Spivak's Calculus Solutions

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Chapter 1

Basic Properties of Numbers

1.1.

(i) Suppose that $ax = a$ and $a \neq 0$, then there exists a number a^{-1}. Multiplying a^{-1} on both sides yields

$$
\begin{aligned}
(a^{-1}a) \cdot x &= a^{-1}a \\
x &= 1
\end{aligned}
$$

as desired.

(ii) Applying the distributive property on $(x - y)(x + y)$ makes

$$
\begin{aligned}
&(x - y)(x + y) = (x - y)x + (x - y)y \\
&= x^2 - yx + xy - y^2 = x^2 - y^2
\end{aligned}
$$

(iii) If we have $x^2 = y^2$ then we certainly have $0 = x^2 - y^2$. By (ii) we have $0 = (x - y)(x + y)$, this implies that $x - y = 0$ or $x + y = 0$, this is equivalent to saying that $x = y$ or $x = -y$.

(iv) Same method as (ii):

$$
\begin{aligned}
&(x - y)(x^2 + xy + y^2) = (x - y)x^2 + (x - y)xy + (x - y)y^2 \\
&= x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3 \\
&= x^3 - y^3
\end{aligned}
$$

(v) We prove this by induction, the base case $n = 2$ is already proven in (ii). Suppose $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ is true. Then we equivalently have $x^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1})+y^n$. We now prove the n+1 case:

$$\begin{aligned} x^{n+1} - y^{n+1} &= x \cdot x^n - y^{n+1} \\ &= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + xy^n - y^{n+1} \\ &= (x-y)(x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}) + (x-y)y^n \\ &= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n) \end{aligned}$$

The resulting relation concludes the finite induction, thus $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi) We know from (iv) that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, by letting $a = x$ and $b = -y$ we get $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

1.2. Multiplying by the multiplicative inverse of $x-y$ works only when $x-y \neq 0$, that is $x \neq y$, however, the hypothesis explicitly states $x = y$. So it is not possible to find the multiplicative inverse of $x-y$ and thus the step is invalid.

1.3. (i) Say we have $\frac{a}{b}$ and $b \neq 0$ then the same fraction can be written as ab^{-1}. Suppose we also have a variable c such that $c \neq 0$, then we have $ab^{-1} \cdot (cc^{-1})$ and consequently $(ac)(b^{-1}c^{-1}) = \frac{ac}{bc}$. The final equality holds by (iii) which is proven below.

(ii) By (i) $\frac{ad}{bd} + \frac{bc}{db} = ad(bd)^{-1} + bc(bd)^{-1} = (ad+bc)(bd)^{-1} = \frac{ad+bc}{bd}$

(iii) ab exists if $a, b \neq 0$. Let $x = (ab)^{-1}$, then

$$\begin{aligned} &x(ab) = (ab)^{-1}(ab) = (xa)b = 1 && \text{(Multiply } x \text{ with } ab) \\ &(xa)(bb^{-1}) = b^{-1} = xa = b^{-1} && \text{(Multiply by } b^{-1}) \\ &x(aa^{-1}) = b^{-1}a^{-1} = x && \text{(Multiply by } a^{-1}) \end{aligned}$$

(iv) Suppose $b, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1}) \cdot (cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(v) Suppose $b, c, d \neq 0$, then $\frac{a}{b} \big/ \frac{c}{d} = (ab^{-1})(cd^{-1})^{-1} = (ab^{-1})(c^{-1}d) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(vi) Suppose $b, d \neq 0$. Assume $\frac{a}{b} = \frac{c}{d}$, multiplying by bd on both side yields the relation $ad = bc$. For the converse multiply $ad = bc$ by $(bd)^{-1}$.

1.4. (i) $4-x<3-2x \iff (4-4)+(-x+2x)<(3-4)+(2x-2x) \iff x<-1$.

(ii) $5-x^2<8 \iff -3<x^2$. Note that $x^2 \geq 0$ and for every single value of x, so our solution is every x.

(iii) $5-x^2<-2 \iff 7<x^2 \iff \sqrt{7}<x$ or $-\sqrt{7}>x$.

(iv) The product is positive when $x-1>0$ and $x-3>0$ or when $x-1<0$ and $x-3<0$, that is when $x>3$ or when $x<1$.

(v) Complete the square $x^2-2x+2=(x-1)^2+1$. The product $(x-1)^2$ is always positive, and since we have the $+1$ as well in the inequality, this inequality must be true for every single x.

(vi) The inequality is equivalent to $x^2+x-1>0$. Completing the square $(x+\frac{1}{2})^2>\frac{5}{4}$. If $x \geq -\frac{1}{2}$ then $x>\frac{-1+\sqrt{5}}{2}$. If $x<-\frac{1}{2}$ then $x<\frac{-1-\sqrt{5}}{2}$. Thus, the solution is $x>\frac{-1+\sqrt{5}}{2}$ and $x<\frac{-1-\sqrt{5}}{2}$.

(vii) Equivalently we have $(x-\frac{1}{2})^2>\frac{25}{4}$. If $x \geq \frac{1}{2}$ then $x>3$ if $x<\frac{1}{2}$ then $x<-2$. The solution set is $x>3$ and $x<-2$.

(viii) Equivalently $(x+\frac{1}{2})^2+\frac{3}{4}>0$. This is true for every x because $(x+\frac{1}{2}) \geq$ and $\frac{3}{4}>0$. Adding them gives $(x+\frac{1}{2})^2+\frac{3}{4}>0$.

(ix) Let $b=(x+5)(x-3)$. Then b is positive if $x>3$ or $x<-5$ and negative if $-5<x<3$. Let $a=x-\pi$. a is positive if $x>\pi$. ab is positive if both a and b are positive or if both are negative. So ab is positive if $x>\pi$ (b must be positive because $x>3$). ab is negative if $-5<x<3$ (This implies $x<\pi$).

(x) If $x>\sqrt[3]{2}$ and $x>\sqrt{2}$ then the product is positive, thus the first solution is $x>\sqrt{2}$. If $x<\sqrt[3]{2}$ and $x<\sqrt{2}$ then the product is positive. The second solution is $x<\sqrt[3]{2}$.

(xi) Apply $\log_2$ on both sides: $x<3$.

(xii) Suppose $x<1$, we will show this is a solution. We have $3^x<3^1=3$, adding $x<1$ to the inequality we get $x+3^x<3+1=4$. Since both 3^x and x are strictly increasing expressions finding the inequality $x<1$ suffices as all real solutions.

(xiii) Noting that $x \neq 0$ and $x \neq 1$. Expanding the fractions we get $\frac{1-x}{x(1-x))} + \frac{x}{x(1-x)} = \frac{1}{x(1-x)} > 0$. The solutions depends on if the denominator is positive. Thus $x(1-x) > 0$ has the same solution set. The solutions are $0 < x < 1$.

(xiv) Note $x \neq -1$. Expand by $(x+1)$: $\frac{(x-1)(x+1)}{(x+1)^2} > 0$. Since the denominator is always positive we can multiply this on both sides, $x^2 - 1 > 0$, Thus $x < -1$ and $x > 1$.

1.5. (i) Suppose $a < b$ and $c < d$ then we have $b - a > 0$ and $d - c > 0$ by property 11 $(b-a) + (d-c) > 0$ which is the same as $b + d > a + c$.

(ii) Suppose $a < b$ then $0 < b - a \iff -b < (b-b) - a = -b < -a$.

(iii) Suppose $a < b$ and $c < d$, by (ii): $-c < -d$, then by (i) we have $a - d < b - d$.

(iv) Suppose $a < b$ then $b - a > 0$. Assume $c > 0$, Using (P12) we know that $c(b-a) > 0$ and consequently $bc - ac > 0 \iff bc > ac$.

(v) Suppose $a < b$ then $b - a > 0$. Assume $c < 0$, then by (ii) we have $-c > 0$. Using P12 we know that $-c(b-a) > 0$ and consequently $ac - bc > 0 \iff ac > bc$.

(vi) Since $a > 1 > 0$ we apply (iv) by letting $c = a$. Thus $a^2 > a$.

(vii) Because a is positive, it follows by applying (iv) to $a < 1$ that $a^2 < a$.

(viii) Using (iv), multiply $a < b$ with c and $c < d$ with b. This means that we have $ac < bc$ and $bc < bd$, this is the same as $ac < bc < bd$, thus $ac < bd$.

(ix) Using (viii) we multiply the same inequality twice, $a^2 < b^2$.

(x) Suppose $a, b \geq 0$, we prove the contra-positive, therefore $a \geq b$. Multiply by a and b respectively gives two inequalities $a^2 \geq ab$ and $ab \geq b^2$ which is the same as $a^2 \geq ab \geq b^2$. This concludes the contra-positive proof because $a^2 \geq b^2$ is the logical opposite of $a^2 < b^2$.

1.6. (a) The base case is $n = 2$ which was proven in problem 1.5. Assume $x^n < y^n$ for $0 \leq x < y$. By problem 1.5. (viii) we have $x \cdot x^n < y \cdot y^n \iff x^{n+1} < y^{n+1}$. The induction is complete, thus if $0 \leq x, y$ then $x^n < y^n$ for $n = 1, 2, \ldots$.

(b) Suppose $x < y$ and $n = 2k + 1$, We have three cases.

(i) $x, y \geq 0$, this case has been proven in (a).

(ii) $x \leq 0$ and $y \geq 0$. Consider x^n, because n is odd, it has the following property, $x^{2k+1} = x \cdot (x^k)^2 < 0$, because x is negative and $(x^k)^2$ is positive. However $y^n > 0$ because y is positive. This means we have $x^n < 0 < y^n$.

(iii) $x, y < 0$, by the inequality we have $-x > 0$ and $-y > 0$. We also have $-y < -x$, by (a) we have $(-y)^n < (-x)^n \iff -y^n < -x^n$ because n is odd. Finally we have $x^n < y^n$.

(c) Suppose $x^n = y^n \iff x^n - y^n = 0 = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ Then either $x - y = 0$ or $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} = 0$ In the first case $x = y$, in the second case we first note that $x^n = y^n$ implies that x and y has the same sign and thus $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} \geq 0$ where the equality holds only when $x, y = 0$ then $x = y$ is still true.

(d) Let n be an even positive integer. Next we prove the contra-positive, suppose $|x| \neq |y|$ ($x = y$ or $x = -y$ is the same as saying $|x| = |y|$). Consequently this means either $|x| < |y|$ or $|x| > |y|$. By (a) this means that either $|x|^n < |y|^n$ or $|x|^n > |y|^n$. Because n is even this is equivalent to $x^n < y^n$ or $x^n > y^n$ which is the logical complement of $x^n = y^n$.

1.7. Suppose $0 < a < b$, multiply by a then $a^2 < ab \iff a < \sqrt{ab}$. Next consider $(a - b)^2 > 0$ which is equivalent to $a^2 + b^2 + 2ab > 4ab \iff \frac{a+b}{2} > \sqrt{ab}$, this means that we have $a < \sqrt{ab} < \frac{a+b}{2}$ now remains the final inequality. By the premise we have $a - b < 0 \iff a + b < 2b \iff \frac{a+b}{2} < b$. We conclude by stating $a < \sqrt{ab} < \frac{a+b}{2} < b$.

***1.8.** (P10) Let $b = 0$ in P'10, then for every a one of the following properties apply

(i) $a = 0$

(ii) $a < 0$

(iii) $a > 0$

Because the collection P contains all the numbers x such that $x > 0$, we can see that (iii) states that a belongs to P. (ii) is equivalent to $-a > 0$, thus -a is in P.

(P11) Suppose x and y are in P then $0 < x$ and $0 < y$. By P'12 (Let a=0) we have $x < y + x$. By P'11 we get $0 < y + x$ which is in P.

(P12) Suppose x and y are in P then $0 < x$ and $0 < y$. Using P'13 we get $0 < xy$, this means that xy is in P.

1.9. (i) $\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$.

(ii) Triangle inequality states that $|a+b| - |a| - |b| \leq 0$. Therefore $|a| + |b| - |a + b|$.

(iii) Triangle inequality gives $|(a + b) + c| - |a + b| - |c| \leq 0 \iff |a + b| + |c| - |a + b + c|. \geq 0$. Our solution is therefore $|a + b| + |c| - |a + b + c|$.

(iv) $x^2 - 2xy + y^2 = (x - y)^2 \geq 0$, thus $x^2 - 2xy + y^2$.

(v) $\sqrt{2} + \sqrt{3} + \sqrt{5} - \sqrt{7}$

1.10. (i) Suppose $a + b \geq 0$ and $b \geq 0$ then we have $a + b - b = a$. Suppose $a + b \geq 0$ and $b < 0$ then $a + b + b = a + 2b$. Suppose $a + b < 0$ and $b \geq 0$ then $-a - b - b = -(a + 2b)$. Suppose $a + b < 0$ and $b < 0$ then $-a - b + b = -a$.

(ii) If $0 \geq x \geq 1$ then $1 - x$. If $-1 \geq x < 0$ then $1 + x$. If $1 < x$ then $x - 1$ then $-x - 1$.

(iii) If $x \geq 0$ then $x - x^2$, if $x < 0$ then $-x - x^2$.

(iv) If $a \geq 0$ then a, if $a < 0$ then $3a$.

1.11. (i) Suppose $x - 3 > 0$ then $x - 3 = 8 \iff x = 11$. Suppose $x - 3 < 0$ then $3 - x = 8 \iff x = -5$.

(ii) Suppose $x - 3 \geq 0$ then $3 \leq x < 11$. Suppose $x - 3 < 0$ then $-5 < x < 3$. Combining both inequalities $-5 < x < 11$.

(iii) Suppose $x + 4 \geq 0$ then $x < -2$, so $-4 \leq x < -2$. If $x + 4 < 0$ then $-6 < x < -4$. Combining both inequalities gives $-6 < x < -2$.

(iv) Suppose $x \leq 2$ then $x - 1 + x - 2 > 1 \iff x > 2$. This means $x > 2$ is always a solution. Suppose $1 \leq x < 2$, then $x - 1 - x + 2 > 1 \iff 1 > 1$, which can not be true. Suppose $x < 1$, then $1 - x - x + 2 > 1 \iff x < 1$. The solution is $x < 1$ and $x > 2$.

(v) Suppose $x \geq 1$ then $x-1+x+1 < 2 \iff x < 1$ which is a contradiction. Suppose $-1 \leq x < 1$ then $1 - x + x + 1 < 2 \iff 2 < 2$, also contradiction. Suppose $x < -1$ then $1 - x - x - 1 < 2 \iff x > -1$, an x that satisfies the inequality is nonexistent.

(vi) Suppose $x \geq 1$ then $x-1+x+1 < 1 \iff x < \frac{1}{2}$ which is a contradiction. Suppose $-1 \leq x < 1$ then $1 - x + x + 1 < 1 \iff 2 < 1$,also a contradiction. Suppose $x < -1$ then $1 - x - x - 1 < 1 \iff x > -\frac{1}{2}$, similarly to (iv), there are no x that satisfy the inequality.

(vii) We have $x - 1 = 0 \iff x = 1$ or $x + 1 = 0 \iff x = -1$.

(viii) Suppose $x \geq 1$ then $(x-1)(x+2) = 3 \iff x^2+x-5 = 0 \iff (x+\frac{1}{2})^2 = \frac{21}{4} \implies x = \frac{-1+\sqrt{21}}{2}$. Suppose $-2 \leq x < 1$ then $(1-x)(x+2) = 3$ which is a polynomial with complex roots thus no solutions there. Suppose $x < -2$, then we get the same polynomial as in the first case because $(-1)^2 = 1$, so the other root is $x = \frac{-1-\sqrt{21}}{2}$ which is less than -2 because $\frac{-1-\sqrt{21}}{2} < \frac{-1-\sqrt{16}}{2} = \frac{-5}{2} < -2$. To conclude $x = \frac{-1\pm\sqrt{21}}{2}$

1.12. (i) $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 \iff |xy| = |x| \cdot |y|$

(ii) Consider $\left|\frac{1}{x}\right|$ for $x \neq 0$. This is the same as $\sqrt{(\frac{1}{x})^2} = \sqrt{\frac{1}{x^2}} = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$.

(iii) Suppose $y \neq 0$ then $\left|\frac{x}{y}\right| = \sqrt{(\frac{x}{y})^2} = \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|}$

(iv) Suppose a, b are real numbers, then the triangle inequality is $|a + b| \leq |a| + |b|$. Let $a = x$ and $b = -y$ then $|x - y| \leq |x| + |-y| = |x| + |y|$. The final equality is proven by $|-y| = \sqrt{(-y)^2} = \sqrt{(-1)^2y^2} = \sqrt{y^2}$.

(v) Using the triangle inequality $|x| = |(x - y) + y| \leq |x - y| + |y| \iff |x| - |y| \leq |x - y|$

(vi) There are two cases from the inequality, $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |y - x|$, note that the last absolute value comes from the fact $|x - y| = |y - x|$. Both inequalities are identical to (v) (the second inequality has the variables interchanged).

(vii) We have $|(x + y) + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$. Doing the case work for the equality is left to the reader.

1.13. We start by proving for max, let $x \geq y$ then $\max(x,y) = \frac{x+y+x-y}{2} = x$ Likewise if $y \geq x$ then $\max(x,y) = y$. Similar reasoning shows that the formula for $\min(x,y)$ is valid. Next we use substitution and get $\max(x,y,z) = \max(x, \max(y,z)) = \frac{y+z+2x+|y-z|+|y+z+2x+|y-z||}{4}$ and $\min(x,y,z) = \min(x, \min(y,z)) = \frac{y+z+2x+|y-z|-|y+z+2x+|y-z||}{4}$.

1.14.

(a) Suppose $a \geq 0$ then we have $a = -(-a)$. The case for $a \leq 0$ is then obvious because we have $(-a) \geq 0$ which can be used on the previously proven fact.

(b) ($\Rightarrow$) Suppose $-b \leq a \leq b$, this implies $a \leq b$ and $-b \leq a \iff -a \leq b$ and consequently $|a| \leq b$.
($\Leftarrow$) Suppose $|a| \leq b$ then $a \leq b$ and $-a \leq b \iff -b \leq a$, thus $-b \leq a \leq b$. Now we prove the last part. Suppose $|a| \leq |a|$ then by the previously proven theorem we have $-|a| \leq a \leq |a|$.

(c) As proven earlier, for every a, b we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Add these together gives $-(|a| + |b|) \leq a + b \leq |a| + |b|$, applying the theorem from (b) on $(|a| + |b|)$ and $(a + b)$ we get $|a + b| \leq |a| + |b|$.

***1.15.** We prove first that if $x = y$ and $x, y \neq 0$. The inequality is then $x^2 + x^2 + x^2 > 0 \iff x^2 > 0$ which is true because $x \neq 0$.

Suppose $x \neq y$, then the left side of inequality is equivalent to $(x^2 + xy + y^2) = \frac{x^3 - y^3}{(x-y)}$. Suppose $x > y$ then $x^3 - y^3 > 0$ by problem 6 (b), since both the numerator and denominator are positive we know that $\frac{x^3-y^3}{(x-y)} > 0$. Next we assume $x < y$ which implies $x^3 - y^3 < 0$ by problem 6 (b). This means the numerator and denominator are both negative, thus $\frac{x^3-y^3}{(x-y)} > 0$. In every case the inequality is positive, thus we have proven that $x^2 + xy + y^2 > 0$.

To prove that the second inequality holds we follow the same steps, suppose $x = y$ which means the inequality is $5x^4 > 0$. Next suppose $x \neq y$ then we have $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5-y^5}{x-y}$. Suppose $x - y > 0$ then $x^5 - y^5 > 0$ which implies $\frac{x^5-y^5}{x-y} > 0$. Assume $x - y < 0$ then $x^5 - y^5 < 0$ which implies $\frac{x^5-y^5}{x-y} > 0$.

***1.16.**

(a) $(x+y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 \iff xy = 0$ which implies $x = 0$ or $y = 0$. Next we have $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 \iff x^2y + xy^2 = 0 = xy(x+y)$. Which implies either $x = 0$ or $y = 0$ or $x = -y$.

(b) Consider $3(x+y)^2 = 3x^2+6xy+3y^3 \geq 0$, since $x, y \neq 0$ we have $x^2 > 0$ and $y^2 > 0$, adding these inequalities makes $4x^2 + 6xy + 4y^2 > 0$. If $x, y = 0$ then the statement would be false.

(c) Equivalently we have $4x^3y + 6x^2y^2 + 4y^3x = xy(4x^2 + 6xy + 4y^2)$, left side indicates that it is equal to zero when $x = 0$ or $y = 0$. Thus $(x + y)^4 = x^4 + y^4$ when $x = 0$ or $y = 0$.

(d) Subtract with $x^5 + y^5$ and since $xy \neq 0$ we divide by $5xy$ this makes $x^3+2x^2y+2xy^2+y^3 = 0 \iff (x+y)^3 = x^2y+y^2x = xy(x+y)$. Suppose $x+y \neq 0$ then $xy = (x+y)^2 \iff x^2+xy+y^2 = 0$, this implies $x, y = 0$ by letting $p = x^2+xy+y^2 \iff 2p = 2x^2+2xy+2y^2 = x^2+y^2+(x+y)^2$, it then follows all the terms have to be zero because they are either zero or positive, $x = 0$ and $y = 0$, this contradicts the fact that $xy = 0$, thus it must be the case that $x = -y$.

Assume this time that $x = 0$ then $(x + y)^5 = x^5 + y^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \iff y^5 = y^5$. By interchanging x with y in the last sentence it follows that $x = 0$ or $y = 0$. To conclude, the solutions are $x = -y$ or $x = 0$ or $y = 0$. My guess is that the same solutions apply to $(x + y)^n = x^n + y^n$ if n is odd and $x = 0$ or $y = 0$ if n is even.

1.17. (a) $2x^2 - 3x + 4 = 2(x - \frac{3}{4})^2 + y \implies y = \frac{32}{8} - \frac{9}{8} = \frac{23}{8}$

(b) Subtract $2(y + 1)^2$ this makes $x^2 - 3x$. Let $x^2 - 3x = (x - \frac{3}{2}) + z$ then $z = -\frac{9}{4}$, z is the smallest value.

(c) Let m be the minimum number for a simple second degree polynomial, then it follows that $x^2+bx+c = 0 = (x+\frac{b}{2})^2+m = x^2+bx+\frac{b^2}{4}+m \iff m = c - \frac{b^2}{4}$

We have $x^2 + 4xy + 5y^2 - 4x - 6y + 7 = x^2 + (4y - 4)x + 5y^2 - 6y + 7$ The minimum is thus $m = 5y^2 - 6y + 7 - 4(y^2 - 2y + 1) = y^2 + 2y + 3 = (y + 1)^2 + 2$. This implies that 2 is in fact the minimum value.

1.18. (a) $x = \frac{-b\pm\sqrt{b^2-4c}}{2} \iff (2x+b)^2 = b^2-4c \iff 4x^2+4xb+b^2-b^2+4c = 0 \iff x^2 + bx + c = 0$.

(b) We complete the square, $x^2 + bx + c = 0 \iff 4(x + \frac{b}{2})^2 = b^2 - 4c$ this follows that $(x + \frac{b}{2})^2 \geq 0$, but $b^2 - 4c < 0$ which is a contradiction. It also follows that $x^2 + bx + c > 0$ which means there are no real values of x that satisfy the equation.

(c) We complete the square $(x + \frac{y}{2})^2 + \frac{3y^2}{4}$. Since $\frac{3y^2}{4} > 0$ because $y \neq 0$ it must be the case that $(x + \frac{y}{2})^2 + \frac{3y^2}{4} > 0$ which is the same as $x^2 + xy + y^2 > 0$

(d) Completing the square makes $(x + \frac{\alpha y}{2})^2 + y^2(1 - \frac{\alpha^2}{4})$. The left term has the property $(x + \frac{\alpha y}{2})^2 \geq 0$ (just let $x = -\frac{\alpha y}{2}$). This means the right term must be positive. Let $1 - \frac{\alpha^2}{4} > 0$ which implies $-2 < \alpha < 2$.

(e) $ax^2 + bx + c = a(x^2 + \frac{bx}{a}) + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a^2}$. Since $a > 0$ the minimum must be when $x + \frac{b}{a} = 0$, so the minimum is $c - \left(\frac{b}{2a}\right)^2$. (The first case is just $a = 1$)

1.19. (a) Suppose $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ then the equality holds if $\lambda({y_1}^2 + {y_2}^2) = \sqrt{\lambda^2({y_1}^2 + {y_2}^2)}\sqrt{({y_1}^2 + {y_2}^2)} \iff \lambda = |\lambda|$. Seems to be some kind of error (edition 3) because it does not hold if λ is negative. Let's assume $\lambda \geq 0$. The then equality holds. The equality also holds if $y_1 = y_2 = 0$ because both factors on both sides are equal to zero.

Assume y_1 and y_2 is not equal to zero. Then there does not exist a λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$, the problems states that this implies $\lambda^2({y_1}^2 + {y_2}^2) - 2\lambda(x_1 y_1 + x_2 y_2) + ({x_1}^2 + {x_2}^2) > 0$. This equation is of the form $\lambda^2 + b\lambda + c > 0$ and since there does not exist any λ we have $b^2 < 4ac$ by noticing that dividing by a in the equation $ax^2 + bx + c = 0$ you can apply problem 18 (b), that is $(x_1 y_1 + x_2 y_2)^2 < ({y_1}^2 + {y_2}^2)({x_1}^2 + {x_2}^2)$. This follows that $|x_1 y_1 + x_2 y_2| < \sqrt{{y_1}^2 + {y_2}^2}\sqrt{{x_1}^2 + {x_2}^2}$

To conclude we have

$$x_1 y_1 + x_2 y_2 \leq |x_1 y_1 + x_2 y_2| \leq \sqrt{{y_1}^2 + {y_2}^2}\sqrt{{x_1}^2 + {x_2}^2}.$$

(b) We start with $(x - y)^2 \geq 0 \iff 2xy \leq x^2 + y^2$. Suppose $x_1, x_2, y_1, y_2 \neq 0$ and let $x = \frac{x_i}{\sqrt{{x_1}^2 + {x_2}^2}}$, $y = \frac{y_i}{\sqrt{{y_1}^2 + {y_2}^2}}$ for $i = 1, 2$. It follows that

$$\begin{cases} \dfrac{2x_1 y_1}{\sqrt{{x_1}^2 + {x_2}^2}\sqrt{{y_1}^2 + {y_2}^2}} \leq \dfrac{{x_1}^2}{{x_1}^2 + {x_2}^2} + \dfrac{{y_1}^2}{{y_1}^2 + {y_2}^2} \\ \dfrac{2x_2 y_2}{\sqrt{{x_1}^2 + {x_2}^2}\sqrt{{y_1}^2 + {y_2}^2}} \leq \dfrac{{x_2}^2}{{x_1}^2 + {x_2}^2} + \dfrac{{y_2}^2}{{y_1}^2 + {y_2}^2} \end{cases}$$

Add both inequalities together, then it follows that $x_1y_1 + x_2y_2 \leq \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2}$.

If we assume $x_i = 0$ or $y_i = 0$ for $i = 1, 2$ then either all the terms will be zero or the resulting inequality is for example $0 \leq y_1{}^2$ (let $x_1 = 0$).

(c) $(x_1{}^2 + x_2{}^2)(y_1{}^2 + y_2{}^2)$

$$\begin{aligned}
&= (x_1y_1)^2 + 2(x_1y_1)(x_2y_2) + (x_2y_2)^2 + (x_2y_1)^2 - 2(x_2y_1)(x_1y_2) + (x_1y_2)^2 \\
&= (x_1y_1 + x_2y_2)^2 + (x_2y_1 - x_1y_2)^2 \geq (x_1y_1 + x_2y_2)^2 \\
&\iff \sqrt{x_1{}^2 + x_2{}^2}\sqrt{y_1{}^2 + y_2{}^2} \geq |x_1y_1 + x_2y_2| \geq x_1y_1 + x_2y_2
\end{aligned}$$

(d) The problem is constructed to waste time, see (a) where we already proved it. It shows that if $y_1 = 0$ and $y_2 = 0$ or there exists a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ then the equality holds, otherwise $|x_1y_1 + x_2y_2| < \sqrt{y_1{}^2 + y_2{}^2}\sqrt{x_1{}^2 + x_2{}^2}$.

1.20. Add both inequalities, $|x-x_0|+|y-y_0| < \varepsilon$. We apply the triangle inequality which makes $|(x + y) - (x_0 + y_0)| \leq |x - x_0| + |y - y_0| < \varepsilon$. For the second inequality, notice that that $|y - y_0| = |y_0 - y|$. So the triangle inequality makes $|(x - y) - (x_0 - y_0)| \leq |x - x_0| + |y_0 - y| < \varepsilon$.

***1.21.** Suppose $|x - x_0| < \frac{\varepsilon}{2(|y_0|+1)}$, then $2|x - x_0|(|y_0| + 1) < \varepsilon$. Now assume $|y - y_0| < \frac{\varepsilon}{2(|y_0|+1)}$ then $2|y - y_0|(|x_0| + 1) < \varepsilon$. Sum the two similar inequalities

$$\begin{aligned}
&2|x - x_0|(|y_0| + 1) + 2|y - y_0|(|x_0| + 1) < 2\varepsilon \\
&|x - x_0|(|y_0| + 1) + |y - y_0|(|x_0| + 1) < \varepsilon \\
&|y_0||x - x_0| + |x - x_0| + |x_0||y - y_0| + |y - y_0| < \varepsilon
\end{aligned}$$

Now suppose $|x-x_0| < 1$ then we have $|y-y_0||x-x_0| < |y-y_0|$. Continuing on the expression above we get

$$\begin{aligned}
&> |y_0||x - x_0| + |x_0||y - y_0| + |y - y_0| \\
&> (|y_0| + |y - y_0|)(|x - x_0|) + |x_0||y - y_0| \\
&\geq |y||x - x_0| + |x_0||y - y_0| \geq |xy - x_0y + x_0y - x_0y_0| = |xy - x_0y_0|
\end{aligned}$$

Therefore we have $|xy - x_0y_0| < \varepsilon$.

***1.22.** We first prove that $y \neq 0$. Suppose $|y - y_0| < \frac{|y_0|}{2}$ then by problem 12, we get $|y_0| < 2|y|$ by problem 12. By supposing $y = 0$ we get a contradiction because $0 < |y_0|$ thus it must be the case that $y \neq 0$.

Now we prove the latter. Suppose $|y - y_0| < \frac{\varepsilon |y_0|^2}{2}$. Then

$$\begin{aligned} |y - y_0| &< \varepsilon |y_0||y| \\ \left| \frac{y_0 - y}{y_0 y} \right| &< \varepsilon \\ \left| \frac{1}{y_0} - \frac{1}{y} \right| &< \varepsilon \end{aligned}$$

as desired.

***1.23.** We begin first by using problem 21. We can then state that if $y \neq 0$, $y_0 \neq 0$, $\left| \frac{1}{y} - \frac{1}{y_0} \right| < \frac{\varepsilon}{2(|x_0|+1)}$ and $|x - x_0| < \min\left(\frac{\varepsilon}{2(\left|\frac{1}{y_0}\right|+1)}, 1 \right)$ then we have $\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \varepsilon$. Now we need to modify the hypothesis. We have that $y_0 \neq 0$ and $|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon |y_0|^2}{2} \right)$ implies $y \neq 0$ and the hypothesis earlier.

To conclude, $y_0 = 0$, $|y - y_0| < \min\left(\frac{|y_0|}{2}, \frac{\varepsilon |y_0|^2}{2} \right)$ and $|x - x_0| < \min\left(\frac{\varepsilon}{2(\left|\frac{1}{y_0}\right|+1)}, 1 \right)$ implies $y \neq 0$ and $\left| \frac{x}{y} - \frac{x_0}{y_0} \right| < \varepsilon$.

***1.24.**

(a) We prove the base case (k=2) with the associative law, $(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$. Next we suppose $P(k)$: $(a_1 + \cdots + a_k) + a_{k+1} = a_1 + \cdots + a_{k+1}$, then we prove for $P(k+1)$:

$$\begin{aligned} (a_1 + \cdots + a_{k+1}) + a_{k+2} &= [(a_1 + \cdots + a_k) + a_{k+1}] + a_{k+2} \\ (a_1 + \cdots + a_k) + (a_{k+1} + a_{k+2}) &= a_1 + \cdots + k_{k+2} \end{aligned}$$

This concludes the induction.

(b) We will prove this by induction on n, suppose $n \geq k$ and $(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n$. The base case is $n = k+1$ which was proven in the previous problem. We will now show the equality holds for $n+1$, we have

$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_{n+1})$
$= (a_1 + \cdots + a_k) + ((a_{k+1} + \cdots + a_n) + a_{n+1})$
$= ((a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n)) + a_{n+1}$
$= (a_1 + \cdots + a_n) + a_{n+1}$
$= a_1 + \cdots + a_{n+1}$

We have now proven that for $n \geq k$ it follows that

$$(a_1 + \cdots + a_k) + (a_{k+1} + \cdots + a_n) = a_1 + \cdots + a_n.$$

(c) We will show that $s(a_1, \ldots, a_k) = s(a_1) + \cdots + s(a_k)$ by induction on k. Let the base case be $k = 1$, then we obviously have an equality. Now we assume $s(a_1, \ldots, a_k) = s(a_1) + \cdots + s(a_k)$ and now prove for the $k + 1$ case.
$s(a_1, \ldots, a_{k+1}) = s(a_1, \ldots, a_k) + s(a_{k+1})$
$\qquad = s(a_1) + \cdots + s(a_{k+1})$

Because $s(a_1) + \cdots + s(a_k) = a_1 + \cdots + a_k$, our proof is done.

1.25. We suppose the rules of addition and multiplication given in the problem we then prove it is a field.

(i) Testing each case is tedious and will not be contained here, but we find that $a + (b + c) = (a + b) + c$ works.

(ii) Suppose $a = 0$ then $0+0 = 0+0 = 0$, and $a = 1$ implies $1+0 = 0+1 = 0$

(iii) If $a = 0$ then then let $-a = 0$ and if $a = 1$ then $-a = 1$.

(iv) This works by exhaustion.

(v) If at least one variable is zero, then $0 = 0$, otherwise $1 \cdot (1 \cdot) = (1 \cdot 1) \cdot 1 \iff 1 = 1$

(vi) Suppose $a = 0$ then $1 \cdot 0 = 1 \cdot 0 = 0$, suppose $a = 1$ then $1 \cdot 1 = 1 \cdot 1 = 1$

(vii) $a = 0$ is not allowed so we only prove for the $a = 1$ case which makes $a^{-1} = 1$.

(viii) If at least one variable is equal to zero then we have $0 = 0$, otherwise $1 \cdot 1 = 1 \cdot 1$

(ix) Suppose $a = 0$ then $0 \cdot (b + c) = 0 \cdot b + 0 \cdot c = 0$. Suppose $a = 1$ then $1 \cdot (b + c) = 1 \cdot b + 1 \cdot c = b + c$

Chapter 2

Numbers of various sorts

2.1. (i) The formula is clearly true for $n = 1$. Suppose

$$1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

then

$$1^2 + \cdots + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + n^2 + 2n + 1$$
$$\frac{2n^3 + 3n^2 + n + 6n^6 + 12n + 6}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

(ii) The base case $n = 1$ is obviously true, suppose

$$(1 + \cdots + n)^2 = 1^3 + \cdots + n^3$$

then

$$((1 + \cdots + n) + (n+1))^2 = (1 + \cdots + n)^2 + 2(1 + \cdots + n)(n+1) + (n+1)^2$$
$$= (1^3 + \cdots + n^3) + n(n+1)^2 + (n+1)^2 = 1^3 + \cdots + n^3 + (n+1)^3$$

2.2. (i) We solve by rewriting the sum,

$$1 + 3 + 5 + \cdots + (2n-1) = 1 + \cdots + 2n - (2 + 4 + \cdots 2n)$$
$$1 + \cdots + 2n - 2(1 + 2 + \cdots n) = n(2n+1) - n(n+1)$$
$$= n^2$$

Thus $\sum_{i=1}^{n}(2i-1) = n^2$.

(ii) Using similar methods as before:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = 1^2 + \cdots + (2n)^2 - (2^2 + 4^2 + \cdots + (2n)^2)$$

$$1^2 + \cdots + (2n)^2 - 4(1^2 + 2^2 + \cdots + n^2) = \frac{2n(2n+1)(4n+1)}{6} - \frac{4n(n+1)(2n+1)}{6}$$
$$\frac{2n[8n^2 + 6n + 1 - 2(2n^2 + 3n + 1)]}{6} = \frac{8n^3 - 2n}{6}$$

We conclude that $\sum_{i=1}^{n}(2i-1)^2 = \frac{8m^3-2n}{6}$

2.3. (a) The easiest way seems to be starting at the right side.

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$
$$\frac{kn!}{k!(n-k+1)!} + \frac{(n-k+1)n!}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$$

(b) We perform induction on n. Let the base case be $n = 1$ then by the definition of the binomial coefficients we have two cases, $\binom{1}{0}$ and $\binom{1}{1}$, both of these are equal to one by definition.
TODO: Is this base case actually valid? $n = 1$ does not use every part of the definition. If the definition can be different for numbers other than the base case does it invalidate the proof? Must there be multiple base cases involved testing each part of the definition?

Suppose $\binom{n}{k}$ is a natural number for every $0 \leq k \leq n$, then it follows that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ is a natural number because $\binom{n}{k-1}$ and $\binom{n}{k}$ are both natural numbers.

(c) Suppose we can chose any number only once from $1, 2, \ldots, n$. First we have n different choices, then $n-1$ choices and so on. If we do this n times we eventually only have one number left to choose. The numbers have $n!$ different sequences from which we can choose. Now we want to count the number of integers we can choose with only k choices. If $k = 1$ then there is only one choice, n. If $k = 2$ then we have $n \cdot (n-1)$, if we continue this we notice that this is the same as cutting the smaller factors in $n!$. Thus the number of ways we can do this is $\frac{n!}{(n-k)!}$. This is also known as the numbers of permutations of n of length k. Notice that the order in which these numbers are chosen are important. However, in a set the order does in fact not matter. The number of permutations of length k is $k!$, so finally we divide the permutations formula by k which means that we have $\frac{n!}{k!(n-k)!}$ and by the definition of the binomial coefficients this is the same as $\binom{n}{k}$. Because $\binom{n}{k}$ is the number of sets with exactly k integers chosen by n we have that $\binom{n+1}{k}$ is the number of sets with exactly k integers chosen by $n+1$. This implies that the case for $n+1$ is also a finite amount.

(d) Base case: $n = 1$ then $(a+b)^1 = \sum_{k=0}^{1} \binom{n}{k} a^{n-k} b^k = a + b$. Before we continue, a new notation must be introduced, $\sum_{0 \le k \le n} a_k$, this means the sum over k from 1 to n. Suppose

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n.$$

then

$$\sum_{0 \le k \le n+1} \binom{n+1}{k} a^{n+1-k} b^k = a^{n+1} + b^{n+1} + \sum_{1 \le k \le n} \binom{n+1}{k} a^{n-k+1} b^k$$

$$a^{n+1} + \sum_{1 \le k \le n} \binom{n}{k} a^{n-k+1} b^k + \sum_{1 \le k \le n+1} \binom{n}{k-1} a^{n-k+1} b^k$$

$$\sum_{0 \le k \le n} \binom{n}{k} a^{n-k+1} b^k + \sum_{0 \le k-1 \le n} \binom{n}{k-1} a^{n-(k-1)} b^{(k-1)+1}$$

(Substitute $k-1$ for k)

$$a \sum_{0 \le k \le n} \binom{n}{k} a^{n-k} b^k + b \sum_{0 \le k \le n} \binom{n}{k} a^{n-k} b^k$$

$$(a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^{n+1}$$

To conclude, we have proven the equality

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^n$$

(e) Using similar steps as above we can derive the transformation

$$\sum_{k=0}^{n+1} a_k \binom{n+1}{k} = \sum_{k=0}^{n} a_k \binom{n}{k} + \sum_{k=0}^{n} a_{k+1} \binom{n}{k}$$

(i) For the sake of base case, let $n = 0$, then we have $\binom{0}{0} = 1$. Suppose $\sum_{j=0}^{n} \binom{n}{j} = 2^n$ then

$$\sum_{0 \leq j \leq n+1} \binom{n+1}{j} = \sum_{0 \leq j \leq n} \binom{n}{j} + \sum_{0 \leq j \leq n} \binom{n}{j}$$
$$2 \sum_{0 \leq j \leq n} \binom{n}{j} = 2^{n+1}$$

(ii) The base case $n = 0$ does not seem to work, therefore we try $n = 1$ then we have $\binom{n}{1} - \binom{1}{1} = 1 - 1 = 0$. Suppose $\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$. Then by replacing n with $n + 1$ we do the same transformation as earlier

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{k} = \sum_{j=0}^{n} (-1)^j \binom{n}{k} + \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{k}$$
$$\sum_{j=0}^{n} (-1)^j \binom{n}{k} - \sum_{j=0}^{n} (-1)^j \binom{n}{k} = 0.$$

(iii) Subtract (i) with (ii), then

$$\binom{n}{0} - \binom{n}{0} + \binom{n}{1} + \binom{n}{1} + \cdots = 2^n - 0$$

The end of the sum depends on whether n is odd or even so we don't explicitly write it down. Notice that the all the even binomials cancel out, thus we have

$$2 \sum_{l \text{ odd}} \binom{n}{l} = 2^n \iff \sum_{l \text{ odd}} \binom{n}{l} = 2^{n-1}$$

(iv) Subtracting (i) with (iii), it follows that we only have the even binomials left, the other part of the formula is then $2^n - 2^{n-1} = 2^{n-1}$. It follows that

$$\sum_{l \text{ even}} \binom{n}{l} = 2^{n-1}.$$

2.4. (i) We first need to prove an important property of sums,

$$\left(\sum_{i=0}^{n} a_i\right)\left(\sum_{j=0}^{m} b_j\right) = (a_1 + \cdots + a_n)(b_1 + \cdots + b_m)$$

$$= a_1(b_1 + \cdots + b_m) + \cdots + a_n(b_1 + \cdots + b_m) \quad \text{(Distributive property)}$$

$$= \sum_{i=0}^{n}\left(a_i \sum_{j=0}^{m} b_j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j \qquad (a_i \text{ is a constant in the } b_j \text{ sum})$$

Note that we can interchange the summation symbols if we were to apply the distributive property as $b_1(a_1 + \cdots + a_n) + \cdots b_m(a_1 + \cdots + a_n)$ and then continue with similar steps, thus we have

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j = \sum_{j=0}^{m}\sum_{i=0}^{n} a_i b_j.$$

Now consider the polynomial $(1+x)^n(1+x)^m$, we can write this as

$$\sum_{l=0}^{n+m} \binom{n+m}{l} x^l \tag{1}$$

and

$$\left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)\left(\sum_{j=0}^{m} \binom{m}{j} x^j\right) = \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} x^{i+j} \tag{2}$$

The important thing is to realize that both sums represent the same polynomial. We must now use the fact that if a polynomial is equal for any x then they must have the same coefficients. This theorem can be proven as a corollary to the problem 3.7. (c).

The coefficients for x^l is $\binom{n+m}{l}$ as stated above (1). By the theorem recently stated we know that the coefficients in the other sum must be the same. Therefore, it makes sense to gather all the coefficients to each indeterminate. To do this we will use the equality

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{i+j} = \sum_{l=0}^{n+m}\sum_{i=0}^{n} a_l \qquad \text{where } l = i + k$$

Proof. Let $l = i + j$, then

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{i+j} = \sum_{0\le i\le n}\sum_{0\le l-i\le m} a_l \qquad \text{(Substitute } j = l - i\text{)}$$

$$\sum_{0\le i\le n}\sum_{0\le l\le m+n} a_l = \sum_{l=0}^{m+n}\sum_{i=0}^{n} a_l. \qquad \text{(Add } 0 \le i \le n \text{ to the left bound)}$$

Applying the transformation to (2) we get the following equalities

$$\sum_{l=0}^{n+m}\sum_{i=0}^{n}\binom{n}{i}\binom{m}{l-i}x^l = \sum_{l=0}^{n+m}\binom{n+m}{l}x^l$$

$$\sum_{i=0}^{n}\binom{n}{i}\binom{m}{l-i} = \binom{n+m}{l}$$

The last equality holds by the theorem stated earlier that the if two polynomials are equal then they have the same coefficients.

(ii) By letting $n = l = m$ in the equality before we get

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

The last equality holds because

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

2.5. We will not prove this inductively because it is trivial. Suppose $S = 1 + r + \cdots + r^n$ then

$$rS+1 = 1+r+r^2+\cdots r^{n+1} = S+r^{n+1} \iff 1-r^{n+1} = S(1-r) \iff S = \frac{1-r^{n+1}}{1-r}$$