Spivak's Calculus Solutions

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Contents

Chapter 1

Basic Properties of Numbers

1.1. (i) *Suppose that* $ax = a$ *and* $a \neq 0$, *then there exists a number* a^{-1}. *Multiplying* a^{-1} *on both sides yields*

$$\begin{aligned}(a^{-1}a) \cdot x &= a^{-1}a \\ x &= 1\end{aligned}$$

as desired.

(ii) *We use the distributive property on* $(x - y)(x + y)$, *this can be done by letting* $a = x - y$*:*

$$\begin{aligned}&(x - y)(x + y) = a(x + y) \\ &= ax + ay = (x - y)x + (x - y)y \\ &= x^2 - yx + xy - y^2 = x^2 - y^2\end{aligned}$$

(iii) *If we have* $x^2 = y^2$ *then we certainly have* $x^2 - y^2 = 0$. *By (ii) we know that* $0 = (x - y)(x + y)$, *this implies that* $x - y = 0$ *or* $x + y = 0$, *this is equivalent to saying that* $x = y$ *or* $x = -y$.

(iv) *Same method as (ii):*

$$\begin{aligned}&a(x^2 + xy + y^2) = ax^2 + axy + ay^2 \\ &= (x - y)x^2 + (x - y)xy + (x - y)y^2 \\ &= x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3 \\ &= x^3 - y^3\end{aligned}$$

(v) *We prove this by induction, the base case $n = 2$ is already proven in (ii). Suppose $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})$ is true. Then we equivalently have $x^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})+y^n$. We now prove the n+1 case:*

$$\begin{aligned}
&x^{n+1} - y^{n+1} = x \cdot x^n - y^{n+1} \\
&= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + xy^n - y^{n+1} \\
&= (x-y)(x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}) + (x-y)y^n \\
&= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n)
\end{aligned}$$

The resulting relation concludes the finite induction, thus $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi) *We know from (iv) that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, by letting $a = x$ and $b = -y$ we get $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.*

1.2. *Multiplying by the multiplicative inverse of $x-y$ works only when $x-y \neq 0$, that is $x \neq y$, however, the hypothesis explicitly states $x = y$. So it is not possible to find the multiplicative inverse of $x-y$ and thus the step is invalid.*

1.3. (i) *Say we have $\frac{a}{b}$ and $b \neq 0$ then the same fraction can be written as ab^{-1}. Suppose we also have a variable c such that $c \neq 0$, then we have $ab^{-1} \cdot (cc^{-1})$ and consequently $(ac)(b^{-1}c^{-1}) = \frac{ac}{bc}$. The final equality holds by (iii) which is proven below.*

(ii) *By (i)* $\frac{ad}{bd} + \frac{bc}{db} = ad(bd)^{-1} + bc(bd)^{-1} = (ad+bc)(bd)^{-1} = \frac{ad+bc}{bd}$

(iii) *ab exists if $a, b \neq 0$. Let $x = (ab)^{-1}$, then*

$$\begin{aligned}
&x(ab) = (ab)^{-1}(ab) = (xa)b = 1 && \textit{(Multiply } x \textit{ with } ab\textit{)} \\
&(xa)(bb^{-1}) = b^{-1} = xa = b^{-1} && \textit{(Multiply by } b^{-1}\textit{)} \\
&x(aa^{-1}) = b^{-1}a^{-1} = x && \textit{(Multiply by } a^{-1}\textit{)}
\end{aligned}$$

(iv) *Suppose $b, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1}) \cdot (cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$*

(v) *Suppose $b, c, d \neq 0$, then $\frac{a}{b} / \frac{c}{d} = (ab^{-1})(cd^{-1})^{-1} = (ab^{-1})(c^{-1}d) = (ac)(bd)^{-1} = \frac{ac}{bd}$*

(vi) *Suppose $b, d \neq 0$. Assume $\frac{a}{b} = \frac{c}{d}$, multiplying by bd on both side yields the relation $ad = bc$. For the converse multiply $ad = bc$ by $(bd)^{-1}$.*

1.4. (i) $4-x < 3-2x \iff (4-4)+(-x+2x) < (3-4)+(2x-2x) \iff x < -1$.

(ii) $5 - x^2 < 8 \iff -3 < x^2$. *Note that $x^2 \geq 0$ and for every single value of x, so our solution is every x.*

(iii) $5 - x^2 < -2 \iff 7 < x^2 \iff \sqrt{7} < x$ *or* $-\sqrt{7} > x$.

(iv) *The product is positive when $x - 1 > 0$ and $x - 3 > 0$ or when $x - 1 < 0$ and $x - 3 < 0$, that is when $x > 3$ or when $x < 1$.*

(v) *Complete the square $x^2 - 2x + 2 = (x-1)^2 + 1$. The product $(x-1)^2$ is always positive, and since we have the $+1$ as well in the inequality, this inequality must be true for every single x.*

(vi) *The inequality is equivalent to $x^2 + x - 1 > 0$. Completing the square $(x + \frac{1}{2})^2 > \frac{5}{4}$. If $x \geq -\frac{1}{2}$ then $x > \frac{-1+\sqrt{5}}{2}$. If $x < -\frac{1}{2}$ then $x < \frac{-1-\sqrt{5}}{2}$. Thus, the solution is $x > \frac{-1+\sqrt{5}}{2}$ and $x < \frac{-1-\sqrt{5}}{2}$.*

(vii) *Equivalently we have $(x - \frac{1}{2})^2 > \frac{25}{4}$. If $x \geq \frac{1}{2}$ then $x > 3$ if $x < \frac{1}{2}$ then $x < -2$. The solution set is $x > 3$ and $x < -2$.*

(viii) *Equivalently $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$. This is true for every x because $(x + \frac{1}{2}) \geq$ and $\frac{3}{4} > 0$. Adding them gives $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$.*

(ix) *Let $b = (x+5)(x-3)$. Then b is positive if $x > 3$ or $x < -5$ and negative if $-5 < x < 3$. Let $a = x - \pi$. a is positive if $x > \pi$. ab is positive if both a and b are positive or if both are negative. So ab is positive if $x > \pi$ (b must be positive because $x > 3$). ab is negative if $-5 < x < 3$ (This implies $x < \pi$).*

(x) *If $x > \sqrt[3]{2}$ and $x > \sqrt{2}$ then the product is positive, thus the first solution is $x > \sqrt{2}$. If $x < \sqrt[3]{2}$ and $x < \sqrt{2}$ then the product is positive. The second solution is $x < \sqrt[3]{2}$.*

(xi) *Apply $\log_2$ on both sides: $x < 3$.*

(xii) *Suppose $x < 1$, we will show this is a solution. We have $3^x < 3^1 = 3$, adding $x < 1$ to the inequality we get $x + 3^x < 3 + 1 = 4$. Since both*

3^x and x are strictly increasing expressions finding the inequality $x < 1$ suffices as all real solutions.

(xiii) Noting that $x \neq 0$ and $x \neq 1$. Expanding the fractions we get $\frac{1-x}{x(1-x))} + \frac{x}{x(1-x)} = \frac{1}{x(1-x)} > 0$. The solutions depends on if the denominator is positive. Thus $x(1-x) > 0$ has the same solution set. The solutions are $0 < x < 1$.

(xiv) Note $x \neq -1$. Expand by $(x+1)$: $\frac{(x-1)(x+1)}{(x+1)^2} > 0$. Since the denominator is always positive we can multiply this on both sides, $x^2 - 1 > 0$, Thus $x < -1$ and $x > 1$.

1.5. *(i) Suppose $a < b$ and $c < d$ then we have $b-a > 0$ and $d-c > 0$ by property 11 $(b-a)+(d-c) > 0$ which is the same as $b+d > a+c$.*

(ii) Suppose $a < b$ then $0 < b-a \iff -b < (b-b)-a = -b < -a$.

(iii) Suppose $a < b$ and $c < d$, by (ii): $-c < -d$, then by (i) we have $a-d < b-d$.

(iv) Suppose $a < b$ then $b-a > 0$. Assume $c > 0$, Using (P12) we know that $c(b-a) > 0$ and consequently $bc - ac > 0 \iff bc > ac$.

(v) Suppose $a < b$ then $b-a > 0$. Assume $c < 0$, then by (ii) we have $-c > 0$. Using P12 we know that $-c(b-a) > 0$ and consequently $ac - bc > 0 \iff ac > bc$.

(vi) Since $a > 1 > 0$ we apply (iv) by letting $c = a$. Thus $a^2 > a$.

(vii) Because a is positive, it follows by applying (iv) to $a < 1$ that $a^2 < a$.

(viii) Using (iv), multiply $a < b$ with c and $c < d$ with b. This means that we have $ac < bc$ and $bc < bd$, this is the same as $ac < bc < bd$, thus $ac < bd$.

(ix) Using (viii) we multiply the same inequality twice, $a^2 < b^2$.

(x) Suppose $a, b \geq 0$, we prove the contra-positive, therefore $a \geq b$. Multiply by a and b respectively gives two inequalities $a^2 \geq ab$ and $ab \geq b^2$ which is the same as $a^2 \geq ab \geq b^2$. This concludes the contra-positive proof because $a^2 \geq b^2$ is the logical opposite of $a^2 < b^2$.

1.6. (a) *The base case is* $n = 2$ *which was proven in problem 1.5. Assume* $x^n < y^n$ *for* $0 \leq x < y$*. By problem 1.6. (viii) we have* $x \cdot x^n < y \cdot y^n \iff x^{n+1} < y^{n+1}$*. The induction is complete, thus if* $0 \leq x, y$ *then* $x^n < y^n$ *for* $n = 1, 2, \ldots$*.*

(b) *Suppose* $x < y$ *and* $n = 2k + 1$*, We have three cases.*

(i) $x, y \geq 0$*, this case has been proven in (a).*

(ii) $x \leq 0$ *and* $y \geq 0$*. Consider* x^n*, because* n *is odd, it has the following property,* $x^{2k+1} = x \cdot (x^k)^2 < 0$*, because* x *is negative and* $(x^k)^2$ *is positive. However* $y^n > 0$ *because* y *is positive. This means we have* $x^n < 0 < y^n$*.*

(iii) $x, y < 0$*, by the inequality we have* $-x > 0$ *and* $-y > 0$*. We also have* $-y < -x$*, by (a) we have* $(-y)^n < (-x)^n \iff -y^n < -x^n$ *because* n *is odd. Finally we have* $x^n < y^n$*.*

(c) *Suppose* $x^n = y^n \iff x^n - y^n = 0 = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ *Then either* $x - y = 0$ *or* $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} = 0$ *In the first case* $x = y$*, in the second case we first note that* $x^n = y^n$ *implies that* x *and* y *has the same sign and thus* $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} \geq 0$ *where the equality holds only when* $x, y = 0$ *then* $x = y$ *is still true.*

(d) *Let* n *be an even positive integer. Next we prove the contra-positive, suppose* $|x| \neq |y|$ *(*$x = y$ *or* $x = -y$ *is the same as saying* $|x| = |y|$*). Consequently this means either* $|x| < |y|$ *or* $|x| > |y|$*. By (a) this means that either* $|x|^n < |y|^n$ *or* $|x|^n > |y|^n$*. Because* n *is even this is equivalent to* $x^n < y^n$ *or* $x^n > y^n$ *which is the logical complement of* $x^n = y^n$*.*

1.7. *Suppose* $0 < a < b$*, multiply by* a *then* $a^2 < ab \iff a < \sqrt{ab}$*. Next consider* $(a-b)^2 > 0$ *which is equivalent to* $a^2 + b^2 + 2ab > 4ab \iff \frac{a+b}{2} > \sqrt{ab}$*, this means that we have* $a < \sqrt{ab} < \frac{a+b}{2}$ *now remains the final inequality. By the premise we have* $b - a > 0 \iff b + a > 2a \iff \frac{b+a}{2} > a$*. We conclude by stating* $a < \sqrt{ab} < \frac{a+b}{2} < b$*.*

1.8. (P10) *Let* $b = 0$ *in P'10, then for every* a *one of the following properties apply*

(i) $a = 0$

(ii) $a < 0$

(iii) $a > 0$

Because the collection P *contains all the numbers* x *such that* $x > 0$*, we can see that (iii) states that* a *belongs to* P*. (ii) is equivalent to* $-a > 0$*, thus -a is in* P*.*

(P11) Suppose x *and* y *are in* P *then* $0 < x$ *and* $0 < y$*. By P'12 (Let a=0) we have* $x < y + x$*. By P'11 we get* $0 < y + x$ *which is in* P*.*

(P12) Suppose x *and* y *are in* P *then* $0 < x$ *and* $0 < y$*. Using P'13 we get* $0 < xy$*, this means that* xy *is in* P*.*

1.9. *(i)* $\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$.

(ii) Triangle inequality states that $|a + b| - |a| - |b| \leq 0$*. Therefore* $|a| + |b| - |a + b|$.

(iii) Triangle inequality gives $|(a + b) + c| - |a + b| - |c| \leq 0 \iff |a + b| + |c| - |a + b + c|. \geq 0$*. Our solution is therefore* $|a + b| + |c| - |a + b + c|$.

(iv) $x^2 - 2xy + y^2 = (x - y)^2 \geq 0$*, thus* $x^2 - 2xy + y^2$.

(v) $\sqrt{2} + \sqrt{3} + \sqrt{5} - \sqrt{7}$

1.10. *(i) Suppose* $a + b \geq 0$ *and* $b \geq 0$ *then we have* $a + b - b = a$*. Suppose* $a + b \geq 0$ *and* $b < 0$ *then* $a + b + b = a + 2b$*. Suppose* $a + b < 0$ *and* $b \geq 0$ *then* $-a - b - b = -(a + 2b)$*. Suppose* $a + b < 0$ *and* $b < 0$ *then* $-a - b + b = -a$.

(ii) If $0 \geq x \geq 1$ *then* $1 - x$*. If* $-1 \geq x < 0$ *then* $1 + x$*. If* $1 < x$ *then* $x - 1$ *then* $-x - 1$.

(iii) If $x \geq 0$ *then* $x - x^2$*, if* $x < 0$ *then* $-x - x^2$.

(iv) If $a \geq 0$ *then* a*, if* $a < 0$ *then* $3a$.

1.11. *(i) Suppose* $x - 3 > 0$ *then* $x - 3 = 8 \iff x = 11$*. Suppose* $x - 3 < 0$ *then* $3 - x = 8 \iff x = -5$.

(ii) Suppose $x - 3 \geq 0$ *then* $3 \leq x < 11$*. Suppose* $x - 3 < 0$ *then* $-5 < x < 3$*. Combining both inequalities* $-5 < x < 11$.

(iii) *Suppose* $x+4 \geq 0$ *then* $x < -2$, *so* $-4 \leq x < -2$. *If* $x+4 < 0$ *then* $-6 < x < -4$. *Combining both inequalities gives* $-6 < x < -2$.

(iv) *Suppose* $x \leq 2$ *then* $x-1+x-2 > 1 \iff x > 2$. *This means* $x > 2$ *is always a solution. Suppose* $1 \leq x < 2$, *then* $x-1-x+2 > 1 \iff 1 > 1$, *which can not be true. Suppose* $x < 1$, *then* $1-x-x+2 > 1 \iff x < 1$. *The solution is* $x < 1$ *and* $x > 2$.

(v) *Suppose* $x \geq 1$ *then* $x-1+x+1 < 2 \iff x < 1$ *which is a contradiction. Suppose* $-1 \leq x < 1$ *then* $1-x+x+1 < 2 \iff 2 < 2$, *also contradiction. Suppose* $x < -1$ *then* $1-x-x-1 < 2 \iff x > -1$, *an* x *that satisfies the inequality is nonexistent.*

(vi) *Suppose* $x \geq 1$ *then* $x-1+x+1 < 1 \iff x < \frac{1}{2}$ *which is a contradiction. Suppose* $-1 \leq x < 1$ *then* $1-x+x+1 < 1 \iff 2 < 1$,*also a contradiction. Suppose* $x < -1$ *then* $1-x-x-1 < 1 \iff x > -\frac{1}{2}$, *similarly to (iv), there are no* x *that satisfy the inequality.*

(vii) *We have* $x-1=0 \iff x=1$ *or* $x+1=0 \iff x=-1$.

(viii) *Suppose* $x \geq 1$ *then* $(x-1)(x+2)=3 \iff x^2+x-5=0 \iff (x+\frac{1}{2})^2 = \frac{21}{4} \implies x = \frac{-1+\sqrt{21}}{2}$. *Suppose* $-2 \leq x < 1$ *then* $(1-x)(x+2)=3$ *which is a polynomial with complex roots thus no solutions there. Suppose* $x < -2$, *then we get the same polynomial as in the first case because* $(-1)^2 = 1$, *so the other root is* $x = \frac{-1-\sqrt{21}}{2}$ *which is less than* -2 *because* $\frac{-1-\sqrt{21}}{2} < \frac{-1-\sqrt{16}}{2} = \frac{-5}{2} < -2$. *To conclude* $x = \frac{-1\pm\sqrt{21}}{2}$

1.12. (i) $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 \iff |xy| = |x| \cdot |y|$

(ii) *Consider* $\left|\frac{1}{x}\right|$ *for* $x \neq 0$. *This is the same as* $\sqrt{(\frac{1}{x})^2} = \sqrt{\frac{1}{x^2}} = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$.

(iii) *Suppose* $y \neq 0$ *then* $\left|\frac{x}{y}\right| = \sqrt{(\frac{x}{y})^2} = \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|}$

(iv) *Suppose* a, b *are real numbers, then the triangle inequality is* $|a+b| \leq |a|+|b|$. *Let* $a = x$ *and* $b = -y$ *then* $|x-y| \leq |x| + |-y| = |x| + |y|$. *The final equality is proven by* $|-y| = \sqrt{(-y)^2} = \sqrt{(-1)^2y^2} = \sqrt{y^2}$.

(v) *Using the triangle inequality* $|x| = |(x-y)+y| \leq |x-y| + |y| \iff |x| - |y| \leq |x-y|$

(vi) *There are two cases from the inequality,* $|x|-|y| \leq |x-y|$ *and* $|y|-|x| \leq |y-x|$, *note that the last absolute value comes from the fact* $|x-y| = |y-x|$. *Both inequalities are identical to (v) (the second inequality has the variables interchanged).*

(vii) *We have* $|(x+y)+z| \leq |x+y|+|z| \leq |x|+|y|+|z|$. *Doing the case work for the equality is left to the reader.*

1.13. *We start by proving for* max, *let* $x \geq y$ *then* $\max(x,y) = \frac{x+y+x-y}{2} = x$ *Likewise if* $y \geq x$ *then* $\max(x,y) = y$. *Similar reasoning shows that the formula for* $\min(x,y)$ *is valid. Next we use substitution and get* $\max(x,y,z) = \max(x, \max(y,z)) = \frac{y+z+2x+|y-z|+|y+z+2x+|y-z||}{4}$ *and* $\min(x,y,z) = \min(x, \min(y,z)) = \frac{y+z+2x+|y-z|-|y+z+2x+|y-z||}{4}$.

1.14.

(a) *Suppose* $a \geq 0$ *then we have* $a = -(-a)$. *The case for* $a \leq 0$ *is then obvious because we have* $(-a) \geq 0$ *which can be used on the previously proven fact.*

(b) $(\Rightarrow)$ *Suppose* $-b \leq a \leq b$, *this implies* $a \leq b$ *and* $-b \leq a \iff -a \leq b$ *and consequently* $|a| \leq b$.
$(\Leftarrow)$ *Suppose* $|a| \leq b$ *then* $a \leq b$ *and* $-a \leq b \iff -b \leq a$, *thus* $-b \leq a \leq b$. *Now we prove the last part. Suppose* $|a| \leq |a|$ *then by the previously proven theorem we have* $-|a| \leq a \leq |a|$.

(c) *As proven earlier, for every* a, b *we have* $-|a| \leq a \leq |a|$ *and* $-|b| \leq b \leq |b|$. *Add these together gives* $-(|a|+|b|) \leq a+b \leq |a|+|b|$, *applying the theorem from (b) on* $(|a|+|b|)$ *and* $(a+b)$ *we get* $|a+b| \leq |a|+|b|$.

1.15. *We prove first that if* $x = y$ *and* $x, y \neq 0$. *The inequality is then* $x^2+x^2+x^2 > 0 \iff x^2 > 0$ *which is true because* $x \neq 0$.

Suppose $x \neq y$, *then the left side of inequality is equivalent to* $(x^2+xy+y^2) = \frac{x^3-y^3}{(x-y)}$. *Suppose* $x > y$ *then* $x^3 - y^3 > 0$ *by problem 6 (b), since both the numerator and denominator are positive we know that* $\frac{x^3-y^3}{(x-y)} > 0$. *Next we assume* $x < y$ *which implies* $x^3 - y^3 < 0$ *by problem 6 (b). This means the numerator and denominator are both negative, thus* $\frac{x^3-y^3}{(x-y)} > 0$. *In every case the inequality is positive, thus we have proven that* $x^2+xy+y^2 > 0$.

To prove that the second inequality holds we follow the same steps, suppose $x = y$ *which means the inequality is* $5x^4 > 0$. *Next suppose*

$x \neq y$ then we have $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5-y^5}{x-y}$. Suppose $x - y > 0$ then $x^5 - y^5 > 0$ which implies $\frac{x^5-y^5}{x-y} > 0$. Assume $x - y < 0$ then $x^5 - y^5 < 0$ which implies $\frac{x^5-y^5}{x-y} > 0$.

1.16. *(a) $(x+y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 \iff xy = 0$ which implies $x = 0$ or $y = 0$.*