Spivak's Calculus Solutions

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Contents

Chapter 1

Basic Properties of Numbers

1.1.

(i) Suppose that $ax = a$ and $a \neq 0$, then there exists a number a^{-1}. Multiplying a^{-1} on both sides yields

$$\begin{aligned}(a^{-1}a) \cdot x &= a^{-1}a \\ x &= 1\end{aligned}$$

as desired.

(ii) We use the distributive property on $(x-y)(x+y)$, this can be done by letting $a = x - y$:

$$\begin{aligned}&(x-y)(x+y) = a(x+y) \\ &= ax + ay = (x-y)x + (x-y)y \\ &= x^2 - yx + xy - y^2 = x^2 - y^2\end{aligned}$$

(iii) If we have $x^2 = y^2$ then we certainly have $x^2 - y^2 = 0$. By (ii) we know that $0 = (x-y)(x+y)$, this implies that $x - y = 0$ or $x + y = 0$, this is equivalent to saying that $x = y$ or $x = -y$.

(iv) Same method as (ii):

$$\begin{aligned}&a(x^2 + xy + y^2) = ax^2 + axy + ay^2 \\ &= (x-y)x^2 + (x-y)xy + (x-y)y^2 \\ &= x^3 - yx^2 + x^2y - xy^2 + xy^2 - y^3 \\ &= x^3 - y^3\end{aligned}$$

(v) We prove this by induction, the base case $n = 2$ is already proven in (ii). Suppose $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ is true. Then we equivalently have $x^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + y^n$. We now prove the n+1 case:

$$
\begin{aligned}
&x^{n+1} - y^{n+1} = x \cdot x^n - y^{n+1} \\
&= x(x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) + xy^n - y^{n+1} \\
&= (x-y)(x^n + x^{n-1}y + \cdots + x^2y^{n-2} + xy^{n-1}) + (x-y)y^n \\
&= (x-y)(x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n)
\end{aligned}
$$

The resulting relation concludes the finite induction, thus $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$.

(vi) We know from (iv) that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, by letting $a = x$ and $b = -y$ we get $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

1.2. Multiplying by the multiplicative inverse of $x - y$ works only when $x - y \neq 0$, that is $x \neq y$, however, the hypothesis explicitly states $x = y$. So it is not possible to find the multiplicative inverse of $x - y$ and thus the step is invalid.

1.3.

(i) Say we have $\frac{a}{b}$ and $b \neq 0$ then the same fraction can be written as ab^{-1}. Suppose we also have a variable c such that $c \neq 0$, then we have $ab^{-1} \cdot (cc^{-1})$ and consequently $(ac)(b^{-1}c^{-1}) = \frac{ac}{bc}$. The final equality holds by (iii) which is proven below.

(ii) By (i) $\frac{ad}{bd} + \frac{bc}{db} = ad(bd)^{-1} + bc(bd)^{-1} = (ad+bc)(bd)^{-1} = \frac{ad+bc}{bd}$

(iii) ab exists if $a, b \neq 0$. Let $x = (ab)^{-1}$, then

$$
\begin{aligned}
x(ab) &= (ab)^{-1}(ab) = (xa)b = 1 && \text{(Multiply } x \text{ with } ab\text{)} \\
(xa)(bb^{-1}) &= b^{-1} = xa = b^{-1} && \text{(Multiply by } b^{-1}\text{)} \\
x(aa^{-1}) &= b^{-1}a^{-1} = x && \text{(Multiply by } a^{-1}\text{)}
\end{aligned}
$$

(iv) Suppose $b, d \neq 0$, then $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1}) \cdot (cd^{-1}) = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(v) Suppose $b, c, d \neq 0$, then $\frac{a}{b}/\frac{c}{d} = (ab^{-1})(cd^{-1})^{-1} = (ab^{-1})(c^{-1}d) = (ac)(bd)^{-1} = \frac{ac}{bd}$

(vi) Suppose $b, d \neq 0$. Assume $\frac{a}{b} = \frac{c}{d}$, multiplying by bd on both side yields the relation $ad = bc$. For the converse multiply $ad = bc$ by $(bd)^{-1}$.

1.4.

(i) $4 - x < 3 - 2x \iff (4-4) + (-x+2x) < (3-4) + (2x-2x) \iff x < -1$.

(ii) $5 - x^2 < 8 \iff -3 < x^2$. Note that $x^2 \geq 0$ and for every single value of x, so our solution is every x.

(iii) $5 - x^2 < -2 \iff 7 < x^2 \iff \sqrt{7} < x$ or $-\sqrt{7} > x$.

(iv) The product is positive when $x-1 > 0$ and $x-3 > 0$ or when $x-1 < 0$ and $x-3 < 0$, that is when $x > 3$ or when $x < 1$.

(v) Complete the square $x^2 - 2x + 2 = (x-1)^2 + 1$. The product $(x-1)^2$ is always positive, and since we have the $+1$ as well in the inequality, this inequality must be true for every single x.

(vi) The inequality is equivalent to $x^2 + x - 1 > 0$. Completing the square $(x + \frac{1}{2})^2 > \frac{5}{4}$. If $x \geq -\frac{1}{2}$ then $x > \frac{-1+\sqrt{5}}{2}$. If $x < -\frac{1}{2}$ then $x < \frac{-1-\sqrt{5}}{2}$. Thus, the solution is $x > \frac{-1+\sqrt{5}}{2}$ and $x < \frac{-1-\sqrt{5}}{2}$.

(vii) Equivalently we have $(x - \frac{1}{2})^2 > \frac{25}{4}$. If $x \geq \frac{1}{2}$ then $x > 3$ if $x < \frac{1}{2}$ then $x < -2$. The solution set is $x > 3$ and $x < -2$.

(viii) Equivalently $(x+\frac{1}{2})^2 + \frac{3}{4} > 0$. This is true for every x because $(x+\frac{1}{2}) \geq$ and $\frac{3}{4} > 0$. Adding them gives $(x + \frac{1}{2})^2 + \frac{3}{4} > 0$.

(ix) Let $b = (x+5)(x-3)$. Then b is positive if $x > 3$ or $x < -5$ and negative if $-5 < x < 3$. Let $a = x - \pi$. a is positive if $x > \pi$. ab is positive if both a and b are positive or if both are negative. So ab is positive if $x > \pi$ (b must be positive because $x > 3$). ab is negative if $-5 < x < 3$ (This implies $x < \pi$).

(x) If $x > \sqrt[3]{2}$ and $x > \sqrt{2}$ then the product is positive, thus the first solution is $x > \sqrt{2}$. If $x < \sqrt[3]{2}$ and $x < \sqrt{2}$ then the product is positive. The second solution is $x < \sqrt[3]{2}$.

(xi) Apply $\log_2$ on both sides: $x < 3$.

(xii) Suppose $x < 1$, we will show this is a solution. We have $3^x < 3^1 = 3$, adding $x < 1$ to the inequality we get $x + 3^x < 3 + 1 = 4$. Since both 3^x and x are strictly increasing expressions finding the inequality $x < 1$ suffices as all real solutions.

(xiii) Noting that $x \neq 0$ and $x \neq 1$. Expanding the fractions we get $\frac{1-x}{x(1-x))} + \frac{x}{x(1-x)} = \frac{1}{x(1-x)} > 0$. The solutions depends on if the denominator is positive. Thus $x(1-x) > 0$ has the same solution set. The solutions are $0 < x < 1$.

(xiv) Note $x \neq -1$. Expand by $(x+1)$: $\frac{(x-1)(x+1)}{(x+1)^2} > 0$. Since the denominator is always positive we can multiply this on both sides, $x^2 - 1 > 0$, Thus $x < -1$ and $x > 1$.

1.5.

(i) Suppose $a < b$ and $c < d$ then we have $b - a > 0$ and $d - c > 0$ by property 11 $(b - a) + (d - c) > 0$ which is the same as $b + d > a + c$.

(ii) Suppose $a < b$ then $0 < b - a \iff -b < (b - b) - a = -b < -a$.

(iii) Suppose $a < b$ and $c < d$, by (ii): $-c < -d$, then by (i) we have $a - d < b - d$.

(iv) Suppose $a < b$ then $b - a > 0$. Assume $c > 0$, Using (P12) we know that $c(b - a) > 0$ and consequently $bc - ac > 0 \iff bc > ac$.

(v) Suppose $a < b$ then $b - a > 0$. Assume $c < 0$, then by (ii) we have $-c > 0$. Using P12 we know that $-c(b - a) > 0$ and consequently $ac - bc > 0 \iff ac > bc$.

(vi) Since $a > 1 > 0$ we apply (iv) by letting $c = a$. Thus $a^2 > a$.

(vii) Because a is positive, it follows by applying (iv) to $a < 1$ that $a^2 < a$.

(viii) Using (iv), multiply $a < b$ with c and $c < d$ with b. This means that we have $ac < bc$ and $bc < bd$, this is the same as $ac < bc < bd$, thus $ac < bd$.

(ix) Using (viii) we multiply the same inequality twice, $a^2 < b^2$.

(x) Suppose $a, b \geq 0$, we prove the contra-positive, therefore $a \geq b$. Multiply by a and b respectively gives two inequalities $a^2 \geq ab$ and $ab \geq b^2$ which is the same as $a^2 \geq ab \geq b^2$. This concludes the contra-positive proof because $a^2 \geq b^2$ is the logical opposite of $a^2 < b^2$.

1.6.

(a) The base case is $n = 2$ which was proven in problem 1.5. Assume $x^n < y^n$ for $0 \leq x < y$. By problem 1.6. (viii) we have $x \cdot x^n < y \cdot y^n \iff x^{n+1} < y^{n+1}$. The induction is complete, thus if $0 \leq x, y$ then $x^n < y^n$ for $n = 1, 2, \ldots$.

(b) Suppose $x < y$ and $n = 2k + 1$, We have three cases.

   (i) $x, y \geq 0$, this case has been proven in (a).

   (ii) $x \leq 0$ and $y \geq 0$. Consider x^n, because n is odd, it has the following property, $x^{2k+1} = x \cdot (x^k)^2 < 0$, because x is negative and $(x^k)^2$ is positive. However $y^n > 0$ because y is positive. This means we have $x^n < 0 < y^n$.

   (iii) $x, y < 0$, by the inequality we have $-x > 0$ and $-y > 0$. We also have $-y < -x$, by (a) we have $(-y)^n < (-x)^n \iff -y^n < -x^n$ because n is odd. Finally we have $x^n < y^n$.

(c) Suppose $x^n = y^n \iff x^n - y^n = 0 = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$ Then either $x - y = 0$ or $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} = 0$ In the first case $x = y$, in the second case we first note that $x^n = y^n$ implies that x and y has the same sign and thus $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1} \geq 0$ where the equality holds only when $x, y = 0$ then $x = y$ is still true.

(d) Let n be an even positive integer. Next we prove the contra-positive, suppose $|x| \neq |y|$ ($x = y$ or $x = -y$ is the same as saying $|x| = |y|$). Consequently this means either $|x| < |y|$ or $|x| > |y|$. By (a) this means that either $|x|^n < |y|^n$ or $|x|^n > |y|^n$. Because n is even this is equivalent to $x^n < y^n$ or $x^n > y^n$ which is the logical complement of $x^n = y^n$.

1.7. Suppose $0 < a < b$, multiply by a then $a^2 < ab \iff a < \sqrt{ab}$. Next consider $(a-b)^2 > 0$ which is equivalent to $a^2 + b^2 + 2ab > 4ab \iff \frac{a+b}{2} > \sqrt{ab}$, this means that we have $a < \sqrt{ab} < \frac{a+b}{2}$ now remains the final inequality. By the premise we have $b - a > 0 \iff b + a > 2a \iff \frac{b+a}{2} > a$. We conclude by stating $a < \sqrt{ab} < \frac{a+b}{2} < b$.

*** 1.8.**

(P10) Let $b = 0$ in P'10, then for every a one of the following properties apply

(i) $a = 0$

(ii) $a < 0$

(iii) $a > 0$

Because the collection P contains all the numbers x such that $x > 0$, we can see that (iii) states that a belongs to P. (ii) is equivalent to $-a > 0$, thus -a is in P.

(P11) Suppose x and y are in P then $0 < x$ and $0 < y$. By P'12 (Let a=0) we have $x < y + x$. By P'11 we get $0 < y + x$ which is in P.

(P12) Suppose x and y are in P then $0 < x$ and $0 < y$. Using P'13 we get $0 < xy$, this means that xy is in P.

1.9. (i) $\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}$.

(ii) Triangle inequality states that $|a + b| - |a| - |b| \leq 0$. Therefore $|a| + |b| - |a + b|$.

(iii) Triangle inequality gives $|(a + b) + c| - |a + b| - |c| \leq 0 \iff |a + b| + |c| - |a + b + c|. \geq 0$. Our solution is therefore $|a + b| + |c| - |a + b + c|$.

(iv) $x^2 - 2xy + y^2 = (x - y)^2 \geq 0$, thus $x^2 - 2xy + y^2$.

(v) $\sqrt{2} + \sqrt{3} + \sqrt{5} - \sqrt{7}$

1.10.

(i) Suppose $a+b \geq 0$ and $b \geq 0$ then we have $a+b-b=a$. Suppose $a+b \geq 0$ and $b<0$ then $a+b+b=a+2b$. Suppose $a+b<0$ and $b \geq 0$ then $-a-b-b=-(a+2b)$. Suppose $a+b<0$ and $b<0$ then $-a-b+b=-a$.

(ii) If $0 \geq x \geq 1$ then $1-x$. If $-1 \geq x<0$ then $1+x$. If $1<x$ then $x-1$ then $-x-1$.

(iii) If $x \geq 0$ then $x-x^2$, if $x<0$ then $-x-x^2$.

(iv) If $a \geq 0$ then a, if $a<0$ then $3a$.

1.11. (i) Suppose $x-3>0$ then $x-3=8 \iff x=11$. Suppose $x-3<0$ then $3-x=8 \iff x=-5$.

(ii) Suppose $x-3 \geq 0$ then $3 \leq x<11$. Suppose $x-3<0$ then $-5<x<3$. Combining both inequalities $-5<x<11$.

(iii) Suppose $x+4 \geq 0$ then $x<-2$, so $-4 \leq x<-2$. If $x+4<0$ then $-6<x<-4$. Combining both inequalities gives $-6<x<-2$.

(iv) Suppose $x \leq 2$ then $x-1+x-2>1 \iff x>2$. This means $x>2$ is always a solution. Suppose $1 \leq x<2$, then $x-1-x+2>1 \iff 1>1$, which can not be true. Suppose $x<1$, then $1-x-x+2>1 \iff x<1$. The solution is $x<1$ and $x>2$.

(v) Suppose $x \geq 1$ then $x-1+x+1<2 \iff x<1$ which is a contradiction. Suppose $-1 \leq x<1$ then $1-x+x+1<2 \iff 2<2$, also contradiction. Suppose $x<-1$ then $1-x-x-1<2 \iff x>-1$, an x that satisfies the inequality is nonexistent.

(vi) Suppose $x \geq 1$ then $x-1+x+1<1 \iff x<\frac{1}{2}$ which is a contradiction. Suppose $-1 \leq x<1$ then $1-x+x+1<1 \iff 2<1$,also a contradiction. Suppose $x<-1$ then $1-x-x-1<1 \iff x>-\frac{1}{2}$, similarly to (iv), there are no x that satisfy the inequality.

(vii) We have $x-1=0 \iff x=1$ or $x+1=0 \iff x=-1$.

(viii) Suppose $x \geq 1$ then $(x-1)(x+2) = 3 \iff x^2 + x - 5 = 0 \iff (x + \frac{1}{2})^2 = \frac{21}{4} \implies x = \frac{-1+\sqrt{21}}{2}$. Suppose $-2 \leq x < 1$ then $(1-x)(x+2) = 3$ which is a polynomial with complex roots thus no solutions there. Suppose $x < -2$, then we get the same polynomial as in the first case because $(-1)^2 = 1$, so the other root is $x = \frac{-1-\sqrt{21}}{2}$ which is less than -2 because $\frac{-1-\sqrt{21}}{2} < \frac{-1-\sqrt{16}}{2} = \frac{-5}{2} < -2$. To conclude $x = \frac{-1\pm\sqrt{21}}{2}$

1.12.

(i) $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 \iff |xy| = |x| \cdot |y|$

(ii) Consider $\left|\frac{1}{x}\right|$ for $x \neq 0$. This is the same as $\sqrt{(\frac{1}{x})^2} = \sqrt{\frac{1}{x^2}} = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$.

(iii) Suppose $y \neq 0$ then $\left|\frac{x}{y}\right| = \sqrt{(\frac{x}{y})^2} = \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|}$

(iv) Suppose a, b are real numbers, then the triangle inequality is $|a+b| \leq |a| + |b|$. Let $a = x$ and $b = -y$ then $|x-y| \leq |x| + |-y| = |x| + |y|$. The final equality is proven by $|-y| = \sqrt{(-y)^2} = \sqrt{(-1)^2y^2} = \sqrt{y^2}$.

(v) Using the triangle inequality $|x| = |(x-y)+y| \leq |x-y| + |y| \iff |x| - |y| \leq |x-y|$

(vi) There are two cases from the inequality, $|x| - |y| \leq |x-y|$ and $|y| - |x| \leq |y-x|$, note that the last absolute value comes from the fact $|x-y| = |y-x|$. Both inequalities are identical to (v) (the second inequality has the variables interchanged).

(vii) We have $|(x+y)+z| \leq |x+y| + |z| \leq |x| + |y| + |z|$. Doing the case work for the equality is left to the reader.

1.13. We start by proving for max, let $x \geq y$ then $\max(x,y) = \frac{x+y+x-y}{2} = x$ Likewise if $y \geq x$ then $\max(x,y) = y$. Similar reasoning shows that the formula for $\min(x,y)$ is valid. Next we use substitution and get $\max(x,y,z) = \max(x, \max(y,z)) = \frac{y+z+2x+|y-z|+|y+z+2x+|y-z||}{4}$ and $\min(x,y,z) = \min(x, \min(y,z)) = \frac{y+z+2x+|y-z|-|y+z+2x+|y-z||}{4}$.

1.14.

(a) Suppose $a \geq 0$ then we have $a = -(-a)$. The case for $a \leq 0$ is then obvious because we have $(-a) \geq 0$ which can be used on the previously proven fact.

(b) $(\Rightarrow)$ Suppose $-b \leq a \leq b$, this implies $a \leq b$ and $-b \leq a \iff -a \leq b$ and consequently $|a| \leq b$.
$(\Leftarrow)$ Suppose $|a| \leq b$ then $a \leq b$ and $-a \leq b \iff -b \leq a$, thus $-b \leq a \leq b$. Now we prove the last part. Suppose $|a| \leq |a|$ then by the previously proven theorem we have $-|a| \leq a \leq |a|$.

(c) As proven earlier, for every a, b we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Add these together gives $-(|a| + |b|) \leq a + b \leq |a| + |b|$, applying the theorem from (b) on $(|a| + |b|)$ and $(a + b)$ we get $|a + b| \leq |a| + |b|$.

1.15. We prove first that if $x = y$ and $x, y \neq 0$. The inequality is then $x^2 + x^2 + x^2 > 0 \iff x^2 > 0$ which is true because $x \neq 0$.

Suppose $x \neq y$, then the left side of inequality is equivalent to $(x^2 + xy + y^2) = \frac{x^3 - y^3}{(x-y)}$. Suppose $x > y$ then $x^3 - y^3 > 0$ by problem 6 (b), since both the numerator and denominator are positive we know that $\frac{x^3-y^3}{(x-y)} > 0$. Next we assume $x < y$ which implies $x^3 - y^3 < 0$ by problem 6 (b). This means the numerator and denominator are both negative, thus $\frac{x^3-y^3}{(x-y)} > 0$. In every case the inequality is positive, thus we have proven that $x^2 + xy + y^2 > 0$.

To prove that the second inequality holds we follow the same steps, suppose $x = y$ which means the inequality is $5x^4 > 0$. Next suppose $x \neq y$ then we have $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = \frac{x^5-y^5}{x-y}$. Suppose $x - y > 0$ then $x^5 - y^5 > 0$ which implies $\frac{x^5-y^5}{x-y} > 0$. Assume $x - y < 0$ then $x^5 - y^5 < 0$ which implies $\frac{x^5-y^5}{x-y} > 0$.

1.16. (a) $(x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 \iff xy = 0$ which implies $x = 0$ or $y = 0$