AM-GM Inequality Done Right

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1 Statement

Given the positive reals $a_1, a_2, \cdots, a_n$ the following inequality holds

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

with the equality if and only if $a_1 = a_2 = \cdots = a_n$.

2 Cauchy's Forward-backward induction proof

The FB (Forward-backward) induction is a rather rare variant of proof by induction based on the idea of first proving the base case as usual such as $P(2)$. Then proving for arbitrary large n such that if $P(n)$ is true then $P(2n)$ which follows recursively that $P(2^k)$ must be true. The final step is to show that $P(n) \Rightarrow P(n-1)$ which will prove that $P(n)$ holds for any $n \geq 2$.

i). Base case.
ii). $P(n) \Rightarrow P(2n)$.
iii). $P(n) \Rightarrow P(n-1)$.

Step i): The base case is when $n = 2$ because this is the smallest n for which there is an inequality, $n = 1$ would give us an equality. Given $n = 2$, the AM-GM inequality implies:

$$\frac{a+b}{2} \geq \sqrt{ab} \tag{1}$$

To prove this we square both sides which yields

$$\begin{aligned} \frac{a^2 + 2ab + b^2}{4} \geq ab &\iff a^2 + 2ab + b^2 \geq 4ab \\ a^2 - 2ab + b^2 \geq 0 &\iff (a-b)^2 \geq 0. \end{aligned}$$

Because $(a-b)^2 \geq 0$ is obviously true, we conclude that $\frac{a+b}{2} \geq \sqrt{ab}$ must be true. $\square$

Step ii): To perform this step, we first suppose $P(n)$ is true then we prove that $P(2n)$ must be true. Suppose for positive reals $a_1, a_2, \cdots, a_n$ that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

holds. Then equivalently we have

$$a_1 + a_2 + \cdots + a_n \geq n \sqrt[n]{a_1 a_2 \cdots a_n} \tag{2}$$

We start with the sum $a_1 + \cdots + a_{2n}$ and then prove for $2n$:

$$a_1 + \cdots + a_n + a_{n+1} + \cdots + a_{2n} \geq n \sqrt[n]{a_1 \cdots a_n} + n \sqrt[n]{a_{n+1} \cdots a_{2n}}$$

Here we use the fact that the sum can be split up in to two parts $a_1 + \cdots + a_n$ and $a_{n+1} + \cdots + a_{2n}$, then we apply the inequality from equation 2 on each part. Recall the base case inequality (1), this can be slightly modified to the form $a + b \geq 2\sqrt{ab}$ which we can apply directly to the inequality above.

$$n \sqrt[n]{a_1 \cdots a_n} + n \sqrt[n]{a_{n+1} \cdots a_{2n}} \geq 2\sqrt{n^2 \sqrt[n]{a_1 \cdots a_{2n}}} = 2n \sqrt[2n]{a_1 \cdots a_{2n}}$$

This means that we have the following inequality

$$a_1 + \cdots + a_{2n} \geq 2n \sqrt[2n]{a_1 \cdots a_{2n}}$$

We have now shown that if

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

then

$$\frac{a_1 + \cdots + a_{2n}}{2n} \geq \sqrt[2n]{a_1 \cdots a_{2n}}.$$

$\square$

Step *iii*): Suppose for positive reals $a_1, a_2, \cdots, a_n$ that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \tag{3}$$

To prove P(n-1) we start with $\frac{a_1+\cdots+a_{n-1}}{n-1}$ and then prove the right side of the inequality. In order to perform the induction we must somehow turn the expression in to a form such that we can apply inequality 3. To do this we may let

$$\frac{a_1 + \cdots + a_{n-1}}{n-1} = \frac{a_1 + \cdots + a_n}{n}$$

and then solve for a_n. This means that we can express $\frac{a_1+\cdots+a_{n-1}}{n-1}$ in a way such that we can use the induction assumption (3).

$$\begin{aligned}
&\frac{a_1 + \cdots + a_{n-1}}{n-1} = \frac{a_1 + \cdots + a_n}{n} \\
\iff &n(a_1 + \cdots + a_{n-1}) = (n-1)(a_1 + \cdots + a_n) \\
\iff &\frac{a_1 + \cdots + a_{n-1}}{n-1} = a_n
\end{aligned}$$

Because we have

$$\frac{a_1 + \cdots + a_{n-1}}{n-1} = \frac{a_1 + \cdots + a_n}{n}$$

when $\frac{a_1+\cdots+a_{n-1}}{n-1} = a_n$ it means that we have

$$\begin{aligned}
\frac{a_1+\cdots+a_n}{n} &\geq \sqrt[n]{a_1\cdots a_n} \\
\frac{a_1+\cdots+a_{n-1}}{n-1} &\geq \sqrt[n]{a_1\cdots a_{n-1}\cdot\frac{a_1+\cdots+a_{n-1}}{n-1}} \\
\left(\frac{a_1+\cdots+a_{n-1}}{n-1}\right)^n &\geq a_1\cdots a_{n-1}\cdot\frac{a_1+\cdots+a_{n-1}}{n-1} \\
\left(\frac{a_1+\cdots+a_{n-1}}{n-1}\right)^{n-1} &\geq a_1\cdots a_{n-1} \\
\frac{a_1+\cdots+a_{n-1}}{n-1} &\geq \sqrt[n-1]{a_1\cdots a_{n-1}}
\end{aligned}$$

Which proves that $n-1$ must also be true. $\square$

We have now completed the forward-backward induction proof which proves the AM-GM inequality for $n \geq 2$, the $n = 1$ case is trivial as $\frac{a}{1} = a^1$, thus we have proven that for positive reals $a_1, a_2, \cdots, a_n$:

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n} \qquad \text{for } n \geq 1$$

$\square$