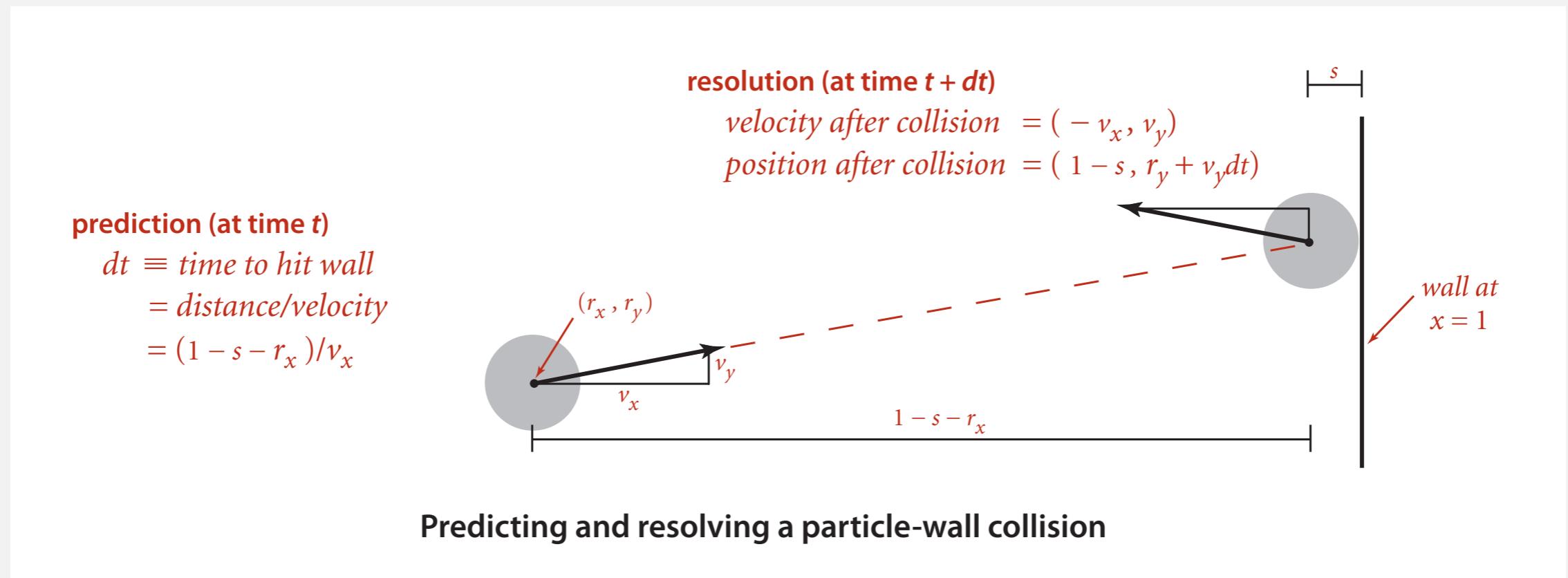
**Collision resolution.** If collision occurs, update colliding particle(s) according to laws of elastic collisions.



# Particle-wall collision

## Collision prediction and resolution.

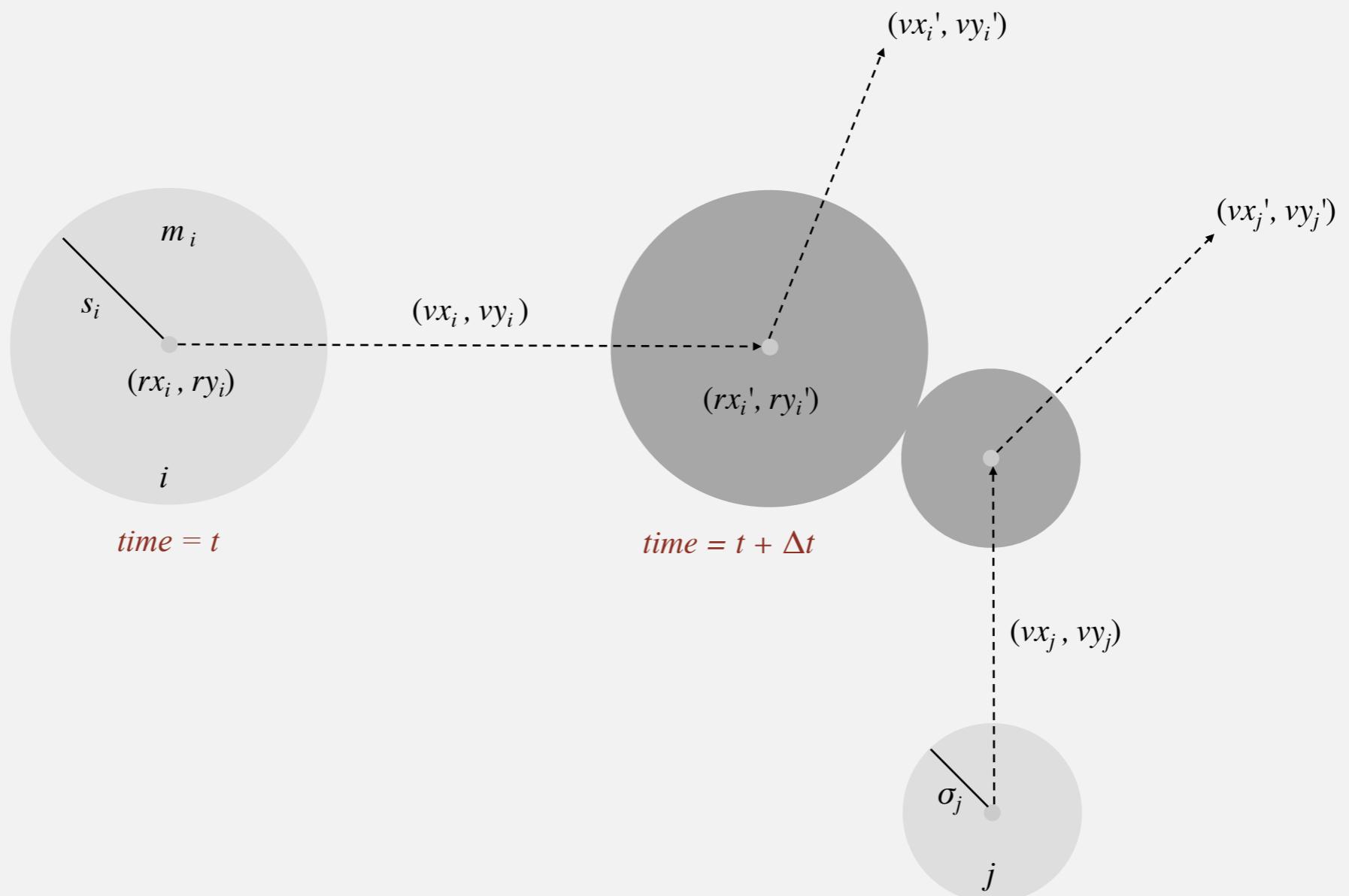
- Particle of radius  $s$  at position  $(r_x, r_y)$ .
- Particle moving in unit box with velocity  $(v_x, v_y)$ .
- Will it collide with a vertical wall? If so, when?



# Particle-particle collision prediction

## Collision prediction.

- Particle  $i$ : radius  $s_i$ , position  $(rx_i, ry_i)$ , velocity  $(vx_i, vy_i)$ .
- Particle  $j$ : radius  $s_j$ , position  $(rx_j, ry_j)$ , velocity  $(vx_j, vy_j)$ .
- Will particles  $i$  and  $j$  collide? If so, when?



# Particle-particle collision prediction

## Collision prediction.

- Particle  $i$ : radius  $s_i$ , position  $(rx_i, ry_i)$ , velocity  $(vx_i, vy_i)$ .
- Particle  $j$ : radius  $s_j$ , position  $(rx_j, ry_j)$ , velocity  $(vx_j, vy_j)$ .
- Will particles  $i$  and  $j$  collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \geq 0, \\ \infty & \text{if } d < 0, \\ -\frac{\Delta v \cdot \Delta r + \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d = (\Delta v \cdot \Delta r)^2 - (\Delta v \cdot \Delta v) (\Delta r \cdot \Delta r - s^2), \quad s = s_i + s_j$$

$$\Delta v = (\Delta vx, \Delta vy) = (vx_i - vx_j, vy_i - vy_j)$$

$$\Delta r = (\Delta rx, \Delta ry) = (rx_i - rx_j, ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + (\Delta vy)^2$$

$$\Delta r \cdot \Delta r = (\Delta rx)^2 + (\Delta ry)^2$$

$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + (\Delta vy)(\Delta ry)$$

**Important note:** This is physics, so we won't be testing you on it!

# Particle-particle collision resolution

Collision resolution. When two particles collide, how does velocity change?

$$\begin{aligned} vx_i' &= vx_i + Jx / m_i \\ vy_i' &= vy_i + Jy / m_i \\ vx_j' &= vx_j - Jx / m_j \\ vy_j' &= vy_j - Jy / m_j \end{aligned}$$

Newton's second law  
(momentum form)

$$Jx = \frac{J \Delta rx}{s}, \quad Jy = \frac{J \Delta ry}{s}, \quad J = \frac{2 m_i m_j (\Delta v \cdot \Delta r)}{s (m_i + m_j)}$$

impulse due to normal force  
(conservation of energy, conservation of momentum)

Important note: This is physics, so we won't be testing you on it!

# Particle data type skeleton

```
public class Particle
{
    private double rx, ry;          // position
    private double vx, vy;          // velocity
    private final double radius;    // radius
    private final double mass;      // mass
    private int count;              // number of collisions

    public Particle( ... ) { ... }

    public void move(double dt) { ... }
    public void draw() { ... }

    public double timeToHit(Particle that) { }
    public double timeToHitVerticalWall() { }                                predict collision
    public double timeToHitHorizontalWall() { }                                with particle or wall

    public void bounceOff(Particle that) { }
    public void bounceOffVerticalWall() { }                                    resolve collision
    public void bounceOffHorizontalWall() { }                                with particle or wall

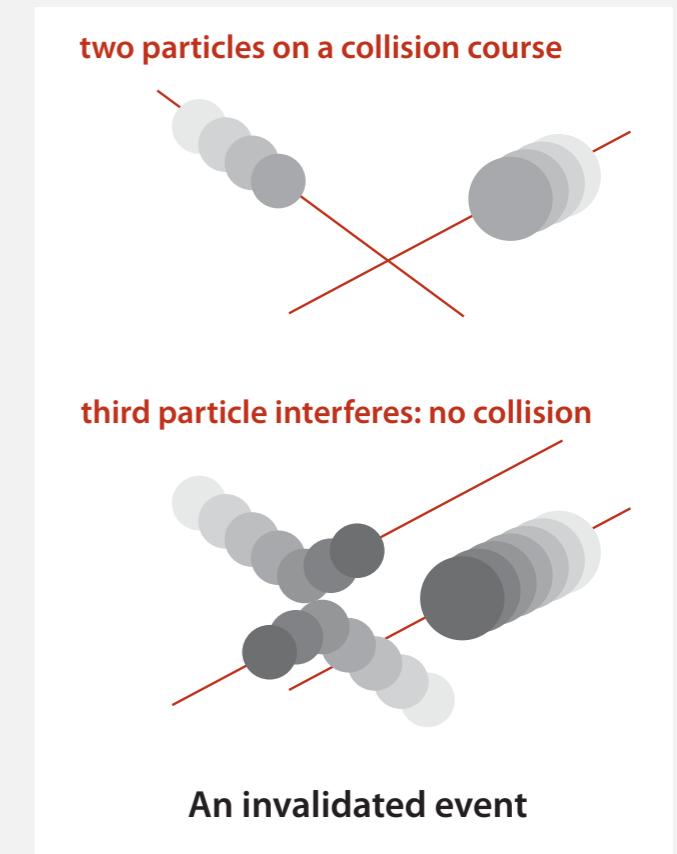
}
```

# Collision system: event-driven simulation main loop

## Initialization.

- Fill PQ with all potential particle-wall collisions.
- Fill PQ with all potential particle-particle collisions.

“potential” since collision is invalidated  
if some other collision intervenes



## Main loop.

- Delete the impending event from PQ (min priority =  $t$ ).
- If the event has been invalidated, ignore it.
- Advance all particles to time  $t$ , on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

# Event data type

## Conventions.

- Neither particle null  $\Rightarrow$  particle-particle collision.
- One particle null  $\Rightarrow$  particle-wall collision.
- Both particles null  $\Rightarrow$  redraw event.

```
private static class Event implements Comparable<Event>
{
    private final double time;                      // time of event
    private final Particle a, b;                    // particles involved in event
    private final int countA, countB;   // collision counts of a and b

    public Event(double t, Particle a, Particle b)      create event
    { ... }

    public int compareTo(Event that)
    { return this.time - that.time; }                  ordered by time

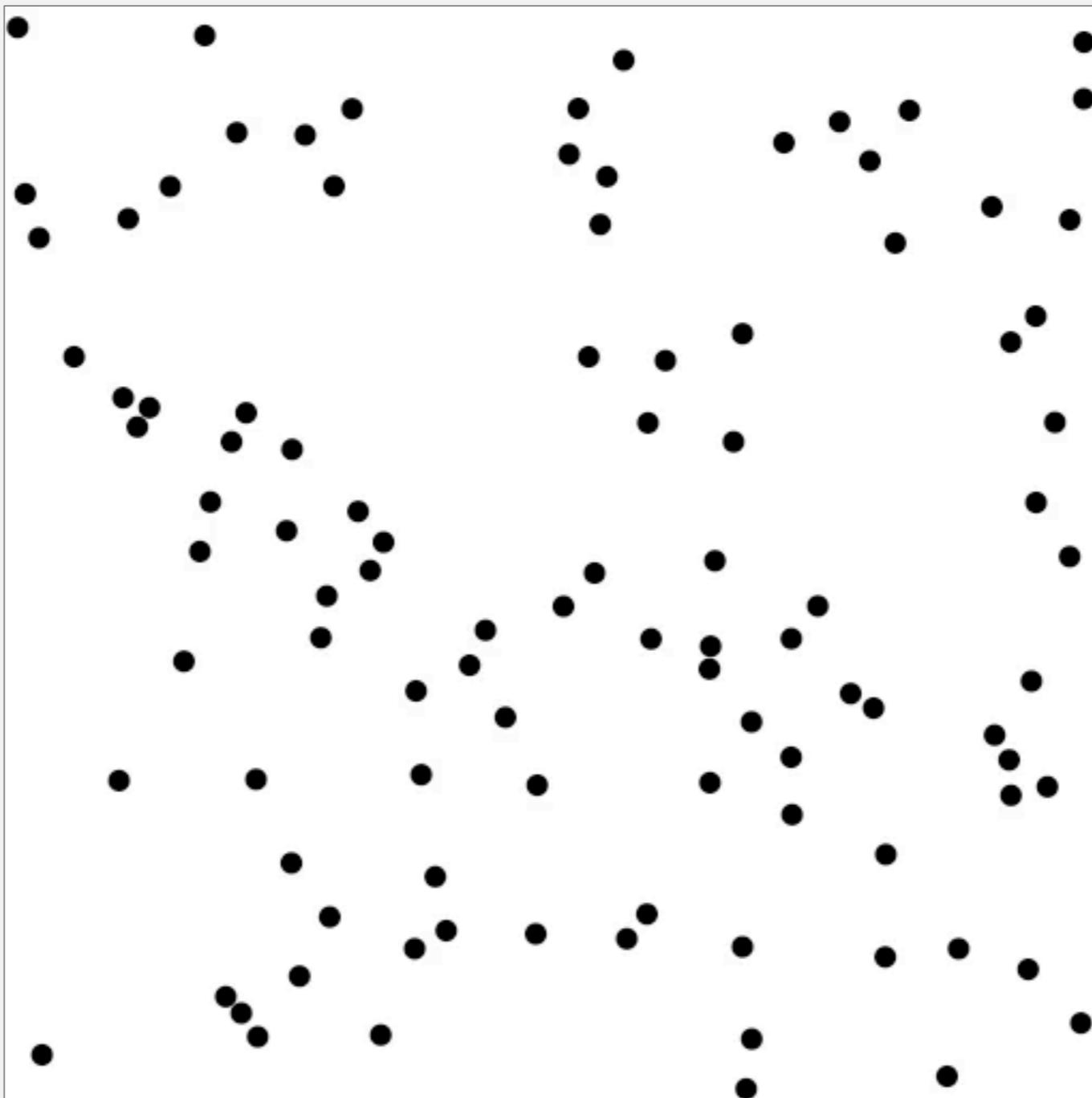
    public boolean isValid()
    { ... }
}
```

valid if no intervening collisions  
(compare collision counts)

# Particle collision simulation: example 1

---

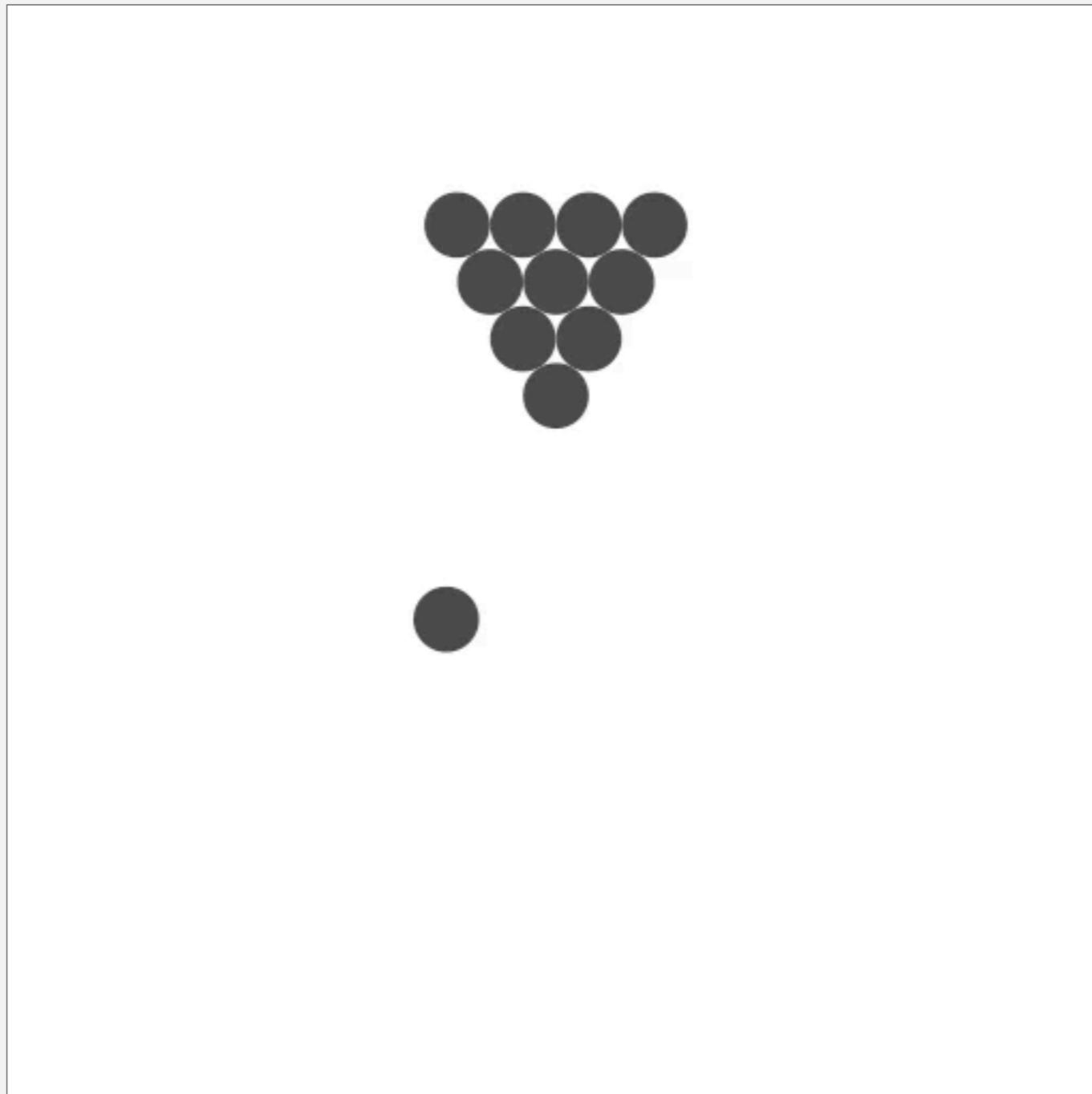
```
% java CollisionSystem 100
```



# Particle collision simulation: example 2

---

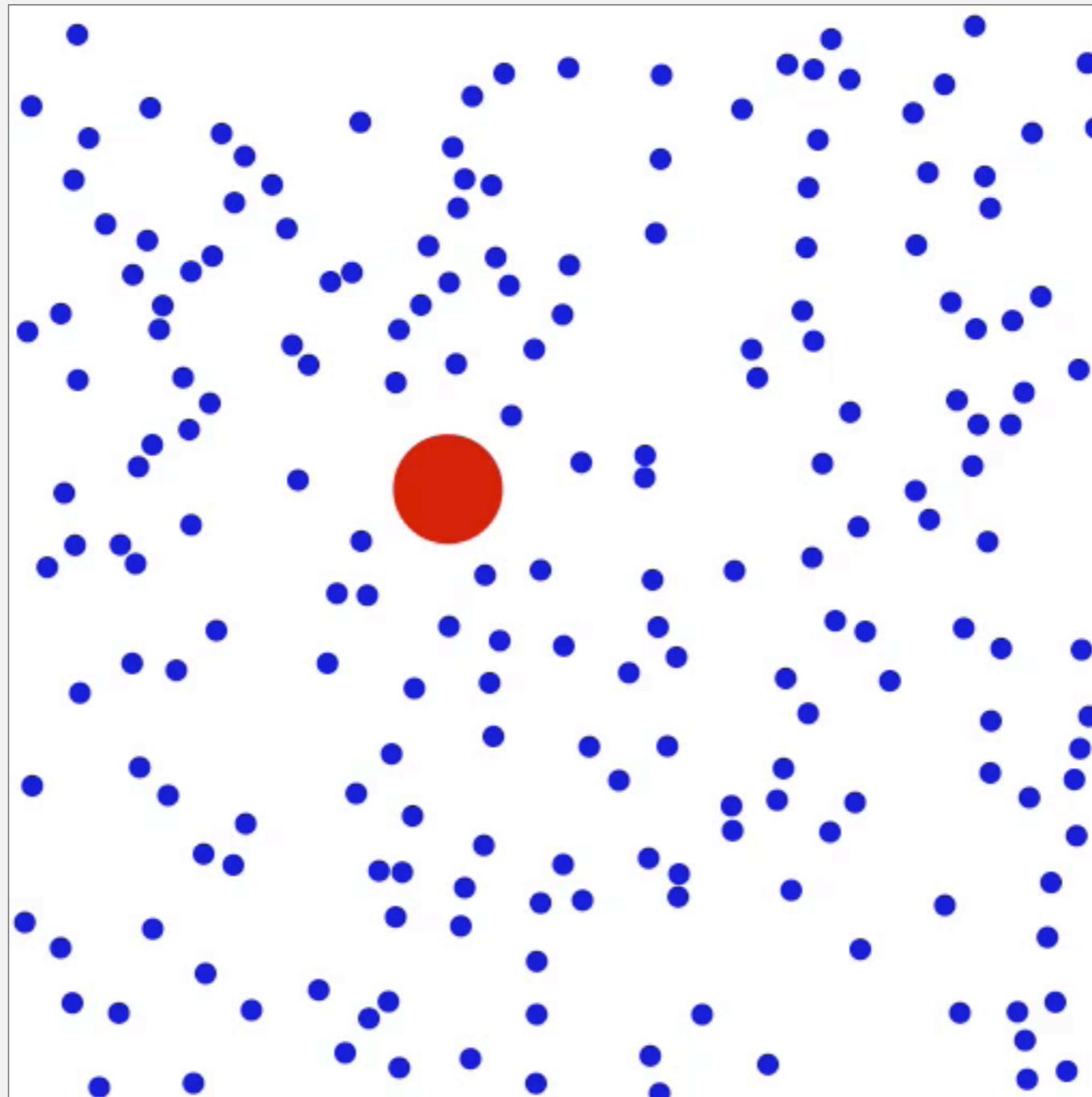
```
% java CollisionSystem < billiards.txt
```



# Particle collision simulation: example 3

---

```
% java CollisionSystem < brownian.txt
```



# Particle collision simulation: example 4

---

```
% java CollisionSystem < diffusion.txt
```

