Spivak's Calculus Solutions

Sebastian Miles

Contents

Chapter 1

Basic Properties of Numbers

1.1.

(i) Suppose that $ax = a$ and $a \neq 0$, then there exists a number a^{-1}. Multiplying a^{-1} on both sides yields

$$\begin{aligned}(a^{-1}a) \cdot x &= a^{-1}a \\ x &= 1\end{aligned}$$

as desired.

(ii) We use the distributive property on $(x-y)(x+y)$, this can be done by letting $a = x - y$:

$$\begin{aligned}&(x-y)(x+y) = a(x+y) \\ &= ax + ay = (x-y)x + (x-y)y \\ &= x^2 - yx + xy - y^2 = x^2 - y^2\end{aligned}$$

(iii) If we have $x^2 = y^2$ then we certainly have $x^2 - y^2 = 0$. By (ii) we know that $0 = (x-y)(x+y)$, this implies that $x - y = 0$ or $x + y = 0$, this is equivalent to saying that $x = y$ or $x = -y$.

(iv) Same method as (ii):

$$\begin{aligned}&a(x^2 + xy + y^2) = ax^2 + axy + ay^2 \\ &= (x-y)x^2 + (x-y)xy + (x-y)y^2 \\ &= x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3 \\ &= x^3 - y^3\end{aligned}$$

(v) We prove this by induction, the base case $n = 2$ is already proven in (ii). Suppose $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ is true. Then we equivalently have $x^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + y^n$. We now prove the n+1 case:

$$\begin{aligned}
&x^{n+1} - y^{n+1} = x \cdot x^n - y^{n+1} \\
&= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + xy^n - y^{n+1} \\
&= (x-y)(x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}) + (x-y)y^n \\
&= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n)
\end{aligned}$$

The resulting relation concludes the finite induction, thus $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi) We know from (iv) that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, by letting $a = xandb = -y$ we get $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

1.2. Multiplying by the multiplicative inverse of $x - y$ works only when $x - y \neq 0$, that is $x \neq y$, however, the hypothesis explicitly states $x = y$. So it is not possible to find the multiplicative inverse of $x - y$ and thus the step is invalid.

1.3.

(i) Say we have $\frac{a}{b}$ and $b \neq 0$ then the same fraction can be written as ab^{-1}. Suppose we also have a variable c such that $c \neq 0$, then we have $ab^{-1} \cdot (cc^{-1})$ and consequently $(ac)(b^{-1}c^{-1}) = \frac{ac}{bc}$. The final equality holds by (iii) which is proven below.

(ii) By (i) $\frac{ad}{bd} + \frac{bc}{db} = ad(bd)^{-1} + bc(bd)^{-1} = (ad+bc)(bd)^{-1} = \frac{ad+bc}{bd}$

(iii) ab exists if $a, b \neq 0$. Let $x = (ab)^{-1}$, then

$$\begin{aligned}
&x(ab) = (ab)^{-1}(ab) = (xa)b = 1 && \texttt{(Multiply } x \texttt{ with } ab\texttt{)} \\
&(xa)(bb^{-1}) = b^{-1} = xa = b^{-1} && \texttt{(Multiply by } b^{-1}\texttt{)} \\
&x(aa^{-1}) = b^{-1}a^{-1} = x && \texttt{(Multiply by } a^{-1}\texttt{)}
\end{aligned}$$

(iv) Suppose $b, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1}) \cdot (cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(v) Suppose $b, c, d \neq 0$, then $\frac{a}{b}/\frac{c}{d} = (ab^{-1})(cd^{-1})^{-1} = (ab^{-1})(c^{-1}d) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(vi) Suppose $b, d \neq 0$. Assume $\frac{a}{b} = \frac{c}{d}$, multiplying by bd on both side yields the relation $ad = bc$. For the converse multiply $ad = bc$ by $(bd)^{-1}$.

1.4.

(i) $4 - x < 3 - 2x \iff (4-4) + (-x+2x) < (3-4) + (2x-2x) \iff x < -1$.

(ii) $5 - x^2 < 8 \iff -3 < x^2$. Note that $x^2 \geq 0$ and for every single value of x, so our solution is every x.

(iii) $5 - x^2 < -2 \iff 7 < x^2 \iff \sqrt{7} < x$ or $-\sqrt{7} > x$.

(iv) The product is positive when $x-1 > 0$ and $x-3 > 0$ or when $x-1 < 0$ and $x-3 < 0$, that is when $x > 3$ or when $x < 1$.

(v) Complete the square $x^2 - 2x + 2 = (x-1)^2 + 1$. The product $(x-1)^2$ is always positive, and since we have the $+1$ as well in the inequality, this inequality must be true for every single x.

(vi) The inequality is equivalent to $x^2 + x - 1 > 0$. Completing the square $(x + \frac{1}{2})^2 > \frac{5}{4}$. If $x \geq -\frac{1}{2}$ then $x > \frac{-1+\sqrt{5}}{2}$. If $x < -\frac{1}{2}$ then $x < \frac{-1-\sqrt{5}}{2}$. Thus, the solution is $x > \frac{-1+\sqrt{5}}{2}$ and $x < \frac{-1-\sqrt{5}}{2}$.

(vii) Equivalently we have $(x - \frac{1}{2})^2 > \frac{25}{4}$. If $x \geq \frac{1}{2}$ then $x > 3$ if $x < \frac{1}{2}$ then $x < -2$. The solution set is $x > 3$ and $x < -2$.

(viii) Equivalently $(x+\frac{1}{2})^2 + \frac{3}{4} > 0$. This is true for every x because $(x+\frac{1}{2}) \geq$ and $\frac{3}{4} > 0$. Adding them gives $(x+\frac{1}{2})^2 + \frac{3}{4} > 0$.

(ix) Let $b = (x+5)(x-3)$. Then b is positive if $x > 3$ or $x < -5$ and negative if $-5 < x < 3$. Let $a = x - \pi$. a is positive if $x > \pi$. ab is positive if both a and b are positive or if both are negative. So ab is positive if $x > \pi$ (b must be positive because $x > 3$). ab is negative if $-5 < x < 3$ (This implies $x < \pi$).

(x) If $x > \sqrt[3]{2}$ and $x > \sqrt{2}$ then the product is positive, thus the first solution is $x > \sqrt{2}$. If $x < \sqrt[3]{2}$ and $x < \sqrt{2}$ then the product is positive. The second solution is $x < \sqrt[3]{2}$.

(xi) Apply $\log_2$ on both sides: $x < 3$.

(xii) Suppose $x < 1$, we will show this is a solution. We have $3^x < 3^1 = 3$, adding $x < 1$ to the inequality we get $x + 3^x < 3 + 1 = 4$. Since both 3^x and x are strictly increasing expressions finding the inequality $x < 1$ suffices as all real solutions.

(xiii) Noting that $x \neq 0$ and $x \neq 1$. Expanding the fractions we get $\frac{1-x}{x(1-x))} + \frac{x}{x(1-x)} = \frac{1}{x(1-x)} > 0$. The solutions depends on if the denominator is positive. Thus $x(1-x) > 0$ has the same solution set. The solutions are $0 < x < 1$.

(xiv) Note $x \neq -1$. Expand by $(x+1)$: $\frac{(x-1)(x+1)}{(x+1)^2} > 0$. Since the denominator is always positive we can multiply this on both sides, $x^2 - 1 > 0$, Thus $x < -1$ and $x > 1$.