Spivak's Calculus Solutions

Sebastian Miles

Contents

Chapter 1

Basic Properties of Numbers

1.1.

(i) Suppose that $ax = a$ and $a \neq 0$, then there exists a number a^{-1}. Multiplying a^{-1} on both sides yields

$$\begin{aligned}(a^{-1}a) \cdot x &= a^{-1}a \\ x &= 1\end{aligned}$$

as desired.

(ii) Applying the distributive property on $(x-y)(x+y)$ makes

$$\begin{aligned}&(x-y)(x+y) = (x-y)x + (x-y)y \\ &= x^2 - yx + xy - y^2 = x^2 - y^2\end{aligned}$$

(iii) If we have $x^2 = y^2$ then we certainly have $0 = x^2 - y^2$. By (ii) we have $0 = (x-y)(x+y)$, this implies that $x - y = 0$ or $x + y = 0$, this is equivalent to saying that $x = y$ or $x = -y$.

(iv) Same method as (ii):

$$\begin{aligned}&(x-y)(x^2+xy+y^2) = (x-y)x^2 + (x-y)xy + (x-y)y^2 \\ &= x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3 \\ &= x^3 - y^3\end{aligned}$$

(v) We prove this by induction, the base case $n = 2$ is already proven in (ii). Suppose $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ is true. Then we equivalently have $x^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})+y^n$. We now prove the n+1 case:

$$\begin{aligned}
x^{n+1} - y^{n+1} &= x \cdot x^n - y^{n+1} \\
&= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + xy^n - y^{n+1} \\
&= (x-y)(x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}) + (x-y)y^n \\
&= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n)
\end{aligned}$$

The resulting relation concludes the finite induction, thus $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi) We know from (iv) that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, by letting $a = x$ and $b = -y$ we get $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

1.2. Multiplying by the multiplicative inverse of $x-y$ works only when $x-y \neq 0$, that is $x \neq y$, however, the hypothesis explicitly states $x = y$. So it is not possible to find the multiplicative inverse of $x-y$ and thus the step is invalid.

1.3. (i) Say we have $\frac{a}{b}$ and $b \neq 0$ then the same fraction can be written as ab^{-1}. Suppose we also have a variable c such that $c \neq 0$, then we have $ab^{-1} \cdot (cc^{-1})$ and consequently $(ac)(b^{-1}c^{-1}) = \frac{ac}{bc}$. The final equality holds by (iii) which is proven below.

(ii) By (i) $\frac{ad}{bd} + \frac{bc}{db} = ad(bd)^{-1} + bc(bd)^{-1} = (ad+bc)(bd)^{-1} = \frac{ad+bc}{bd}$

(iii) ab exists if $a, b \neq 0$. Let $x = (ab)^{-1}$, then

$$\begin{aligned}
&x(ab) = (ab)^{-1}(ab) = (xa)b = 1 && \text{(Multiply } x \text{ with } ab\text{)} \\
&(xa)(bb^{-1}) = b^{-1} = xa = b^{-1} && \text{(Multiply by } b^{-1}\text{)} \\
&x(aa^{-1}) = b^{-1}a^{-1} = x && \text{(Multiply by } a^{-1}\text{)}
\end{aligned}$$

(iv) Suppose $b, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1}) \cdot (cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(v) Suppose $b, c, d \neq 0$, then $\frac{a}{b} \big/ \frac{c}{d} = (ab^{-1})(cd^{-1})^{-1} = (ab^{-1})(c^{-1}d) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(vi) Suppose $b, d \neq 0$. Assume $\frac{a}{b} = \frac{c}{d}$, multiplying by bd on both side yields the relation $ad = bc$. For the converse multiply $ad = bc$ by $(bd)^{-1}$.

1.4. (i) $4-x<3-2x \iff (4-4)+(-x+2x)<(3-4)+(2x-2x) \iff x<-1$.

(ii) $5-x^2<8 \iff -3<x^2$. Note that $x^2 \geq 0$ and for every single value of x, so our solution is every x.

(iii) $5-x^2<-2 \iff 7<x^2 \iff \sqrt{7}<x$ or $-\sqrt{7}>x$.

(iv) The product is positive when $x-1>0$ and $x-3>0$ or when $x-1<0$ and $x-3<0$, that is when $x>3$ or when $x<1$.

(v) Complete the square $x^2-2x+2=(x-1)^2+1$. The product $(x-1)^2$ is always positive, and since we have the $+1$ as well in the inequality, this inequality must be true for every single x.

(vi) The inequality is equivalent to $x^2+x-1>0$. Completing the square $(x+\frac{1}{2})^2>\frac{5}{4}$. If $x \geq -\frac{1}{2}$ then $x>\frac{-1+\sqrt{5}}{2}$. If $x<-\frac{1}{2}$ then $x<\frac{-1-\sqrt{5}}{2}$. Thus, the solution is $x>\frac{-1+\sqrt{5}}{2}$ and $x<\frac{-1-\sqrt{5}}{2}$.

(vii) Equivalently we have $(x-\frac{1}{2})^2>\frac{25}{4}$. If $x \geq \frac{1}{2}$ then $x>3$ if $x<\frac{1}{2}$ then $x<-2$. The solution set is $x>3$ and $x<-2$.

(viii) Equivalently $(x+\frac{1}{2})^2+\frac{3}{4}>0$. This is true for every x because $(x+\frac{1}{2}) \geq$ and $\frac{3}{4}>0$. Adding them gives $(x+\frac{1}{2})^2+\frac{3}{4}>0$.

(ix) Let $b=(x+5)(x-3)$. Then b is positive if $x>3$ or $x<-5$ and negative if $-5<x<3$. Let $a=x-\pi$. a is positive if $x>\pi$. ab is positive if both a and b are positive or if both are negative. So ab is positive if $x>\pi$ (b must be positive because $x>3$). ab is negative if $-5<x<3$ (This implies $x<\pi$).

(x) If $x>\sqrt[3]{2}$ and $x>\sqrt{2}$ then the product is positive, thus the first solution is $x>\sqrt{2}$. If $x<\sqrt[3]{2}$ and $x<\sqrt{2}$ then the product is positive. The second solution is $x<\sqrt[3]{2}$.

(xi) Apply $\log_2$ on both sides: $x<3$.

(xii) Suppose $x<1$, we will show this is a solution. We have $3^x<3^1=3$, adding $x<1$ to the inequality we get $x+3^x<3+1=4$. Since both 3^x and x are strictly increasing expressions finding the inequality $x<1$ suffices as all real solutions.

(xiii) Noting that $x \neq 0$ and $x \neq 1$. Expanding the fractions we get $\frac{1-x}{x(1-x))} + \frac{x}{x(1-x)} = \frac{1}{x(1-x)} > 0$. The solutions depends on if the denominator is positive. Thus $x(1-x) > 0$ has the same solution set. The solutions are $0 < x < 1$.

(xiv) Note $x \neq -1$. Expand by $(x+1)$: $\frac{(x-1)(x+1)}{(x+1)^2} > 0$. Since the denominator is always positive we can multiply this on both sides, $x^2 - 1 > 0$, Thus $x < -1$ and $x > 1$.

1.5. (i) Suppose $a < b$ and $c < d$ then we have $b - a > 0$ and $d - c > 0$ by property 11 $(b-a) + (d-c) > 0$ which is the same as $b + d > a + c$.

(ii) Suppose $a < b$ then $0 < b - a \iff -b < (b-b) - a = -b < -a$.

(iii) Suppose $a < b$ and $c < d$, by (ii): $-c < -d$, then by (i) we have $a - d < b - d$.

(iv) Suppose $a < b$ then $b - a > 0$. Assume $c > 0$, Using (P12) we know that $c(b-a) > 0$ and consequently $bc - ac > 0 \iff bc > ac$.

(v) Suppose $a < b$ then $b - a > 0$. Assume $c < 0$, then by (ii) we have $-c > 0$. Using P12 we know that $-c(b-a) > 0$ and consequently $ac - bc > 0 \iff ac > bc$.

(vi) Since $a > 1 > 0$ we apply (iv) by letting $c = a$. Thus $a^2 > a$.

(vii) Because a is positive, it follows by applying (iv) to $a < 1$ that $a^2 < a$.

(viii) Using (iv), multiply $a < b$ with c and $c < d$ with b. This means that we have $ac < bc$ and $bc < bd$, this is the same as $ac < bc < bd$, thus $ac < bd$.

(ix) Using (viii) we multiply the same inequality twice, $a^2 < b^2$.

(x) Suppose $a, b \geq 0$, we prove the contra-positive, therefore $a \geq b$. Multiply by a and b respectively gives two inequalities $a^2 \geq ab$ and $ab \geq b^2$ which is the same as $a^2 \geq ab \geq b^2$. This concludes the contra-positive proof because $a^2 \geq b^2$ is the logical opposite of $a^2 < b^2$.

1.6. (a) The base case is $n = 2$ which was proven in problem 1.5. Assume $x^n < y^n$ for $0 \leq x < y$. By problem 1.5. (viii) we have $x \cdot x^n < y \cdot y^n \iff x^{n+1} < y^{n+1}$. The induction is complete, thus if $0 \leq x, y$ then $x^n < y^n$ for $n = 1, 2, \ldots$.

(b) Suppose $x < y$ and $n = 2k+1$, We have three cases.

(i) $x, y \geq 0$, this case has been proven in (a).

(ii) $x \leq 0$ and $y \geq 0$. Consider x^n, because n is odd, it has the following property, $x^{2k+1} = x \cdot (x^k)^2 < 0$, because x is negative and $(x^k)^2$ is positive. However $y^n > 0$ because y is positive. This means we have $x^n < 0 < y^n$.

(iii) $x, y < 0$, by the inequality we have $-x > 0$ and $-y > 0$. We also have $-y < -x$, by (a) we have $(-y)^n < (-x)^n \iff -y^n < -x^n$ because n is odd. Finally we have $x^n < y^n$.

(c) Suppose $x^n = y^n \iff x^n - y^n = 0 = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ Then either $x - y = 0$ or $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} = 0$ In the first case $x = y$, in the second case we first note that $x^n = y^n$ implies that x and y has the same sign and thus $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} \geq 0$ where the equality holds only when $x, y = 0$ then $x = y$ is still true.

(d) Let n be an even positive integer. Next we prove the contra-positive, suppose $|x| \neq |y|$ ($x = y$ or $x = -y$ is the same as saying $|x| = |y|$). Consequently this means either $|x| < |y|$ or $|x| > |y|$. By (a) this means that either $|x|^n < |y|^n$ or $|x|^n > |y|^n$. Because n is even this is equivalent to $x^n < y^n$ or $x^n > y^n$ which is the logical complement of $x^n = y^n$.

1.7. Suppose $0 < a < b$, multiply by a then $a^2 < ab \iff a < \sqrt{ab}$. Next consider $(a-b)^2 > 0$ which is equivalent to $a^2 + b^2 + 2ab > 4ab \iff \frac{a+b}{2} > \sqrt{ab}$, this means that we have $a < \sqrt{ab} < \frac{a+b}{2}$ now remains the final inequality. By the premise we have $a - b < 0 \iff a + b < 2b \iff \frac{a+b}{2} < b$. We conclude by stating $a < \sqrt{ab} < \frac{a+b}{2} < b$.

***1.8.** (P10) Let $b = 0$ in P'10, then for every a one of the following properties apply

(i) $a = 0$

(ii) $a < 0$

(iii) $a > 0$

Because the collection P contains all the numbers x such that $x > 0$, we can see that (iii) states that a belongs to P. (ii) is equivalent to $-a > 0$, thus -a is in P.

(P11) Suppose x and y are in P then $0 < x$ and $0 < y$. By P'12 (Let a=0) we have $x < y + x$. By P'11 we get $0 < y + x$ which is in P.

(P12) Suppose x and y are in P then $0 < x$ and $0 < y$. Using P'13 we get $0 < xy$, this means that xy is in P.

1.9. (i) $\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$.

(ii) Triangle inequality states that $|a+b| - |a| - |b| \leq 0$. Therefore $|a| + |b| - |a+b|$.

(iii) Triangle inequality gives $|(a+b)+c| - |a+b| - |c| \leq 0 \iff |a+b| + |c| - |a+b+c|. \geq 0$. Our solution is therefore $|a+b| + |c| - |a+b+c|$.

(iv) $x^2 - 2xy + y^2 = (x-y)^2 \geq 0$, thus $x^2 - 2xy + y^2$.

(v) $\sqrt{2} + \sqrt{3} + \sqrt{5} - \sqrt{7}$

1.10. (i) Suppose $a + b \geq 0$ and $b \geq 0$ then we have $a + b - b = a$. Suppose $a + b \geq 0$ and $b < 0$ then $a + b + b = a + 2b$. Suppose $a + b < 0$ and $b \geq 0$ then $-a - b - b = -(a + 2b)$. Suppose $a + b < 0$ and $b < 0$ then $-a - b + b = -a$.

(ii) If $0 \geq x \geq 1$ then $1 - x$. If $-1 \geq x < 0$ then $1 + x$. If $1 < x$ then $x - 1$ then $-x - 1$.

(iii) If $x \geq 0$ then $x - x^2$, if $x < 0$ then $-x - x^2$.

(iv) If $a \geq 0$ then a, if $a < 0$ then $3a$.

1.11. (i) Suppose $x - 3 > 0$ then $x - 3 = 8 \iff x = 11$. Suppose $x - 3 < 0$ then $3 - x = 8 \iff x = -5$.

(ii) Suppose $x - 3 \geq 0$ then $3 \leq x < 11$. Suppose $x - 3 < 0$ then $-5 < x < 3$. Combining both inequalities $-5 < x < 11$.

(iii) Suppose $x + 4 \geq 0$ then $x < -2$, so $-4 \leq x < -2$. If $x + 4 < 0$ then $-6 < x < -4$. Combining both inequalities gives $-6 < x < -2$.

(iv) Suppose $x \leq 2$ then $x - 1 + x - 2 > 1 \iff x > 2$. This means $x > 2$ is always a solution. Suppose $1 \leq x < 2$, then $x - 1 - x + 2 > 1 \iff 1 > 1$, which can not be true. Suppose $x < 1$, then $1 - x - x + 2 > 1 \iff x < 1$. The solution is $x < 1$ and $x > 2$.

(v) Suppose $x \geq 1$ then $x-1+x+1 < 2 \iff x < 1$ which is a contradiction. Suppose $-1 \leq x < 1$ then $1-x+x+1 < 2 \iff 2 < 2$, also contradiction. Suppose $x < -1$ then $1-x-x-1 < 2 \iff x > -1$, an x that satisfies the inequality is nonexistent.

(vi) Suppose $x \geq 1$ then $x-1+x+1 < 1 \iff x < \frac{1}{2}$ which is a contradiction. Suppose $-1 \leq x < 1$ then $1-x+x+1 < 1 \iff 2 < 1$,also a contradiction. Suppose $x < -1$ then $1-x-x-1 < 1 \iff x > -\frac{1}{2}$, similarly to (iv), there are no x that satisfy the inequality.

(vii) We have $x-1=0 \iff x=1$ or $x+1=0 \iff x=-1$.

(viii) Suppose $x \geq 1$ then $(x-1)(x+2)=3 \iff x^2+x-5=0 \iff (x+\frac{1}{2})^2 = \frac{21}{4} \implies x = \frac{-1+\sqrt{21}}{2}$. Suppose $-2 \leq x < 1$ then $(1-x)(x+2)=3$ which is a polynomial with complex roots thus no solutions there. Suppose $x < -2$, then we get the same polynomial as in the first case because $(-1)^2 = 1$, so the other root is $x = \frac{-1-\sqrt{21}}{2}$ which is less than -2 because $\frac{-1-\sqrt{21}}{2} < \frac{-1-\sqrt{16}}{2} = \frac{-5}{2} < -2$. To conclude $x = \frac{-1\pm\sqrt{21}}{2}$

1.12. (i) $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 \iff |xy| = |x| \cdot |y|$

(ii) Consider $\left|\frac{1}{x}\right|$ for $x \neq 0$. This is the same as $\sqrt{(\frac{1}{x})^2} = \sqrt{\frac{1}{x^2}} = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$.

(iii) Suppose $y \neq 0$ then $\left|\frac{x}{y}\right| = \sqrt{(\frac{x}{y})^2} = \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|}$

(iv) Suppose a, b are real numbers, then the triangle inequality is $|a+b| \leq |a|+|b|$. Let $a = x$ and $b = -y$ then $|x-y| \leq |x| + |-y| = |x| + |y|$. The final equality is proven by $|-y| = \sqrt{(-y)^2} = \sqrt{(-1)^2y^2} = \sqrt{y^2}$.

(v) Using the triangle inequality $|x| = |(x-y)+y| \leq |x-y| + |y| \iff |x| - |y| \leq |x-y|$

(vi) There are two cases from the inequality, $|x| - |y| \leq |x-y|$ and $|y| - |x| \leq |y-x|$, note that the last absolute value comes from the fact $|x-y| = |y-x|$. Both inequalities are identical to (v) (the second inequality has the variables interchanged).

(vii) We have $|(x+y)+z| \leq |x+y| + |z| \leq |x|+|y|+|z|$. Doing the case work for the equality is left to the reader.

1.13. We start by proving for max, let $x \geq y$ then $\max(x, y) = \frac{x+y+x-y}{2} = x$ Likewise if $y \geq x$ then $\max(x, y) = y$. Similar reasoning shows that the formula for $\min(x, y)$ is valid. Next we use substitution and get $\max(x, y, z) = \max(x, \max(y, z)) = \frac{y+z+2x+|y-z|+|y+z+2x+|y-z||}{4}$ and $\min(x, y, z) = \min(x, \min(y, z)) = \frac{y+z+2x+|y-z|-|y+z+2x+|y-z||}{4}$.

1.14. (a) Suppose $a \geq 0$ then we have $a = -(-a)$. The case for $a \leq 0$ is then obvious because we have $(-a) \geq 0$ which can be used on the previously proven fact.

(b) ($\Rightarrow$) Suppose $-b \leq a \leq b$, this implies $a \leq b$ and $-b \leq a \iff -a \leq b$ and consequently $|a| \leq b$.
($\Leftarrow$) Suppose $|a| \leq b$ then $a \leq b$ and $-a \leq b \iff -b \leq a$, thus $-b \leq a \leq b$. Now we prove the last part. Suppose $|a| \leq |a|$ then by the previously proven theorem we have $-|a| \leq a \leq |a|$.

(c) As proven earlier, for every a, b we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Add these together gives $-(|a| + |b|) \leq a + b \leq |a| + |b|$, applying the theorem from (b) on $(|a| + |b|)$ and $(a + b)$ we get $|a + b| \leq |a| + |b|$.

***1.15.** We prove first that if $x = y$ and $x, y \neq 0$. The inequality is then $x^2 + x^2 + x^2 > 0 \iff x^2 > 0$ which is true because $x \neq 0$.

Suppose $x \neq y$, then the left side of inequality is equivalent to $(x^2 + xy + y^2) = \frac{x^3-y^3}{(x-y)}$. Suppose $x > y$ then $x^3 - y^3 > 0$ by problem 6 (b), since both the numerator and denominator are positive we know that $\frac{x^3-y^3}{(x-y)} > 0$. Next we assume $x < y$ which implies $x^3 - y^3 < 0$ by problem 6 (b). This means the numerator and denominator are both negative, thus $\frac{x^3-y^3}{(x-y)} > 0$. In every case the inequality is positive, thus we have proven that $x^2 + xy + y^2 > 0$.

To prove that the second inequality holds we follow the same steps, suppose $x = y$ which means the inequality is $5x^4 > 0$. Next suppose $x \neq y$ then we have $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5-y^5}{x-y}$. Suppose $x - y > 0$ then $x^5 - y^5 > 0$ which implies $\frac{x^5-y^5}{x-y} > 0$. Assume $x - y < 0$ then $x^5 - y^5 < 0$ which implies $\frac{x^5-y^5}{x-y} > 0$.

***1.16.** (a) $(x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 \iff xy = 0$ which implies $x = 0$ or $y = 0$. Next we have $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 \iff x^2y + xy^2 = 0 = xy(x + y)$. Which implies either $x = 0$ or $y = 0$ or $x = -y$.

(b) Consider $3(x+y)^2 = 3x^2+6xy+3y^3 \geq 0$, since $x, y \neq 0$ we have $x^2 > 0$ and $y^2 > 0$, adding these inequalities makes $4x^2 + 6xy + 4y^2 > 0$. If $x, y = 0$ then the statement would be false.

(c) Equivalently we have $4x^3y + 6x^2y^2 + 4y^3x = xy(4x^2 + 6xy + 4y^2)$, left side indicates that it is equal to zero when $x = 0$ or $y = 0$. Thus $(x + y)^4 = x^4 + y^4$ when $x = 0$ or $y = 0$.

(d) Subtract with $x^5 + y^5$ and since $xy \neq 0$ we divide by $5xy$ this makes $x^3+2x^2y+2xy^2+y^3 = 0 \iff (x+y)^3 = x^2y+y^2x = xy(x+y)$. Suppose $x+y \neq 0$ then $xy = (x+y)^2 \iff x^2+xy+y^2 = 0$, this implies $x, y = 0$ by letting $p = x^2+xy+y^2 \iff 2p = 2x^2+2xy+2y^2 = x^2+y^2+(x+y)^2$, it then follows all the terms have to be zero because they are either zero or positive, $x = 0$ and $y = 0$, this contradicts the fact that $xy = 0$, thus it must be the case that $x = -y$.

Assume this time that $x = 0$ then $(x + y)^5 = x^5 + y^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \iff y^5 = y^5$. By interchanging x with y in the last sentence it follows that $x = 0$ or $y = 0$. To conclude, the solutions are $x = -y$ or $x = 0$ or $y = 0$. My guess is that the same solutions apply to $(x + y)^n = x^n + y^n$ if n is odd and $x = 0$ or $y = 0$ if n is even.

1.17. (a) $2x^2 - 3x + 4 = 2(x - \frac{3}{4})^2 + y \implies y = \frac{32}{8} - \frac{9}{8} = \frac{23}{8}$

(b) Subtract $2(y + 1)^2$ this makes $x^2 - 3x$. Let $x^2 - 3x = (x - \frac{3}{2}) + z$ then $z = -\frac{9}{4}$, z is the smallest value.

(c) Let m be the minimum number for a simple second degree polynomial, then it follows that $x^2+bx+c = 0 = (x+\frac{b}{2})^2+m = x^2+bx+\frac{b^2}{4}+m \iff m = c - \frac{b^2}{4}$

We have $x^2 + 4xy + 5y^2 - 4x - 6y + 7 = x^2 + (4y - 4)x + 5y^2 - 6y + 7$ The minimum is thus $m = 5y^2 - 6y + 7 - 4(y^2 - 2y + 1) = y^2 + 2y + 3 = (y + 1)^2 + 2$. This implies that 2 is in fact the minimum value.

1.18. (a) $x = \frac{-b\pm\sqrt{b^2-4c}}{2} \iff (2x+b)^2 = b^2-4c \iff 4x^2+4xb+b^2-b^2+4c = 0 \iff x^2 + bx + c = 0$.

(b) We complete the square, $x^2 + bx + c = 0 \iff 4(x + \frac{b}{2})^2 = b^2 - 4c$ this follows that $(x + \frac{b}{2})^2 \geq 0$, but $b^2 - 4c < 0$ which is a contradiction. It also follows that $x^2+bx+c>0$ which means there are no real values of x that satisfy the equation.

(c) We complete the square $(x+\frac{y}{2})^2+\frac{3y^2}{4}$. Since $\frac{3y^2}{4}>0$ because $y\neq 0$ it must be the case that $(x+\frac{y}{2})^2+\frac{3y^2}{4}>0$ which is the same as $x^2+xy+y^2>0$

(d) Completing the square makes $(x+\frac{\alpha y}{2})^2+y^2(1-\frac{\alpha^2}{4})$. The left term has the property $(x+\frac{\alpha y}{2})^2\geq 0$ (just let $x=-\frac{\alpha y}{2}$). This means the right term must be positive. Let $1-\frac{\alpha^2}{4}>0$ which implies $-2<\alpha<2$.

(e) $ax^2+bx+c=a(x^2+\frac{bx}{a})+c=a(x+\frac{b}{2a})^2+c-\frac{b^2}{4a^2}$. Since $a>0$ the minimum must be when $x+\frac{b}{a}=0$, so the minimum is $c-\left(\frac{b}{2a}\right)^2$. (The first case is just $a=1$)

1.19. (a) Suppose $x_1=\lambda y_1$ and $x_2=\lambda y_2$ then the equality holds if $\lambda({y_1}^2+{y_2}^2)=\sqrt{\lambda^2({y_1}^2+{y_2}^2)}\sqrt{({y_1}^2+{y_2}^2)}\iff\lambda=|\lambda|$. Seems to be some kind of error (edition 3) because it does not hold if λ is negative. Let's assume $\lambda\geq 0$. The then equality holds. The equality also holds if $y_1=y_2=0$ because both factors on both sides are equal to zero.

Assume y_1 and y_2 is not equal to zero. Then there does not exist a λ such that $x_1=\lambda y_1$ and $x_2=\lambda y_2$, the problems states that this implies $\lambda^2({y_1}^2+{y_2}^2)-2\lambda(x_1y_1+x_2y_2)+({x_1}^2+{x_2}^2)>0$. This equation is of the form $\lambda^2+b\lambda+c>0$ and since there does not exist any λ we have $b^2<4ac$ by noticing that dividing by a in the equation $ax^2+bx+c=0$ you can apply problem 18 (b), that is $(x_1y_1+x_2y_2)^2<({y_1}^2+{y_2}^2)({x_1}^2+{x_2}^2)$. This follows that $|x_1y_1+x_2y_2|<\sqrt{{y_1}^2+{y_2}^2}\sqrt{{x_1}^2+{x_2}^2}$

To conclude we have

$$x_1y_1+x_2y_2\leq|x_1y_1+x_2y_2|\leq\sqrt{{y_1}^2+{y_2}^2}\sqrt{{x_1}^2+{x_2}^2}.$$

(b) We start with $(x-y)^2\geq 0\iff 2xy\leq x^2+y^2$. Suppose $x_1,x_2,y_1,y_2\neq 0$ and let $x=\frac{x_i}{\sqrt{{x_1}^2+{x_2}^2}}$, $y=\frac{y_i}{\sqrt{{y_1}^2+{y_2}^2}}$ for $i=1,2$. It follows that

$$\begin{cases}\dfrac{2x_1y_1}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}}\leq\dfrac{{x_1}^2}{{x_1}^2+{x_2}^2}+\dfrac{{y_1}^2}{{y_1}^2+{y_2}^2}\\[2ex]\dfrac{2x_2y_2}{\sqrt{{x_1}^2+{x_2}^2}\sqrt{{y_1}^2+{y_2}^2}}\leq\dfrac{{x_2}^2}{{x_1}^2+{x_2}^2}+\dfrac{{y_2}^2}{{y_1}^2+{y_2}^2}\end{cases}$$

Add both inequalities together, then it follows that $x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}$.

If we assume $x_i = 0$ or $y_i = 0$ for $i = 1, 2$ then either all the terms will be zero or the resulting inequality is for example $0 \leq y_1{}^2$ (let $x_1 = 0$).

(c) $(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2)$

$$\begin{aligned}
&= (x_1y_1)^2 + 2(x_1y_1)(x_2y_2) + (x_2y_2)^2 + (x_2y_1)^2 - 2(x_2y_1)(x_1y_2) + (x_1y_2)^2 \\
&= (x_1y_1 + x_2y_2)^2 + (x_2y_1 - x_1y_2)^2 \geq (x_1y_1 + x_2y_2)^2 \\
&\iff \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2} \geq |x_1y_1 + x_2y_2| \geq x_1y_1 + x_2y_2
\end{aligned}$$

(d) The problem is constructed to waste time, see (a) where we already proved it. It shows that if $y_1 = 0$ and $y_2 = 0$ or there exists a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ then the equality holds, otherwise $|x_1y_1 + x_2y_2| < \sqrt{y_1{}^2 + y_2{}^2}\sqrt{x_1{}^2 + x_2{}^2}$.

1.20. Add both inequalities, $|x-x_0|+|y-y_0| < \varepsilon$. We apply the triangle inequality which makes $|(x + y) - (x_0 + y_0)| \leq |x - x_0| + |y - y_0| < \varepsilon$. For the second inequality, notice that that $|y - y_0| = |y_0 - y|$. So the triangle inequality makes $|(x - y) - (x_0 - y_0)| \leq |x - x_0| + |y_0 - y| < \varepsilon$.

***1.21.** Suppose $|x - x_0| < \frac{\varepsilon}{2(|y_0|+1)}$, then $2|x - x_0|(|y_0| + 1) < \varepsilon$. Now assume $|y - y_0| < \frac{\varepsilon}{2(|y_0|+1)}$ then $2|y - y_0|(|x_0| + 1) < \varepsilon$. Sum the two similar inequalities

$$\begin{aligned}
&2|x - x_0|(|y_0| + 1) + 2|y - y_0|(|x_0| + 1) < 2\varepsilon \\
&|x - x_0|(|y_0| + 1) + |y - y_0|(|x_0| + 1) < \varepsilon \\
&|y_0||x - x_0| + |x - x_0| + |x_0||y - y_0| + |y - y_0| < \varepsilon
\end{aligned}$$

Now suppose $|x-x_0| < 1$ then we have $|y-y_0||x-x_0| < |y-y_0|$. Continuing on the expression above we get

$$\begin{aligned}
&> |y_0||x - x_0| + |x_0||y - y_0| + |y - y_0| \\
&> (|y_0| + |y - y_0|)(|x - x_0|) + |x_0||y - y_0| \\
&\geq |y||x - x_0| + |x_0||y - y_0| \geq |xy - x_0y + x_0y - x_0y_0| = |xy - x_0y_0|
\end{aligned}$$

Therefore we have $|xy - x_0y_0| < \varepsilon$.

***1.22.** We first prove that $y \neq 0$. Suppose $|y - y_0| < \frac{|y_0|}{2}$ then by problem 12, we get $|y_0| < 2|y|$ by problem 12. By supposing $y = 0$ we get a contradiction because $0 < |y_0|$ thus it must be the case that $y \neq 0$.

Now we prove the latter. Suppose $|y - y_0| < \frac{\varepsilon |y_0|^2}{2}$. Then

$$|y - y_0| < \varepsilon |y_0||y|$$
$$\left|\frac{y_0 - y}{y_0 y}\right| < \varepsilon$$
$$\left|\frac{1}{y_0} - \frac{1}{y}\right| < \varepsilon$$

as desired.

***1.23.** We begin first by using problem 21. We can then state that if $y \neq 0$, $y_0 \neq 0$, $\left|\frac{1}{y} - \frac{1}{y_0}\right| < \frac{\varepsilon}{2(|x_0|+1)}$ and $|x - x_0| < \min\left(\frac{\varepsilon}{2(\left|\frac{1}{y_0}\right|+1)}, 1\right)$ then we have $\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon$. Now we need to modify the hypothesis. We have that $y_0 \neq 0$ and $|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon |y_0|^2}{2}\right)$ implies $y \neq 0$ and the hypothesis earlier.

To conclude, $y_0 = 0$, $|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon |y_0|^2}{2}\right)$ and $|x - x_0| < \min\left(\frac{\varepsilon}{2(\left|\frac{1}{y_0}\right|+1)}, 1\right)$ implies $y \neq 0$ and $\left|\frac{x}{y} - \frac{x_0}{y_0}\right| < \varepsilon$.

***1.24.** (a) We prove the base case (k=2) with the associative law, $(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$. Next we suppose $P(k)$: $(a_1 + \cdots + a_k) + a_{k+1} = a_1 + \cdots + a_{k+1}$, then we prove for $P(k+1)$:

$$(a_1 + \cdots + a_{k+1}) + a_{k+2} = [(a_1 + \cdots + a_k) + a_{k+1}] + a_{k+2}$$
$$(a_1 + \cdots + a_k) + (a_{k+1} + a_{k+2}) = a_1 + \cdots + k_{k+2}$$

This concludes the induction.

(b) We will prove this by induction on n, suppose $n \geq k$ and $(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n$. The base case is $n = k+1$ which was proven in the previous problem. We will now show the equality holds for $n+1$, we have

$$\begin{aligned}
&(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_{n+1}) \\
&= (a_1 + \cdots + a_k) + ((a_{k+1} + \cdots + a_n) + a_{n+1}) \\
&= ((a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n)) + a_{n+1} \\
&= (a_1 + \cdots + a_n) + a_{n+1} \\
&= a_1 + \cdots + a_{n+1}
\end{aligned}$$

We have now proven that for $n \geq k$ it follows that

$$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n.$$

(c) We will show that $s(a_1, \ldots, a_k) = s(a_1) + \cdots + s(a_k)$ by induction on k. Let the base case be $k = 1$, then we obviously have an equality. Now we assume $s(a_1, \ldots, a_k) = s(a_1) + \cdots + s(a_k)$ and now prove for the $k + 1$ case.

$$\begin{aligned}
s(a_1, \ldots, a_{k+1}) &= s(a_1, \ldots, a_k) + s(a_{k+1}) \\
&= s(a_1) + \cdots + s(a_{k+1})
\end{aligned}$$

Because $s(a_1) + \cdots + s(a_k) = a_1 + \cdots + a_k$, our proof is done.

1.25. We suppose the rules of addition and multiplication given in the problem we then prove it is a field.

(i) Testing each case is tedious and will not be contained here, but we find that $a + (b + c) = (a + b) + c$ works.

(ii) Suppose $a = 0$ then $0+0 = 0+0 = 0$, and $a = 1$ implies $1+0 = 0+1 = 0$

(iii) If $a = 0$ then then let $-a = 0$ and if $a = 1$ then $-a = 1$.

(iv) This works by exhaustion.

(v) If at least one variable is zero, then $0 = 0$, otherwise $1\cdot(1\cdot) = (1\cdot1)\cdot1 \iff 1 = 1$

(vi) Suppose $a = 0$ then $1 \cdot 0 = 1 \cdot 0 = 0$, suppose $a = 1$ then $1 \cdot 1 = 1 \cdot 1 = 1$

(vii) $a = 0$ is not allowed so we only prove for the $a = 1$ case which makes $a^{-1} = 1$.

(viii) If at least one variable is equal to zero then we have $0 = 0$, otherwise $1 \cdot 1 = 1 \cdot 1$

(ix) Suppose $a = 0$ then $0 \cdot (b + c) = 0 \cdot b + 0 \cdot c = 0$. Suppose $a = 1$ then $1 \cdot (b + c) = 1 \cdot b + 1 \cdot c = b + c$

Chapter 2

Numbers of various sorts

2.1. (i) The formula is clearly true for $n = 1$. Suppose

$$1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

then

$$1^2 + \cdots + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + n^2 + 2n + 1$$
$$\frac{2n^3 + 3n^2 + n + 6n^6 + 12n + 6}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

(ii) The base case $n = 1$ is obviously true, suppose

$$(1 + \cdots + n)^2 = 1^3 + \cdots + n^3$$

then

$$\begin{aligned}((1 + \cdots + n) + (n+1))^2 &= (1 + \cdots + n)^2 + 2(1 + \cdots + n)(n+1) + (n+1)^2 \\ &= (1^3 + \cdots + n^3) + n(n+1)^2 + (n+1)^2 = 1^3 + \cdots + n^3 + (n+1)^3\end{aligned}$$

2.2. (i) We solve by rewriting the sum,

$$\begin{aligned}1 + 3 + 5 + \cdots + (2n-1) &= 1 + \cdots + 2n - (2 + 4 + \cdots 2n) \\ 1 + \cdots + 2n - 2(1 + 2 + \cdots n) &= n(2n+1) - n(n+1) \\ &= n^2\end{aligned}$$

Thus $\sum_{i=1}^{n}(2i-1) = n^2$.

(ii) Using similar methods as before:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = 1^2 + \cdots + (2n)^2 - (2^2 + 4^2 + \cdots + (2n)^2)$$

$$1^2+\cdots+(2n)^2-4(1^2+2^2+\cdots+n^2)=\frac{2n(2n+1)(4n+1)}{6}-\frac{4n(n+1)(2n+1)}{6}$$
$$\frac{2n[8n^2+6n+1-2(2n^2+3n+1)]}{6}=\frac{8n^3-2n}{6}$$

We conclude that $\sum_{i=1}^{n}(2i-1)^2=\frac{8m^3-2n}{6}$

2.3. (a) The easiest way seems to be starting at the right side.

$$\binom{n}{k-1}+\binom{n}{k}=\frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!}$$
$$\frac{kn!}{k!(n-k+1)!}+\frac{(n-k+1)n!}{k!(n-k+1)!}=\frac{(n+1)!}{k!(n+1-k)!}=\binom{n+1}{k}$$

(b) We perform induction on n. Let the base case be $n=1$ then by the definition of the binomial coefficients we have two cases, $\binom{1}{0}$ and $\binom{1}{1}$, both of these are equal to one by definition.
TODO: Is this base case actually valid? $n=1$ does not use every part of the definition. If the definition can be different for numbers other than the base case does it invalidate the proof? Must there be multiple base cases involved testing each part of the definition?

Suppose $\binom{n}{k}$ is a natural number for every $0\le k\le n$, then it follows that $\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}$ is a natural number because $\binom{n}{k-1}$ and $\binom{n}{k}$ are both natural numbers.

(c) Suppose we can chose any number only once from $1,2,\ldots,n$. First we have n different choices, then $n-1$ choices and so on. If we do this n times we eventually only have one number left to choose. The numbers have $n!$ different sequences from which we can choose. Now we want to count the number of integers we can choose with only k choices. If $k=1$ then there is only one choice, n. If $k=2$ then we have $n\cdot(n-1)$, if we continue this we notice that this is the same as cutting the smaller factors in $n!$. Thus the number of ways we can do this is $\frac{n!}{(n-k)!}$. This is also known as the numbers of permutations of n of length k. Notice that the order in which these numbers are chosen are important. However, in a set the order does in fact not matter. The number of permutations of length k is $k!$, so finally we divide the permutations formula by k which means that we have $\frac{n!}{k!(n-k)!}$ and by the definition of the binomial coefficients this is the same as $\binom{n}{k}$. Because $\binom{n}{k}$ is the number of sets with exactly k integers chosen by n we have that $\binom{n+1}{k}$ is the number of sets with exactly k integers chosen by $n+1$. This implies that the case for $n+1$ is also a finite amount.

(d) Base case: $n = 1$ then $(a+b)^1 = \sum_{k=0}^{1} \binom{n}{k} a^{n-k} b^k = a + b$. Before we continue, a new notation must be introduced, $\sum_{0 \leq k \leq n} a_k$, this means the sum over k from 1 to n. Suppose

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n.$$

then

$$\sum_{0 \leq k \leq n+1} \binom{n+1}{k} a^{n+1-k} b^k = a^{n+1} + b^{n+1} + \sum_{1 \leq k \leq n} \binom{n+1}{k} a^{n-k+1} b^k$$

$$a^{n+1} + \sum_{1 \leq k \leq n} \binom{n}{k} a^{n-k+1} b^k + \sum_{1 \leq k \leq n+1} \binom{n}{k-1} a^{n-k+1} b^k$$

$$\sum_{0 \leq k \leq n} \binom{n}{k} a^{n-k+1} b^k + \sum_{0 \leq k-1 \leq n} \binom{n}{k-1} a^{n-(k-1)} b^{(k-1)+1}$$

(Substitute $k-1$ for k)

$$a \sum_{0 \leq k \leq n} \binom{n}{k} a^{n-k} b^k + b \sum_{0 \leq k \leq n} \binom{n}{k} a^{n-k} b^k$$

$$(a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^{n+1}$$

To conclude, we have proven the equality

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^n$$

(e) Using similar steps as above we can derive the transformation

$$\sum_{k=0}^{n+1} a_k \binom{n+1}{k} = \sum_{k=0}^{n} a_k \binom{n}{k} + \sum_{k=0}^{n} a_{k+1} \binom{n}{k}$$

(i) For the sake of base case, let $n = 0$, then we have $\binom{0}{0} = 1$. Suppose $\sum_{j=0}^{n} \binom{n}{j} = 2^n$ then

$$\sum_{0 \le j \le n+1} \binom{n+1}{j} = \sum_{0 \le j \le n} \binom{n}{j} + \sum_{0 \le j \le n} \binom{n}{j}$$
$$2 \sum_{0 \le j \le n} \binom{n}{j} = 2^{n+1}$$

(ii) The base case $n = 0$ does not seem to work, therefore we try $n = 1$ then we have $\binom{n}{1} - \binom{1}{1} = 1 - 1 = 0$. Suppose $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$. Then by replacing n with $n + 1$ we do the same transformation as earlier

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{k} = \sum_{j=0}^{n} (-1)^j \binom{n}{k} + \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{k}$$
$$\sum_{j=0}^{n} (-1)^j \binom{n}{k} - \sum_{j=0}^{n} (-1)^j \binom{n}{k} = 0.$$

(iii) Subtract (i) with (ii), then

$$\binom{n}{0} - \binom{n}{0} + \binom{n}{1} + \binom{n}{1} + \cdots = 2^n - 0$$

The end of the sum depends on whether n is odd or even so we don't explicitly write it down. Notice that the all the even binomials cancel out, thus we have

$$2 \sum_{l \text{ odd}} \binom{n}{l} = 2^n \iff \sum_{l \text{ odd}} \binom{n}{l} = 2^{n-1}$$

(iv) Subtracting (i) with (iii), it follows that we only have the even binomials left, the other part of the formula is then $2^n - 2^{n-1} = 2^{n-1}$. It follows that

$$\sum_{l \text{ even}} \binom{n}{l} = 2^{n-1}.$$

2.4. (i) We first need to prove an important property of sums,

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{j=0}^{m} b_j\right) = (a_1 + \cdots + a_n)(b_1 + \cdots + b_m)$$

$$= a_1(b_1 + \cdots + b_m) + \cdots + a_n(b_1 + \cdots + b_m) \quad \text{(Distributive property)}$$

$$= \sum_{i=0}^{n}\left(a_i \sum_{j=0}^{m} b_j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j \qquad (a_i \text{ is a constant in the } b_j \text{ sum})$$

Note that we can interchange the summation symbols if we were to apply the distributive property as $b_1(a_1 + \cdots + a_n) + \cdots b_m(a_1 + \cdots + a_n)$ and then continue with similar steps, thus we have

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j = \sum_{j=0}^{m}\sum_{i=0}^{n} a_i b_j.$$

Now consider the polynomial $(1+x)^n(1+x)^m$, we can write this as

$$\sum_{l=0}^{n+m}\binom{n+m}{l}x^l \tag{1}$$

and

$$\left(\sum_{i=0}^{n}\binom{n}{i}x^i\right)\left(\sum_{j=0}^{m}\binom{m}{j}x^j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}x^{i+j} \tag{2}$$

The important thing is to realize that both sums represent the same polynomial. We must now use the fact that if a polynomial is equal for any x then they must have the same coefficients. This theorem can be proven as a corollary to the problem 3.7. (c).

The coefficients for x^l is $\binom{n+m}{l}$ as stated above (1). By the theorem recently stated we know that the coefficients in the other sum must be the same. Therefore, it makes sense to gather all the coefficients to each indeterminate. To do this we will use the equality

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{i+j} = \sum_{l=0}^{n+m}\sum_{i=0}^{n} a_l \qquad \text{where } l = i + k$$

Proof. Let $l = i + j$, then

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{i+j} = \sum_{0\le i\le n}\ \sum_{0\le l-i\le m} a_l \qquad \text{(Substitute } j = l - i\text{)}$$

$$\sum_{0\le i\le n}\ \sum_{0\le l\le m+n} a_l = \sum_{l=0}^{m+n}\sum_{i=0}^{n} a_l. \qquad \text{(Add } 0 \le i \le n \text{ to the left bound)}$$

Applying the transformation to (2) we get the following equalities

$$\sum_{l=0}^{n+m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{l-i}x^l = \sum_{l=0}^{n+m}\binom{n+m}{l}x^l$$
$$\sum_{i=0}^{n}\binom{n}{i}\binom{m}{l-i} = \binom{n+m}{l}$$

The last equality holds by the theorem stated earlier that the if two polynomials are equal then they have the same coefficients.

(ii) By letting $n = l = m$ in the equality before we get

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$
$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

The last equality holds because

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

2.5. We will not prove this inductively because it is trivial. Suppose $S = 1 + r + \cdots + r^n$ then

$$rS+1 = 1+r+r^2+\cdots r^{n+1} = S+r^{n+1} \iff 1-r^{n+1} = S(1-r) \iff S = \frac{1-r^{n+1}}{1-r}$$

Chapter 3

Functions

3.1.

3.2.

3.3.

3.4.

3.5.

3.6. (a) We wish to create a polynomial such that has all the roots at every x_j, and that when $x = x_i$ the polynomial is equal to one. The hint provides us with a polynomial that abides the roots. We try modifying this polynomial by applying an unknown product

$$f_i(x) = P \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n} (x - x_j)$$

Since we know that $f_i(x_i) = 1$ we get the following equation

$$P \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j) = 1 \iff P = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{x_i - x_j}$$

Notice that this is okay since x_i and x_j are distinct by the hypothesis. Substitute the P and we find a solution by combining the products

$$f_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

(b) Since each f_i is equal to zero at ever x_j except for x_i which makes a one we have that $a_i f_i$ gives a polynomial that intersects the point (x_i, a_i). The final function f can be derived by simply summing all the f_j, that is

$$f(x) = \sum_{j=1}^{n} a_j f_j = a_j \sum_{j=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

3.7. (a) Base case $n = 0$ then we have $f(x) = p_0$ and then just let $g(x) = 0$ and $b = p_0$. Base case $n = 1$ then $f(x) = p_1 x + p_0$, let $g(x) = p_1$ thus $p_1 x + p_0 = p_1 x - ap_1 + b \iff p_0 + ap_1$.
Now for induction, assume for $f_n(x) = p_n x^n + \cdots + p_1 x + p_0$ and any a that there exists a function $g(x)$ and a number b such that

$$p_n x^n + \cdots + p_0 = (x - a)g(x) + b.$$

Now consider the polynomial

$$f_{n+1}(x) = p_{n+1} x^{n+1} + \cdots + p_0 \tag{1}$$

then by doing one step of long division we get

$$\begin{aligned} p_{n+1} x^{n+1} + \cdots + p_0 &= (x - a)(p_{n+1} x^n) + p_n x^n + \cdots + p_1 x + p_0 + p_{n+1} a \\ (x - a)(p_{n+1} x^n) + (x - a)g(x) + b &= (x - a)(p_{n+1} x^n + g(x)) + b \\ & \qquad \text{(Apply (1))} \end{aligned}$$

This concludes the induction because $p_{n+1} + g(x)$ is a polynomial which implies f_{n+1} must be true.

(b) Spivak has likely forgotten to mention that f is a polynomial, we will just assume it is. By the last problem we have that for any a there is a polynomial $g(x)$ and b such that $f(x) = (x - a)g(x) + b$, since we have $f(a) = 0$ we certainly have

$$f(a) = (a - a)g(x) + b = b = 0.$$

Thus, this means $f(x) = (x - a)g(x)$

(c) The base case is a polynomial of degree 1, that is $a_0 + a_1 x$. The only root is $x = -\frac{a_0}{a_1}$. Suppose for induction that $a_0 + a_1 x + \cdots + a_n x^n$ has at most n distinct roots, then let x_1 be a root to the polynomial

$$a_0 + \cdots + a_{n+1} x^{n+1} = (x - x_1)g(x)$$

The equality holds by the theorem in (b) which also states that $g(x)$ is polynomial, the important thing to notice is that $g(x)$ is of degree n. By the induction assumption we have that $g(x)$ has at most n roots, then $f(x)$ has at most one more additional root at x_1 if this is distinct from the other roots.

(d) Suppose $f_n(x)$ is polynomial function with n roots then let r be a number such that it is not a root of $f_n(x)$, then consider $f_{n+1} = (x-r)f_n(x)$ which means there is a polynomial with one degree greater than $f_n(x)$.

Now we find the even degree function, since the degree is even there exists a least value or maximum value for every even degree polynomial (This can easily be proven with limits by factoring out $a_n x^n$ in a polynomial of the form $a_n x^n + \cdots a_0$ and considering $\lim_{x\to\pm\infty} f_n(x)$). Since both variations exists, we choose the a polynomial f_n such that it has n roots and a minimum l. Now we create a function $h(x) = f_n(x) - l + 1$ This ensures that the minimum value is exactly one unit above zero.

Next suppose n is odd then we have the function

$$\sum_{i=1}^{n} x^i$$

the function has only one real root at $x = 0$ because if we suppose $x \neq 0$ and try to find any other solutions we get $x^{n-1} + \cdots + x + 1 = 0$ which is a polynomial without any real roots because $n - 1$ is even. The proof of claim requires knowledge from chapter 7 and 8.

3.8. Note that $c = 0$ because otherwise it would imply there is a root to $cx + d$ which means we divide by zero, we also have $d \neq 0$. Now we get the equation

$$f(f(x)) = \frac{a\frac{ax+b}{d} + b}{d} = a^2x + ab + bd = x$$

This implies that $a^2 = 1$ thus $a = \pm 1$, we have two cases:
Case 1:$x - b + bd = x \iff b = bd$, if $b = 0$ then any $d \neq 0$ works otherwise if $b \neq 0$ then $d = 1$.
Case 2: $x + b + db = x \iff -b = bd$, if $b = 0$ then any $d \neq 0$ works otherwise if $b \neq 0$ then $d = -1$.

Chapter 4

Graphs

Chapter 5

Limits

5.1. (i) Let $f(x) = x^2 - 1$ and $g(x) = \frac{1}{x+1}$, we get $\lim_{x\to 1} f(x) = 0$ and $\lim_{x\to 1} g(x) = \frac{1}{2}$. Then we get

$$\lim_{x\to 1}(f \cdot g)(x) = 0.$$

(ii) Expanding $x^3 - 2^3$ and noting that $x \neq 2$

$$\lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2} x^2 + 2x + 4 = 12.$$

(iii) Expanding using

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$$

we get

$$\lim_{x\to y}(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) = y^n.$$

(iv) Same thing here except we get x^n.

(v) Expand the fraction by the conjugate of the numerator

$$\lim_{h\to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} = \lim_{h\to 0} \frac{h}{h(\sqrt{a+h} + \sqrt{a})}$$
$$\lim_{h\to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

5.2. (i) Note that $1-x=(1-\sqrt{x})(1+\sqrt{x})$. Then, the limit is

$$\lim_{x\to 1}\frac{1}{1+\sqrt{x}}=\frac{1}{2}$$

(ii) Expand the fraction by $1+\sqrt{1-x^2}$. This makes

$$\lim_{x\to 0}\frac{x^2}{x(1+\sqrt{1-x^2})}=\lim_{x\to 0}\frac{x}{1+\sqrt{1-x^2}}=0$$

(iii) Expand this in the same way:

$$\lim_{x\to 0}\frac{x^2}{x^2(1+\sqrt{1-x^2})}=\lim_{x\to 0}\frac{1}{1+\sqrt{1-x^2}}=\frac{1}{2}$$

5.3. (i) Note that

$$|x^4-a^4|=|x-a|\cdot|x^3+ax^2+a^2x+a^3|<\varepsilon$$

So we let

$$\delta=\frac{\varepsilon}{|x^3+ax^2+a^2x+a^3|}$$

(ii) We start by plugging in $f(x)$ and l,

$$\left|\frac{1}{x}-1\right|<\varepsilon$$
$$\iff |x-1|<|x|\varepsilon$$

Thus we let $\delta=|x|\varepsilon$

(iii) This is closely related to (i) and (ii). Taking a close look at the proof of part one of theorem 2 indicates that we have $\delta=min(\delta_1,\delta_2)$. So we find δ_1,δ_2 for x^4 and $\frac{1}{x}$ respectively. Let

$$\delta_1=\frac{\varepsilon}{2|x^3+ax^2+a^2x+a^3|}$$

and

$$\delta_2=\frac{|x|\varepsilon}{2}.$$

Then $0<|x-1|<\delta$ implies $|x^4-1|<\frac{\varepsilon}{2}$ and $|\frac{1}{x}-1|<\frac{\varepsilon}{2}$, and consequently

$$\left|\left(x^4+\frac{1}{x}\right)-2\right|\le\left|x^4-1\right|+\left|\frac{1}{x}-1\right|<\varepsilon$$

(iv) This is similar to the above problem except it follows part two of theorem 2.

(v) $\delta = \varepsilon^2$

(vi) $\delta = (\sqrt{x} + 1)\varepsilon$

5.4.

5.5.

5.6.

5.7.

5.8. (a) Consider $f(x) = \frac{1}{x-a}$ and $g(x) = \frac{-1}{x-a}$, then $\lim_{x\to a}[f(x) + g(x)] = 0$

(b) Consider
$$\lim_{x\to a}[f(x) + g(x)] - \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

(c) Suppose for contradiction that $\lim_{x\to a}[f(x)+g(x)]$ exists. Since $\lim_{x\to a} f(x)$ exists by the hypothesis and using (b) we get a contradiction that $\lim_{x\to a} g(x)$ exists and does not exist. Thus $\lim_{x\to a}[f(x) + g(x)]$ does not exist.

(d) Let $f(x) = 0$ and $g(x) = \frac{1}{x-a}$ then $\lim_{x\to a} f(x)g(x) = 0$ but $\lim_{x\to a} g(x)$ does not exist.

5.9. There is a $\delta > 0$ such that for every $\varepsilon > 0$, if for any x, $0 < |x - a| < \delta$ then $|f(x) - l| < \varepsilon$. Now let $x = a + h$ then we equivalently have if $0 < |h| < \delta$ then, $|f(a+h)-l| < \varepsilon$ so to conclude we have $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$.

5.10.

5.11.

5.12. (a) Suppose $f(x) \leq g(x)$ for every x and that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. This means there exists a $\delta > 0$ so that for every $\frac{\varepsilon}{2} > 0$ and every x, if $0 < |x - a| < \delta$ then $|f(x) - l| < \frac{\varepsilon}{2}$ and $|g(x) - m| < \frac{\varepsilon}{2}$. Consequently we have
$$f(x) - l > -\frac{\varepsilon}{2}$$

and

$$g(x) - m < \frac{\varepsilon}{2}.$$

Now we also have that

$$l - \frac{\varepsilon}{2} < f(x) \leq g(x) < m + \frac{\varepsilon}{2}$$

$$l - m < \varepsilon \iff l \leq m$$

(b) ?

(c) We would have

$$l - \frac{\varepsilon}{2} < f(x) < g(x) < m + \frac{\varepsilon}{2}$$

which still implies

$$l \leq m$$

so not necessarily.

Chapter 6

Continuous Functions

6.1. (i) The function has a discontinuity at $x = 2$, the limit as $x \to 2$ is 4 so we define

$$F(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{for } x \neq 2 \\ 4, & \text{otherwise.} \end{cases}$$

(ii) Since the limit at $x = 0$ does not even exist, there is nothing we can do here.

(iii) $F(x) = 0$

(iv) The limit at every rational x is zero as proven earlier in the book. So f is discontinuous everywhere. The answer is no.

6.2.

6.3. (a) Suppose $|f(x)| \leq |x|$, if we let $x = 0$ then $|f(0)| \leq 0$ and by noting that the absolute value has the property $|f(0)| \geq 0$ we must in fact have $f(0) = 0$. Now let $\delta = \varepsilon$ then for every $\epsilon > 0$ and any x, if

$$|x - 0| < \delta$$

then

$$|f(x)| \leq |x| < \varepsilon$$

which is the same as

$$|f(x) - f(0)| < \varepsilon.$$

Thus $f(x)$ is continuous at 0.

(b) $f : \emptyset \to \emptyset$

(c) By letting $x = 0$ we find that by using similar arguments to a) that $f(0) = 0$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every x if

$$|x - a| < \delta$$

then,

$$|g(x)| < \varepsilon$$

and then we certainly have $|f(x)| < \varepsilon$ and consequently

$$|f(x) - f(0)| < \varepsilon,$$

so f is continuous at zero.

6.4. Let

$$f(x) = \begin{cases} 1, & \text{x irrational} \\ -1, & \text{x rational.} \end{cases}$$

The limit of $f(x)$ does not exist, but the function $|f(x)|$ is a constant function, thus it is continuous.

6.5.

$$g(x) = \begin{cases} 0, & \text{x irrational} \\ x, & \text{otherwise} \end{cases}$$

The function $g(x)$ is continuous at precisely one point, that is 0. Now we just let the function f we are looking for to be $f(x) = g(x - a)$.

6.6. (a)

$$f(x) = \begin{cases} \frac{1}{k}, & \frac{1}{k+1} < x < \frac{1}{k}\text{for k=1,2,...} \\ 0, & \text{otherwise} \end{cases}$$

()Not confirmed)

(b)

$$f(x) = \prod_{k=1}^{\infty} \frac{1}{x - \frac{1}{k}}.$$

Even though infinite polynomials have not been defined I conjecture that this implies the function is not continuous at $x = 0$ because $k \to \infty$.

6.7. Let $y = 0$, then it follows that $f(0) = 0$. Let $y = -x$ then we obtain $f(-x) = -f(x)$. There exists an $\delta > 0$ such that if $|t| < \delta$ then $|f(t)| < \varepsilon$. Let $t = x - a$, this is the same as the following:
For every x if $|x - a| < \delta$ then

$$|f(x) + f(-a)| < \varepsilon \iff |f(x) - f(a)| < \varepsilon$$

which is the same as stating that f is continuous everywhere.

6.8. By theorem 1 it is the case that $f + \alpha$ is continuous at a and $(f + \alpha)(a) = \alpha \neq 0$. For every $\varepsilon > 0$ there exists a $\delta > 0$ so that for every x satisfying $|x - a| < \delta$ it follows that

$$|f(x) - f(a)| < \varepsilon,$$

which is the same as

$$|\alpha| > |f(x) + \alpha + (-\alpha)| \leq |f(x) + \alpha| + |\alpha|$$

$$|f(x) + \alpha| < 0$$

This last inequality implies $f + \alpha$ is non zero. We conclude the proof by noting that $|x - a| < \delta$ is the same as $x - \delta < a < x + \delta$

6.9.

(a) This is proven directly by stating the logical negation of the epsilon-delta continuity definition.

(b) Deconstructing $|f(x) - f(a)| > \varepsilon$ yields $f(x) > f(a) + \varepsilon$ or $f(a) - \varepsilon > f(x)$

6.10.

(a) For every $\varepsilon > 0$ there exists an $\delta > 0$ such that for every x satisfying $|x - a| < \delta$ it follows that $|f(x) - f(a)| < \varepsilon$. Using one of the standard triangle inequalities it follows directly that $||f(x)| - |f(a)|| \leq< |f(x) - f(a)| < \varepsilon$.

(b) We have $f(x) = E(x) + O(x)$ and $f(-x) = E(x) - O(x)$, thus $O(x) = \frac{f(x) - f(-x)}{2}$ which is continuous by theorem 1. We also have $E(x) = \frac{f(-x) + f(x)}{2}$ of which is also continuous by theorem 1.

(c) Recalling the formula for $\max(x, y)$ and $\min(x, y)$ on exercise 13 (Chapter 1) and using theorem 1-2 proves the statement

(d) Define

$$g(x) = \begin{cases} f(x), & f(x) > 0 \\ 0, & f(x) \leq 0 \end{cases}$$

$$h(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0 \end{cases}$$

Testing the cases $f(x) > 0$ and $f(x) \leq 0$ will show that $f(x) = g - h$ for every x. It is also easy to see that g, h are continuous and non-negative.

6.11. Consider the function $f(x) = \frac{1}{x}$, it is continuous at every $a \neq 0$. Suppose g is continuous at any $g(a) \neq 0$. It follows directly from theorem 2 that $\frac{1}{g}$ is continuous at any a so long $g(a) \neq 0$.

6.12. (a) Consider the function $G(x) = g(x)$ for $x \neq a$, and $G(a) = l$. G is then continuous at a, so

$$\lim_{x \to a} G(x) = G(a) = l.$$

It is then the case that f is continuous at $l = G(a)$, using theorem 2 we know that $f \circ G$ must be continuous at a, that is

$$\lim_{x \to a} f(G(x)) = f(G(a)).$$

Notice that in the limit, $G(x) = g(x)$ because $x \neq a$. This means that we have

$$\lim_{x \to a} f(g(x)) = f(l).$$

(b) Suppose $f(x) = 0$ for $x \neq l$, and $f(l) = 1$. Also suppose $g(x) \neq l$ for $x \neq a$. Then $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$ implies $0 = 1$ which is obviously a contradiction.

6.13. Let

$$g(x) = \begin{cases} f(x), a \leq x \leq b \\ f(a), x < a \\ f(b), x > b \end{cases}$$

Chapter 7

Three Hard Theorems

7.1. (i) Bounded below and above. f takes on the minimum at $x = 0$

(ii) Bounded below and above. f does not take on maximum or minimum

(iii) Function takes on the minimum value at $x = 0$ and only bounded below

(iv) Same as (iii)

(v) If $a \leq -1$ then the set $[-a-1, a+1]$ is the empty set and thus it is not bounded. If $a > -1$ then since x^2 and $a+2$ is bounded below, f must be bounded below. We also have that f takes on the minimum value of 0 at $x = 0$. The function is also certainly bounded above because of the interval. Whether f takes on the maximum value depends on if $a + 2 \geq x^2$ when x is maximum, thus

$$\begin{aligned} a + 2 &\geq a^2 + 2a + 1 \\ 0 &\geq a^2 + a - 1 \\ \frac{-1-\sqrt{5}}{2} \leq a &\leq \frac{-1-\sqrt{5}}{2} \end{aligned}$$

But we must also remember $a > -1$, thus $-1 < a \leq \frac{-1-\sqrt{5}}{2}$ means that f takes on the maximum value.

If $\frac{-1-\sqrt{5}}{2} < a$ then f does not take on the maximum value.

(vi) The main difference from (v) is the fact that x^2 may now take on the maximum value—which was the only reason f did not take on the maximum value. The conclusion is therefore, if $a < -1$ then f is not bounded and does not take on any values. If $a \geq -1$ then f is bounded and takes on the maximum and maximum value.

(vii) f is bounded below 0 and is bounded above 1. The function takes on the minimum value when x is irrational and takes on the maximum value when $q = 1$, that is only when $x = 1, 0$.

(viii) f is bounded below and above. f does not take on a minimum value, but takes on the maximum when x is irrational or $x = 1, 0$.

(ix) f is bounded and takes on a maximum value of 1 and minimum value of -1

(x) If $a < 1$ then f does not take on any values and consequently is not bounded. Suppose otherwise, then clearly f is bounded below and takes on the minimum of 0. If a is rational, then the maximum is a. If a is irrational then f does not take on the maximum value.

(xi) If $0 > a$ then the function does not take on any values and thus the function is neither bounded above or below. If $a = 0$ then the function contains a single point of which is the maximum and minimum value.

If $a > 0$ then f is continuous and it takes on the minimum value and maximum value by theorem 3 and 7. Consequently this means f is bounded above and below.

(xii) Same thing here as in (xi)

7.2. (i) Let $n = -2$ then $f(n) = -3$ and $f(n+1) = 3$

(ii) Let $n = -5$ then $f(n) = -9$ and $f(n+1) = 249$

(iii) Let $n = -1$

(iv) Let $n = 0$

7.3. (i) Let $f(x) = x^{179} + \frac{163}{1+x^2+\sin^2 x}$. Setting $x = 0$ makes $f(0) = 163$. Using a calculator we find that $f(1) \approx 61.2$. Since f is continuous on this interval we may use theorem 5 to conclude that there exists an x such that $f(x) = 119$

(ii) Let $f(x) = \sin x - x$ we then find a, b such that $f(a) < -1 < f(b)$. Let $b = 0$ then $f(b) = 0$ Let $a = \pi$ then $f(a) = -\pi$. There is therefore an x such that $\sin x - x = -1$.

7.4. (i) We have $n \geq k$. Let $a_1, \ldots, a_k$ be distinct real numbers. Let

$$f(x) = \prod_{p=1}^{k}(x - a_p)\prod_{q=1}^{n-k}(x + i(-1)^q)$$

Then the polynomial has degree $k+n-k = n$ and k real roots and $n-k$ complex roots which are not counted as a real root.

(ii) Let $f(x) = h(x)g(x)$ such that $g(x)$ does not have any roots and $h(x)$ has all the roots. Then h is of degree k and g is of degree $n-k$.

Suppose for contradiction that $n-k$ is odd, then by theorem 9 g must have a root which is a contradiction. Thus $n-k$ is even.

7.5. The function must be constant because between any two rationals there exists a real number and for the function to reach more than one rational it would have to discontinuous.

7.6.

$$x^2 + (f(x))^2 = 1 \iff (f(x))^2 = 1 - x^2 \iff f(x) = \pm\sqrt{1-x^2}$$

Either case works because $(-f(x))^2 = (f(x))^2$.

7.7.

$$(f)^2 = x^2 \iff (f-x)(f+x) = 0 \iff f = x \text{ or } f = -x$$

Therefore, two functions.

7.8.

$$f^2 = g^2 \iff (f-g)(f+g) = 0 \iff f = g \text{ or } f = -g$$

7.9. (a) The function has a minimum at $x = a$ because $f(x) \neq 0$ for every $x \neq a$

(b) If $x > a$ then $f(x) > 0$ for all x. If $x < a$ then $f(x) < 0$ for all x.

(c) Consider the function $f(x) = x^4 - y^4 = 0$, all the the roots are $x = \pm y$. Since
$$x^4 - y^4 = (x-y)(x^3 + x^2y + xy^2 + y^3)$$
we can divide by $x - y$ to remove the $x = y$ root and thus the function
$$g(x) = x^3 + x^2y + xy^2 + y^3$$
has one unique root at $x = -y$. Consequently if $x > -y$ then $g(x) > 0$ and if $x < -y$ then $g(x) < 0$

7.10. Let $h(x) = f(x) - g(x)$ then $h(a) < 0 < h(b)$, by theorem 1 there is an x such that $h(x) = 0$, and consequently $f(x) = g(x)$.

Chapter 8

Least Upper Bounds

8.1. If maximum or minimum is not mentioned, then they do not exist

(i) Maximum is 1, infimum is 0.

(ii) Maximum is 1, minimum is -1.

(iii) Maximum is 1, minimum is 1.

(iv) Minimum is 0, supremum is $\sqrt{2}$.

(v) Unbounded.

(vi) The infimum is $\frac{-1-\sqrt{5}}{2}$ and the supremum is $\frac{-1+\sqrt{5}}{2}$.

(vii) The infimum is $\frac{-1-\sqrt{5}}{2}$ and the supremum is 0.

(viii) The infimum is -1 and the maximum is 1.5.

8.2. (a) Let

$$-A = \{ -x : x \in A \}$$

Suppose $A \neq \emptyset$, then there must exist an $a \in A$. Thus $-a \in -A$ by definition.

Suppose A is bounded below, then there is a $y \in \mathbb{R}$ such that $y \leq x$ for any $x \in A$. Equivalently we have $-y \geq -x$. This means that $-A$ is bounded above. Consequently there is a least upper bound $\sup(-A) = \alpha$ for $-A$. By definition $-x \leq \alpha \leq b$ for any upper bound

b for $-A$. It follows that $-b \leq -\alpha \leq x$ since $-b$ is any lower bound it means that $-\alpha = -\sup(-A)$ is the greatest lower bound.

(b) Suppose A is a nonempty set that is bounded below. Let B be the set containing every lower bound of A. Since A is bounded below, the set B is nonempty. Since B contains every upper bound it is certainly true that for every $x \in A$ and $y \in B$ we have $x \geq y$, that is B is bounded above by every element of A. Since B is bounded above there must be a least upper bound α (by property 13). Now it remains to show that α is a greatest lower bound.

Since A is a subset of the set of upper bounds for B we have in particular that for any $a \in A$ it is true that $\alpha \leq a$. This means α is a lower bound for A which satisfies (1) for the definition of a greatest lower bound.

Now, because α is an upper bound it is certainly true that for any $b \in B$ it follows that $\alpha \geq b$. Since B contains every upper bound for A this means (2) is satisfied. Thus we have shown that $\sup B = \alpha = \inf A$.

8.3. (a) There is not necessarily a second smallest x, consider a function which intersects the horizontal axis once, then there is only one x satisfying $f(x) = 0$.