Fixed-length contract problems

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April 2020

The contract problems are principal-agent problems based on a model where an investor (principal) and a manager (agent) enter into a fixed-length contract where the investor pays the manager to manage a firm. The investor benefits from increasing firm value and pays the manager for his work. The catch is that the manager can appropriate some firm value for himself, reporting a different (lower) firm value to the investor. The problem is how the investor should make the payment contract to incentivize the manager not to appropriate too much.

1 The model

We assume that the investor and manager enter into a contract of fixed length $T \in \mathbb{N}$, where in each period $t = 1, \ldots, T$, the manager can appropriate some amount $a_t \geq 0$ of firm value, and the investor pays the manager a quantity l_t based on the firm value the manager reports (the actual firm value minus the amount appropriated).

The manager's continuation utility at a time t (expected gain from continuing through the end of the contract) is, for $t = 0, 1, \ldots, T-1$,

$$W_t = E_t \sum_{s=t+1}^{T} e^{-\rho(s-t)} \left(l_s + \theta a_s\right)$$

where E_t is the expected value operator given information at time t, T is the length of the contract, ρ is the manager's rate (greater than the riskless rate), l_t is the payment from the investor to the manager in period t, a_t is the amount of firm value appropriated by the manager in period t, and $\theta \in [0, 1]$ represents the proportion of appropriated firm value that the manager can actually use (stealing is not totally efficient). Note that l_t and a_t are defined only for integer values of t, since the manager is paid and has the opportunity to appropriate firm value only in discrete periods.

The most important manager's continuation utility to consider is W_0, which is the expected gain to the manager of entering into to the contract. We can assume that the manager will not accept any contract where $W_0 < W_{min}$, where W_{min} is some nonnegative threshold, which we can think of as the manager's opportunity cost of entering into the contract.

The investor's continuation utility for $t = 0, 1, \ldots, T-1$ is

$$V_t = E_t \sum_{s=t+1}^{T} e^{-r(s-t)} \left(\Delta \hat{Y}_s - l_s \right)$$

Here $\hat{Y}_t$ is the reported firm value at time s. Specifically,

$$\hat{Y}_t = Y_t - \sum_{s=1}^{\lfloor t \rfloor} a_s,$$

where Y_t is the true firm value, which we model as a jump-diffusion process

$$dY_t = Y_t(\mu dt + \sigma dZ_t + dJ_t)$$

(μ is a growth rate parameter, σ is a volatility parameter, Z_t is a Brownian motion, and $J_t = \sum_{i=1}^{N_t} X_i$, where N_t is a Poisson process with parameter λ and the X_is are $\mathrm{N}(m, s^2)$ random variables.)

$\Delta \hat{Y}_t = \hat{Y}_t - \hat{Y}_{t-1} = Y_t - Y_{t-1} - a_t$, and r is the riskless rate.

1.1 Contracts with linear payments

If we assume that the investor chooses a payment scheme of the form

$$l_t = \alpha + \beta \hat{Y}_t + \delta \Delta \hat{Y}_t$$

then we can make some statements about the optimal strategies on the parts of the investor and the manager.

Since $\hat{Y}_t = Y_t - \sum_{s=1}^{t} a_s$, we can rewrite l_t as

$$l_t = \alpha + \beta \left(Y_t - \sum_{s=1}^{t} a_s \right) + \delta \left(Y_t - Y_{t-1} - a_t \right)$$

The cost (in lost payment) to the manager of appropriating a_t in time period t is the negative of all the terms in the above expression that contain a_t summed over the current and future time periods, which is

$$\delta a_t + \sum_{s=t}^{T} e^{\rho(s-t)} \beta a_t$$

The manager will only appropriate firm value if the cost is less than the benefit (θa_t), i.e., cancelling out a_t (which we can do when $a_t > 0$) if

$$\delta + \sum_{s=t}^{T} e^{\rho(s-t)} \beta < \theta$$

If that condition is met, then the manager will appropriate as much as he can (if there are no restrictions in place, the manager will just rob the investor blind). It's realistic to assume that there is some maximum amount that the manager can appropriate in a given period; let's call that a_{max}. We also assume that a_t cannot be negative. So we have

$$a_t = a_{max} 1 \left\{ \delta + \sum_{s=t}^{T} e^{\rho(s-t)} \beta < \theta \right\}$$

Having chosen a_t for $t = 1, \ldots, T$, the manager will (as explained above) accept a contract where $W_0 \geq W_{min}$.

The problem of interest is how the investor can choose the parameters α, β, and δ so as to maximize V_0 given that the manager follows the optimal appropriation strategy and requires $W_0 \geq W_{min}$. The code in `contract_with_jumps.py` contains functions that calculate W_0 and Y_0, visualize V_0 for various payment strategies, and estimate the values of α, β, and δ that maximize V_0.

1.2 Contracts with quadratic payments

If we assume that the investor chooses a payments scheme of the form

$$l_t = \alpha + \beta \hat{Y}_t + \gamma \hat{Y}_t^2 + \delta \Delta \hat{Y}_t$$

then the basic problem remains the same, but the manager's optimal appropriation strategy is considerably more complicated. Expanding l_t as before, we get

$$\begin{aligned} l_t &= \alpha + \beta\left(Y_t - \sum_{s=1}^{t} a_s\right) + \gamma\left(Y_t - \sum_{s=1}^{t} a_s\right)^2 + \delta\left(Y_t - Y_{t-1} - a_t\right) \\ &= \alpha + \beta\left(Y_t - \sum_{s=1}^{t} a_s\right) + \gamma\left(Y_t^2 - 2Y_t\sum_{s=1}^{t} a_s + \left[\sum_{s=1}^{t} a_s\right]^2\right) + \delta\left(Y_t - Y_t - a_t\right) \end{aligned}$$

so the cost of appropriating a_t is

$$\delta a_t + \sum_{s=t}^{T} e^{-\rho(s-t)}\left(\gamma\left[2Y_s a_t - a_t^2 - 2a_t\sum_{u=1}^{s} a_u 1\{u \neq t\}\right] + \beta a_t\right)$$

so the manager should appropriate a_t if (factoring out a_t)

$$\delta + \sum_{s=t}^{T} e^{-\rho(s-t)}\left(\gamma\left[2Y_s - a_t - 2\sum_{u=1}^{s} a_u 1\{u \neq t\}\right] + \beta\right) < \theta$$

You can see that the decision to appropriate a_t depends both upon the current and future values of Y_t and upon the *other* appropriation decisions, with the decision to appropriate at some other time making appropriation at the current time less costly.

Of course, the manager does not actually know Y_s for $s > t$; the manager has to rely on the *expected* cost of appropriation, so the actual inequality to be considered is

$$\delta + \sum_{s=t+1}^{T} e^{-\rho(s-t)}\left(\gamma\left[2E_t Y_{s-1} - a_t - 2\sum_{u=1}^{s-1} E_t a_u 1\{u \neq t\}\right] + \beta\right) < \theta$$

Notice that it will always be desirable to appropriate if the manager is allowed to appropriate enough, so we again impose the restriction that $a_t \in [0, a_{max}]$. We have to treat a_u as a random variable for $u > t$.

Also notice that the cost of appropriation will fall as t increases, and will only rise if Y_{s-1} turns out to be significantly larger than its expected value at time t. This suggests that we can reasonably make the assumption that if $a_t = a_{max}$ then $E_t a_u = a_{max}$ for every $u > t$. $E_t a_u$ is of course already known when $u \leq t$. The code in `contract_quadratic_payments.py` computes expected utilities for the manager and investor and determines the investor's optimal payment scheme (α, β, γ, and δ coefficients), assuming that the manager appropriates following this strategy. However, it is likely that this strategy is not actually optimal in cases where Y_t is very volatile (in which case Y_s may frequently exceed its expected value as calculated in a previous time period). Determining an exactly optimal strategy seems potentially computationally intractable, suggesting that we may need a different approach to more robustly model this type of contract.

2 A modified model

The model discussed in the previous section has some shortcomings. First, as seen in the discussion of a quadratic payment scheme, it is difficult to determine the manager's optimal appropriation strategy when the payment scheme used is a nonlinear function of current and past reported firm values. Second, the evolution of the firm value does not respond to appropriation: since we modeled the firm value Y_t as

$$dY_t = Y_t(\mu dt + \sigma dZ_t + dJ_t),$$

the growth rate of the firm can still be high even if the reported value $\hat{Y}_t$ is substantially less than Y_t. To be more realistic, we would expect the firm's growth prospects to be decreased when the manager appropriates value.

To address these shortcomings, we can model the firm value as

$$dY_t = \hat{Y}_t(\mu dt + \sigma dZ_t + dJ_t)$$

where again

$$\hat{Y}_t = Y_t - \sum_{s=1}^{\lfloor t \rfloor} a_s,$$

so the firm value's growth rate and volatility is associated with the reported value – the value remaining after appropriation by the manager. We also use a different form of payment function:

$$l_t = l\left(\hat{Y}_t - E_{t-1}[\hat{Y}_t | a_t = 0]\right)$$

Here l is an arbitrary function, so this payment scheme can be thought of as the investor paying the manager based on the difference between the reported firm value and the value that the investor expected, based on information in the previous period, if the manager did not appropriate anything in the current period. This allows the investor to punish the manager for reported firm values significantly below what was expected (indicating that it is likely that $a_t > 0$) and reward the manager otherwise.

At a given period t, if the l function is analytically well-behaved it is not too difficult for the manager to determine the appropriation amount a_t that will maximize the manager's total gain $l_t + \theta a_t$ for that period. Although the $\hat{Y}_t$ sequence has jumps at integer values due to the a_t, it behaves just like a regular jump-diffusion process on any interval $[t-1, t)$, with the transition between $\hat{Y}_{t-1}$ and $\hat{Y}_t$ being

$$\hat{Y}_t = \hat{Y}_{t-1} \exp\left\{\mu - \frac{1}{2}\sigma^2 + \sigma(Z_t - Z_{t-1}) + (J_t - J_{t-1})\right\} - a_t.$$

Notice that $Z_t - Z_{t-1} \sim \mathrm{N}(0,1)$ and $J_t - J_{t-1} \sim \sum_{i=1}^{N_t'} X_i'$, with N_t' being a Poisson(λ) random variable, and the $X_i' \sim \mathrm{N}(m, s^2)$. It is therefore possible to calculate $E_{t-1}[\hat{Y}_t | a_t = 0]$ as

$$E_{t-1}[\hat{Y}_t | a_t = 0] = \hat{Y}_{t-1} \exp\left\{\mu + \lambda\left(e^{m+\frac{1}{2}s^2} - 1\right)\right\}.$$

To keep the notation from getting unwieldy, we'll define the (stochastic) "transition ratio" R_t as

$$R_t = \exp\left\{\mu - \frac{1}{2}\sigma^2 + \sigma(Z_t - Z_{t-1}) + (J_t - J_{t-1})\right\}$$

so that the evolution of $\hat{Y}_t$ at integer values of t can be written as $\hat{Y}_t = \hat{Y}_{t-1}R_t - a_t$. We'll define the (constant) "expected transition ratio" R^* as

$$R^* = \exp\left\{\mu + \lambda\left(e^{m+\frac{1}{2}s^2} - 1\right)\right\}$$

so we can write $E_{t-1}[\hat{Y}_t | a_t = 0] = \hat{Y}_{t-1}R^*$. Based on this, the manager's payoff in period t is

$$l_t + \theta a_t = l\left(\hat{Y}_{t-1}[R_t - R^*] - a_t\right) + \theta a_t.$$

If we again impose the restriction that $a_t \in [0, a_{max}]$, then the manager can choose a_t to maximize the above quantity as long as it is defined for $a_t \in [0, a_{max}]$. However, this is not necessarily what the manager should do at every step. At each t, the manager wants to maximize gain from the current period *plus* the expected continuation utility:

$$l_t + \theta a_t + W_t = E_t \sum_{s=t}^{T} e^{-\rho(s-t)} \left(l_s + \theta a_s \right)$$

The decision to appropriate $a_t > 0$ will decrease $\hat{Y}_s$ for $s \geq t$, which affects not only l_t but also l_s and the optimal a_s for each $s > t$. This might look a bit intractable, but we can make use of a couple of key observations to figure out the manager's optimal strategy:

First, the manager must decide on an amount for a_t using only the information available at time t. As discussed above, the firm value at times $t-1$ and t are relevant to the manager's decision (since they are needed to compute the difference $\hat{Y}_t - E_{t-1}[\hat{Y}_t | a_t = 0]$ that the current payment is based on). The manager would also like to know how the firm value will evolve in future periods, but this information is of course not available – the only information the manager can use is the current and past firm value, the terms of the contract, and the time $T-t$ remaining in the contract. For each t and given the parameters of the contract, it should therefore be possible to express the optimal a_t as some function of $\hat{Y}_{t-1}$ and the transition ratio R_t, since those are the stochastic quantities that factor into the payoff $l_t + \theta a_t$ for period t, and the best estimates of future $\hat{Y}$ values are based on $\hat{Y}_t$, which is equal to $\hat{Y}_{t-1} R_t - a_t$. That is, for a given contract, it should be that there is some function ϕ_t such that

$$\phi_t(\hat{Y}_{t-1}, R_t) = \text{argmax}_{a_t} \left\{ l_t + \theta a_t + W_t : a_t \in [0, a_{max}] \right\}$$

assuming that future appropriation is also optimal. Since analogous functions should exist for all future periods $s = t+1, \ldots, T$ as well, the ϕ_t function has a sort of recursive nature:

$$\phi_t(\hat{Y}_{t-1}, R_t) = \text{argmax}_{a_t} \left\{ E_t \sum_{s=t}^{T} e^{-\rho(s-t)} \left(l_s + \theta a_s \right) : a_t \in [0, a_{max}] \right\}$$

where

$$l_s + \theta a_s = l\left(\hat{Y}_{s-1}[R_s - R^*] - \phi_s(\hat{Y}_{s-1}, R_s) \right) + \theta \phi_s(\hat{Y}_{s-1}, R_s)$$

for each $s = t+1, \ldots, T$. This suggests that the manager can determine the optimal strategy by working backward, starting by determining ϕ_T, then ϕ_{T-1}, and so on.

2.1 Finding the ϕ_t (optimal appropriation) functions

The code in `contract_modified.py` contains classes/functions meant to find the optimal appropriation strategies for a given contract (as discussed above) and compute the manager's and investor's starting utilities, W_0 and V_0, respectively, assuming that the manager follows the optimal appropriation strategy. The `fit_approp_rules` method of the `ModifiedContract` class attempts to model the ϕ_t optimal appropriation functions by assuming that each ϕ_t can be approximated as

$$\phi_t \approx \frac{a_{max}}{1 + e^{\psi_t}},$$

where ψ_t is a polynomial function of $\hat{Y}_{t-1}$ and R_t. To do this, I work backward from $t = T$ to $t = 1$, at each stage estimating the quantity

$$W_t = E_t \sum_{s=t}^{T} e^{-\rho(s-t)} \left(l_s + \theta a_s \right),$$

where as above

$$l_s + \theta a_s = l \left(\hat{Y}_{s-1}[R_s - R^*] - \phi_s(\hat{Y}_{s-1}, R_s) \right) + \theta \phi_s(\hat{Y}_{s-1}, R_s), \quad s = t, \ldots, T$$

by computing this sum for a large number of simulated paths and choosing the approximate ϕ_t that maximizes the average of W_t over all simulated paths. This is of course very computationally expensive, but should yield reasonable results if the number of simulated paths is high enough and the degree of polynomial ψ_t used is sufficient to capture the behavior of true optimal strategy.

The `ModifiedContract` class also has an option to compute appropriation quantities by simply maximizing $l_t + \theta a_t$ for every t. This is much more computationally efficient, although theoretically it should not generally yield optimal results for the manager. In practice, though, this shortcut method tends to meet or exceed the performance (in terms of utility for the manager) of the exact method unless a large number of paths are simulated and a high-degree polynomial is used to approximate the appropriation rule.