

IDENTITIES IN ITERATED RASCAL TRIANGLES

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ABSTRACT. In this manuscript, we introduce new binomial identities in iterated Rascal triangles, uncovering a connection between Vandermonde convolution and iterated Rascal numbers. Additionally, we present novel identities involving the finite differences of iterated Rascal numbers and binomial coefficients. The manuscript also offers a proof of the row sums conjecture for iterated Rascal triangles. Furthermore, we establish and explore the relationship between iterated Rascal triangles and $(1, q)$ -binomial coefficients, highlighting connections to relevant OEIS sequences. All results are supported by supplementary Mathematica programs for validation.

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Date: September 28, 2024.

2010 Mathematics Subject Classification. 11B25, 11B99.

Key words and phrases. Pascal's triangle, Rascal triangle, Binomial coefficients, Binomial identities, Binomial theorem, Generalized Rascal triangles, Iterated rascal triangles, Iterated rascal numbers, Vandermonde convolution.

8. Addendum 2: Mathematica documentation

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1. INTRODUCTION

Rascal triangle is Pascal-like numeric triangle developed in 2010 by three middle school students, Alif Anggoro, Eddy Liu, and Angus Tulloch [1]. During math classes they were challenged to provide the next row for the following number triangle

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & & & \\
 & & & 1 & & 1 & \\
 & & 1 & & 2 & & 1 \\
 & 1 & & 3 & & 3 & & 1 \\
 & & & & & & & \dots
 \end{array}$$

The teacher anticipated that the next row would match Pascal’s triangle, such as “1 4 6 4 1”, by applying the binomial coefficient recurrence rule $South = East + West$. However, Anggoro, Liu, and Tulloch proposed that the next row should be “1 4 5 4 1”. Instead of using Pascal’s triangle rule $South = East + West$, they derived this new row using a relation they termed the diamond formula

$$South = \frac{East \cdot West + 1}{North} \quad (1.1)$$

By applying the recurrence relation from equation (1.1), the students successfully generated an entirely new triangular sequence, now referred to as the Rascal triangle.

Sources: <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	5	4	1					
5	1	5	7	7	5	1				
6	1	6	9	10	9	6	1			
7	1	7	11	13	13	11	7	1		
8	1	8	13	16	17	16	13	8	1	
9	1	9	15	19	21	21	19	15	9	1

Table 1. Rascal triangle. Sequence [A077028](#) in OEIS [2].

For example, the fourth row is “1 4 5 4 1” because $4 = \frac{1 \cdot 3 + 1}{1}$ and $5 = \frac{3 \cdot 3 + 1}{2}$. Moreover, the Rascal triangle, as presented in table (1), represents the first and foundational instance of a new family of Pascal-like triangles. This family, known as *iterated Rascal triangles*, was first introduced by J. Gregory in her master’s thesis [3].

We define the k -th element in the n -th row of an iterated Rascal triangle as $\binom{n}{k}_i$, where i represents the number of iterations. The integer sequence produced by $\binom{n}{k}_i$ is referred to as an *iterated Rascal triangle* Ri , and each $\binom{n}{k}_i$ is termed an *iterated Rascal number*. Therefore, the Rascal triangle shown in table (1) corresponds to the iterated Rascal triangle $R1$, generated by the formula $\binom{n}{k}_1 = k(n - k) + 1$. While the iterated Rascal number $\binom{n}{k}_i$ is defined by the diamond rule (1.1), which differs from the standard binomial coefficient recurrence, it still maintains a significant connection with the binomial coefficients $\binom{n}{k}$, as demonstrated by

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} \quad (1.2)$$

For example, $\binom{7}{4}_3 = 35$, $\binom{12}{7}_5 = 792$, $\binom{11}{5}_5 = 462$.

Example 1.1. Rascal triangle $R2$ generated by $\binom{n}{k}_2$

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	19	15	6	1			
7	1	7	21	31	31	21	7	1		
8	1	8	28	46	53	46	28	8	1	
9	1	9	36	64	81	81	64	36	9	1

Table 2. Rascal triangle R2. Sequence [A374378](#) in OEIS [\[4\]](#).

Example 1.2. Rascal triangle R3 generated by $\binom{n}{k}_3$

n/k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	69	56	28	8	1	
9	1	9	36	84	121	121	84	36	9	1

Table 3. Rascal triangle R3. Sequence [A374452](#) in OEIS [\[5\]](#).

Since then, a lot of work has been done over the topic of rascal triangles. Numerous identities and relations have been revealed. For instance, a few combinatorial interpretations of rascal numbers provided at [\[6\]](#), in particular, these interpretations establish a relation between rascal numbers and combinatorics of binary words. Several generalization approaches were proposed, namely generalized and iterated rascal triangles [\[7, 8\]](#). In particular, the

concept of iterated rascal numbers establishes a close connection between rascal numbers and binomial coefficients.

2. BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Prior we begin our discussion it is worth to introduce a few preliminary facts and statements. Define the iterated rascal number [3, eq. 3.2]

Definition 2.1. *Iterated rascal number*

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{m} \quad (2.1)$$

The first important thing to notice is that the iterated Rascal number is a special case of the Vandermonde convolution. Consider the Vandermonde convolution [9]

$$\binom{a+b}{r} = \sum_{m=0}^r \binom{a}{m} \binom{b}{r-m}$$

Implies

$$\binom{n}{k} = \sum_{m=0}^k \binom{n-k}{m} \binom{k}{k-m}$$

Thus,

$$\binom{n}{k}_i = \sum_{m=0}^i \binom{n-k}{m} \binom{k}{k-m} \quad (2.2)$$

Meaning that iterated rascal number is partial case of Vandermonde convolution of $\binom{n}{k}$ with the upper summation bound equals to i . Without further hesitation consider our findings.

Proposition 2.2. *(Column identity.) Iterated rascal triangle equals to Pascal's triangle up to i -th column. For every $k \leq i$*

$$\binom{n}{k}_i = \binom{n}{k} \quad (2.3)$$

For example, we notice that

- Triangle R1 generated by $\binom{n}{k}_1$ is equivalent to Pascal's triangle for columns $k = 0, 1$. See (1).

- Triangle R2 generated by $\binom{n}{k}_2$ is equivalent to Pascal's triangle for columns $k = 0, 1, 2$. See (2).
- Triangle R3 generated by $\binom{n}{k}_3$ is equivalent to Pascal's triangle for columns $k = 0, 1, 2, 3$. See (3).

Then for every $k \leq i$ binomial identity follows

$$\binom{n}{i-k}_i = \binom{n}{i-k}$$

Applying the symmetry of binomial coefficients, we obtain

$$\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$$

Proof. Proof of proposition (2.2) is given by [3, proposition 6.0.1]. \square

Proposition 2.3. (*Row identity.*) *Iterated rascal triangle equals to Pascal's triangle up to $2i + 1$ -th row. For every $n \leq 2i + 1$*

$$\binom{n}{k}_i = \binom{n}{k}$$

For example, we notice that

- Triangle R1 generated by $\binom{n}{k}_1$ is equivalent to Pascal's triangle up to 3-rd row, see (1).
- Triangle R2 generated by $\binom{n}{k}_2$ is equivalent to Pascal's triangle up to 5-th row, see (2).
- Triangle R3 generated by $\binom{n}{k}_3$ is equivalent to Pascal's triangle up to 7-rd row, see (3).

Therefore, for every $i \geq 0$ and $n \geq 0$

$$\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k} \quad (2.4)$$

Equation (2.4) is of interest because in contrast to rascal column identity (2.3) it gives relation over k for each i , so that it is true for all cases in i, k : $i < k$, $i = k$ and $k > i$. In particular, equation (2.4) implies the row sums identity in iterated rascal triangles

$$\sum_{k=0}^{\infty} \binom{2i+1-n}{k}_i = 2^{2i+1-n}$$

Given $n = 0$ we obtain

$$\sum_{k=0}^{\infty} \binom{2i+1}{k}_i = 2^{2i+1}$$

and so on. Taking $t = 2i + 1$ in (2.4) yields

$$\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$$

Moreover, equation (2.4) gives Vandermonde-like identity, by definition

$$\binom{2i+1-n}{k} = \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m} \quad (2.5)$$

In particular, given $n = 0, 1$ equation (2.5) yields the following Vandermonde-like identities

$$\begin{aligned} \binom{2i+1}{k} &= \sum_{m=0}^i \binom{2i+1-k}{m} \binom{k}{m} \\ \binom{2i}{k} &= \sum_{m=0}^i \binom{2i-k}{m} \binom{k}{m} \end{aligned}$$

Proof. Proof of proposition (2.3). We have to prove that for every i, k

$$\sum_{m=0}^k \binom{2i+1-n-k}{m} \binom{k}{m} - \sum_{m=0}^i \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

For the case $k < i$ proof is given by [3, proposition 6.0.1]. For the case $k = i$ proof is trivial.

Thus, the remaining case is $k > i$ yields

$$\sum_{m=i+1}^k \binom{2i+1-n-k}{m} \binom{k}{m} = 0 \quad (2.6)$$

If (2.6) is true for each $k > i$, then its sum over k should be zero as well. Introducing sum in k to (2.6) we get

$$\sum_k \sum_{m=i+1}^k \binom{2i+1-n-k}{m} \binom{k}{m} = 0$$

The sum $\sum_k \binom{2i+1-n-k}{m} \binom{k}{m}$ appears to match the equation (5.26) in Concrete mathematics [10, eq. 5.26]

$$\sum_{k=0}^{\ell} \binom{\ell-k}{m} \binom{q+k}{n} = \binom{\ell+q+1}{m+n+1} \quad (2.7)$$

Therefore,

$$\sum_k \binom{2i+1-n-k}{m} \binom{k}{m} = \binom{2i+2-n}{2m+1}$$

Thus, our main assumption is equivalent to

$$\sum_{m=i+1}^k \sum_k \binom{2i+1-n-k}{m} \binom{k}{m} \equiv \sum_{m=i+1}^k \binom{2i+2-n}{2m+1}$$

Hence, we have to prove that

$$\sum_{m=i+1}^k \binom{2i+2-n}{2m+1} = 0 \quad (2.8)$$

Substituting $m = i+1+m$ into (2.8), we get

$$\sum_{m=0}^k \binom{2i+2-n}{2(i+1+m)+1} = \sum_{m=0}^k \binom{2i+2-n}{2i+3+2m} = 0$$

Which is indeed true because $\binom{2i+2-n}{2i+3+2m} = 0$ for every $m, n \geq 0$. Thus, the proposition (2.4) is true. □

Now, let's smoothly switch our focus to finite differences of binomial coefficients and iterated rascal numbers. Considering the table of differences $\binom{n}{k} - \binom{n}{k}_3$

	0				k=4																
	0	0																			
	0	0	0																		
	0	0	0	0																	
	0	0	0	0	0			A000332													
	0	0	0	0	0	0															
	0	0	0	0	0	0	0														
	0	0	0	0	0	0	0	0													
	0	0	0	0	1	0	0	0	0									n=8			
	0	0	0	0	5	5	0	0	0												
Out[67]=	0	0	0	0	15	26	15	0	0	0	0										
	0	0	0	0	35	81	81	35	0	0	0	0									
	0	0	0	0	70	196	262	196	70	0	0	0	0								
	0	0	0	0	126	406	658	658	406	126	0	0	0	0							
	0	0	0	0	210	756	1414	1716	1414	756	210	0	0	0	0						
	0	0	0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0					
	0	0	0	0	495	2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0				
	0	0	0	0	715	3267	8184	14232	18525	18525	14232	8184	3267	715	0	0	0	0			
	0	0	0	0	1001	4862	13101	24816	35697	40186	35697	24816	13101	4862	1001	0	0	0	0		
	0	0	0	0	1365	7007	20163	41217	64713	80587	80587	64713	41217	20163	7007	1365	0	0	0	0	
	0	0	0	0	1820	9828	30030	65780	111705	152020	168230	152020	111705	65780	30030	9828	1820	0	0	0	0

We can spot that having $i = 3$ the fourth column gives binomial coefficient $\binom{n}{4}$ with offset $(0, 3)$. Indeed, this rule is true for every i .

$$\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$$
$$\sum_{m=0}^i \binom{n+i}{m} \binom{i}{i-m} - \sum_{m=0}^{i-1} \binom{n+i}{m} \binom{i}{i-m} = \binom{n+i}{i} \binom{i}{0}$$
☐
$$\binom{n+2i}{n+i} - \binom{n+2i}{n+i}_{i-1} = \binom{n+i}{n}$$

Taking $j = n + i$ gives

$$\binom{j+i}{j} - \binom{j+i}{j}_{i-1} = \binom{j}{j-i}$$

By symmetry

$$\binom{j+i}{i} - \binom{j+i}{i}_{i-1} = \binom{j}{i}$$

Proposition (2.4) can be generalized even further, for every $i < k$ and $i > k$.

Proposition 2.5. *(Finite difference of binomial coefficients and iterated rascal numbers for $i < k$.) For every $i < k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=i+1}^k \binom{n-k}{m} \binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. □

Proposition 2.6. *(Finite difference of binomial coefficients and iterated rascal numbers for $i > k$.) For every $i > k$*

$$\binom{n}{k} - \binom{n}{k}_i = \sum_{m=k+1}^i \binom{n-k}{m} \binom{k}{k-m}$$

Proof. It is true by means of Vandermonde convolution. □

3. Q-BINOMIAL IDENTITIES IN ITERATED RASCAL TRIANGLES

Consider the table of differences of binomial coefficients and iterated rascal numbers one more time as there is another pattern we can spot.

```
In[67]:= Grid[Table[Binomial[n, k] - RascalNumber[n, k, 3], {n, 0, 20}, {k, 0, n}], Frame -> All]
```

0																				
0	0																			
0	0	0																		
0	0	0	0																	
0	0	0	0	0																
0	0	0	0	0	0															
0	0	0	0	0	0	0														
0	0	0	0	0	0	0	0													
0	0	0	0	0	1	0	0	0	0											
0	0	0	0	0	5	5	0	0	0	0										
0	0	0	0	0	15	26	15	0	0	0	0									
0	0	0	0	0	35	81	81	35	0	0	0	0								
0	0	0	0	0	70	196	262	196	70	0	0	0	0							
0	0	0	0	0	126	406	658	658	406	126	0	0	0	0						
0	0	0	0	0	210	756	1414	1716	1414	756	210	0	0	0	0					
0	0	0	0	0	330	1302	2730	3830	3830	2730	1302	330	0	0	0	0				
0	0	0	0	0	495	2112	4872	7680	8885	7680	4872	2112	495	0	0	0	0			
0	0	0	0	0	715	3267	8184	14232	18525	18525	14232	8184	3267	715	0	0	0	0		
0	0	0	0	0	1001	4862	13101	24816	35697	40186	35697	24816	13101	4862	1001	0	0	0	0	
0	0	0	0	0	1365	7007	20163	41217	64713	80587	80587	64713	41217	20163	7007	1365	0	0	0	0
0	0	0	0	0	1820	9828	30030	65780	111705	152020	168230	152020	111705	65780	30030	9828	1820	0	0	0

Figure 2. Difference $\binom{n}{k} - \binom{n}{k}_3$. Highlighted column is $(1, 5)$ -binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]^5$. Sequence **A096943** in the OEIS [12].

The $(1, q)$ -binomial coefficients $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]^q$ are special kind of binomial coefficients defined by

Definition 3.1. $(1, q)$ -Binomial coefficient

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]^q = \begin{cases} q & \text{if } k = 0, n = 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k > n \\ \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right]^q + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right]^q & \end{cases} \quad (3.1)$$

Indeed, the relation shown in Figure (2) is true for every i , so that it establishes a relation between $(1, q)$ -binomial coefficients and iterated rascal numbers.

Proposition 3.2. (Relation between iterated rascal numbers and $(1, q)$ -binomial coefficients.)

For every $i \geq 0$

$$\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[\begin{smallmatrix} i+2+j \\ i+2 \end{smallmatrix} \right]^{i+2}$$

Taking $t = i + 2$ in (3.2) yields

$$\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[\begin{matrix} t+j \\ t \end{matrix} \right]^t$$

In particular,

- Having $i = 1$ proposition (3.2) gives the OEIS sequence [A006503](#) [13] such that third column of $(1, 3)$ -Pascal triangle [A095660](#) [14].
- Having $i = 3$ proposition (3.2) gives the OEIS sequence [A096943](#) [12] such that third column of $(1, 5)$ -Pascal triangle [A096940](#) [15].
- Having $i = 5$, the proposition (3.2) yields the OEIS sequence [A097297](#) [16] such that seventh column of $(1, 6)$ -Pascal triangle

4. ROW SUMS CONJECTURE

In [8] the authors propose the following conjecture for row sums of iterated rascal triangles.

Conjecture 4.1. (*Conjecture 7.5 in [8].*) For every i

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

Proof. Rewrite conjecture statement explicitly as

$$\sum_{k=0}^{4i+3} \sum_{m=0}^i \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2}$$

Rearranging sums and omitting summation bounds yields

$$\sum_{m=0}^i \sum_k \binom{4i+3-k}{m} \binom{k}{m} = 2^{4i+2} \quad (4.1)$$

We can observe the pattern (2.7) in equation (4.1). Thus, the sum $\sum_k \binom{4i+3-k}{m} \binom{k}{m}$ equals to

$$\sum_k \binom{4i+3-k}{m} \binom{k}{m} = \binom{4i+4}{2m+1}$$

Therefore, conjecture (4.1) is equivalent to

$$\sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

Note that

$$\sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^{2i+1} \left[\binom{4i+3}{2m+1} + \binom{4i+3}{2m} \right] = 2^{4i+3}$$

So that

$$\frac{1}{2} \sum_{m=0}^{2i+1} \binom{4i+4}{2m+1} = \sum_{m=0}^i \binom{4i+4}{2m+1} = 2^{4i+2}$$

This completes the proof. \square

Proposition 4.2. *For every i*

$$\sum_{k=0}^{4i+3} \binom{4i+3}{k}_i = 2^{4i+2}$$

In particular, equation (2.7) assumes the following identity in row sums of iterated rascal triangles

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^i \binom{n+1}{2m+1}$$

Decomposing $\binom{n+1}{2m+1}$ in above equation yields

Proposition 4.3. *(Iterated rascal triangles row sums.) For every i*

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^{2i+1} \binom{n}{m}$$

Proof.

$$\sum_{k=0}^n \binom{n}{k}_i = \sum_{m=0}^i \binom{n+1}{2m+1} = \sum_{m=0}^i \binom{n}{2m} + \binom{n}{2m+1} = \sum_{m=0}^{2i+1} \binom{n}{m}$$

\square

5. CONCLUSIONS

In this manuscript we have discussed new binomial identities in iterated rascal triangles (2.4), (2.4), (2.5), revealing a connection between the Vandermonde convolution formula and iterated rascal numbers. We also present Vandermonde-like binomial identities (2.5). Furthermore, we establish a relation between iterated rascal triangles and $(1, q)$ -binomial coefficients (3.2). Some of the results accepted for publication in *Mathematical gazette*. Supplementary Mathematica scripts to be found at [17].

6. ACKNOWLEDGEMENTS

Author is grateful to Oleksandr Kulkov, Markus Scheuer, Amelia Gibbs for their valuable feedback and suggestions regarding the conjecture (4.1) at MathStackExchange discussion [18].

REFERENCES

- [1] Anggoro, Alif and Liu, Eddy and Tulloch, Angus. The Rascal Triangle. *The College Mathematics Journal*, 41(5):393–395, 2010. <https://doi.org/10.4169/074683410X521991>.
- [2] Sloane, N. J. A. The Rascal triangle read by rows. Entry A077028 in The On-Line Encyclopedia of Integer Sequences, 2002. <https://oeis.org/A077028>.
- [3] Gregory, Jena M. Iterated rascal triangles. *Theses and Dissertations. 1050.*, 2022. <https://scholarworks.utrgv.edu/etd/1050/>.
- [4] Kolosov, Petro. Iterated rascal triangle R2, Entry A374378 in The On-Line Encyclopedia of Integer Sequences, 2024. <https://oeis.org/A374378>.
- [5] Petro Kolosov. Iterated rascal triangle r3, entry a374452 in the on-line encyclopedia of integer sequences, 2024. <https://oeis.org/A374452>.
- [6] Amelia Gibbs and Brian K. Miceli. Two Combinatorial Interpretations of Rascal Numbers. *arXiv preprint arXiv:2405.11045*, 2024. <https://arxiv.org/abs/2405.11045>.
- [7] Hotchkiss, Philip K. Generalized Rascal Triangles. *arXiv preprint arXiv:1907.11159*, 2019. <https://arxiv.org/abs/1907.11159>.
- [8] Gregory, Jena and Kronholm, Brandt and White, Jacob. Iterated rascal triangles. *Aequationes mathematicae*, pages 1–18, 2023. <https://doi.org/10.1007/s00010-023-00987-6>.

- [9] George E. Andrews, Richard Askey, and Ranjan Roy. *Special Functions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999. <https://doi.org/10.1017/CB09781107325937>.
- [10] Graham, Ronald L. and Knuth, Donald E. and Patashnik, Oren. *Concrete mathematics: A foundation for computer science (second edition)*. Addison-Wesley Publishing Company, Inc., 1994. <https://archive.org/details/concrete-mathematics>.
- [11] Sloane, N. J. A. Binomial coefficient $\text{binomial}(n,4)$. Entry A000332 in The On-Line Encyclopedia of Integer Sequences, 2009. <https://oeis.org/A000332>.
- [12] Sloane, N. J. A. Sixth column of $(1,5)$ -Pascal triangle. Entry A096943 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096943>.
- [13] Sloane, N. J. A. Entry A006503 in The On-Line Encyclopedia of Integer Sequences, 1995. <https://oeis.org/A006503>.
- [14] Sloane, N. J. A. Pascal $(1,3)$ triangle. Entry A095660 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A095660>.
- [15] Sloane, N. J. A. Pascal $(1,5)$ triangle. Entry A096940 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A096940>.
- [16] Sloane, N. J. A. Seventh column ($m=6$) of $(1,6)$ -Pascal triangle. Entry A097297 in The On-Line Encyclopedia of Integer Sequences, 2004. <https://oeis.org/A097297>.
- [17] Kolosov, Petro. Identities in Iterated Rascal Triangles Mathematica programs, 2024. <https://github.com/kolosovpetro/IdentitiesInRascalTriangle>.
- [18] Kolosov, Petro. "Iterated rascal triangle row sums" on MathStackExchange, 2024. <https://math.stackexchange.com/q/4941336/463487>.

Version: 1.2.2-tags-v1-2-1.2+tags/v1.2.1.5270749

7. ADDENDUM 1: RELATED OEIS SEQUENCES

- Having $i = 2$ and $k = 4$: $\binom{n}{k} - \binom{n}{k}_i$ gives Fifth column ($m = 4$) of $(1,4)$ -Pascal triangle <https://oeis.org/A095667>
- Having $i = 2$ and $k = 3$: $\binom{n}{k} - \binom{n}{k}_i$ gives Tetrahedral (or triangular pyramidal) numbers: $a(n) = C(n+2, 3) = n * (n+1) * (n+2)/6$. <https://oeis.org/A000292>
- Having $i = 1$ and $k = 2$: $\binom{n}{k} - \binom{n}{k}_i$ gives Triangular numbers: $a(n) = \text{binomial}(n+1, 2) = n * (n+1)/2 = 0 + 1 + 2 + \dots + n$ <https://oeis.org/A000217>

- Having $i = 0$ and $k = 3$: $\binom{n}{k} - \binom{n}{k}_i$ gives Fourth column ($r=3$) of FS(3) staircase array. <https://oeis.org/A062748>
- Having $i = 0$ and $k = 6$: $\binom{n}{k} - \binom{n}{k}_i$ gives $a(n) = \text{binomial}(n, 6) - 1$. <https://oeis.org/A124089>
- Having $i = 0$ and $k = 7$: $\binom{n}{k} - \binom{n}{k}_i$ gives $a(n) = \text{binomial}(n, 7) - 1$. <https://oeis.org/A124090>
- Having $i = 0$ and $k = 2$: $\binom{n}{k} - \binom{n}{k}_i$ gives $a(n) = n * (n + 3)/2$. <https://oeis.org/A000096>
- Having $i = 0$ and $k = 9$: $\binom{n}{k} - \binom{n}{k}_i$ gives *One less than number of n -multisets chosen from a 10-set*. <https://oeis.org/A035927>
- Having $i = 3$ and $k = 4$: $\binom{n}{k} - \binom{n}{k}_i$ gives Binomial coefficient $\text{binomial}(n, 4) = n * (n - 1) * (n - 2) * (n - 3)/24$. <https://oeis.org/A000332>
- Having $i = 4$ and $k = 5$: $\binom{n}{k} - \binom{n}{k}_i$ gives Binomial coefficients $C(n, 5)$. <https://oeis.org/A000389>

8. ADDENDUM 2: MATHEMATICA DOCUMENTATION

Mathematica programs documentation. See [17].

- ColumnIdentity1[20, 20] validates $\binom{n}{i-k}_i = \binom{n}{i-k}$
- ColumnIdentity2[20, 20] validates $\binom{n}{n-i+k}_i = \binom{n}{n-i+k}$
- RowIdentity1[5] validates $\binom{2i+1-n}{k}_i = \binom{2i+1-n}{k}$, see (2.4)
- RowIdentity2[12, 5] validates $\binom{t-n}{k}_{t-i-1} = \binom{t-n}{k}$
- RowColumnDifferenceIdentity1[10, 20] validates $\binom{n+2i}{i} - \binom{n+2i}{i}_{i-1} = \binom{n+i}{i}$, see (2.4).
- RowColumnDifferenceIdentity2[10, 20] validates $\binom{j+i}{i} - \binom{j+i}{i}_{i-1} = \binom{j}{i}$
- OneQPascalIdentity1[10, 20] validates $\binom{2i+3+j}{i+2} - \binom{2i+3+j}{i+2}_i = \left[\binom{i+2+j}{i+2} \right]^{i+2}$, see (3.2).
- OneQPascalIdentity2[10, 20] validates $\binom{2t-1+j}{t} - \binom{2t-1+j}{t}_{t-2} = \left[\binom{t+j}{t} \right]^t$

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