Spivak's Calculus Solutions

Sebastian Miles

Contents

Chapter 1

Basic Properties of Numbers

1.1.

(i) Suppose that $ax = a$ and $a \neq 0$, then there exists a number a^{-1}. Multiplying a^{-1} on both sides yields

$$\begin{aligned}(a^{-1}a) \cdot x &= a^{-1}a\\ x &= 1\end{aligned}$$

as desired.

(ii) We use the distributive property on $(x-y)(x+y)$, this can be done by letting $a = x - y$:

$$\begin{aligned}&(x-y)(x+y) = a(x+y)\\ &= ax + ay = (x-y)x + (x-y)y\\ &= x^2 - yx + xy - y^2 = x^2 - y^2\end{aligned}$$

(iii) If we have $x^2 = y^2$ then we certainly have $x^2 - y^2 = 0$. By (ii) we know that $0 = (x-y)(x+y)$, this implies that $x - y = 0$ or $x + y = 0$, this is equivalent to saying that $x = y$ or $x = -y$.

(iv) Same method as (ii):

$$\begin{aligned}&a(x^2 + xy + y^2) = ax^2 + axy + ay^2\\ &= (x-y)x^2 + (x-y)xy + (x-y)y^2\\ &= x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3\\ &= x^3 - y^3\end{aligned}$$

(v) We prove this by induction, the base case $n = 2$ is already proven in (ii). Suppose $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})$ is true. Then we equivalently have $x^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})+y^n$. We now prove the n+1 case:

$$\begin{aligned}
&x^{n+1} - y^{n+1} = x \cdot x^n - y^{n+1} \\
&= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + xy^n - y^{n+1} \\
&= (x-y)(x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}) + (x-y)y^n \\
&= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n)
\end{aligned}$$

The resulting relation concludes the finite induction, thus $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi) We know from (iv) that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, by letting $a = xandb = -y$ we get $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

1.2. Multiplying by the multiplicative inverse of $x - y$ works only when $x - y \neq 0$, that is $x \neq y$, however, the hypothesis explicitly states $x = y$. So it is not possible to find the multiplicative inverse of $x - y$ and thus the step is invalid.

1.3. (i) Say we have $\frac{a}{b}$ and $b \neq 0$