Spivak's Calculus Solutions

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Contents

Chapter 1

Basic Properties of Numbers

1.1.

(i) Suppose that $ax = a$ and $a \neq 0$, then there exists a number a^{-1}. Multiplying a^{-1} on both sides yields

$$\begin{aligned}(a^{-1}a) \cdot x &= a^{-1}a \\ x &= 1\end{aligned}$$

as desired.

(ii) Applying the distributive property on $(x-y)(x+y)$ makes

$$\begin{aligned}&(x-y)(x+y) = (x-y)x + (x-y)y \\ &= x^2 - yx + xy - y^2 = x^2 - y^2\end{aligned}$$

(iii) If we have $x^2 = y^2$ then we certainly have $0 = x^2 - y^2$. By (ii) we have $0 = (x-y)(x+y)$, this implies that $x - y = 0$ or $x + y = 0$, this is equivalent to saying that $x = y$ or $x = -y$.

(iv) Same method as (ii):

$$\begin{aligned}&(x-y)(x^2 + xy + y^2) = (x-y)x^2 + (x-y)xy + (x-y)y^2 \\ &= x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3 \\ &= x^3 - y^3\end{aligned}$$

(v) We prove this by induction, the base case $n = 2$ is already proven in (ii). Suppose $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ is true. Then we equivalently have $x^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})+y^n$. We now prove the n+1 case:

$$\begin{aligned} x^{n+1} - y^{n+1} &= x \cdot x^n - y^{n+1} \\ &= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + xy^n - y^{n+1} \\ &= (x-y)(x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}) + (x-y)y^n \\ &= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n) \end{aligned}$$

The resulting relation concludes the finite induction, thus $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi) We know from (iv) that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, by letting $a = x$ and $b = -y$ we get $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

1.2. Multiplying by the multiplicative inverse of $x-y$ works only when $x-y \neq 0$, that is $x \neq y$, however, the hypothesis explicitly states $x = y$. So it is not possible to find the multiplicative inverse of $x-y$ and thus the step is invalid.

1.3. (i) Say we have $\frac{a}{b}$ and $b \neq 0$ then the same fraction can be written as ab^{-1}. Suppose we also have a variable c such that $c \neq 0$, then we have $ab^{-1} \cdot (cc^{-1})$ and consequently $(ac)(b^{-1}c^{-1}) = \frac{ac}{bc}$. The final equality holds by (iii) which is proven below.

(ii) By (i) $\frac{ad}{bd} + \frac{bc}{db} = ad(bd)^{-1} + bc(bd)^{-1} = (ad+bc)(bd)^{-1} = \frac{ad+bc}{bd}$

(iii) ab exists if $a, b \neq 0$. Let $x = (ab)^{-1}$, then

$$\begin{aligned} &x(ab) = (ab)^{-1}(ab) = (xa)b = 1 && \text{(Multiply } x \text{ with } ab) \\ &(xa)(bb^{-1}) = b^{-1} = xa = b^{-1} && \text{(Multiply by } b^{-1}) \\ &x(aa^{-1}) = b^{-1}a^{-1} = x && \text{(Multiply by } a^{-1}) \end{aligned}$$

(iv) Suppose $b, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1}) \cdot (cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(v) Suppose $b, c, d \neq 0$, then $\frac{a}{b} \big/ \frac{c}{d} = (ab^{-1})(cd^{-1})^{-1} = (ab^{-1})(c^{-1}d) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(vi) Suppose $b, d \neq 0$. Assume $\frac{a}{b} = \frac{c}{d}$, multiplying by bd on both side yields the relation $ad = bc$. For the converse multiply $ad = bc$ by $(bd)^{-1}$.

1.4. (i) $4-x<3-2x \iff (4-4)+(-x+2x)<(3-4)+(2x-2x) \iff x<-1$.

(ii) $5-x^2<8 \iff -3<x^2$. Note that $x^2 \geq 0$ and for every single value of x, so our solution is every x.

(iii) $5-x^2<-2 \iff 7<x^2 \iff \sqrt{7}<x$ or $-\sqrt{7}>x$.

(iv) The product is positive when $x-1>0$ and $x-3>0$ or when $x-1<0$ and $x-3<0$, that is when $x>3$ or when $x<1$.

(v) Complete the square $x^2-2x+2=(x-1)^2+1$. The product $(x-1)^2$ is always positive, and since we have the $+1$ as well in the inequality, this inequality must be true for every single x.

(vi) The inequality is equivalent to $x^2+x-1>0$. Completing the square $(x+\frac{1}{2})^2>\frac{5}{4}$. If $x \geq -\frac{1}{2}$ then $x>\frac{-1+\sqrt{5}}{2}$. If $x<-\frac{1}{2}$ then $x<\frac{-1-\sqrt{5}}{2}$. Thus, the solution is $x>\frac{-1+\sqrt{5}}{2}$ and $x<\frac{-1-\sqrt{5}}{2}$.

(vii) Equivalently we have $(x-\frac{1}{2})^2>\frac{25}{4}$. If $x \geq \frac{1}{2}$ then $x>3$ if $x<\frac{1}{2}$ then $x<-2$. The solution set is $x>3$ and $x<-2$.

(viii) Equivalently $(x+\frac{1}{2})^2+\frac{3}{4}>0$. This is true for every x because $(x+\frac{1}{2}) \geq$ and $\frac{3}{4}>0$. Adding them gives $(x+\frac{1}{2})^2+\frac{3}{4}>0$.

(ix) Let $b=(x+5)(x-3)$. Then b is positive if $x>3$ or $x<-5$ and negative if $-5<x<3$. Let $a=x-\pi$. a is positive if $x>\pi$. ab is positive if both a and b are positive or if both are negative. So ab is positive if $x>\pi$ (b must be positive because $x>3$). ab is negative if $-5<x<3$ (This implies $x<\pi$).

(x) If $x>\sqrt[3]{2}$ and $x>\sqrt{2}$ then the product is positive, thus the first solution is $x>\sqrt{2}$. If $x<\sqrt[3]{2}$ and $x<\sqrt{2}$ then the product is positive. The second solution is $x<\sqrt[3]{2}$.

(xi) Apply $\log_2$ on both sides: $x<3$.

(xii) Suppose $x<1$, we will show this is a solution. We have $3^x<3^1=3$, adding $x<1$ to the inequality we get $x+3^x<3+1=4$. Since both 3^x and x are strictly increasing expressions finding the inequality $x<1$ suffices as all real solutions.

(xiii) Noting that $x \neq 0$ and $x \neq 1$. Expanding the fractions we get $\frac{1-x}{x(1-x))} + \frac{x}{x(1-x)} = \frac{1}{x(1-x)} > 0$. The solutions depends on if the denominator is positive. Thus $x(1-x) > 0$ has the same solution set. The solutions are $0 < x < 1$.

(xiv) Note $x \neq -1$. Expand by $(x+1)$: $\frac{(x-1)(x+1)}{(x+1)^2} > 0$. Since the denominator is always positive we can multiply this on both sides, $x^2 - 1 > 0$, Thus $x < -1$ and $x > 1$.

1.5. (i) Suppose $a < b$ and $c < d$ then we have $b - a > 0$ and $d - c > 0$ by property 11 $(b-a) + (d-c) > 0$ which is the same as $b + d > a + c$.

(ii) Suppose $a < b$ then $0 < b - a \iff -b < (b-b) - a = -b < -a$.

(iii) Suppose $a < b$ and $c < d$, by (ii): $-c < -d$, then by (i) we have $a - d < b - d$.

(iv) Suppose $a < b$ then $b - a > 0$. Assume $c > 0$, Using (P12) we know that $c(b-a) > 0$ and consequently $bc - ac > 0 \iff bc > ac$.

(v) Suppose $a < b$ then $b - a > 0$. Assume $c < 0$, then by (ii) we have $-c > 0$. Using P12 we know that $-c(b-a) > 0$ and consequently $ac - bc > 0 \iff ac > bc$.

(vi) Since $a > 1 > 0$ we apply (iv) by letting $c = a$. Thus $a^2 > a$.

(vii) Because a is positive, it follows by applying (iv) to $a < 1$ that $a^2 < a$.

(viii) Using (iv), multiply $a < b$ with c and $c < d$ with b. This means that we have $ac < bc$ and $bc < bd$, this is the same as $ac < bc < bd$, thus $ac < bd$.

(ix) Using (viii) we multiply the same inequality twice, $a^2 < b^2$.

(x) Suppose $a, b \geq 0$, we prove the contra-positive, therefore $a \geq b$. Multiply by a and b respectively gives two inequalities $a^2 \geq ab$ and $ab \geq b^2$ which is the same as $a^2 \geq ab \geq b^2$. This concludes the contra-positive proof because $a^2 \geq b^2$ is the logical opposite of $a^2 < b^2$.

1.6. (a) The base case is $n = 2$ which was proven in problem 1.5. Assume $x^n < y^n$ for $0 \leq x < y$. By problem 1.6. (viii) we have $x \cdot x^n < y \cdot y^n \iff x^{n+1} < y^{n+1}$. The induction is complete, thus if $0 \leq x, y$ then $x^n < y^n$ for $n = 1, 2, \ldots$.

(b) Suppose $x < y$ and $n = 2k+1$, We have three cases.

(i) $x, y \geq 0$, this case has been proven in (a).

(ii) $x \leq 0$ and $y \geq 0$. Consider x^n, because n is odd, it has the following property, $x^{2k+1} = x \cdot (x^k)^2 < 0$, because x is negative and $(x^k)^2$ is positive. However $y^n > 0$ because y is positive. This means we have $x^n < 0 < y^n$.

(iii) $x, y < 0$, by the inequality we have $-x > 0$ and $-y > 0$. We also have $-y < -x$, by (a) we have $(-y)^n < (-x)^n \iff -y^n < -x^n$ because n is odd. Finally we have $x^n < y^n$.

(c) Suppose $x^n = y^n \iff x^n - y^n = 0 = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ Then either $x - y = 0$ or $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} = 0$ In the first case $x = y$, in the second case we first note that $x^n = y^n$ implies that x and y has the same sign and thus $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} \geq 0$ where the equality holds only when $x, y = 0$ then $x = y$ is still true.

(d) Let n be an even positive integer. Next we prove the contra-positive, suppose $|x| \neq |y|$ ($x = y$ or $x = -y$ is the same as saying $|x| = |y|$). Consequently this means either $|x| < |y|$ or $|x| > |y|$. By (a) this means that either $|x|^n < |y|^n$ or $|x|^n > |y|^n$. Because n is even this is equivalent to $x^n < y^n$ or $x^n > y^n$ which is the logical complement of $x^n = y^n$.

1.7. Suppose $0 < a < b$, multiply by a then $a^2 < ab \iff a < \sqrt{ab}$. Next consider $(a-b)^2 > 0$ which is equivalent to $a^2 + b^2 + 2ab > 4ab \iff \frac{a+b}{2} > \sqrt{ab}$, this means that we have $a < \sqrt{ab} < \frac{a+b}{2}$ now remains the final inequality. By the premise we have $b - a > 0 \iff b + a > 2a \iff \frac{b+a}{2} > a$. We conclude by stating $a < \sqrt{ab} < \frac{a+b}{2} < b$.

***1.8.** (P10) Let $b = 0$ in P'10, then for every a one of the following properties apply

(i) $a = 0$

(ii) $a < 0$

(iii) $a > 0$

Because the collection P contains all the numbers x such that $x > 0$, we can see that (iii) states that a belongs to P. (ii) is equivalent to $-a > 0$, thus -a is in P.

(P11) Suppose x and y are in P then $0 < x$ and $0 < y$. By P'12 (Let a=0) we have $x < y + x$. By P'11 we get $0 < y + x$ which is in P.

(P12) Suppose x and y are in P then $0 < x$ and $0 < y$. Using P'13 we get $0 < xy$, this means that xy is in P.

1.9. (i) $\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$.

(ii) Triangle inequality states that $|a+b| - |a| - |b| \leq 0$. Therefore $|a| + |b| - |a+b|$.

(iii) Triangle inequality gives $|(a+b)+c| - |a+b| - |c| \leq 0 \iff |a+b| + |c| - |a+b+c|. \geq 0$. Our solution is therefore $|a+b| + |c| - |a+b+c|$.

(iv) $x^2 - 2xy + y^2 = (x-y)^2 \geq 0$, thus $x^2 - 2xy + y^2$.

(v) $\sqrt{2} + \sqrt{3} + \sqrt{5} - \sqrt{7}$

1.10. (i) Suppose $a + b \geq 0$ and $b \geq 0$ then we have $a + b - b = a$. Suppose $a + b \geq 0$ and $b < 0$ then $a + b + b = a + 2b$. Suppose $a + b < 0$ and $b \geq 0$ then $-a - b - b = -(a + 2b)$. Suppose $a + b < 0$ and $b < 0$ then $-a - b + b = -a$.

(ii) If $0 \geq x \geq 1$ then $1 - x$. If $-1 \geq x < 0$ then $1 + x$. If $1 < x$ then $x - 1$ then $-x - 1$.

(iii) If $x \geq 0$ then $x - x^2$, if $x < 0$ then $-x - x^2$.

(iv) If $a \geq 0$ then a, if $a < 0$ then $3a$.

1.11. (i) Suppose $x - 3 > 0$ then $x - 3 = 8 \iff x = 11$. Suppose $x - 3 < 0$ then $3 - x = 8 \iff x = -5$.

(ii) Suppose $x - 3 \geq 0$ then $3 \leq x < 11$. Suppose $x - 3 < 0$ then $-5 < x < 3$. Combining both inequalities $-5 < x < 11$.

(iii) Suppose $x + 4 \geq 0$ then $x < -2$, so $-4 \leq x < -2$. If $x + 4 < 0$ then $-6 < x < -4$. Combining both inequalities gives $-6 < x < -2$.

(iv) Suppose $x \leq 2$ then $x - 1 + x - 2 > 1 \iff x > 2$. This means $x > 2$ is always a solution. Suppose $1 \leq x < 2$, then $x - 1 - x + 2 > 1 \iff 1 > 1$, which can not be true. Suppose $x < 1$, then $1 - x - x + 2 > 1 \iff x < 1$. The solution is $x < 1$ and $x > 2$.

(v) Suppose $x \geq 1$ then $x-1+x+1 < 2 \iff x < 1$ which is a contradiction. Suppose $-1 \leq x < 1$ then $1 - x + x + 1 < 2 \iff 2 < 2$, also contradiction. Suppose $x < -1$ then $1 - x - x - 1 < 2 \iff x > -1$, an x that satisfies the inequality is nonexistent.

(vi) Suppose $x \geq 1$ then $x-1+x+1 < 1 \iff x < \frac{1}{2}$ which is a contradiction. Suppose $-1 \leq x < 1$ then $1 - x + x + 1 < 1 \iff 2 < 1$,also a contradiction. Suppose $x < -1$ then $1 - x - x - 1 < 1 \iff x > -\frac{1}{2}$, similarly to (iv), there are no x that satisfy the inequality.

(vii) We have $x - 1 = 0 \iff x = 1$ or $x + 1 = 0 \iff x = -1$.

(viii) Suppose $x \geq 1$ then $(x-1)(x+2) = 3 \iff x^2+x-5 = 0 \iff (x+\frac{1}{2})^2 = \frac{21}{4} \implies x = \frac{-1+\sqrt{21}}{2}$. Suppose $-2 \leq x < 1$ then $(1-x)(x+2) = 3$ which is a polynomial with complex roots thus no solutions there. Suppose $x < -2$, then we get the same polynomial as in the first case because $(-1)^2 = 1$, so the other root is $x = \frac{-1-\sqrt{21}}{2}$ which is less than -2 because $\frac{-1-\sqrt{21}}{2} < \frac{-1-\sqrt{16}}{2} = \frac{-5}{2} < -2$. To conclude $x = \frac{-1\pm\sqrt{21}}{2}$

1.12. (i) $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 \iff |xy| = |x| \cdot |y|$

(ii) Consider $\left|\frac{1}{x}\right|$ for $x \neq 0$. This is the same as $\sqrt{(\frac{1}{x})^2} = \sqrt{\frac{1}{x^2}} = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$.

(iii) Suppose $y \neq 0$ then $\left|\frac{x}{y}\right| = \sqrt{(\frac{x}{y})^2} = \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|}$

(iv) Suppose a, b are real numbers, then the triangle inequality is $|a + b| \leq |a| + |b|$. Let $a = x$ and $b = -y$ then $|x - y| \leq |x| + |-y| = |x| + |y|$. The final equality is proven by $|-y| = \sqrt{(-y)^2} = \sqrt{(-1)^2y^2} = \sqrt{y^2}$.

(v) Using the triangle inequality $|x| = |(x - y) + y| \leq |x - y| + |y| \iff |x| - |y| \leq |x - y|$

(vi) There are two cases from the inequality, $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |y - x|$, note that the last absolute value comes from the fact $|x - y| = |y - x|$. Both inequalities are identical to (v) (the second inequality has the variables interchanged).

(vii) We have $|(x + y) + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$. Doing the case work for the equality is left to the reader.

1.13. We start by proving for max, let $x \geq y$ then $\max(x, y) = \frac{x+y+x-y}{2} = x$ Likewise if $y \geq x$ then $\max(x, y) = y$. Similar reasoning shows that the formula for $\min(x, y)$ is valid. Next we use substitution and get $\max(x, y, z) = \max(x, \max(y, z)) = \frac{y+z+2x+|y-z|+|y+z+2x+|y-z||}{4}$ and $\min(x, y, z) = \min(x, \min(y, z)) = \frac{y+z+2x+|y-z|-|y+z+2x+|y-z||}{4}$.

1.14.

(a) Suppose $a \geq 0$ then we have $a = -(-a)$. The case for $a \leq 0$ is then obvious because we have $(-a) \geq 0$ which can be used on the previously proven fact.

(b) ($\Rightarrow$) Suppose $-b \leq a \leq b$, this implies $a \leq b$ and $-b \leq a \iff -a \leq b$ and consequently $|a| \leq b$.
($\Leftarrow$) Suppose $|a| \leq b$ then $a \leq b$ and $-a \leq b \iff -b \leq a$, thus $-b \leq a \leq b$. Now we prove the last part. Suppose $|a| \leq |a|$ then by the previously proven theorem we have $-|a| \leq a \leq |a|$.

(c) As proven earlier, for every a, b we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Add these together gives $-(|a| + |b|) \leq a + b \leq |a| + |b|$, applying the theorem from (b) on $(|a| + |b|)$ and $(a + b)$ we get $|a + b| \leq |a| + |b|$.

***1.15.** We prove first that if $x = y$ and $x, y \neq 0$. The inequality is then $x^2 + x^2 + x^2 > 0 \iff x^2 > 0$ which is true because $x \neq 0$.

Suppose $x \neq y$, then the left side of inequality is equivalent to $(x^2 + xy + y^2) = \frac{x^3-y^3}{(x-y)}$. Suppose $x > y$ then $x^3 - y^3 > 0$ by problem 6 (b), since both the numerator and denominator are positive we know that $\frac{x^3-y^3}{(x-y)} > 0$. Next we assume $x < y$ which implies $x^3 - y^3 < 0$ by problem 6 (b). This means the numerator and denominator are both negative, thus $\frac{x^3-y^3}{(x-y)} > 0$. In every case the inequality is positive, thus we have proven that $x^2 + xy + y^2 > 0$.

To prove that the second inequality holds we follow the same steps, suppose $x = y$ which means the inequality is $5x^4 > 0$. Next suppose $x \neq y$ then we have $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5-y^5}{x-y}$. Suppose $x - y > 0$ then $x^5 - y^5 > 0$ which implies $\frac{x^5-y^5}{x-y} > 0$. Assume $x - y < 0$ then $x^5 - y^5 < 0$ which implies $\frac{x^5-y^5}{x-y} > 0$.

***1.16.**

(a) $(x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 \iff xy = 0$ which implies $x = 0$ or $y = 0$. Next we have $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 \iff x^2y + xy^2 = 0 = xy(x + y)$. Which implies either $x = 0$ or $y = 0$ or $x = -y$.

(b) Consider $3(x+y)^2 = 3x^2+6xy+3y^3 \geq 0$, since $x, y \neq 0$ we have $x^2 > 0$ and $y^2 > 0$, adding these inequalities makes $4x^2 + 6xy + 4y^2 > 0$. If $x, y = 0$ then the statement would be false.

(c) Equivalently we have $4x^3y + 6x^2y^2 + 4y^3x = xy(4x^2 + 6xy + 4y^2)$, left side indicates that it is equal to zero when $x = 0$ or $y = 0$. Thus $(x + y)^4 = x^4 + y^4$ when $x = 0$ or $y = 0$.

(d) Subtract with $x^5 + y^5$ and since $xy \neq 0$ we divide by $5xy$ this makes $x^3+2x^2y+2xy^2+y^3 = 0 \iff (x+y)^3 = x^2y+y^2x = xy(x+y)$. Suppose $x+y \neq 0$ then $xy = (x+y)^2 \iff x^2+xy+y^2 = 0$, this implies $x, y = 0$ by letting $p = x^2+xy+y^2 \iff 2p = 2x^2+2xy+2y^2 = x^2+y^2+(x+y)^2$, it then follows all the terms have to be zero because they are either zero or positive, $x = 0$ and $y = 0$, this contradicts the fact that $xy = 0$, thus it must be the case that $x = -y$.

Assume this time that $x = 0$ then $(x + y)^5 = x^5 + y^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \iff y^5 = y^5$. By interchanging x with y in the last sentence it follows that $x = 0$ or $y = 0$. To conclude, the solutions are $x = -y$ or $x = 0$ or $y = 0$. My guess is that the same solutions apply to $(x + y)^n = x^n + y^n$ if n is odd and $x = 0$ or $y = 0$ if n is even.

1.17. (a) $2x^2 - 3x + 4 = 2(x - \frac{3}{4})^2 + y \implies y = \frac{32}{8} - \frac{9}{8} = \frac{23}{8}$

(b) Subtract $2(y + 1)^2$ this makes $x^2 - 3x$. Let $x^2 - 3x = (x - \frac{3}{2}) + z$ then $z = -\frac{9}{4}$, z is the smallest value.

(c) Let m be the minimum number for a simple second degree polynomial, then it follows that $x^2+bx+c = 0 = (x+\frac{b}{2})^2+m = x^2+bx+\frac{b^2}{4}+m \iff m = c - \frac{b^2}{4}$

We have $x^2 + 4xy + 5y^2 - 4x - 6y + 7 = x^2 + (4y - 4)x + 5y^2 - 6y + 7$ The minimum is thus $m = 5y^2 - 6y + 7 - 4(y^2 - 2y + 1) = y^2 + 2y + 3 = (y + 1)^2 + 2$. This implies that 2 is in fact the minimum value.

1.18. (a) $x = \frac{-b\pm\sqrt{b^2-4c}}{2} \iff (2x+b)^2 = b^2 - 4c \iff 4x^2+4xb+b^2-b^2+4c = 0 \iff x^2 + bx + c = 0$.

(b) We complete the square, $x^2 + bx + c = 0 \iff 4(x + \frac{b}{2})^2 = b^2 - 4c$ this follows that $(x + \frac{b}{2})^2 \geq 0$, but $b^2 - 4c < 0$ which is a contradiction. It also follows that $x^2 + bx + c > 0$ which means there are no real values of x that satisfy the equation.

(c) We complete the square $(x + \frac{y}{2})^2 + \frac{3y^2}{4}$. Since $\frac{3y^2}{4} > 0$ because $y \neq 0$ it must be the case that $(x + \frac{y}{2})^2 + \frac{3y^2}{4} > 0$ which is the same as $x^2 + xy + y^2 > 0$

(d) Completing the square makes $(x + \frac{\alpha y}{2})^2 + y^2(1 - \frac{\alpha^2}{4})$. The left term has the property $(x + \frac{\alpha y}{2})^2 \geq 0$ (just let $x = -\frac{\alpha y}{2}$). This means the right term must be positive. Let $1 - \frac{\alpha^2}{4} > 0$ which implies $-2 < \alpha < 2$.

(e) $ax^2 + bx + c = a(x^2 + \frac{bx}{a}) + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a^2}$. Since $a > 0$ the minimum must be when $x + \frac{b}{a} = 0$, so the minimum is $c - \left(\frac{b}{2a}\right)^2$. (The first case is just $a = 1$)

1.19. (a) Suppose $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ then the equality holds if $\lambda({y_1}^2 + {y_2}^2) = \sqrt{\lambda^2({y_1}^2 + {y_2}^2)}\sqrt{({y_1}^2 + {y_2}^2)} \iff \lambda = |\lambda|$. Seems to be some kind of error (edition 3) because it does not hold if λ is negative. Let's assume $\lambda \geq 0$. The then equality holds. The equality also holds if $y_1 = y_2 = 0$ because both factors on both sides are equal to zero.

Assume y_1 and y_2 is not equal to zero. Then there does not exist a λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, the problems states that this implies $\lambda^2({y_1}^2 + {y_2}^2) - 2\lambda(x_1 y_1 + x_2 y_2) + ({x_1}^2 + {x_2}^2) > 0$. This equation is of the form $\lambda^2 + b\lambda + c > 0$ and since there does not exist any λ we have $b^2 < 4ac$ by noticing that dividing by a in the equation $ax^2 + bx + c = 0$ you can apply problem 18 (b), that is $(x_1 y_1 + x_2 y_2)^2 < ({y_1}^2 + {y_2}^2)({x_1}^2 + {x_2}^2)$. This follows that $|x_1 y_1 + x_2 y_2| < \sqrt{{y_1}^2 + {y_2}^2}\sqrt{{x_1}^2 + {x_2}^2}$

To conclude we have

$$x_1 y_1 + x_2 y_2 \leq |x_1 y_1 + x_2 y_2| \leq \sqrt{{y_1}^2 + {y_2}^2}\sqrt{{x_1}^2 + {x_2}^2}.$$

(b) We start with $(x - y)^2 \geq 0 \iff 2xy \leq x^2 + y^2$. Suppose $x_1, x_2, y_1, y_2 \neq 0$ and let $x = \frac{x_i}{\sqrt{{x_1}^2 + {x_2}^2}}$, $y = \frac{y_i}{\sqrt{{y_1}^2 + {y_2}^2}}$ for $i = 1, 2$. It follows that

$$\begin{cases} \dfrac{2x_1 y_1}{\sqrt{{x_1}^2 + {x_2}^2}\sqrt{{y_1}^2 + {y_2}^2}} \leq \dfrac{{x_1}^2}{{x_1}^2 + {x_2}^2} + \dfrac{{y_1}^2}{{y_1}^2 + {y_2}^2} \\ \dfrac{2x_2 y_2}{\sqrt{{x_1}^2 + {x_2}^2}\sqrt{{y_1}^2 + {y_2}^2}} \leq \dfrac{{x_2}^2}{{x_1}^2 + {x_2}^2} + \dfrac{{y_2}^2}{{y_1}^2 + {y_2}^2} \end{cases}$$

Add both inequalities together, then it follows that $x_1y_1 + x_2y_2 \le \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}$.

If we assume $x_i = 0$ or $y_i = 0$ for $i = 1, 2$ then either all the terms will be zero or the resulting inequality is for example $0 \le y_1{}^2$ (let $x_1 = 0$).

(c) $(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2)$

$$
\begin{aligned}
&= (x_1y_1)^2 + 2(x_1y_1)(x_2y_2) + (x_2y_2)^2 + (x_2y_1)^2 - 2(x_2y_1)(x_1y_2) + (x_1y_2)^2 \\
&= (x_1y_1 + x_2y_2)^2 + (x_2y_1 - x_1y_2)^2 \ge (x_1y_1 + x_2y_2)^2 \\
&\iff \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2} \ge |x_1y_1 + x_2y_2| \ge x_1y_1 + x_2y_2
\end{aligned}
$$

(d) The problem is constructed to waste time, see (a) where we already proved it. It shows that if $y_1 = 0$ and $y_2 = 0$ or there exists a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ then the equality holds, otherwise $|x_1y_1 + x_2y_2| < \sqrt{y_1{}^2 + y_2{}^2}\sqrt{x_1{}^2 + x_2{}^2}$.

1.20. Add both inequalities, $|x - x_0| + |y - y_0| < \varepsilon$. We apply the triangle inequality which makes $|(x + y) - (x_0 + y_0)| \le |x - x_0| + |y - y_0| < \varepsilon$. For the second inequality, notice that that $|y - y_0| = |y_0 - y|$. So the triangle inequality makes $|(x - y) - (x_0 - y_0)| \le |x - x_0| + |y_0 - y| < \varepsilon$.

***1.21.** Suppose $|x - x_0| < \frac{\varepsilon}{2(|y_0|+1)}$, then $2|x - x_0|(|y_0| + 1) < \varepsilon$. Now assume $|y - y_0| < \frac{\varepsilon}{2(|y_0|+1)}$ then $2|y - y_0|(|x_0| + 1) < \varepsilon$. Sum the two similar inequalities

$$
\begin{aligned}
&2|x - x_0|(|y_0| + 1) + 2|y - y_0|(|x_0| + 1) < 2\varepsilon \\
&|x - x_0|(|y_0| + 1) + |y - y_0|(|x_0| + 1) < \varepsilon \\
&|y_0||x - x_0| + |x - x_0| + |x_0||y - y_0| + |y - y_0| < \varepsilon
\end{aligned}
$$

Now suppose $|x - x_0| < 1$ then we have $|y - y_0||x - x_0| < |y - y_0|$. Continuing on the expression above we get

$$
\begin{aligned}
&> |y_0||x - x_0| + |x_0||y - y_0| + |y - y_0| \\
&> (|y_0| + |y - y_0|)(|x - x_0|) + |x_0||y - y_0| \\
&\ge |y||x - x_0| + |x_0||y - y_0| \ge |xy - x_0y + x_0y - x_0y_0| = |xy - x_0y_0|
\end{aligned}
$$

Therefore we have $|xy - x_0y_0| < \varepsilon$.

***1.22.** We first prove that $y \neq 0$. Suppose $|y - y_0| < \frac{|y_0|}{2}$ then by problem 12, we get $|y_0| < 2|y|$ by problem 12. By supposing $y = 0$ we get a contradiction because $0 < |y_0|$ thus it must be the case that $y \neq 0$.

Now we prove the latter. Suppose $|y - y_0| < \frac{\varepsilon |y_0|^2}{2}$. Then

$$|y - y_0| < \varepsilon |y_0||y|$$
$$\left| \frac{y_0 - y}{y_0 y} \right| < \varepsilon$$
$$\left| \frac{1}{y_0} - \frac{1}{y} \right| < \varepsilon$$

as desired.

***1.23.** We begin first by using problem 21. We can then state that if $y \neq 0$, $y_0 \neq 0$, $\left|\frac{1}{y} - \frac{1}{y_0}\right| < \frac{\varepsilon}{2(|x_0|+1)}$ and $|x - x_0| < \min\left(\frac{\varepsilon}{2(\left|\frac{1}{y_0}\right|+1)}, 1\right)$ then we have $\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon$. Now we need to modify the hypothesis. We have that $y_0 \neq 0$ and $|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right)$ implies $y \neq 0$ and the hypothesis earlier.

To conclude, $y_0 = 0$, $|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon|y_0|^2}{2}\right)$ and $|x - x_0| < \min\left(\frac{\varepsilon}{2(\left|\frac{1}{y_0}\right|+1)}, 1\right)$ implies $y \neq 0$ and $\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon$.

***1.24.**

(a) We prove the base case (k=2) with the associative law, $(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$. Next we suppose $P(k)$: $(a_1 + \cdots + a_k) + a_{k+1} = a_1 + \cdots + a_{k+1}$, then we prove for $P(k+1)$:

$$(a_1 + \cdots + a_{k+1}) + a_{k+2} = [(a_1 + \cdots + a_k) + a_{k+1}] + a_{k+2}$$
$$(a_1 + \cdots + a_k) + (a_{k+1} + a_{k+2}) = a_1 + \cdots + k_{k+2}$$

This concludes the induction.

(b) We will prove this by induction on n, suppose $n \geq k$ and $(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n$. The base case is $n = k+1$ which was proven in the previous problem. We will now show the equality holds for $n+1$, we have

$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_{n+1})$
$= (a_1 + \cdots + a_k) + ((a_{k+1} + \cdots + a_n) + a_{n+1})$
$= ((a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n)) + a_{n+1}$
$= (a_1 + \cdots + a_n) + a_{n+1}$
$= a_1 + \cdots + a_{n+1}$

We have now proven that for $n \geq k$ it follows that

$$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n.$$

(c) We will show that $s(a_1, \ldots, a_k) = s(a_1) + \cdots + s(a_k)$ by induction on k. Let the base case be $k = 1$, then we obviously have an equality. Now we assume $s(a_1, \ldots, a_k) = s(a_1) + \cdots + s(a_k)$ and now prove for the $k + 1$ case.

$s(a_1, \ldots, a_{k+1}) = s(a_1, \ldots, a_k) + s(a_{k+1})$
$= s(a_1) + \cdots + s(a_{k+1})$

Because $s(a_1) + \cdots + s(a_k) = a_1 + \cdots + a_k$, our proof is done.

1.25. We suppose the rules of addition and multiplication given in the problem we then prove it is a field.

(i) Testing each case is tedious and will not be contained here, but we find that $a + (b + c) = (a + b) + c$ works.

(ii) Suppose $a = 0$ then $0+0 = 0+0 = 0$, and $a = 1$ implies $1+0 = 0+1 = 0$

(iii) If $a = 0$ then then let $-a = 0$ and if $a = 1$ then $-a = 1$.

(iv) This works by exhaustion.

(v) If at least one variable is zero, then $0 = 0$, otherwise $1\cdot(1\cdot) = (1\cdot1)\cdot1 \iff 1 = 1$

(vi) Suppose $a = 0$ then $1 \cdot 0 = 1 \cdot 0 = 0$, suppose $a = 1$ then $1 \cdot 1 = 1 \cdot 1 = 1$

(vii) $a = 0$ is not allowed so we only prove for the $a = 1$ case which makes $a^{-1} = 1$.

(viii) If at least one variable is equal to zero then we have $0 = 0$, otherwise $1 \cdot 1 = 1 \cdot 1$

(ix) Suppose $a = 0$ then $0 \cdot (b + c) = 0 \cdot b + 0 \cdot c = 0$. Suppose $a = 1$ then $1 \cdot (b + c) = 1 \cdot b + 1 \cdot c = b + c$

Chapter 2

Numbers of various sorts